Directed Algebraic Topology and Concurrency

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- To do so, we are in search for a class that behaves much like the one of isomorphisms
- From now on $\mathcal{C}$ denotes a one-way category

Potential weak isomorphisms

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- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $\mathcal{C}(x, y)$ contains a potential weak isomorphism, then it is a singleton
  Requires the assumption that $\mathcal{C}$ is one-way

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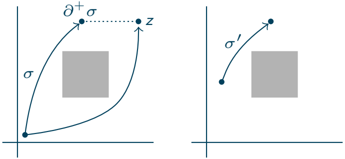

Due to the lower dipath, the σ, z-precomposition is not bijective; yet σ' is a potential weak isomorphism.

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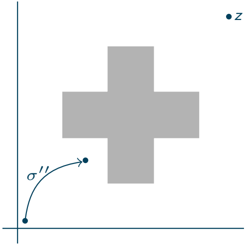

Note that σ'' is a potential weak isomorphism though there exists a morphism from $\partial^{-}\sigma''$ to z but none from $\partial^{+}\sigma''$ to z.

Stability under pushout and pullback

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  - For $n \in \mathbb{N}$ define Σ_{n+1} as the collection of morphisms $\sigma \in \Sigma_n$ s.t. the pushout and the pullback of σ along with all morphisms exist (when sources or targets match) and belong to Σ_n

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- The collection Σ_∞ is stable under the action of $\mathrm{Aut}(\mathcal{C})$

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- Hence we suppose the systems of weak isomorphisms are closed under composition

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- Let $\Sigma_i \subseteq \mathcal{C}_i$ be a family of collections of morphisms, then

$$\boxed{\prod_i \Sigma_i \text{ is a swi of } \prod_i \mathcal{C}_i \text{ iff each } \Sigma_i \text{ is a swi of } \mathcal{C}_i}$$

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- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.

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- Given a one-way category $\mathcal{C}$ we have:

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  - The greatest swi is invariant under the action of $\mathrm{Aut}(\mathcal{C})$

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- A conjecture:

> For all loop-free isothetic region X, $\overrightarrow{\pi_1}X$ satisfies the square filling property

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- Σ-connectedness is an equivalence relation on the objects of $\mathcal{C}$
- The equivalence classes are called a Σ-components

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- [1.] The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.

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$$\begin{tikzcd} & u & \\ x \arrow[ur] & & y \arrow[ul] \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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Diagram 3

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- [4.] $\mathcal{C} = \mathcal{K}$ iff $\mathcal{C}$ is a prelattice, and Σ is the greatest system of weak isomorphisms of $\mathcal{C}$ i.e. all the morphisms in this case.

$$\begin{tikzcd} & u & \\ x \arrow[ur] & & y \arrow[ul] \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} x & & y \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & u & \\ x \arrow[ur] & & y \arrow[ul] \end{tikzcd}$$

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\begin{tikzcd}
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- If $\gamma \sim \delta$ then $\partial^- \gamma \sim \partial^- \delta$ and $\partial^+ \gamma \sim \partial^+ \delta$

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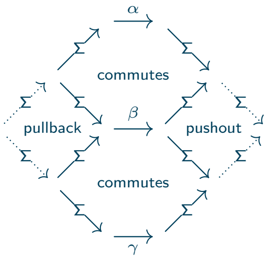

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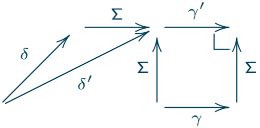

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$$\begin{tikzcd} & \mathcal{C}/\Sigma \arrow[d, "G"] \\ \mathcal{C} \arrow[ur, "Q"] \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

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  - In the diagram below we have $Q(\gamma' \circ \gamma'') = Q(\gamma') \circ Q(\gamma'') = Q(\gamma') \circ Q(\gamma)$ hence the composite $(\gamma' \circ \gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pushout nor on the representatives (γ, σ) and (γ', σ').

$$\begin{tikzcd}
 & & {} \arrow[dl, "\sigma''"'] \arrow[dr, "\gamma''"] & & \\
 & {} \arrow[dl, "\sigma"'] \arrow[dr, "\gamma"] & \text{pushout} & {} \arrow[dl, "\sigma'"'] \arrow[dr, "\gamma'"] & \\
{} & & {} & & {}
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Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$

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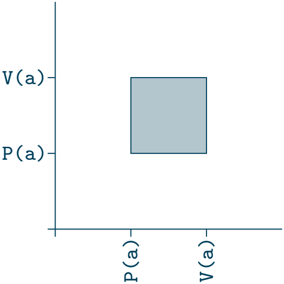

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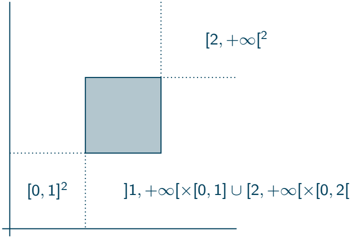

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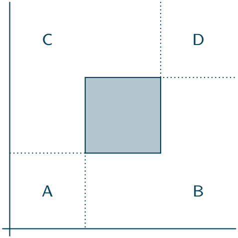

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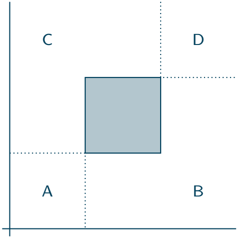

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D				σ

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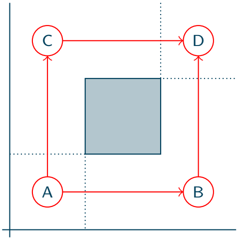

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Two rectangles

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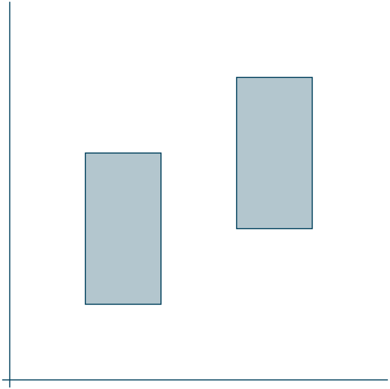

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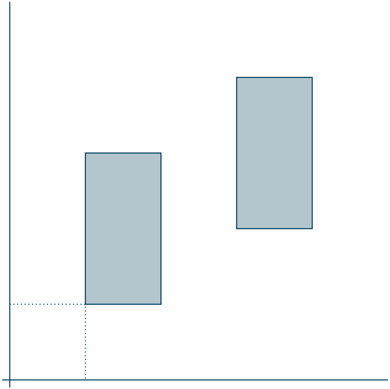

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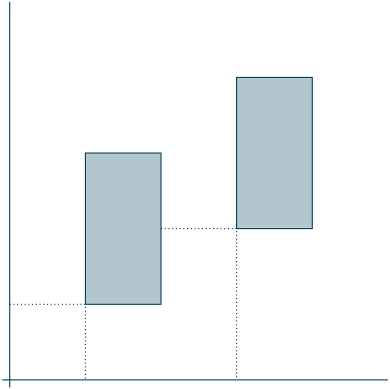

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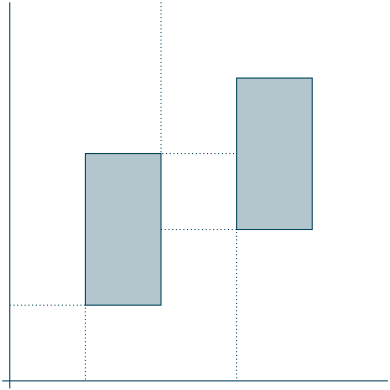

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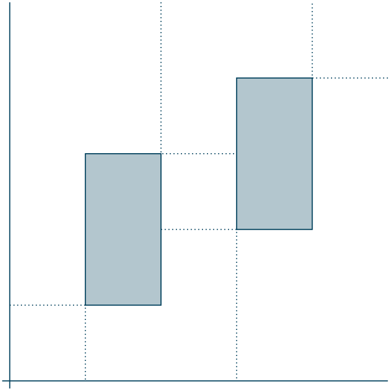

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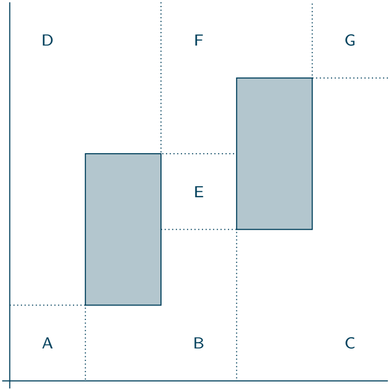

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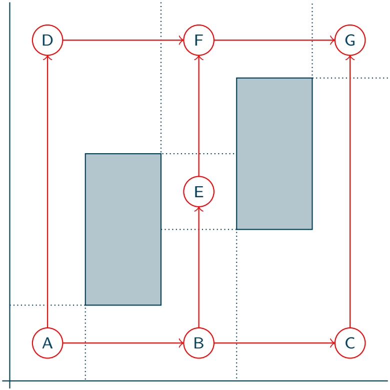

Swiss Flag

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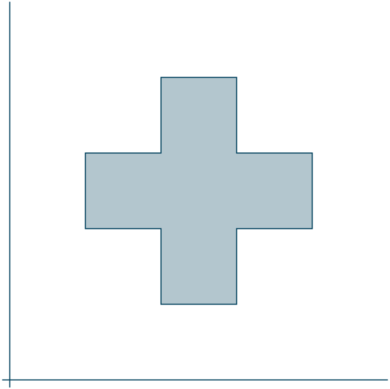

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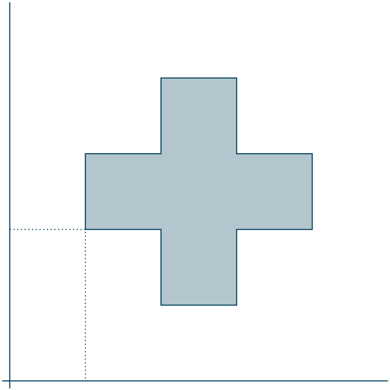

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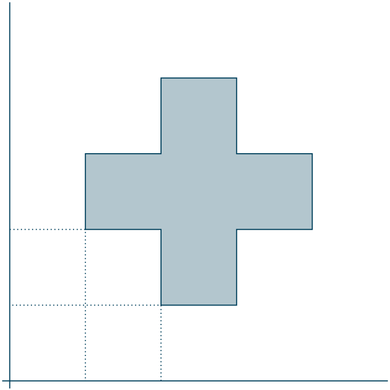

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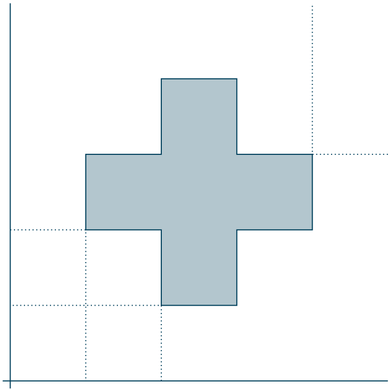

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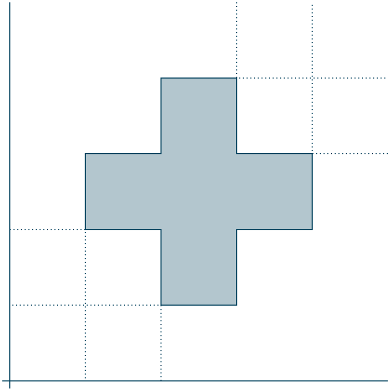

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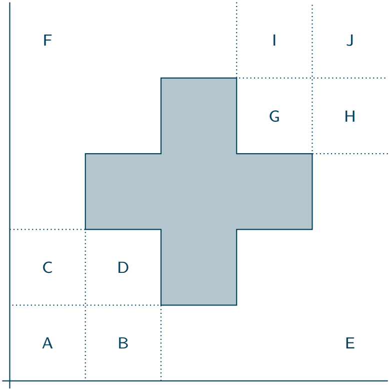

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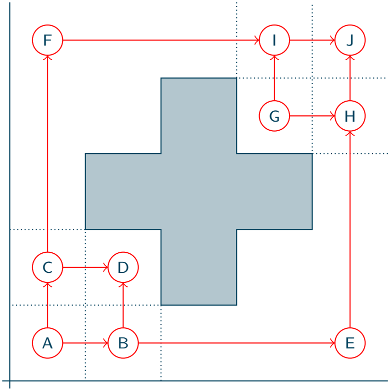

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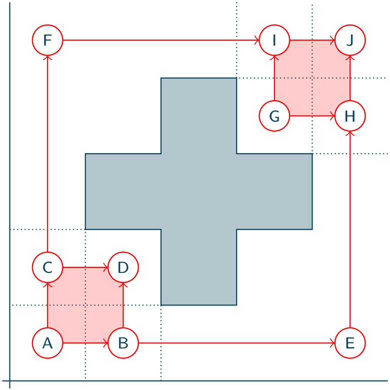

Achronal overlaping square

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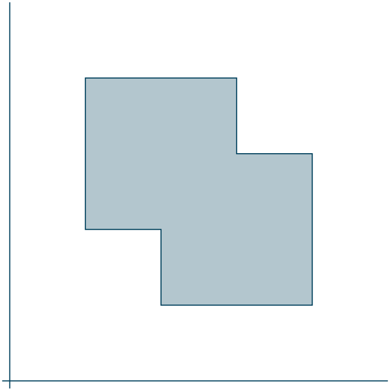

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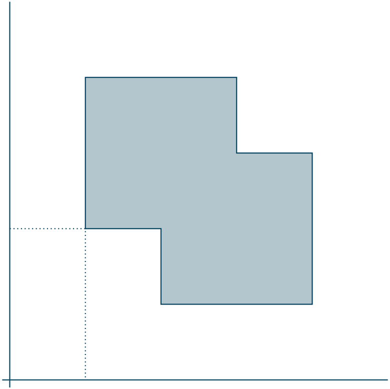

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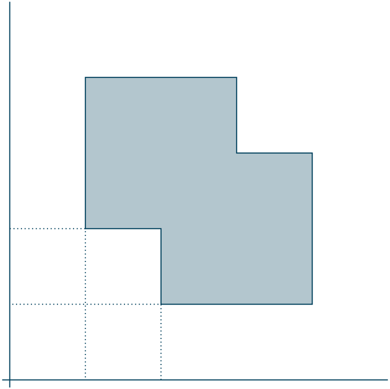

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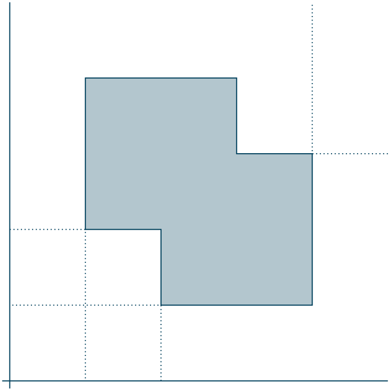

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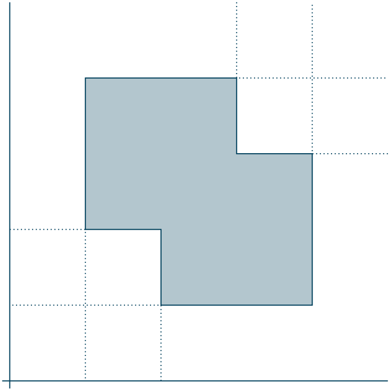

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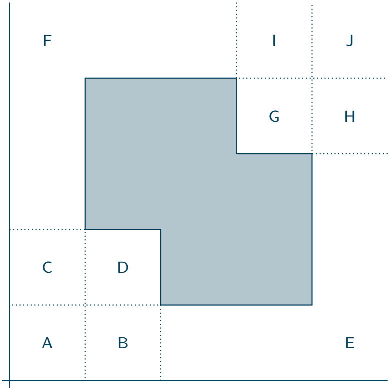

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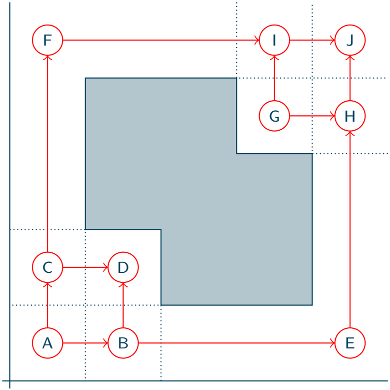

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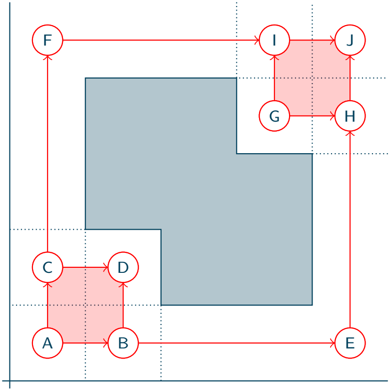

Diagonal overlaping squares

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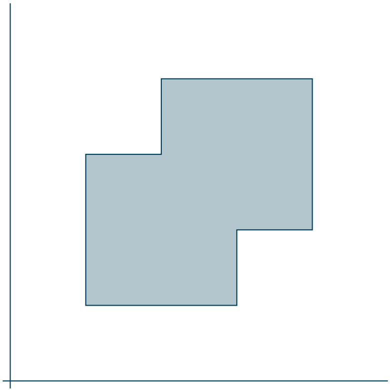

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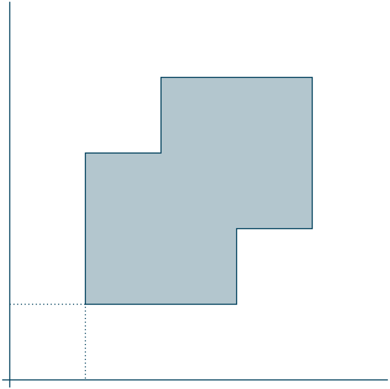

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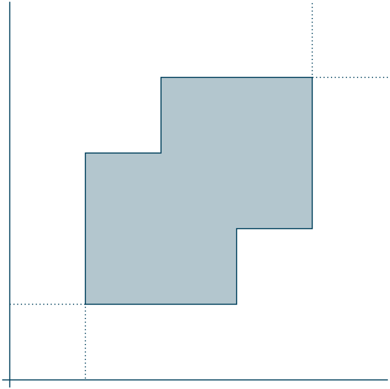

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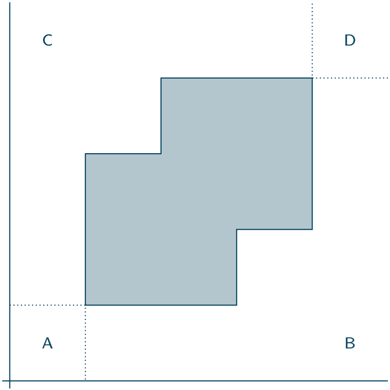

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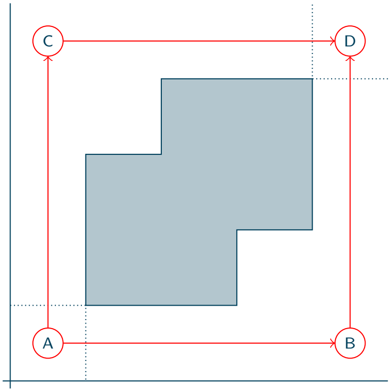

The floating cube

Non potential weak isomorphisms

The floating cube

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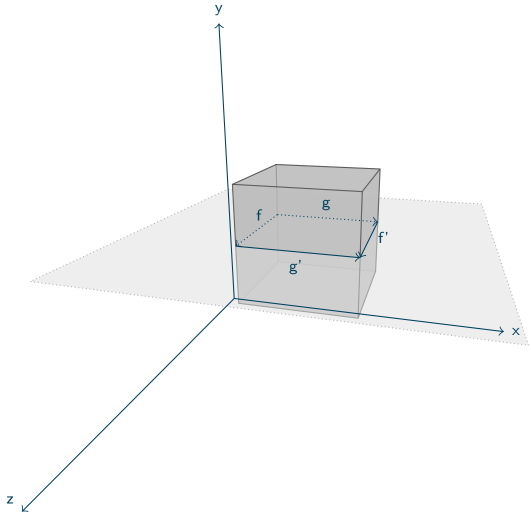

The floating cube

A "vee" that does not extend to a pushout

The floating cube

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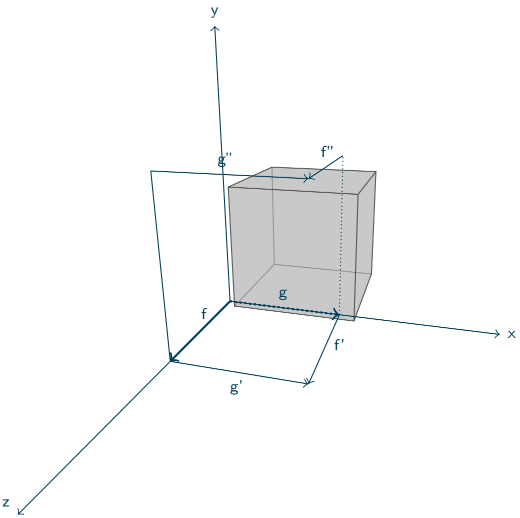
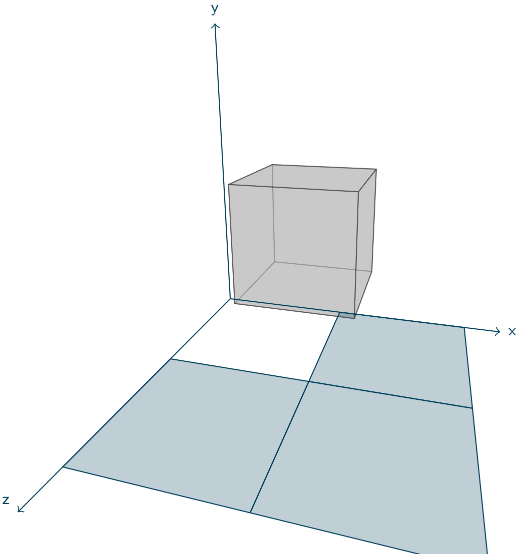


The floating cube

Some pushouts squares

The floating cube

Some pushouts squares



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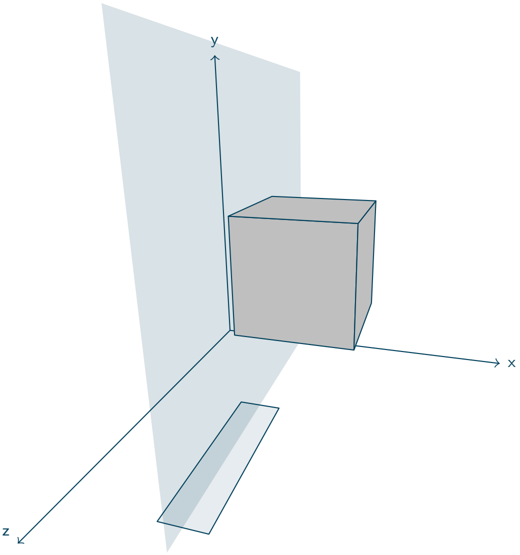

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  - Therefore $f', g' \notin \Sigma$ (anyway they do not preserve the future cones)

The floating cube

boundaries of the components

The floating cube

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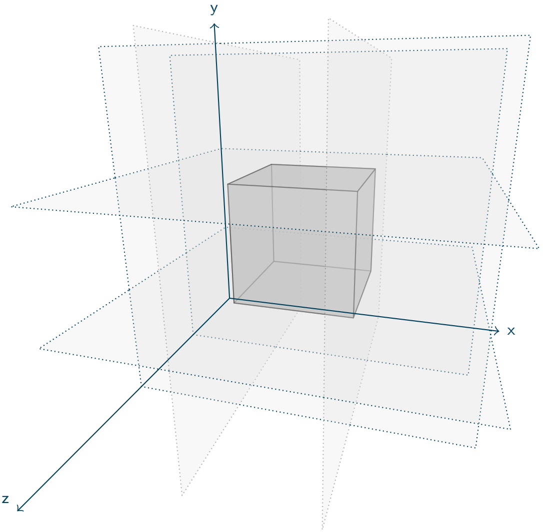

Homotopy of maps

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- If $\eta' : X \times [0, r'] $ is an A-homotopy from g to h then the concatenation

$$\begin{array}{rcl} \eta' \cdot \eta : X \times [0, r + r'] & \to & Y \\ (x, t) & \mapsto & \begin{cases} \eta(x, t) & \text{if } t \leqslant r \\ \eta'(x, t - r) & \text{if } r \leqslant t \end{cases} \end{array}$$

is an A-homotopy from f to g

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- If $\eta' : X \times [0, r']$ is an A-homotopy from g to h then the concatenation

$$\begin{array}{rcl} \eta' \cdot \eta : X \times [0, r + r'] & \to & Y \\ (x, t) & \mapsto & \begin{cases} \eta(x, t) & \text{if } t \leqslant r \\ \eta'(x, t - r) & \text{if } r \leqslant t \end{cases} \end{array}$$

  is an A-homotopy from f to g
- Writting $f \sim g$ when there is an A-homotopy from f to g, we define an equivalence relation over the mappings from X to Y.

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- for $n \geqslant 2$ the n^{th} homotopy group construction extends to a functor $\pi_n : \mathcal{Top} \to \mathcal{Ab}$.
  i.e. for $n \geq 2$, the n^{th} homotopy group of a space is commutative

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  Finite Computability of Postnikov Complexes. E. H. Brown, Jr. Ann. of Math. 65(1). 1957

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- The CW-complexes arises in this way.

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  It is a homeomorphic image.

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- The following product in $\mathcal{Top}$ is not a CW-complex

$$|\mathbb{R} \rightrightarrows \{0\}| \times |\mathbb{N} \rightrightarrows \{0\}|$$

Homotopy equivalences

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- If there exists $f' : Y \to X$ such that $f' \circ f \sim \mathrm{id}_X$ and $f \circ f' \sim \mathrm{id}_Y$, then f (and f') are said to be homotopy equivalences. The spaces X and Y are said to be homotopic. A space (resp. map) that is homotopic with $\{0\}$ (resp. a constant map) is said to be null homotopic

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- For all $n \geqslant 1$, $\mathbb{R}^{n+1} \setminus \{0\}$ is homotopically equivalent to $\mathbb{S}^n$

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> If X and Y are CW-complexes and $f : X \to Y$ induces isomorphisms of n^{th} homotopy groups for all n, then f is a homotopy equivalence

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$$\begin{tikzcd}
 & \mathcal{C}[\mathcal{W}^{-1}] \arrow[d, "G"] \\
\mathcal{C} \arrow[ur, "I"] \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

- The homotopy category is defined as the localization of $\mathcal{Top}$ (or $\mathcal{CGH}$ etc) with respect to the class of homotopy equivalences

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