Directed Algebraic Topology and Concurrency

Emmanuel Haucourt

emmanuel.haucourt@polytechnique.edu

MPRI : Concurrency (2.3)

Wednesday, the 4^{th} of January 2017

Partially ordered spaces

Topology and Order, *L. Nachbin*, 1965

A partially ordered space (or pospace) is a topological space X together with a partial order $\sqsubseteq$ on (the underlying set of) X such that

$$\{(a, b) \in X \times X \mid a \sqsubseteq b\}$$

is a closed subset of $X \times X$.
A pospace morphism is an order-preserving continuous map. Pospaces and their morphisms form the category $\mathcal{PoSp}$.

The underlying space of a pospace is Hausdorff.

Examples

- The real line with standard topology and order.
- Any subset a pospace with the induced topology and order.
- The collection of compact subsets of a metric space equipped with the Hausdorff distance is a metric space.

$$d_H(K, K') \quad = \quad \sup\left\{d(x, K'), d(x', K) \mid x \in K; x' \in K'\right\}$$

$$d(x, K) \quad = \quad \inf\left\{d(x, k) \mid k \in K\right\}$$

  The induced topological space ordered by inclusion is a pospace.
- Problem: there is no pospace on the circle whose collection of directed paths is

$$\left\{\rho(t) \cdot e^{i\theta(t)} \mid \rho, \theta : [0, r] \to \mathbb{R}_+ \text{ increasing}\right\}$$

Ordered atlas

Algebraic topology and concurrency, *L. Fajstrup, É. Goubault, and M. Raußen*, 1998

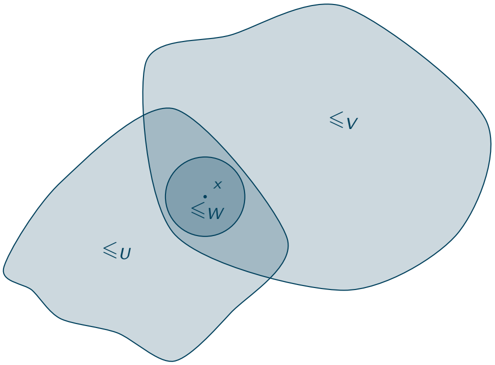
Let X be a Hausdorff space.
An (ordered) chart on X is a pospace U whose underlying space is an open subset of X.
An (ordered) atlas is a collection $\mathcal{U}$ of ordered charts on X such that:

- the underlying spaces of the charts form a basis of the topology of X, and
- for all $U, V \in \mathcal{U}$ for all $x \in U \cap V$ there exists $W \in \mathcal{U}$ such that $x \in W \subseteq U \cap V$ and denoting by $\sqsubseteq_{U|_W}$ the relation induced by $\sqsubseteq_U$ on the underlying set of W, the restrictions of $\sqsubseteq_U$ and $\sqsubseteq_V$ to W match $\sqsubseteq_W$.

$$\sqsubseteq_{U|_W} \quad = \quad \sqsubseteq_W \quad = \quad \sqsubseteq_{V|_W}$$

Any subset of X inherits an ordered atlas from $\mathcal{U}$.

Ordered atlas



Locally ordered space

Two atlases on the same space are compatible when their union is still an atlas.
The relation of compatibility is an equivalence relation.
The union of all the atlases of a given equivalence class is still an atlas
i.e. every equivalence class contains a greatest element for inclusion.
A local pospace is a Hausdorff space togethter with an equivalence class of ordered atlas.

The locally ordered line

Examples of equivalent atlases on $\mathbb{R}$

- $\{(I,\leqslant) \mid I$ open interval of $\mathbb{R}\}$,
- $\{(U,\leqslant) \mid U$ open subset of $\mathbb{R}\}$,
- $\{(U,\sqsubseteq_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq_U y$ stands for $x \leqslant y$ and $[x,y] \subseteq U$,
- $\{(U,\sqsubseteq'_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq'_U y$ is any extension of $\sqsubseteq_U$.

The locally ordered circle

Examples of equivalent atlases on $\mathbb{S}^1$

- $\{(A, \leqslant) \mid A \text{ open arc}\}$ where $\leqslant$ is the order induced by $\mathbb{R}$ and the restriction of the exponential map to an open subinterval of $\{t \in \mathbb{R} \mid e^{it} \in A\}$ of length at most 2π,
- $\{(U, \sqsubseteq_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $x \sqsubseteq_U y$ means that the anticlockwise compact arc from x to y is included in U,
- $\{(U, \sqsubseteq'_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $\sqsubseteq'_U$ is any extension of the partial order $\sqsubseteq_U$.

Local pospace morphisms

An atlas morphism from $\mathcal{U}$ to $\mathcal{V}$ is a map f (between the underlying sets of $\mathcal{U}$ and $\mathcal{V}$) such that for all $x \in \mathrm{dom}(f)$ there exists an ordered chart $U \in \mathcal{U}$ and an ordered chart $V \in \mathcal{V}$ such that $x \in U$ and f induces a pospace morphism from U to V (implicitly $f(U) \subseteq V$).

e.g. $t \in [0,1] \cup [2,3] \quad \mapsto \quad t + 2 \pmod 4 \in [0,1] \cup [2,3]$

Let f be an atlas morphism from $\mathcal{U}$ to $\mathcal{V}$.
If $\mathcal{U} \sim \mathcal{U}'$ and $\mathcal{V} \sim \mathcal{V}'$ then f is also an atlas morphism from $\mathcal{U}'$ to $\mathcal{V}'$.

A local pospace morphism from $(X, [\mathcal{U}]_\sim)$ to $(Y, [\mathcal{V}]_\sim)$ is a map from X to Y inducing an atlas morphism from $\mathcal{U}$ to $\mathcal{V}$.

A local pospace morphism defined over a locally ordered compact interval is called a directed path.

Pospaces as local pospaces

Each pospace $(X, \sqsubseteq)$ can be seen as a local pospace

$$\Big(X, \big\{(U, \sqsubseteq_{|U}) \mid U \text{ open subset of } X\big\}\Big)$$

The resulting functor is:

- faithful
- not injective on object (hence not an embedding)
- not full

Directed loops on local pospaces

A directed path δ on a local pospace X is constant iff its extremities are equal and there exists an ordered chart of some atlas of X that contains the image of δ.

A vortex is a point every neighbourhood of which contains a non-constant directed loop.

A local pospace has no vortex.

Ordered atlas on metric graphs

Let $\mathcal{B}$ be the collection of open balls B of $|G|$ such that

- B is centred at a vertex and its radius is $\leqslant \frac{1}{3}$, or
- $B = \{a\} \times U$ for some arrow a and some open interval $U \subseteq \,]0,1[$ of length $\leqslant \frac{1}{3}$.

Let $B, B' \in \mathcal{B}$ be such that $B \cap B' \neq \emptyset$.

- If B is of the first kind, then so is $B \cap B'$.
- If B, B' are centred at v and v', we have then $v = v'$ and $B \subseteq B'$ or $B' \subseteq B$

Ordered open stars

An element B of $\mathcal{B}$ centred at v of radius $r \leqslant \frac{1}{3}$ is the disjoint union of $\{v\}$ together with

- $\{a\}\times]0,r[$ for each arrow a such that $\partial^{-} a = v$
- $\{a\}\times]1-r,1[$ for each arrow a such that $\partial^{+} a = v$

The partial order on B is characterized by the following constraints:

- each branch $\{a\}\times]1-r,1[$ and $\{a\}\times]0,r[$ inherits its order from $\mathbb{R}$
- $\{v\} \sqsubseteq \{a\}\times]0,r[$ for each arrow a such that $\partial^{-} a = v$
- $\{a\}\times]1-r,1[\sqsubseteq \{v\}$ for each arrow a such that $\partial^{+} a = v$

We have $B \cap B' \neq \emptyset \ \Rightarrow \ B \cap B' \in \mathcal{B}$ and

$$\sqsubseteq_{B_{|_{B\cap B'}}} \quad = \quad \sqsubseteq_{B\cap B'} \quad = \quad \sqsubseteq_{B'_{|_{B\cap B'}}}$$

The metric graph of $|G|$ thus becomes a local pospace.

The locally ordered metric graph construction is functorial.

Cartesian product

in $\mathcal{S}et$

$$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$$

There exist two mappings π_A and π_B

$$\pi_A : A \times B \longrightarrow A \qquad (a, b) \longmapsto a$$

$$\pi_B : A \times B \longrightarrow B \qquad (a, b) \longmapsto b$$

such that for all sets X the following map is a bijection

$$\mathcal{S}et[X, A \times B] \longrightarrow \mathcal{S}et[X, A] \times \mathcal{S}et[X, B]$$

$$h \longmapsto (\pi_A \circ h, \pi_B \circ h)$$

Cartesian product

in a category $\mathcal{C}$

The object c is the Cartesian product (in $\mathcal{C}$) of a and b when there exist two morphisms $\pi_a : c \to a$ and $\pi_b : c \to b$ such that for all objects x of $\mathcal{C}$ the following map is a bijection

$$\begin{array}{ccc} \mathcal{C}[x, c] & \longrightarrow & \mathcal{C}[x, a] \times \mathcal{C}[x, b] \\ h & \longmapsto & (\ \pi_a \circ h\ ,\ \pi_b \circ h\ ) \end{array}$$

When such an object c exists we write $c = a \times b$

Cartesian product in the category of graphs ($\mathcal{Grph}$)

$$\left(\begin{array}{c} A \\ \mathsf{t} \Big\downarrow \Big\downarrow \mathsf{s} \\ V \end{array} \right) \times \left(\begin{array}{c} A' \\ \mathsf{t}' \Big\downarrow \Big\downarrow \mathsf{s}' \\ V' \end{array} \right) \cong \left(\begin{array}{c} A \times A' \\ \mathsf{t}\times\mathsf{t}' \Big\downarrow \Big\downarrow \mathsf{s}\times\mathsf{s}' \\ V \times V' \end{array} \right)$$

The Cartesian product in $\mathcal{Grph}$ is deduced form the Cartesian product in $\mathcal{Set}$

Examples of Cartesian products

- The product of (X, Ω_X) and (Y, Ω_Y) in $\mathcal{Top}$ is given by $X \times Y$ together with unions of subsets of the form $U \times V$ with $U \in \Omega_X$ and $V \in \Omega_Y$. It is the least topology making the projections continuous.
- The product of $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ in $\mathcal{Pos}$ is given by $X \times Y$ and the partial order $\sqsubseteq$ defined by $(x, y) \sqsubseteq (x', y')$ when $x \sqsubseteq_X x'$ and $y \sqsubseteq_Y y'$. It is the greatest partial order such that the projection are poset morphisms.
- The product of $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ in $\mathcal{PoSp}$ is given by $X \times Y$ and the product order $\sqsubseteq_X \times \sqsubseteq_Y$.
- The product of $(X, [\mathcal{U}]_\sim)$ and $(Y, [\mathcal{V}]_\sim)$ in $\mathcal{Lpo}$ is $X \times Y$ together with the collection of ordered charts $U \times V$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{emb}$ does not exist.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{ctr}$ is given by $X \times Y$ together with $d\big((x, y), (x', y')\big) = \max\{d_X(x, x'), d_Y(y, y')\}$.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{top}$ can also be given by $X \times Y$ together with the Euclidean product

$$d\big((x, y), (x', y')\big) = \sqrt{d_X^2(x, x') + d_Y^2(y, y')}$$

- Categories of models of algebraic theories.

Infinite Cartesian product

The product of a family $(A_i)_{i\in\mathcal{I}}$ of objects of a category $\mathcal{C}$, when it exists, is an object

$$\prod_i A_i$$

together with projections

$$\pi_{A_j} : \prod_i A_i \longrightarrow A_j$$

such that the next mapping is a bijection.

$$\begin{array}{ccc} \mathcal{C}(X, \prod_i A_i) & \longrightarrow & \prod_i \mathcal{C}(X, A_i) \\ h & \longmapsto & (\pi_{A_i} \circ h) \end{array}$$

Infinite products of directed circle does not exist in $\mathcal{L}po$.

Canonical partition

$$G : A \underset{\partial^-}{\overset{\partial^+}{\rightrightarrows}} V \qquad \rceil G \lfloor \;=\; V \;\sqcup\; A \times]0,1[$$

$$\rceil G_1 \lfloor \times \cdots \times \rceil G_n \lfloor \;=\; (\; V_1 \sqcup A_1 \times]0,1[\;) \;\times\; \cdots \;\times\; (\; V_n \sqcup A_n \times]0,1[\;)$$

$$\rceil G_1 \lfloor \times \cdots \times \rceil G_n \lfloor \;=\; \bigsqcup_{\substack{\text{points } p \text{ of} \\ G_1,\ldots,G_n}} \{p\} \times]0,1[^{\dim(p_1,\ldots,p_n)}$$

where $p = (p_1, \ldots, p_n)$, $p_i \in V_i \sqcup A_i$, and $\dim p = \#\{ i \in \{1, \ldots, n\} \mid p_i \in A_i \}$

$B_p = \{p\} \times]0,1[^{\dim(p_1,\ldots,p_n)}$ is called a canonical block

The collection of canonical blocks forms the canonical partition of $\rceil G_1 \lfloor \times \cdots \times \rceil G_n \lfloor$.

Justifying de definition of discrete directed paths

Let B_p and $B_{p'}$ be canonical blocks.

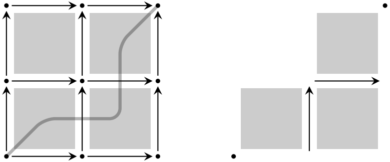
If there exists a directed path starting in B_p, ending in $B_{p'}$, and whose image is contained in $B_p \cup B_{p'}$ then one of the following facts is satisfied:

- for all $i \in \{1, \ldots, n\}$, $p_i = p'_i$ or p_i is the source of the arrow p'_i, or
- for all $i \in \{1, \ldots, n\}$, $p_i = p'_i$ or p'_i is the target of the arrow p_i.

Discretization and lifting

- Given a directed path γ on the local pospace $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$ we have a finite partition $I_0 < \cdots < I_N$ of $\mathrm{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point p^k such that $\gamma(I_k) \subseteq B_{p^k}$.
- The sequence $p^0, \ldots, p^N$ is a directed path on $(G_1, \ldots, G_n)$, it is called the discretization of γ and denoted by $D(\gamma)$.
- Given a directed path δ on $(G_1, \ldots, G_n)$ there exists a directed path γ on $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$ whose discretization is δ, such a directed path γ is said to be a lifting of δ.

Example of discretization



Admissible directed paths and execution traces

on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$

The sequence of multi-instructions of a directed path γ on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$ is that of its discretization of $D(\gamma)$.

A directed path on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$ is admissible (resp. an execution trace) iff so is its discretization.

The action of a directed path γ on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$ on the right of a state σ is that of its discretization of $D(\gamma)$.

Potential function on $\upharpoonright G_1 \downharpoonleft \times \cdots \times \upharpoonright G_n \downharpoonleft$

If the program under consideration is conservative, then we have the potential function

$$F : \upharpoonright G_1 \downharpoonleft \times \cdots \times \upharpoonright G_n \downharpoonleft \times \mathcal{S} \to \{\text{multisets over } \{1, \ldots, n\}\}$$

The function F is constant on each canonical block.

Geometric model

The forbidden region is

$$\coprod_{\substack{\text{forbidden points } p \\ \text{of } (G_1, \ldots, G_n)}} B_p$$

The directed continuous model is

$$|G_1| \times \cdots \times |G_n| \setminus \{\text{forbidden region}\}$$

Geometric models are sound and complete

- Any directed path on a continuous model is admissible.
- Conversely, for each admissible path on a continuous model which meets a forbidden point, there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.

Trade off

More mathematics for more properties?

- Both discrete and geometric models are sound and complete.
- The continuous models satisfy extra properties that are "naturally" expressed in terms of metrics.

The geometric model of a conservative program

The continuous model X of a conservative program whose running processes are $G_1, \ldots, G_n$ is a sub-local pospace of $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$.

The continuous model X inherits a distance d_X from the distances $d_{\upharpoonleft G_i \downharpoonleft}$ of the metric graphs $\upharpoonleft G_i \downharpoonleft$

$$d_X(p, p') \quad = \quad \max\left\{ d_{\upharpoonleft G_i \downharpoonleft}(p_i, p'_i) \mid i \in \{1, \ldots, n\} \right\}$$

The distance d_X is in accordance with the fact that the execution time of the simultaneous execution of many processes is the longest execution time among that of the processes considered individually.

Uniform distance between directed paths

Given a compact Hausdorff space K and a metric space (X, d_X), the set of continuous maps from K to X can be equipped with the uniform distance

$$d(f, g) \quad = \quad \max\{d_X(f(k), g(k)) \mid k \in K\} \ .$$

We consider the case where $K = [0, r]$ is the domain of definition of a directed path and (X, d_X) is the geometric model of a conservative program.

The main theorem

Let B_p and $B_{p'}$ be canonical blocks of the geometric model X of a conservative program.

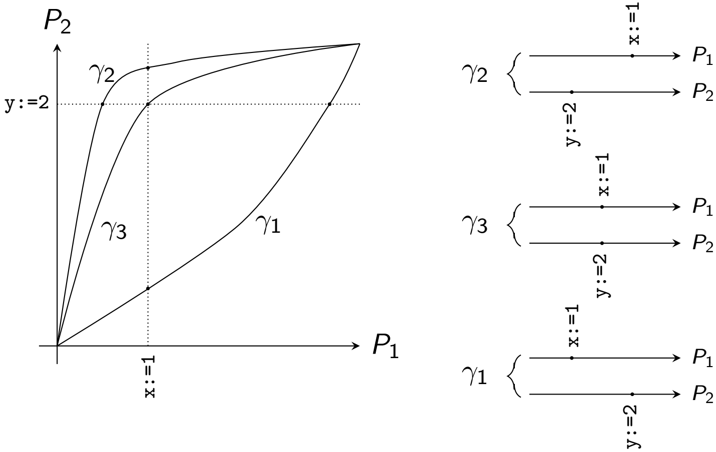
Let $dX^{[0,r]}(B_p, B_{p'})$ be the set of directed paths on X whose sources and targets lie in B_p and $B_{p'}$ respectively.

Let γ be an element of $dX^{[0,r]}(B_p, B_{p'})$.

There exists an open ball Ω of $dX^{[0,r]}(B_p, B_{p'})$, centred in γ, such that all the elements of Ω induce the same action on valuations. Moreover, if γ is an execution trace, then so are all the elements of Ω.

Desynchronization

one of the artifact used in the proof



Standard homotopy of paths

Let γ and δ be two paths on X defined over the segment $[0, r]$

A homotopy from γ to δ is a continuous map h from $[0, r] \times [0, q]$ to X such that

- The mappings $h(0, -) : s \in [0, q] \mapsto h(0, s)$ and $h(r, -) : s \in [0, q] \mapsto h(r, s)$ are constant
- The mappings $h(-, 0) : t \in [0, r] \mapsto h(t, 0)$ and $h(-, q) : s \in [0, r] \mapsto h(t, q)$ are γ and δ

As a consequence we have $\gamma(0) = \delta(0)$ and $\gamma(r) = \delta(r)$.

Uniform distance and Curryfication

Suppose that X is a metric space.

For all compact Hausdorff space K, the homset $\mathcal{Top}(K, X)$ with the (topology induced by the) uniform distance is denoted by X^K

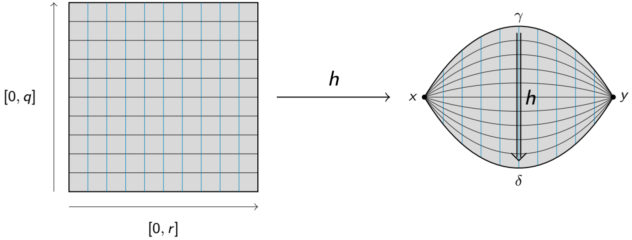
The Curryfication $(\hat{\_})$ induces a homeomorphism from $X^{[0,r]\times[0,q]}$ to $\left(X^{[0,r]}\right)^{[0,q]}$

$$(h : [0, r] \times [0, q] \to X) \to (\hat{h} : [0, q] \to X^{[0,r]})$$

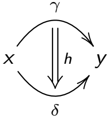
The two faces of homotopies

h is a continuous map from $[0,r] \times [0,q]$ to X i.e. $h \in \mathcal{T}op\big[[0,r] \times [0,q], X\big]$

but is also a path from γ to δ in the space $X^{[0,r]}$ i.e. $h \in \mathcal{T}op\big[[0,q], X^{[0,r]}\big]$



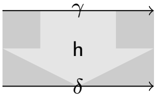
The second point of view leads us to introduce the following notation



Directed homotopy on a locally ordered space

Let $\gamma, \delta \in \mathcal{L}po([0,r], X)$ such that $\partial^- \gamma = \partial^- \delta$ and $\partial^+ \gamma = \partial^+ \delta$.

- A directed homotopy from γ to δ is a homotopy of paths $h : [0,r] \times [0,q] \to X$ that induces a local pospace morphism.



- A weakly directed homotopy from γ to δ is a homotopy of paths $h : [0,r] \times [0,q] \to X$ whose intermediate paths $h(\_, s)$, for $s \in [0,q]$, are directed.
- Any directed homotopy is a weakly directed homotopy. The converse is false.

Theorem

Two weakly dihomotopic paths on the geometric model of a conservative program induce the same action on valuations. Moreover, if one of them is an execution trace, then so is the other.

Proof

By a standard result from general topology and the Curryfication of h, namely

$$\hat{h} : s \in [0,q] \mapsto (t \in [0,r] \mapsto h(t,s) \in X)$$

is a continuous path on $dX^{[0,r]}(p,p')$.

The image of $\hat{h}$ is thus compact, so we cover it with open balls given by the main theorem of geometric models.

By the Lebesgue number theorem there exists a real number $\varepsilon > 0$ such that $|s - s'| \leqslant \varepsilon$ implies that $\hat{h}(s)$ and $\hat{h}(s')$ belong to the same open ball from the covering.

The conclusion follows considering the sequence

$$\hat{h}(0), \hat{h}(\varepsilon), \hat{h}(2\varepsilon), \hat{h}(3\varepsilon), \cdots, \hat{h}(n\varepsilon), \hat{h}(q)$$

where n is the greatest natural number such that $n\varepsilon \leqslant q$.

Concatenation of homotopies

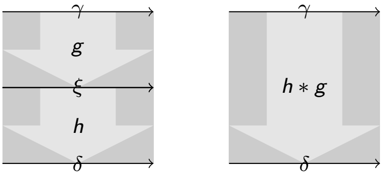
vertical composition

Let $g : [0, r] \times [0, q'] \to X$ and $h : [0, r] \times [0, q] \to X$ be homotopies from γ to ξ and from ξ to δ.

The mapping $h * g : [0, r] \times [0, q + q'] \to X$ defined by

$$h * g(t, s) \quad = \quad \begin{cases} g(t, s) & \text{if } 0 \leqslant s \leqslant q \\ h(t, s - q) & \text{if } q \leqslant s \leqslant q + q' \end{cases}$$

is a homotopy from γ to δ.



If g and h are (weakly) directed homotopies, then so is their concatenation $h * g$.

Homotopy and dihomotopy relations

An anti-directed homotopy from γ to δ is a homotopy of paths $h : [0, r] \times [0, q] \to X$ such that $(t, s) \mapsto h(t, q - s)$ is a directed homotopy from δ to γ.

An elementary homotopy between γ to δ is a homotopy of paths $h : [0, r] \times [0, q] \to X$ obtained as a finite concatenation of directed homotopies and anti-directed homotopies.

Two paths γ and γ' are said to be homotopic when there exists a homotopy between them. We have the equivalence relation $\sim_h$ between paths on a topological space.

They are said to be dihomotopic when there exists an elementary homotopy between them. We have the equivalence relation $\sim_d$ between directed paths on a locally ordered space.

They are said to be weakly dihomotopic when there exists a weakly directed homotopy between them. We have the equivalence relation $\sim_w$ between directed paths on a locally ordered space.

An important remark

If h is a homotopy from γ to γ' on the topological space X and $f : X \to Y$ is a continuous map, then $f \circ h$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the topological space Y.

If h is a (weakly) directed homotopy from γ to γ' on the local pospace space X and $f : X \to Y$ is a local pospace morphism, then $f \circ h$ is a (weakly) directed homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the local pospace space Y.

If $\gamma, \gamma' : [0, r] \to X$ are ((weakly) di)homotopic, then so are $f \circ \gamma, f \circ \gamma' : [0, r] \to Y$.

Reparametrization

An increasing and surjective map $\theta : [0, r] \to [0, r]$ is called a reparametrization.
The mapping

$$h : (t, s) \in [0, r] \times [0, 1] \mapsto \theta(t) + s \cdot (\max(t, \theta(t)) - \theta(t)) \in [0, r]$$

is a directed homotopy from θ to $\max(\mathrm{id}_{[0,r]}, \theta)$.

If $\gamma : [0, r] \to X$ is a directed path on the local pospace X, then $\gamma \circ h$ is a directed homotopy from $\gamma \circ \theta$ to $\gamma \circ \max(\mathrm{id}_{[0,r]}, \theta)$

Therefore γ and $\gamma \circ \theta$ are dihomotopic.

Images of directed paths on a pospace

Theorem

The image of a nonconstant directed path on a pospace is isomorphic to $[0, 1]$.

Corollary

Two directed paths on a posapce having the same image are dihomotopic.

proof:
Suppose that $\mathrm{im}(\gamma) = \mathrm{im}(\gamma')$.
$\phi : [0, r] \rightarrow \mathrm{im}(\gamma)$ a pospace isomorphism.
$\phi^{-1} \circ \gamma$ and $\phi^{-1} \circ \gamma'$ are reparametrization.
We have h an elementary homotopy from $\phi^{-1} \circ \gamma$ to $\phi^{-1} \circ \gamma'$.
Hence $\phi \circ h$ is an elementary homotopy from γ and γ'.

Programs with mutex only

a result by É. Goubault and S. Mimram

Let X be the geometric model of a conservative program whose semaphores have arity 1 (mutex), then two directed paths on X are dihomotopic if and only if they are homotopic.