Directed Algebraic Topology and Concurrency

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- Define a class Σ of morphisms of $\mathcal{C}$ so we can keep one representative in each class of Σ-related objects without loss of information
- To do so, we are in search for a class that behaves much like the one of isomorphisms
- From now on $\mathcal{C}$ denotes a one-way category

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- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $\mathcal{C}(x, y)$ contains a potential weak isomorphism, then it is a singleton
  Requires the assumption that $\mathcal{C}$ is one-way

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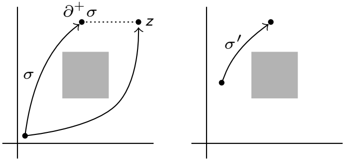

Due to the lower dipath, the σ, z-precomposition is not bijective; yet σ' is a potential weak isomorphism.

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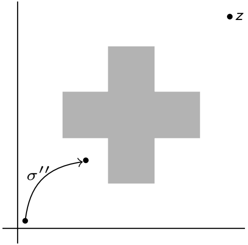

Note that σ'' is a potential weak isomorphism though there exists a morphism from $\partial^- \sigma''$ to z but none from $\partial^+ \sigma''$ to z.

Stability under pushout and pullback

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- The collection Σ_∞ is stable under the action of $\mathrm{Aut}(\mathcal{C})$

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- Hence we suppose the systems of weak isomorphisms are closed under composition

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- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.

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- Given a one-way category $\mathcal{C}$ we have:

$$\boxed{\text{All the systems of weak isomorphisms of } \mathcal{C} \text{ are pure}}$$

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  - $\boxed{\text{The greatest swi is invariant under the action of } \mathrm{Aut}(\mathcal{C})}$

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- A conjecture:

> For all loop-free isothetic region X, $\overrightarrow{\pi_1}X$ satisfies the square filling property

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- The equivalence classes are called a Σ-components

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- [1.] The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.

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A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Let K be a Σ-component of $\mathcal{C}$ and $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:

- [1.] The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.
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$$\begin{tikzcd} & u & \\ x \arrow[ur] & & y \arrow[ul] \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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$$\begin{tikzcd} x & & y \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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- [4.] $\mathcal{C} = \mathcal{K}$ iff $\mathcal{C}$ is a prelattice, and Σ is the greatest system of weak isomorphisms of $\mathcal{C}$ i.e. all the morphisms in this case.

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- If $\gamma \sim \delta$ then $\partial^- \gamma \sim \partial^- \delta$ and $\partial^+ \gamma \sim \partial^+ \delta$

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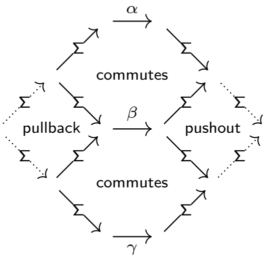

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$$
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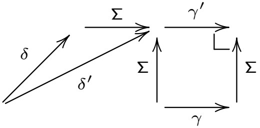

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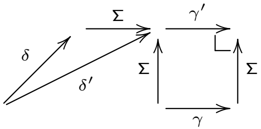

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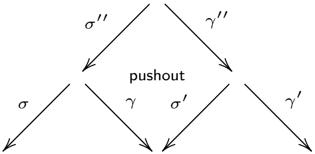
  - In the diagram below we have $Q(\gamma' \circ \gamma'') = Q(\gamma') \circ Q(\gamma'') = Q(\gamma') \circ Q(\gamma)$ hence the composite $(\gamma' \circ \gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pushout nor on the representatives (γ, σ) and (γ', σ').



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$$\boxed{\text{The functor } P \text{ is an equivalence of categories}}$$

- The skeleton of $\mathcal{C}[\Sigma^{-1}]$ is $\mathcal{C}/\Sigma$ and $\mathcal{C}[\Sigma^{-1}]$ is one-way.

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Plane without a square

$x = \mathbb{R}_+^2 \setminus]0, 1[^2$

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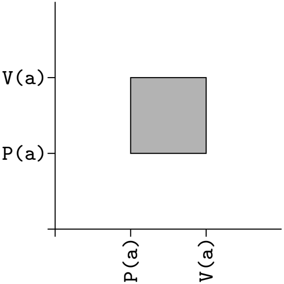

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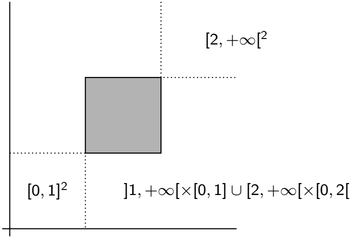

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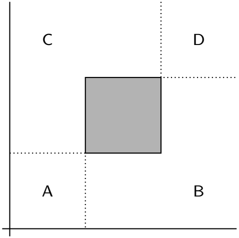

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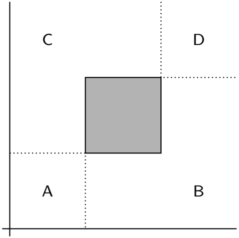

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$\to$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

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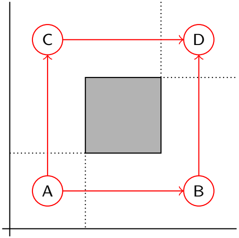

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C			σ	γ'
D				σ

Two rectangles

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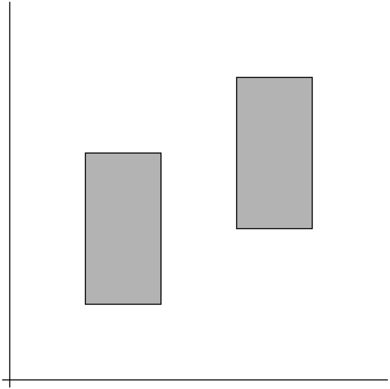

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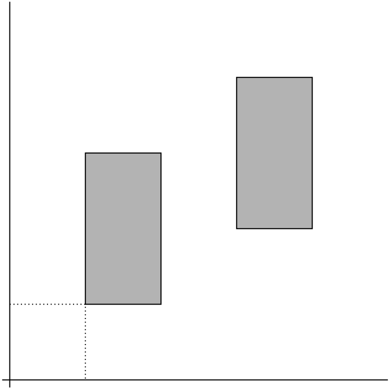

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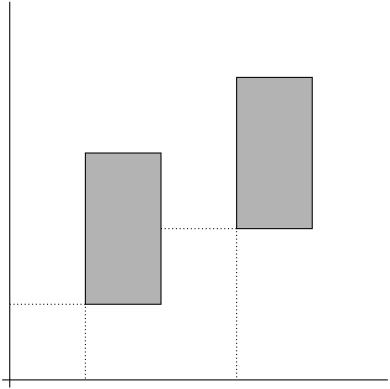

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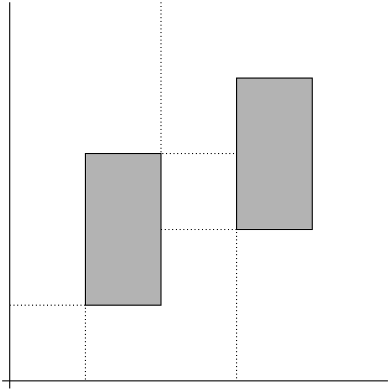

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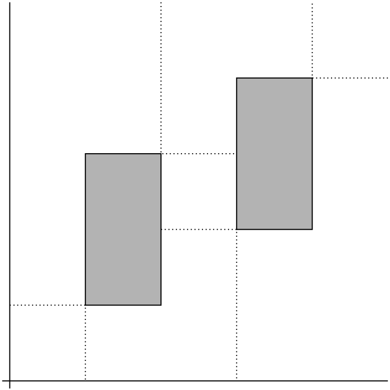

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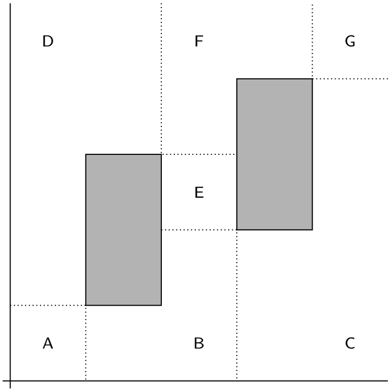

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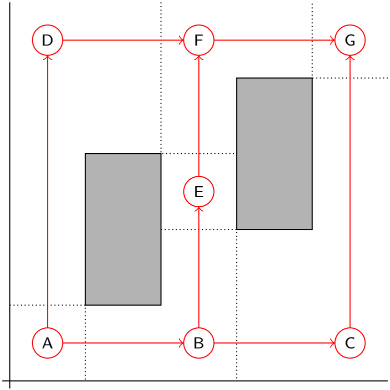

Swiss Flag

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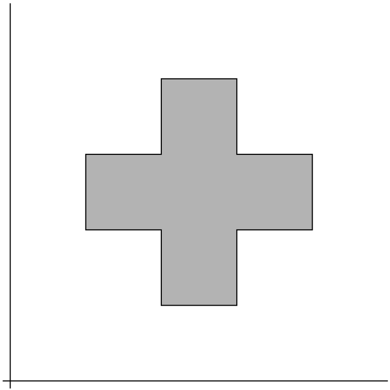

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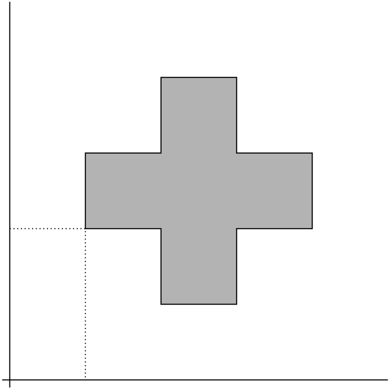

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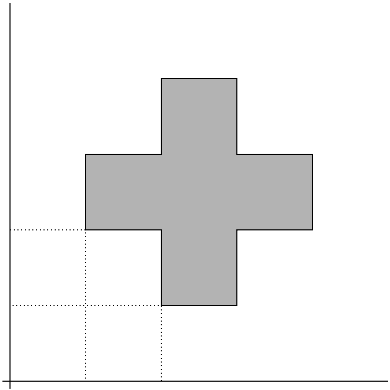

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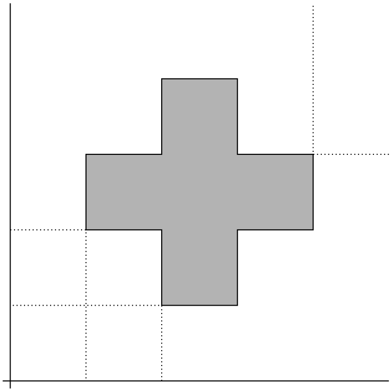

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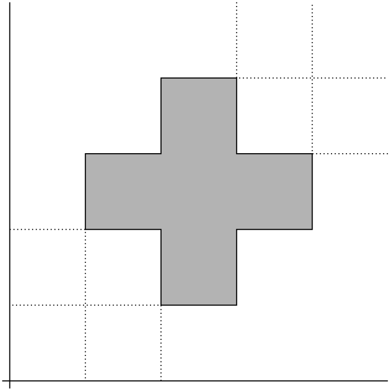

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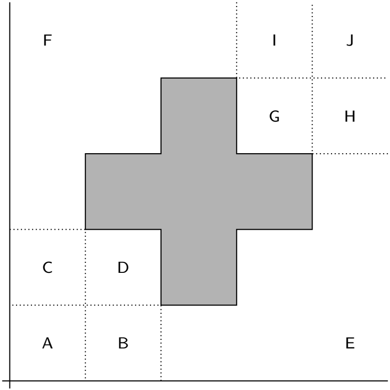

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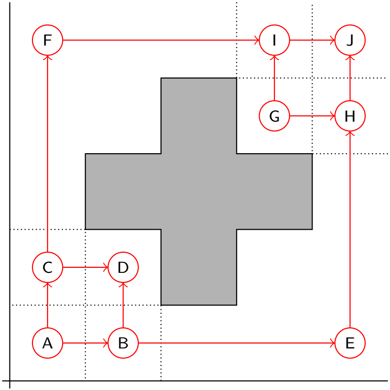

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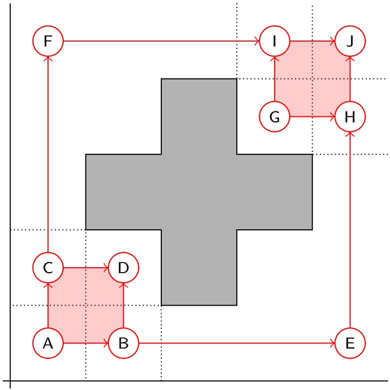

Achronal overlaping square

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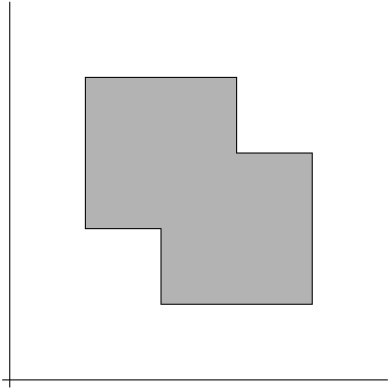

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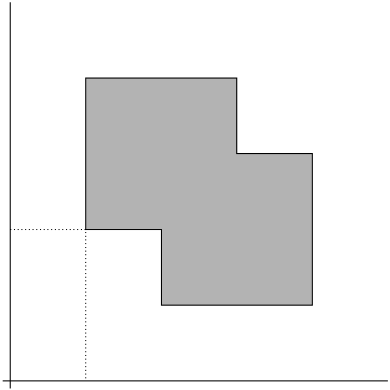

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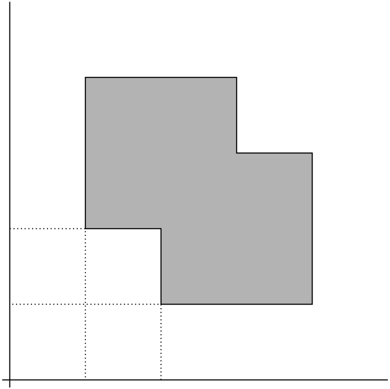

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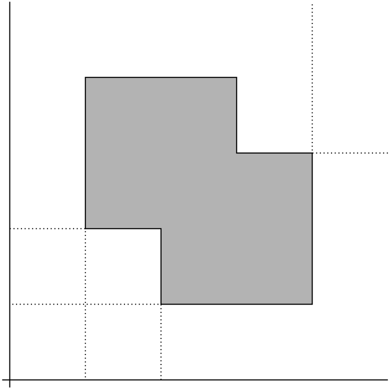

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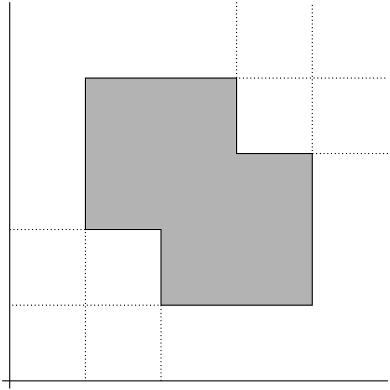

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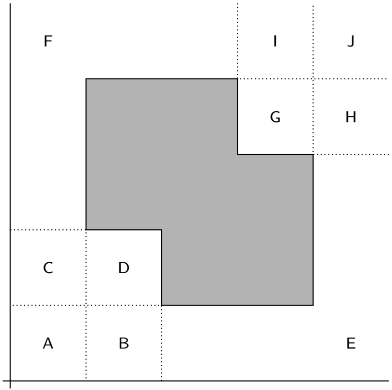

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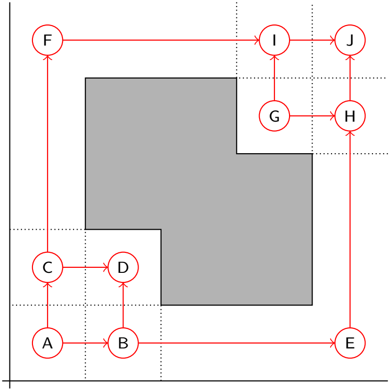

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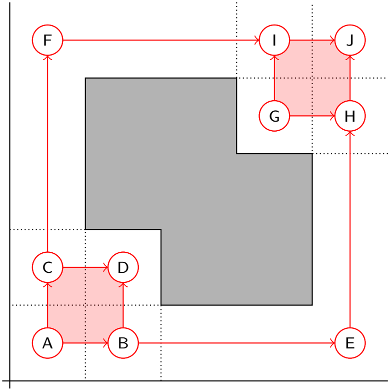

Diagonal overlaping squares

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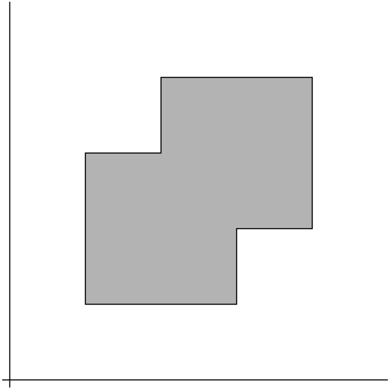

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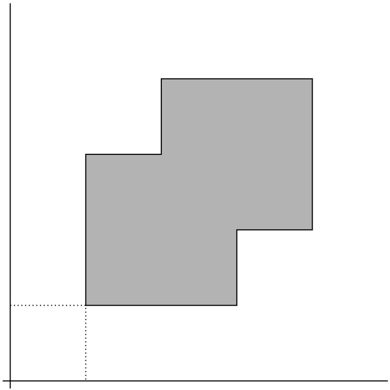

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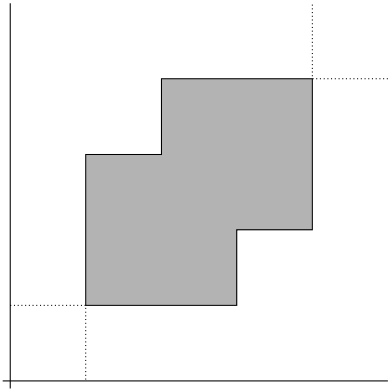

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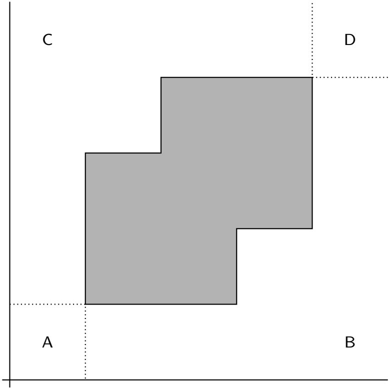

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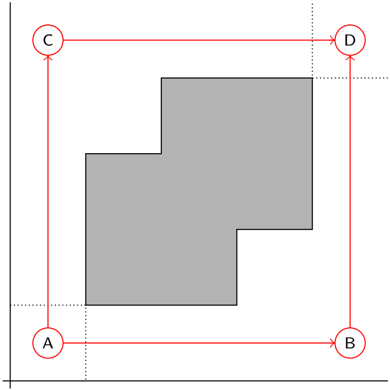

The floating cube

Non potential weak isomorphisms

The floating cube

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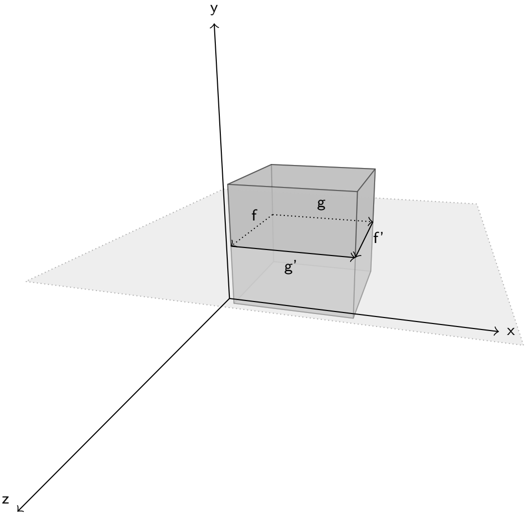

The floating cube

A “vee” that does not extend to a pushout

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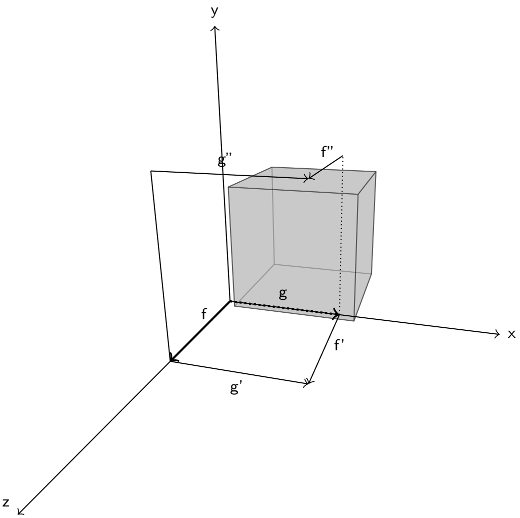
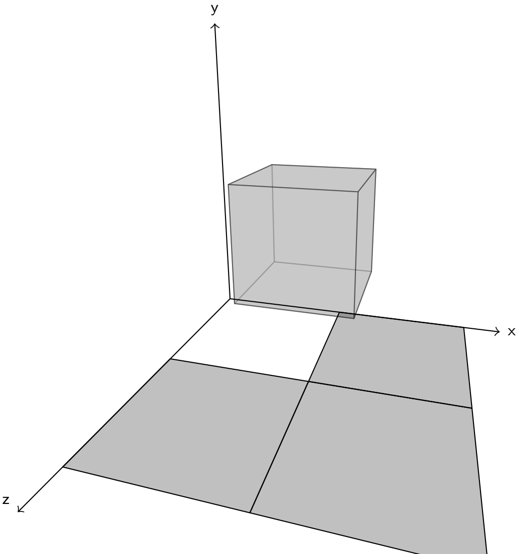


The floating cube

Some pushouts squares

The floating cube

Some pushouts squares



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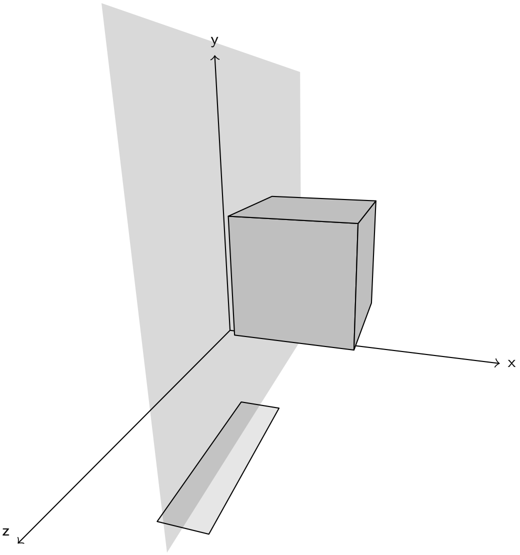

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The floating cube

boundaries of the components

The floating cube

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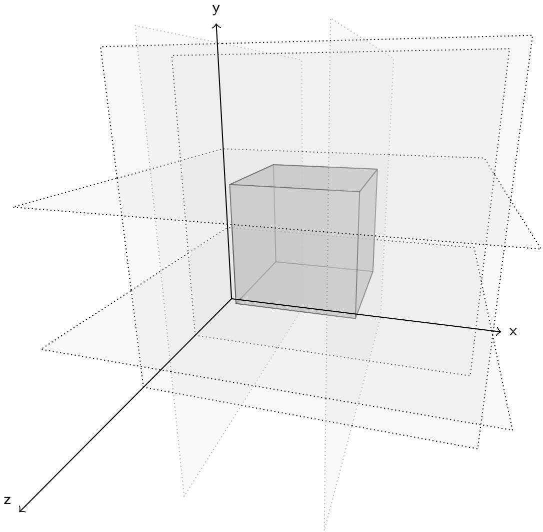

Commutative monoid

of nonempty finite connected loop-free categories

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- Any element of $\mathcal{M}$ freely generated by a graph, is prime

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- Conjecture

$$\boxed{\text{If } P \in \mathcal{N} \text{ is prime and } \overrightarrow{\pi_1}(P) \text{ is not a lattice, then } \overrightarrow{\pi_0}(\overrightarrow{\pi_1}(P)) \text{ is prime}}$$

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- Writting $f \sim g$ when there is an A-homotopy from f to g, we define an equivalence relation over the mappings from X to Y.

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- for $n \geqslant 2$ the n^{th} homotopy group construction extends to a functor $\pi_n : \mathcal{Top} \to \mathcal{Ab}$.
  i.e. for $n \geq 2$, the n^{th} homotopy group of a space is commutative

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  Finite Computability of Postnikov Complexes. E. H. Brown, Jr. Ann. of Math. 65(1). 1957

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- As a standard example we have $Y = [0,1]^n$, A the boundary of Y i.e.

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Attaching spaces

- Let A be a subspace of Y and $f : A \to X$ be a continuous map.
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- The CW-complexes arises in this way.

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- For $x \in \mathcal{I}_n$, an open n-cell is the image of $\{x\} \times]0,1[^n$ under Φ_n.
  It is a homeomorphic image.

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- The following product in $\mathcal{Top}$ is not a CW-complex

$$|\mathbb{R} \rightrightarrows \{0\}| \times |\mathbb{N} \rightrightarrows \{0\}|$$

Homotopy equivalences

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- If there exists $f' : Y \to X$ such that $f' \circ f \sim \mathrm{id}_X$ and $f \circ f' \sim \mathrm{id}_Y$, then f (and f') are said to be homotopy equivalences. The spaces X and Y are said to be homotopic. A space (resp. map) that is homotopic with $\{0\}$ (resp. a constant map) is said to be null homotopic

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- For all $n \geqslant 1$, $\mathbb{R}^{n+1} \setminus \{0\}$ is homotopically equivalent to $\mathbb{S}^n$

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- If X and Y are homotopic, then $\pi_n(X) \cong \pi_n(Y)$ for all n
- Whitehead theorem:

> If X and Y are CW-complexes and $f : X \to Y$ induces isomorphisms of n^{th} homotopy groups for all n, then f is a homotopy equivalence

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- The homotopy category is defined as the localization of $\mathcal{Top}$ (or $\mathcal{CGH}$ etc) with respect to the class of homotopy equivalences

Basic

- *Revêtements et Groupe Fondamental*, Michèle Audin, en ligne, 2004
- *Categories and Groupoids*, P. J. Higgins, Van Nostrand - Reinhold, 1971
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