Directed Algebraic Topology and Concurrency

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MPRI : Concurrency (2.3.1)

Wednesday, the 8^{th} of February 2017

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- Then we have $|\_| \dashv Sing$

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- There exists a precubical set K such that $\rceil K \lfloor \quad \cong \quad \vec{\mathbb{S}}^3$

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- Cubical sets are presheaves over $\square$ i.e. $pc\mathcal{S}et = \mathcal{S}et^{\square^{\mathrm{op}}}$

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- if $w : a \to b$ and $w' : b \to c$ then $w'w$ is obtained by replacing, for $k \in \{0, \ldots, b-1\}$, the occurrence of x_k in w' (if any) by the k^{th} letter of w.

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That construction defines a functor $\rceil\_\lfloor : c\mathcal{S}et \to \mathcal{C}$.

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$$\begin{tikzcd}
 & {|K|} & \\
\{\sigma_{k,n}(x)\} \times [0,1]^{n+1} \arrow[ur, "\phi_{\sigma_{k,n}(x)}"] \arrow[rr, "\mathrm{proj}"'] & & \{x\} \times [0,1]^n \arrow[ul, "\phi_x"']
\end{tikzcd}$$

Hence $\phi_{\sigma_{k,n}(x)}(t_0, \ldots, t_n)$ does not depend on t_k.

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$$\left(\begin{array}{c}\vdots\\ K_{n+1}\\ \partial_k^+ \Big\downarrow \; \Big\uparrow \sigma_k \; \Big\downarrow \partial_k^- \\ K_n \\ \vdots\end{array}\right) \times \left(\begin{array}{c}\vdots\\ K'_{n+1}\\ {\partial_k^+}' \Big\downarrow \; \Big\uparrow \sigma'_k \; \Big\downarrow {\partial_k^-}' \\ K'_n \\ \vdots\end{array}\right) \cong$$

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The Cartesian product in $pc\mathcal{S}et$ is deduced form the Cartesian product in $\mathcal{S}et$

Cartesian product of two segments in $c\mathcal{S}et$

Compute the product $\quad \square_1 \times \square_1$

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The "segment" is $\square(-, 1)$ and the standard n-cube is $\square_n := \square(-, n)$. We have

$$\square_n = \bigotimes_{i=1}^{n} \square_1$$

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Theorem

Nonabelian Algebraic Topology, Brown, R., Higgins, P. J., and Sivera R., EMS, 2011. Proposition 11.1.17, p.372

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Proposition 11.1.17, p.372

For any "topological space" X, the counit at X

$$\varepsilon_X : |Sing\ X| \to X$$

of the adjunction $\text{“}\mathcal{Top}\text{”} \underset{|\_|}{\overset{\mathrm{Sing}}{\rightleftarrows}} c\mathcal{Set}$ is a weak homotopy equivalence.

("topological space" maybe mean compactly generated space here.)

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- There exists a unique functor $\nu_X : G \circ \overrightarrow{\pi_1}(X) \to \Pi_1 \circ U(X)$,

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we would like to known when it is an isomorphism.

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- If the collection of isomorphisms of $\mathcal{C}$ is pure then so is its greatest p.o. and p.b. stable collection of morphisms.

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- [reminder] $f : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories when there exists $g : \mathcal{D} \to \mathcal{C}$ s.t. $g \circ f \cong \mathrm{id}_{\mathcal{C}}$ and $f \circ g \cong \mathrm{id}_{\mathcal{D}}$

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- In fact there is a unique model category structure on $\mathcal{C}at$ whose weak equivalences are the equivalences of category.

Finding a homotopical category structure on *OwCat*

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- The class of isomorphisms of a category $\mathcal{C}$ is stable under the action of $\mathrm{Aut}(\mathcal{C})$ therefore any one-way category should be equipped with its greatest system of weak isomorphisms
- The class of isomorphisms of a category $\mathcal{C}$ is preserved by any functor, so the only functors between one-way categories that should be considered are the ones that preserve the greatest systems of weak isomorphisms.
  These functors are therefore the homotopical ones.

A guiding result

for defining the weak equivalences

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Given a system of weak isomorphisms Σ of a one-way category $\mathcal{C}$ we have:

> the functor $g : \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}[\Sigma'^{-1}]$ is an equivalence of category
> iff the quotient functor $q : \mathcal{C}/\Sigma \to \mathcal{D}/\Sigma'$ is an isomorphism

$$\begin{tikzcd}
\mathcal{C}[\Sigma^{-1}] \arrow[rrr, "h"] \arrow[d, "P_\Sigma"'] & & & \mathcal{D}[\Sigma'^{-1}] \arrow[d, "P_{\Sigma'}"] \\
& \mathcal{C} \arrow[r, "f"] \arrow[ul, "I_\Sigma"'] \arrow[dl, "Q_\Sigma"] & \mathcal{D} \arrow[ur, "I_{\Sigma'}"] \arrow[dr, "Q_{\Sigma'}"'] & \\
\mathcal{C}/\Sigma \arrow[rrr, "g"'] & & & \mathcal{D}/\Sigma'
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& & & \\
\mathcal{C}/\Sigma \arrow[rrr, "g"'] & & & \mathcal{D}/\Sigma'
\end{tikzcd}
$$

When Σ anf Σ' are the greatest systems of weak isomorphisms of $\mathcal{C}$ and $\mathcal{D}$, then we define:

$$\overrightarrow{\pi_0}(f) = g \qquad \text{and} \qquad Ho(f) = h$$

The functors $\overrightarrow{\pi_0}$ and *Ho*

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- In particular we have a functor $\overrightarrow{\pi_0} : \mathcal{O}w\mathcal{C}at_h \to \mathcal{L}f\mathcal{C}at$ with $\overrightarrow{\pi_0}\mathcal{C}$ being the quotient of $\mathcal{C}$ by its greatest system of weak isomorphisms

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "q_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "q_{\mathcal{D}}"] \\ \overrightarrow{\pi_0}\mathcal{C} \arrow[r, "\overrightarrow{\pi_0}f"'] & \overrightarrow{\pi_0}\mathcal{D} \end{tikzcd}$$

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- We also have the functor $Ho : \mathcal{O}w\mathcal{C}at_h \to \mathcal{O}w\mathcal{C}at$ defined by

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "i_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "i_{\mathcal{D}}"] \\
Ho\mathcal{C} \arrow[r, "Ho(f)"'] & Ho\mathcal{D}
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- Then $\mathcal{O}w\mathcal{C}at_h$ is the subcategory of $\mathcal{O}w\mathcal{C}at$ whose morphisms are the functors that preserve the greatest systems of weak isomorphisms

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- Then $\mathcal{O}w\mathcal{C}at_h$ is the subcategory of $\mathcal{O}w\mathcal{C}at$ whose morphisms are the functors that preserve the greatest systems of weak isomorphisms
- The collection of weak equivalences of $\mathcal{O}w\mathcal{C}at_h$ is

$$\mathcal{W} := \left\{ f \mid \overrightarrow{\pi_0} f \text{ is an isomorphism} \right\} = \left\{ f \mid Ho(f) \text{ is an equivalence of category} \right\}$$

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- Then $\mathcal{O}w\mathcal{C}at_h$ is the subcategory of $\mathcal{O}w\mathcal{C}at$ whose morphisms are the functors that preserve the greatest systems of weak isomorphisms
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$$\mathcal{W} := \left\{ f \mid \overrightarrow{\pi_0} f \text{ is an isomorphism} \right\} = \left\{ f \mid Ho(f) \text{ is an equivalence of category} \right\}$$

- $\boxed{\text{Any equivalence of categories between one-way categories is a weak equivalence}}$

The homotopical category $\mathcal{OwCat}_h$

- Then $\mathcal{OwCat}_h$ is the subcategory of $\mathcal{OwCat}$ whose morphisms are the functors that preserve the greatest systems of weak isomorphisms
- The collection of weak equivalences of $\mathcal{OwCat}_h$ is

$$\mathcal{W} := \left\{ f \mid \overrightarrow{\pi_0} f \text{ is an isomorphism} \right\} = \left\{ f \mid Ho(f) \text{ is an equivalence of category} \right\}$$

- $\boxed{\text{Any equivalence of categories between one-way categories is a weak equivalence}}$
- The pushout in $\mathcal{OwCat}_h$ of two copies of $\{0\} \hookrightarrow \{0 < 1\}$ is $\{0, 1\}^2$