Directed Algebraic Topology and Concurrency

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MPRI : Concurrency (2.3)

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Category $\mathcal{C}$

Definition (the "underlying graph" part)

- $\mathrm{Ob}(\mathcal{C})$: collection of objects
- $\mathrm{Mo}(\mathcal{C})$: collection of morphisms
- s, t : mappings source, target as follows

$$\mathrm{Mo}(\mathcal{C}) \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C})$$

- We define the homset $\mathcal{C}(x, y) := \left\{\gamma \in \mathrm{Mo}(\mathcal{C}) \;\middle|\; \partial^{-}\gamma = x \text{ and } \partial^{+}\gamma = y\right\}$

Category $\mathcal{C}$

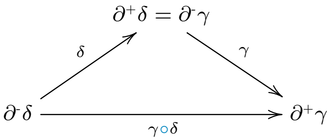
Definition (the "underlying local monoid" part)

- id : provides each object with an identity

$$\mathsf{Mo}(\mathcal{C}) \underset{t}{\overset{s}{\rightrightarrows}} \mathsf{Ob}(\mathcal{C}), \qquad \mathsf{Ob}(\mathcal{C}) \xrightarrow{\ \mathrm{id}\ } \mathsf{Mo}(\mathcal{C})$$

- The binary composition is a partially defined and often denoted by $\circ$

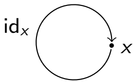
$$\Big\{(\gamma,\delta) \;\Big|\; \gamma,\delta \text{ morphisms of } \mathcal{C} \text{ s.t. } \partial^{-}\gamma = \partial^{+}\delta\Big\} \xrightarrow{\text{composition}} \mathsf{Mo}(\mathcal{C})$$



Category $\mathcal{C}$

Definition (the axioms)

- The composition law is associative
- For all objects x one has $\partial^{-}\mathrm{id}_x = x = \partial^{+}\mathrm{id}_x$



- For all morphisms γ one has $\mathrm{id}_{\partial^{+}\gamma} \circ \gamma = \gamma = \gamma \circ \mathrm{id}_{\partial^{-}\gamma}$

Classical examples

- $\mathcal{Set}$: the category of sets.
- $\mathcal{Mon}$: the category of monoids
- $\mathcal{Cmon}$: the category of commutative monoids
- $\mathcal{Gr}$: the category of groups
- $\mathcal{Pre}$: the category of preordered sets.
- $\mathcal{Pos}$: the category of posets.
- Any preordered set $(X, \sqsubseteq)$ can be seen as a category in which any homset has at most one element.
- Any monoid $(X, \sqsubseteq)$ can be seen as a category with a single object.
- The opposite of a category is obtained by reversing all its arrows (i.e. by swapping the roles of the source and the target)

Some special kinds of morphisms

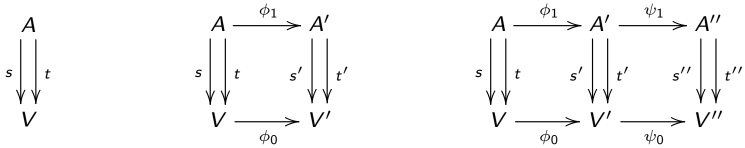
- f is an isomorphism when there exists g such that both $f \circ g$ and $g \circ f$ are identities.
- Two objects related by an isomorphism are said to be isomorphic.
- A groupoid is a category that only has isomorphisms.
- f is a monomorphism when it is left-cancellative i.e. for all g_1, g_2, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.
- f is a epimorphism when it is right-cancellative i.e. for all g_1, g_2, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.
- any isomorphism is both monomorphism and an epimorphism, the converse is false in general (e.g. $\mathcal{Pos}$).
- if $r \circ s = \mathrm{id}$ then r is called a retract/split epimorphism and s is called a section/split monomorphism.

The category of graphs

Grph

The elements of V are the vertices and those of A are the arrows
In particular A and V are sets

Objects | Morphisms | Composition



with $s'(\phi_1(\alpha)) = \phi_0(\partial^- \alpha)$ and $t'(\phi_1(\alpha)) = \phi_0(\partial^+ \alpha)$

Topological spaces

A topological space is a set X and a collection $\Omega_X \subseteq \mathcal{P}(X)$ s.t.

1) $\emptyset \in \Omega_X$ and $X \in \Omega_X$
2) Ω_X is stable under union
3) Ω_X is stable under finite intersection

A continuous map $f : (X, \Omega_X) \to (Y, \Omega_Y)$ is a map $f : X \to Y$ s.t.

$$\forall U \in \Omega_Y \; f^{-1}(U) \in \Omega_X$$

The elements of Ω_X are called the open subsets of X.
The complement of an open subsets is said to be closed.
$\mathcal{B} \subseteq \Omega_X$ is a basis of topology when each open subset is a union of elements of $\mathcal{B}$.

Topological spaces and continuous maps form the category $\mathcal{Top}$

Related definitions

The interior of a subset A of X is the greatest open subset of X contained in A.

The closure of a subset A of X is the least closed subset of X containing A.

A neighbourhood of a subset A of X is a subset of X whose interior contains A.

A topological space X is said to be Hausdorff when for all $x, x' \in X$, if $x \neq x'$ then x and x' have disjoint neighbourhoods.

A subset Q of X is said to be saturated when

$$Q \quad = \quad \bigcap \{ U \mid U \text{ open and } Q \subseteq U \}$$

Every subset of a Hausdorff space is saturated.

Compactness and local compactness

Let X be a topological space.

- An open covering of X is a collection of open subsets of X whose union is X.
- X is said to be compact when every open covering of X admit a finite sub-covering.
- X is said to be locally compact when for every $x \in X$ and every open neighbourhood U of x contains a compact saturated neighbourhood of x.

A Hausdorff space is locally compact iff each of its points admits a compact neighbourhood.

Functors f from $\mathcal{C}$ to $\mathcal{D}$

Definition (preserving the "underlying graph")

A functor $f : \mathcal{C} \to \mathcal{D}$ is defined by two "mappings" Ob(f) and Mo(f) such that

$$\begin{array}{ccc} \mathrm{Mo}(\mathcal{C}) & \overset{\mathrm{s}}{\underset{\mathrm{t}}{\rightrightarrows}} & \mathrm{Ob}(\mathcal{C}) \\ {\scriptstyle \mathrm{Mo}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ob}(f)} \\ \mathrm{Mo}(\mathcal{D}) & \overset{\mathrm{s'}}{\underset{\mathrm{t'}}{\rightrightarrows}} & \mathrm{Ob}(\mathcal{D}) \end{array}$$

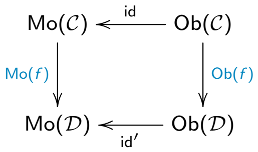
with $\mathrm{s}'(\mathrm{Mo}(f)(\alpha)) = \mathrm{Ob}(f)(\partial^{-}\alpha)$ and $\mathrm{t}'(\mathrm{Mo}(f)(\alpha)) = \mathrm{Ob}(f)(\partial^{+}\alpha)$

Hence it is in particular a morphism of graphs.

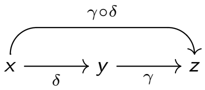
Functors f from $\mathcal{C}$ to $\mathcal{D}$

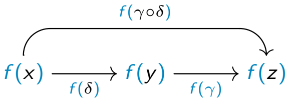
Definition (preserving the "underlying local monoid")

The "mappings" Ob(f) and Mo(f) also make the following diagram commute

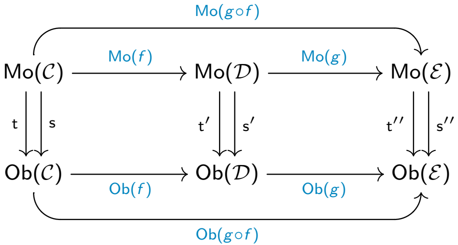


and satisfies $\mathrm{Mo}(f)(\gamma \circ \delta) = \mathrm{Mo}(f)(\gamma) \circ \mathrm{Mo}(f)(\delta)$





Functors compose as morphisms of graphs do



Hence the functors should be thought of as the morphisms of categories

The small categories and their funtors form a (large) category denoted by $\mathcal{C}at$

The overall idea of algebraic topology

Every functor preserves the isomorphisms

Problem: prove the topological spaces X and Y are *not* the same
Strategy: find a functor F defined over $\mathcal{Top}$ such that $F(X) \ncong F(Y)$

The connected component functor

1) A topological space X is said to be connected when its only closed-open subsets are $\emptyset$ and X
2) A $\subseteq$-monotonic union of connected subspaces is connected
3) A topologcial space X is the disjoint sum of its connected components
4) Any connected subset of X is contained in a connected component of X
5) Any continuous direct image of a connected subset of X is connected

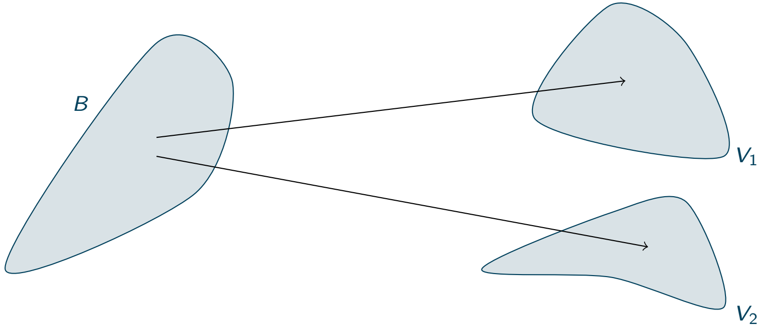
$$\mathcal{T}op \xrightarrow{\ \pi_0\ } \mathcal{S}et$$

$$\begin{array}{ccc} X & & \pi_0(X) \\ \downarrow f & \longmapsto & \downarrow \pi_0(f) \\ Y & & \pi_0(Y) \end{array}$$

An application involving basic (algebraic) topology

The continuous image of a connected space is connected

The image of the space B is entirely contained in a connected component of the space V.



The set of connected components

is a functorial construction

This situation is abstracted by classifying continuous maps from B to V according to which connected component (V_1 or V_2) the single connected components of B (namely B itself) is sent to. There are exactly two set theoretic maps from the singleton $\{B\}$ to the pair $\{V_1, V_2\}$ hence there is at most (in fact exactly) two kinds of continuous maps from B to V.

$$\{B\} \rightrightarrows \{V_1, V_2\}$$

In particular B and V are not homeomorphic.

Application

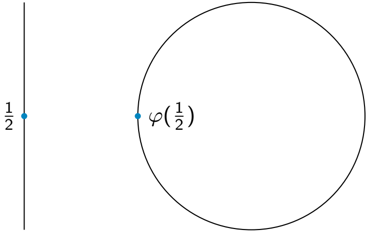
The compact interval and the circle are not homeomorphic

Let $\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ be the Euclidean circle and suppose $\varphi : [0,1] \to \mathbb{S}^1$ is a homeomorphism.

Then φ induces a homeomorphism

$$[0, \tfrac{1}{2}[\,\cup\,]\tfrac{1}{2}, 1] \;\to\; \mathbb{S}^1 \backslash \{\varphi(\tfrac{1}{2})\}$$

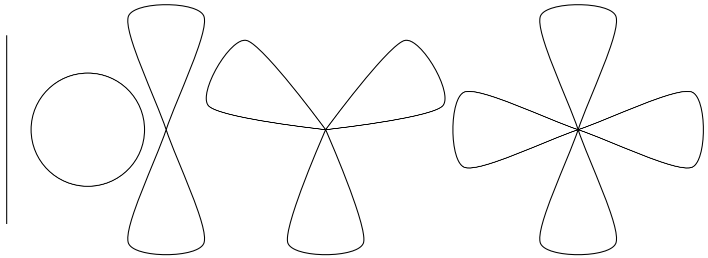
which does not exist!



Generalization

Bouquets of circles

These topological spaces are pairwise not homeomorphic. Why ?



Functors terminology

Given a functor $f : \mathcal{C} \to \mathcal{D}$ and two objects x and y we have the mapping

$$\begin{aligned} f_{x,y} : \ \mathcal{C}[x,y] &\longrightarrow \mathcal{D}[\mathrm{Ob}(f)(x), \mathrm{Ob}(f)(y)] \\ \alpha &\longmapsto \mathrm{Mo}(f)(\alpha) \end{aligned}$$

- f is faithful when for all objects x and y the mapping $f_{x,y}$ is one-to-one (injective)
- f is full when for all objects x and y the mapping $f_{x,y}$ is onto (surjective)
- f is fully faithful when it is full and faithful
- f is an embedding when it is faithful and $\mathrm{Ob}(f)$ is one-to-one

Some small functors

(functor between small categories)

The morphisms of monoids are the functors between small categories with a single object

The morphisms of preordered sets are the functors between small categories whose homsets contain at most one element

The actions of a monoid M over a set X are the functors from M to $\mathcal{Set}$ which sends the only element of M to X

Some full embeddings in $\mathcal{Cat}$

Remark : The full embeddings compose

$\mathcal{Pre} \hookrightarrow \mathcal{Cat}$

$\mathcal{Mon} \hookrightarrow \mathcal{Cat}$

$\mathcal{Pos} \hookrightarrow \mathcal{Pre}$

$\mathcal{Gr} \hookrightarrow \mathcal{Mon}$

$\mathcal{Cmon} \hookrightarrow \mathcal{Mon}$

$\mathcal{Ab} \hookrightarrow \mathcal{Cmon}$

$\mathcal{Ab} \hookrightarrow \mathcal{Gr}$

$\mathcal{Set} \hookrightarrow \mathcal{Pos}$

Some forgetful functors

$(M, *, e) \in \mathcal{Mon} \mapsto M \in \mathcal{Set}$
$(X, \Omega) \in \mathcal{Top} \mapsto X \in \mathcal{Set}$
$(X, \sqsubseteq) \in \mathcal{Pos} \mapsto X \in \mathcal{Set}$

$\mathcal{C} \in \mathcal{Cat} \mapsto \mathrm{Ob}(\mathcal{C}) \in \mathcal{Set}$
$\mathcal{C} \in \mathcal{Cat} \mapsto \mathrm{Mo}(\mathcal{C}) \in \mathcal{Set}$

$$\mathcal{C} \in \mathcal{Cat} \mapsto \left(\mathrm{Mo}(\mathcal{C}) \underset{s}{\overset{t}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C}) \right) \in \mathcal{Grph}$$

Metric spaces

A metric space is a set X together with a mapping $d : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ such that:

- $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leqslant d(x, y) + d(y, z)$

Goal: turn any graph into metric space in a natural way.

Metric space morphisms

- $\mathcal{M}et_{emb}$ $f : X \to Y$ s.t. $\forall x, x' \in X,\ d_Y(f(x), f(x')) \ = \ d_X(x, x')$
- $\mathcal{M}et_{ctr}$ $f : X \to Y$ s.t. $\forall x, x' \in X,\ d_Y(f(x), f(x')) \ \leqslant \ d_X(x, x')$
- $\mathcal{M}et$ $f : X \to Y$ s.t. $\exists r \in \,]0, \infty[\ \forall x, x' \in X,\ d_Y(f(x), f(x')) \ \leqslant \ r \cdot d_X(x, x')$
- $\mathcal{M}et_{top}$ $f : X \to Y$ s.t. $\forall x \in X\ \forall \varepsilon > 0\ \exists \eta > 0,\ f\big(B(x, \eta)\big) \ \subseteq \ B\big(f(x), \varepsilon\big)$

$$\mathcal{M}et_{emb} \hookrightarrow \mathcal{M}et_{ctr} \hookrightarrow \mathcal{M}et \hookrightarrow \mathcal{M}et_{top} \overset{full}{\hookrightarrow} \mathcal{T}op$$

Length spaces

The length $\ell(\gamma)$ of a path $\gamma : [0, r] \to (X, d)$ is the least upper bound of the collection of sums

$$\sum_{i=0}^{n} d\big(\gamma(t_{i+1}), \gamma(t_i)\big)$$

where $n \in \mathbb{N}$ and $0 = t_0 \leqslant \cdots \leqslant t_n = r$.
The metric space (X, d) is a length space when the distance between two points $x, x' \in X$ is the following greatest lower bound

$$\inf \big\{\ell(\gamma) \mid \gamma \text{ is a path from } x \text{ to } x'\big\}$$

A path γ from x to x' such that $\ell(\gamma) = d(x, x')$ is said to be geodesic.
The space is said to be geodesic when any two points are related by a geodesic path.

The Hopf-Rinow theorem

Metric Spaces of Non-Positive Curvature, *M. R. Bridson, and A. Haefliger*, 1999

A metric space is said to be complete when all its Cauchy sequences admit a limit.

Let X be a length space.
If X is complete and locally compact, then

- every closed bounded subset of X is compact, and
- X is a geodesic space.

Isometric embedding in $\mathbb{R}^n$

- $\mathbb{R}^n$ is a geodesic space
- $\mathbb{R}^n \setminus \{0\}$ with the distance inherited from $\mathbb{R}^n$ is a length space, not a geodesic one.
- $\mathbb{R}^n \setminus [0,1]^n$ with the distance inherited from $\mathbb{R}^n$ is not a length space.
- Any metric space (X, d) is associated with a length space (X, d_ℓ) with

$$d_\ell(x, x') \quad = \quad \inf \left\{ \ell(\gamma) \mid \gamma \text{ is a path from } x \text{ to } x' \right\}$$

- Every finite graph with weighted arrows (in $\mathbb{R}_+ \setminus \{0\}$) with can be embedded in $\mathbb{R}^3$.

Neighbours

$$G : A \underset{\partial^+}{\overset{\partial^-}{\rightrightarrows}} V$$

- The underlying set of the metric graph is $A\times]0,1[\ \sqcup V$
- Two points p, p' are said to be neighbours when there is an arrow a such that $p, p' \in \{a\}\times]0,1[\ \sqcup \{\partial^- a, \partial^+ a\}$

Distance between two neighbours

- If $\partial^- a \neq \partial^+ a$ there is a canonical bijection

$$\phi : \{a\} \times]0,1[\ \sqcup \{\partial^- a, \partial^+ a\} \to [0,1]$$

  In that case $d(p,p') = |t - t'|$ with $t = \phi(p)$ and $t' = \phi(p')$.

- If $\partial^- a = \partial^+ a$ there is a canonical bijection

$$\phi : \{a\} \times]0,1[\ \sqcup \{\partial^- a, \partial^+ a\} \to [0,1[$$

  In that case

$$d(p,p') = \min \left\{ |t - t'|, 1 - t + t', 1 - t' + t \right\}$$

  with $t = \phi(p)$ and $t' = \phi(p')$.

Itinerary

An itinerary on $A\times]0,1[\ \sqcup\ V$ is a (finite) sequence $p_0, \ldots, p_q$ of points such that p_k and p_{k+1} are neighbours for $k \in \{0, \ldots, q-1\}$.

The length of that itinerary is

$$\ell(p_0, \ldots, p_q) \quad = \quad \sum_{k=0}^{q-1} d(p_k, p_{k+1})$$

The distance between two points p and p' of $A\times]0,1[\ \sqcup\ V$ is

$$d(p, p') \quad = \quad \inf \left\{ \ell(p_0, \ldots, p_q) \mid p_0, \ldots, p_q \text{ is a itinerary from } p \text{ to } p' \right\}$$

The metric graph associated with G is the metric space

$$\Big(A\times]0,1[\ \sqcup\ V \ ,\ d \Big)$$

The metric graph construction is functorial.

Open balls

The open ball of radius $r < 1$ centered at the vertex v is the set

$$\{v\} \quad \cup \quad \{a \mid \partial^- a = v\}\times]0, r[\quad \cup \quad \{a \mid \partial^+ a = v\}\times]1 - r, 1[$$

For $(a, t) \in \{a\}\times]0, 1[$ the open ball of radius $r \leqslant \min\{t, 1 - t\}$ centered at the vertex (a, t) is the set

$$\{a\}\times]t - r, t + r[$$

That collection of open balls forms a basis of open sets.

Partially ordered spaces

Topology and Order, *L. Nachbin*, 1965

A partially ordered space (or pospace) is a topological space X together with a partial order $\sqsubseteq$ on (the underlying set of) X such that

$$\{(a, b) \in X \times X \mid a \sqsubseteq b\}$$

is a closed subset of $X \times X$.
A pospace morphism is an order-preserving continuous map. Pospaces and their morphisms form the category $\mathscr{PoSp}$.

The underlying space of a pospace is Hausdorff.

Examples

- The real line with standard topology and order.
- Any subset a pospace with the induced topology and order.
- The collection of compact subsets of a metric space equipped with the Hausdorff distance is a metric space.

$$d_H(K, K') = \sup \left\{ d(x, K'), d(x', K) \mid x \in K; x' \in K' \right\}$$

$$d(x, K) = \inf \left\{ d(x, k) \mid k \in K \right\}$$

  The induced topological space ordered by inclusion is a pospace.
- Problem: there is no pospace on the circle whose collection of directed paths is

$$\left\{ \rho(t) \cdot e^{i\theta(t)} \mid \rho, \theta : [0, r] \to \mathbb{R}_+ \text{ increasing} \right\}$$

Ordered atlas

Algebraic topology and concurrency, *L. Fajstrup, É. Goubault, and M. Raußen*, 1998

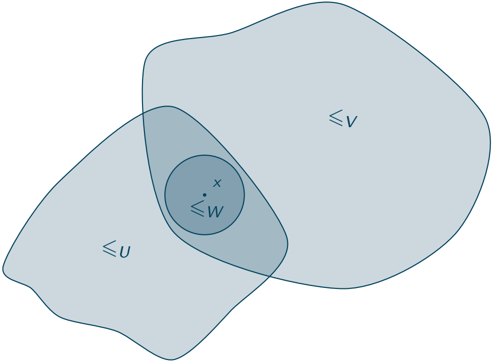
Let X be a Hausdorff space.
An (ordered) chart on X is a pospace U whose underlying space is an open subset of X.
An (ordered) atlas is a collection $\mathcal{U}$ of ordered charts on X such that:

- the underlying spaces of the charts form a basis of the topology of X, and
- for all $U, V \in \mathcal{U}$ for all $x \in U \cap V$ there exists $W \in \mathcal{U}$ such that $x \in W \subseteq U \cap V$ and denoting by $\sqsubseteq_{U|_W}$ the relation induced by $\sqsubseteq_U$ on the underlying set of W, the restrictions of $\sqsubseteq_U$ and $\sqsubseteq_V$ to W match $\sqsubseteq_W$.

$$\sqsubseteq_{U|_W} \quad = \quad \sqsubseteq_W \quad = \quad \sqsubseteq_{V|_W}$$

Any subset of X inherits an ordered atlas from $\mathcal{U}$.

Ordered atlas



Locally ordered space

Two atlases on the same space are compatible when their union is still an atlas.
The relation of compatibility is an equivalence relation.
The union of all the atlases of a given equivalence class is still an atlas
i.e. every equivalence class contains a greatest element for inclusion.
A local pospace is a Hausdorff space togethter with an equivalence class of ordered atlas.

The locally ordered line

Examples of equivalent atlases on $\mathbb{R}$

- $\{(I, \leqslant) \mid I$ open interval of $\mathbb{R}\}$,
- $\{(U, \leqslant) \mid U$ open subset of $\mathbb{R}\}$,
- $\{(U, \sqsubseteq_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq_U y$ stands for $x \leqslant y$ and $[x, y] \subseteq U$,
- $\{(U, \sqsubseteq'_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq'_U y$ is any extension of $\sqsubseteq_U$.

The locally ordered circle

Examples of equivalent atlases on $\mathbb{S}^1$

- $\{(A, \leqslant) \mid A \text{ open arc}\}$ where $\leqslant$ is the order induced by $\mathbb{R}$ and the restriction of the exponential map to an open subinterval of $\{t \in \mathbb{R} \mid e^{it} \in A\}$ of length at most 2π,
- $\{(U, \sqsubseteq_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $x \sqsubseteq_U y$ means that the anticlockwise compact arc from x to y is included in U,
- $\{(U, \sqsubseteq'_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $\sqsubseteq'_U$ is any extension of the partial order $\sqsubseteq_U$.

Local pospace morphisms

An atlas morphism from $\mathcal{U}$ to $\mathcal{V}$ is a map f (between the underlying sets of $\mathcal{U}$ and $\mathcal{V}$) such that for all $x \in \mathrm{dom}(f)$ there exists an ordered chart $U \in \mathcal{U}$ and an ordered chart $V \in \mathcal{V}$ such that $x \in U$ and f induces a pospace morphism from U to V (implicitly $f(U) \subseteq V$).

e.g. $t \in [0,1] \cup [2,3] \quad \mapsto \quad t + 2 \pmod 4 \in [0,1] \cup [2,3]$

Let f be an atlas morphism from $\mathcal{U}$ to $\mathcal{V}$.
If $\mathcal{U} \sim \mathcal{U}'$ and $\mathcal{V} \sim \mathcal{V}'$ then f is also an atlas morphism from $\mathcal{U}'$ to $\mathcal{V}'$.

A local pospace morphism from $(X, [\mathcal{U}]_\sim)$ to $(Y, [\mathcal{V}]_\sim)$ is a map from X to Y inducing an atlas morphism from $\mathcal{U}$ to $\mathcal{V}$.

A local pospace morphism defined over a locally ordered compact interval is called a directed path.

Pospaces as local pospaces

Each pospace $(X, \sqsubseteq)$ can be seen as a local pospace

$$\Big(X, \{(U, \sqsubseteq_{|_U}) \mid U \text{ open subset of } X\}\Big)$$

The resulting functor is:

- faithful
- not injective on object (hence not an embedding)
- not full

Directed loops on local pospaces

A directed path δ on a local pospace X is constant iff its extremities are equal and there exists an ordered chart of some atlas of X that contains the image of δ.

A vortex is a point every neighbourhood of which contains a non-constant directed loop.

A local pospace has no vortex.

Ordered atlas on metric graphs

Let $\mathcal{B}$ be the collection of open balls B of $|G|$ such that

- B is centred at a vertex and its radius is $\leqslant \frac{1}{3}$, or
- $B = \{a\} \times U$ for some arrow a and some open interval $U \subseteq]0,1[$ of length $\leqslant \frac{1}{3}$.

Given $B, B' \in \mathcal{B}$ if B is of the second kind, then so is $B \cap B'$.

If B, B' are centred at v and v' we have

- $v \neq v' \;\Rightarrow\; B \cap B' = \emptyset$ and
- $v = v' \;\Rightarrow\; B \subseteq B'$ or $B' \subseteq B$

Ordered open stars

An element B of $\mathcal{B}$ centred at v of radius $r \leqslant \frac{1}{3}$ is the disjoint union of $\{v\}$ together with

- $\{a\}\times]0, r[$ for each arrow a such that $\partial^- a = v$
- $\{a\}\times]1 - r, 1[$ for each arrow a such that $\partial^+ a = v$

The partial order on B is characterized by the following constraints:

- each branch $\{a\}\times]1 - r, 1[$ and $\{a\}\times]0, r[$ inherits its order from $\mathbb{R}$
- $\{v\} \sqsubseteq \{a\}\times]0, r[$ for each arrow a such that $\partial^- a = v$
- $\{a\}\times]1 - r, 1[\sqsubseteq \{v\}$ for each arrow a such that $\partial^+ a = v$

We have $B \cap B' \neq \emptyset \;\Rightarrow\; B \cap B' \in \mathcal{B}$ and

$$\sqsubseteq_{B|_{B\cap B'}} \quad = \quad \sqsubseteq_{B\cap B'} \quad = \quad \sqsubseteq_{B'|_{B\cap B'}}$$

The metric graph of $|G|$ thus becomes a local pospace.

The locally ordered metric graph construction is functorial.