

Directed Algebraic Topology and Concurrency

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MPRI : Concurrency (2.3.1)

Wednesday, the 8th of February 2017

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- There exists a precubical set K such that $\downarrow K|_{\mathcal{L}po} \cong \mathbb{R}^3$

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- Then \square is the subcategory of Set generated by all the face inclusions and projections.
- Cubical sets are presheaves over \square i.e. $pcSet = Set^{\square^{op}}$

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 - every variable occurs at most once in w , and
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- if $w : a \rightarrow b$ and $w' : b \rightarrow c$ then $w'w$ is obtained by replacing, for $k \in \{0, \dots, b-1\}$, the occurrence of x_k in w' (if any) by the k^{th} letter of w .

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That construction defines a functor $|-| : \mathcal{cSet} \rightarrow \mathcal{C}$.

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$$\begin{array}{ccc}
 & |K| & \\
 \phi_{\sigma_{k,n}(x)} \nearrow & & \nwarrow \phi_x \\
 \{\sigma_{k,n}(x)\} \times [0, 1]^{n+1} & \xrightarrow{\text{proj}} & \{x\} \times [0, 1]^n
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Hence $\phi_{\sigma_{k,n}(x)}(t_0, \dots, t_n)$ does not depend on t_k .

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The Cartesian product in $pcSet$ is deduced from the Cartesian product in Set

Cartesian product of two segments in $cSet$

Compute the product $\square_1 \times \square_1$

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The “segment” is $\square(-, 1)$ and the standard n -cube is $\square_n := \square(-, n)$. We have

$$\square_n = \bigotimes_{i=1}^n \square_1$$

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Theorem

Nonabelian Algebraic Topology, Brown, R., Higgins, P. J., and Sivera R., EMS, 2011.
Proposition 11.1.17, p.372

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For any “topological space” X , the counit at X

$$\varepsilon_X : |\mathit{Sing} X| \rightarrow X$$

of the adjunction $\mathit{Top} \begin{array}{c} \xrightarrow{\mathit{Sing}} \\ \xleftarrow{|\cdot|} \end{array} \mathit{cSet}$ is a weak homotopy equivalence.

(“topological space” maybe mean compactly generated space here.)

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- Denote by $G : \mathit{Cat} \rightarrow \mathit{Grd}$ the left adjoint to the inclusion functor $\mathit{Grd} \hookrightarrow \mathit{Cat}$

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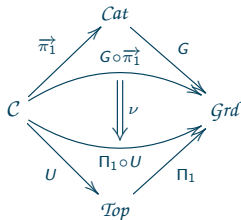
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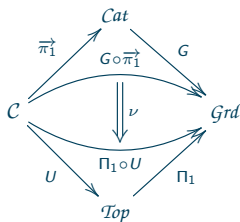
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- In fact there is a unique model category structure on Cat whose weak equivalences are the equivalences of category.

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- The class of isomorphisms of a category \mathcal{C} is preserved by any functor, so the only functors between one-way categories that should be considered are the ones that preserve the greatest systems of weak isomorphisms.
These functors are therefore the homotopical ones.

A guiding result

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Given a system of weak isomorphisms Σ of a one-way category \mathcal{C} we have:

the functor $g : \mathcal{C}[\Sigma^{-1}] \rightarrow \mathcal{D}[\Sigma'^{-1}]$ is an equivalence of category
iff the quotient functor $q : \mathcal{C}/\Sigma \rightarrow \mathcal{D}/\Sigma'$ is an isomorphism

$$\begin{array}{ccccc}
 \mathcal{C}[\Sigma^{-1}] & \xrightarrow{h} & & \xrightarrow{h} & \mathcal{D}[\Sigma'^{-1}] \\
 \downarrow P_\Sigma & \swarrow l_\Sigma & \mathcal{C} \xrightarrow{f} \mathcal{D} & \searrow l_{\Sigma'} & \downarrow P_{\Sigma'} \\
 \mathcal{C}/\Sigma & \xleftarrow{Q_\Sigma} & & \xrightarrow{Q_{\Sigma'}} & \mathcal{D}/\Sigma' \\
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When Σ and Σ' are the greatest systems of weak isomorphisms of \mathcal{C} and \mathcal{D} , then we define:

$$\vec{\pi}_0(f) = g \quad \text{and} \quad Ho(f) = h$$

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- The pushout in $OwCat_h$ of two copies of $\{0\} \hookrightarrow \{0 < 1\}$ is $\{0, 1\}^2$