Directed Algebraic Topology and Concurrency

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MPRI : Concurrency (2.3.1)

Wednesday, the 8^{th} of February 2017

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- There exists a precubical set K such that $\rceil K \lfloor \quad \cong \quad \vec{\mathbb{S}}^3$

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- Then $\square$ is the subcategory of $\mathcal{S}et$ generated by all the face inclusions and projections.
- Cubical sets are presheaves over $\square$ i.e. $pc\mathcal{S}et = \mathcal{S}et^{\square^{\mathrm{op}}}$

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- if $w : a \to b$ and $w' : b \to c$ then $w'w$ is obtained by replacing, for $k \in \{0,\ldots,b-1\}$, the occurrence of x_k in w' (if any) by the k^{th} letter of w.

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That construction defines a functor $\rceil\_\lfloor : c\mathcal{S}et \to \mathcal{C}$.

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$$\begin{tikzcd}
 & {|K|} & \\
\{\sigma_{k,n}(x)\} \times [0,1]^{n+1} \arrow[ur, "\phi_{\sigma_{k,n}(x)}"] \arrow[rr, "\text{proj}"'] & & \{x\} \times [0,1]^{n} \arrow[ul, "\phi_x"']
\end{tikzcd}$$

Hence $\phi_{\sigma_{k,n}(x)}(t_0, \ldots, t_n)$ does not depend on t_k.

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- $\rceil K \lfloor_{d\mathcal{T}op}$ and $\rceil K \lfloor_{d\mathcal{T}op_{\mathsf{f}}}$ may differ.

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The Cartesian product in $\mathcal{pcSet}$ is deduced form the Cartesian product in $\mathcal{Set}$

Cartesian product of two segments in $c\mathcal{S}et$

Compute the product $\square_1 \times \square_1$

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The "segment" is $\square(-, 1)$ and the standard n-cube is $\square_n := \square(-, n)$. We have

$$\square_n = \bigotimes_{i=1}^{n} \square_1$$

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Theorem

Nonabelian Algebraic Topology, Brown, R., Higgins, P. J., and Sivera R., EMS, 2011. Proposition 11.1.17, p.372

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For any "topological space" X, the counit at X

$$\varepsilon_X : |Sing\ X| \to X$$

of the adjunction $\text{“}\mathcal{Top}\text{”} \underset{|\text{-}|}{\overset{\text{Sing}}{\rightleftarrows}} c\mathcal{S}et$ is a weak homotopy equivalence.

("topological space" maybe mean compactly generated space here.)

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we would like to known when it is an isomorphism.

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Conjecture: η_X is an isomorphism when X is an isothetic region.

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- Problem: The fundamental category of a local pospace has no isomorphisms but its identities, hence it is its own skeleton

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- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $\mathcal{C}(x, y)$ contains a potential weak isomorphism, then it is a singleton
  Requires the assumption that $\mathcal{C}$ is one-way

An example

of potential weak isomorphism

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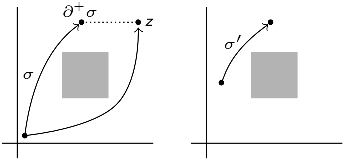

Due to the lower dipath, the σ, z-precomposition is not bijective; yet σ' is a potential weak isomorphism.

An unwanted example

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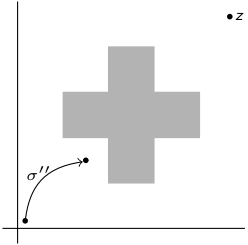

Note that σ'' is a potential weak isomorphism though there exists a morphism from $\partial^{-}\sigma''$ to z but none from $\partial^{+}\sigma''$ to z.

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- The collection Σ_∞ is stable under the action of $\mathrm{Aut}(\mathcal{C})$

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- Hence we suppose the systems of weak isomorphisms are closed under composition

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- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.

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- A conjecture:

> For all loop-free isothetic region X, $\overrightarrow{\pi_1}X$ satisfies the square filling property

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- [3.] If d and u are respectively a greatest lower bound and a least upper bound of the pair $\{x, y\}$, then Diagram 1 is both a pullback and a pushout in $\mathcal{C}$, and all the arrows appearing on the diagram belong to Σ.

$$\begin{tikzcd} & u & \\ x \arrow[ur] & & y \arrow[ul] \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Diagram 1

$$\begin{tikzcd} x & & y \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Diagram 2

$$\begin{tikzcd} & u & \\ x \arrow[ur] & & y \arrow[ul] \end{tikzcd}$$

Diagram 3

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- [3.] If d and u are respectively a greatest lower bound and a least upper bound of the pair $\{x, y\}$, then Diagram 1 is both a pullback and a pushout in $\mathcal{C}$, and all the arrows appearing on the diagram belong to Σ.
- [4.] $\mathcal{C} = \mathcal{K}$ iff $\mathcal{C}$ is a prelattice, and Σ is the greatest system of weak isomorphisms of $\mathcal{C}$ i.e. all the morphisms in this case.

$$\begin{tikzcd} & u & \\ x \arrow[ur] & & y \arrow[ul] \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Diagram 1

$$\begin{tikzcd} x & & y \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Diagram 2

$$\begin{tikzcd} & u & \\ x \arrow[ur] & & y \arrow[ul] \end{tikzcd}$$

Diagram 3

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$$\begin{tikzcd}
 & & x \arrow[r, "\delta"] & y \arrow[dr, "\Sigma"] \arrow[drr, "\Sigma"] & & \\
d \arrow[r, dash, "\cong"] \arrow[urr, "\Sigma"] \arrow[drr, "\Sigma"'] & x \wedge x' \arrow[ur, "\Sigma"'] \arrow[dr, "\Sigma"] & & & y \vee y' \arrow[r, dash, "\cong"] & u \\
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- If $\gamma \sim \delta$ then $\partial^- \gamma \sim \partial^- \delta$ and $\partial^+ \gamma \sim \partial^+ \delta$

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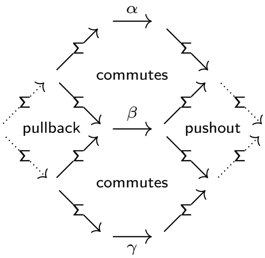

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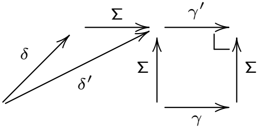

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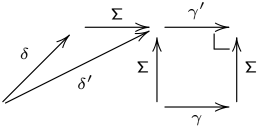

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  - In the diagram below we have $Q(\gamma' \circ \gamma'') = Q(\gamma') \circ Q(\gamma'') = Q(\gamma') \circ Q(\gamma)$ hence the composite $(\gamma' \circ \gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pushout nor on the representatives (γ, σ) and (γ', σ').

$$\begin{tikzcd}
& & {} \arrow[dl, "\sigma''"'] \arrow[dr, "\gamma''"] & & \\
& {} \arrow[dl, "\sigma"'] \arrow[dr, "\gamma"] & \text{pushout} & {} \arrow[dl, "\sigma'"'] \arrow[dr, "\gamma'"] & \\
{} & & {} & & {}
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- The skeleton of $\mathcal{C}[\Sigma^{-1}]$ is $\mathcal{C}/\Sigma$ and $\mathcal{C}[\Sigma^{-1}]$ is one-way.

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  - If $\mathcal{C}/\Sigma$ is infinite the existence of an admissible choice is a open question

Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$

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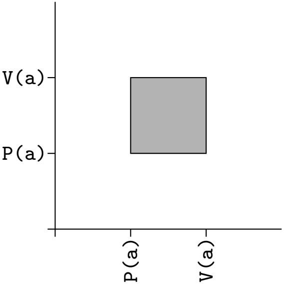

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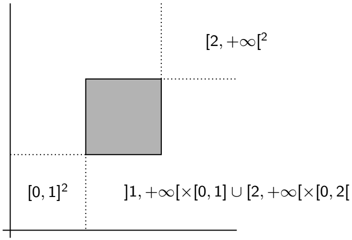

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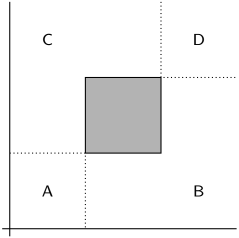

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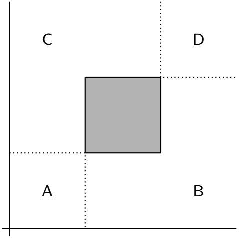

Let x, y such that $x \leqslant^2 y$, then $\overrightarrow{\pi_1}X(x,y)$ only depends on which elements of the partition x and y belong to

$\rightarrow$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

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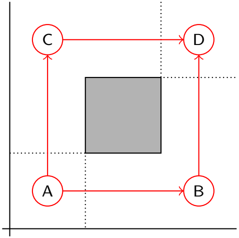

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Two rectangles

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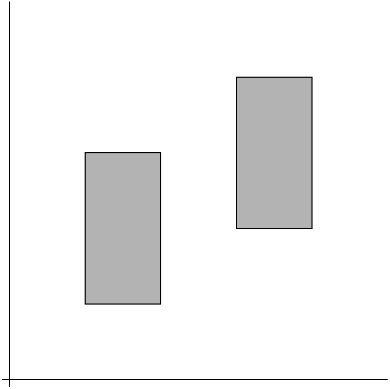

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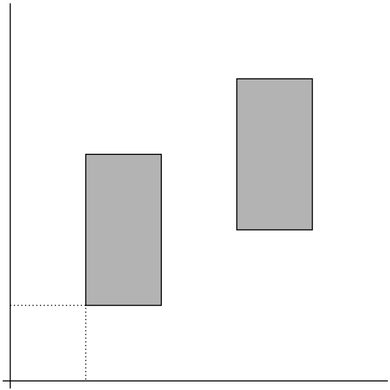

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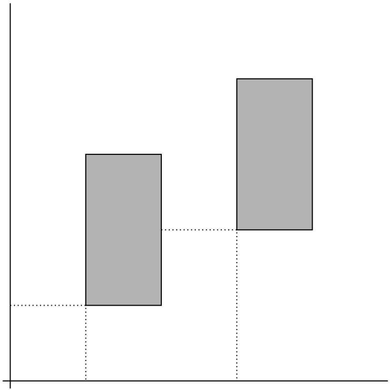

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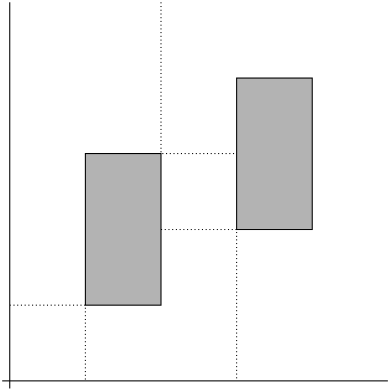

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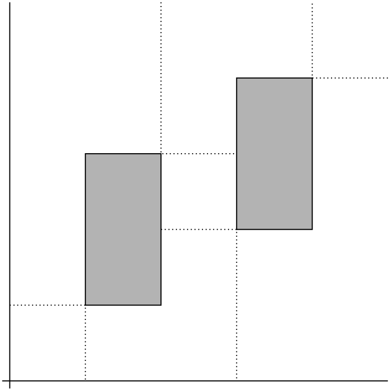

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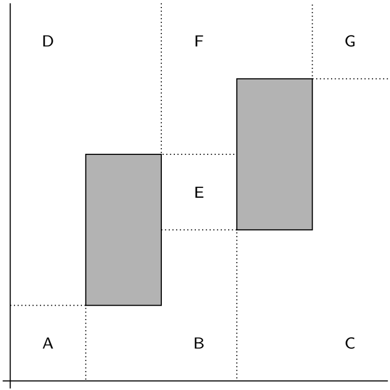

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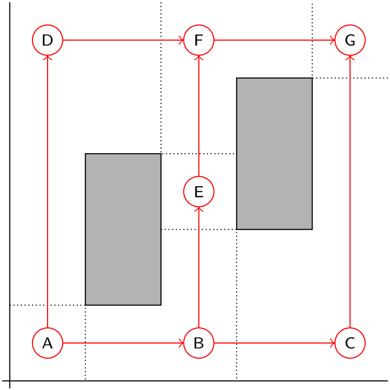

Swiss Flag

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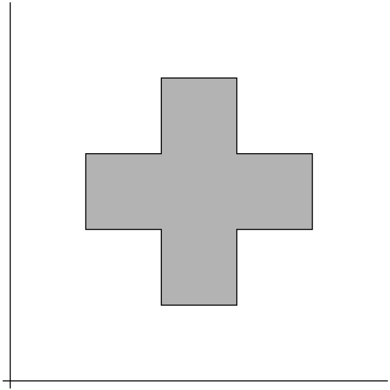

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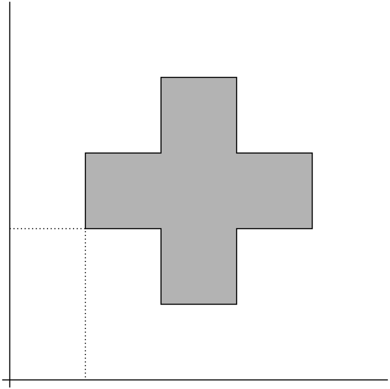

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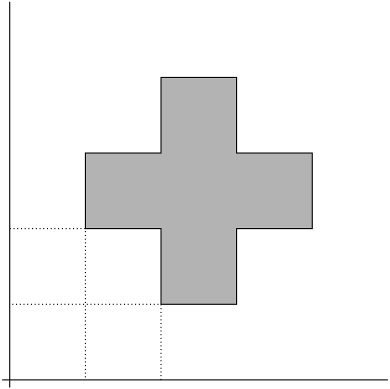

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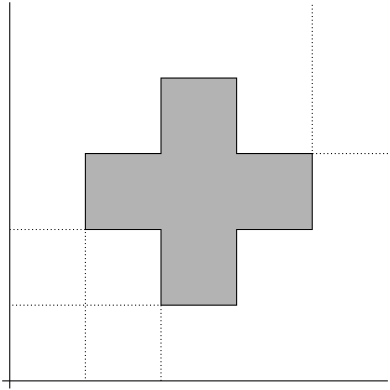

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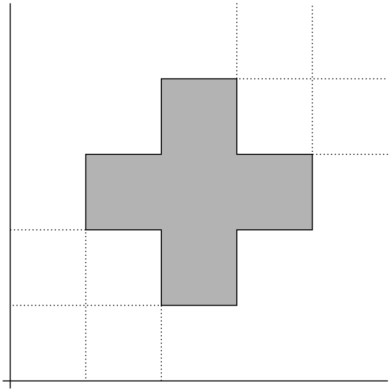

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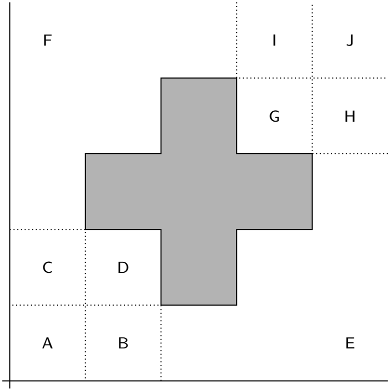

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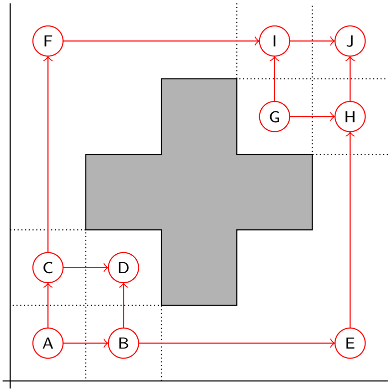

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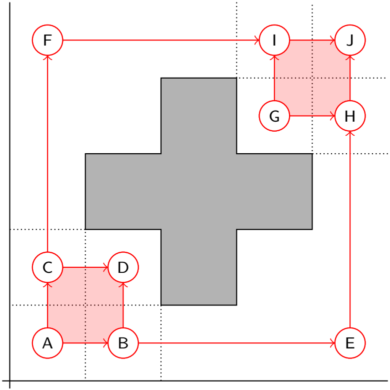

Achronal overlaping square

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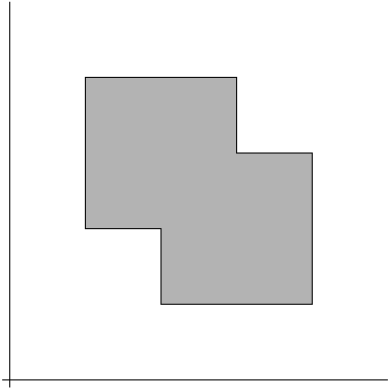

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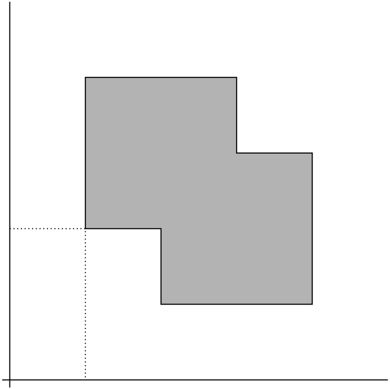

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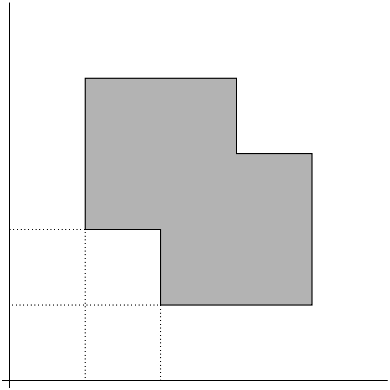

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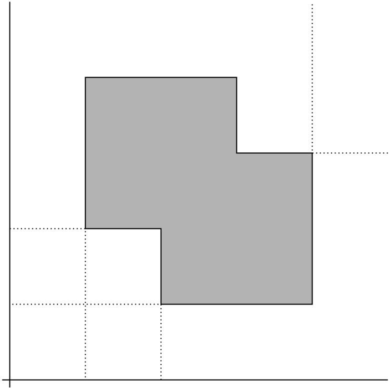

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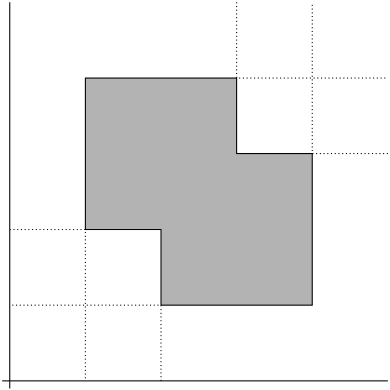

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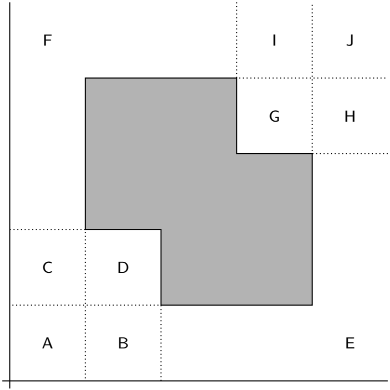

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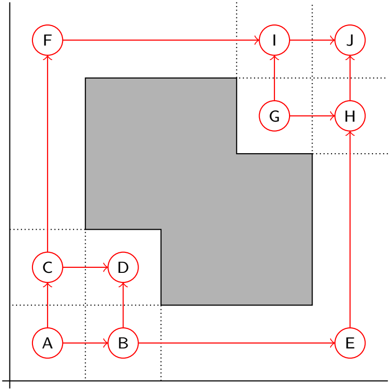

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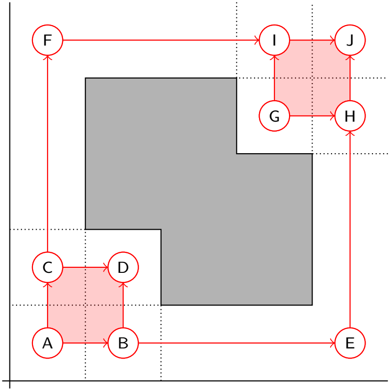

Diagonal overlaping squares

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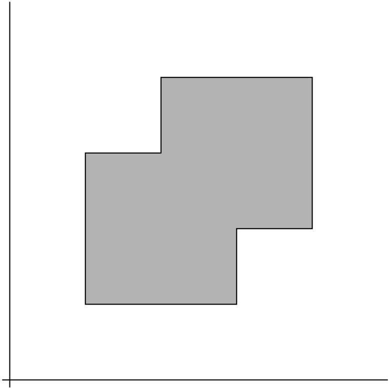

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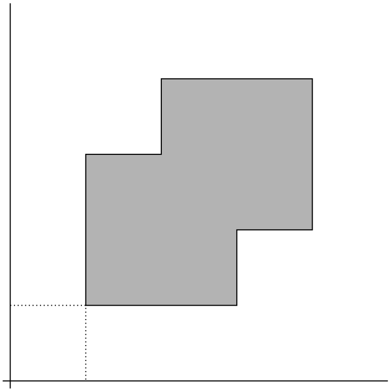

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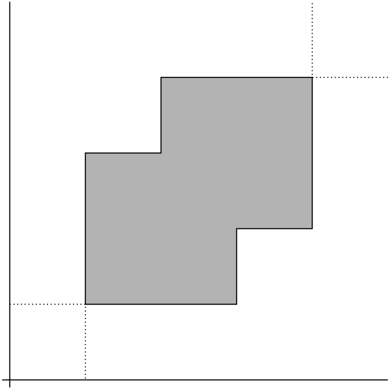

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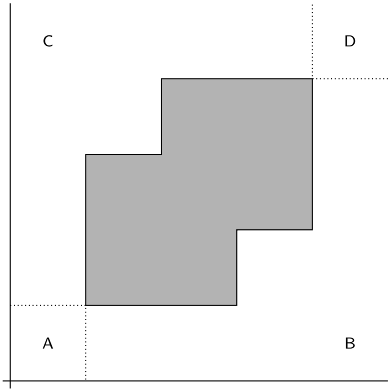

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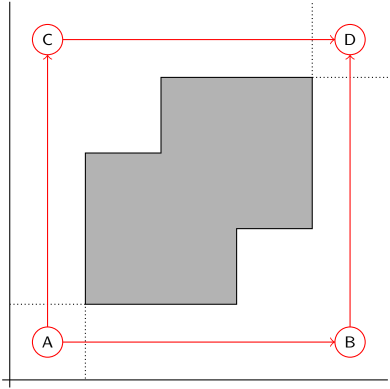

The floating cube

Non potential weak isomorphisms

The floating cube

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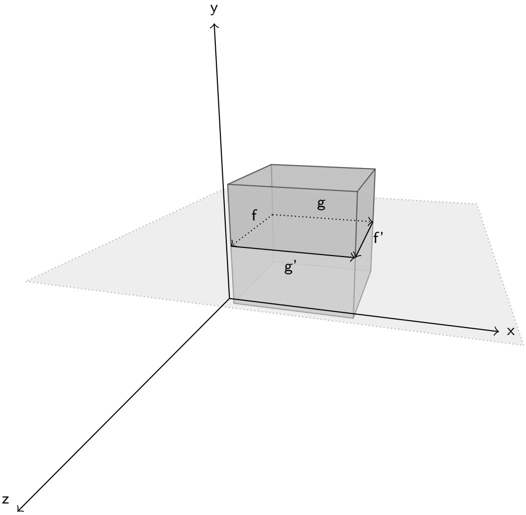

The floating cube

A "vee" that does not extend to a pushout

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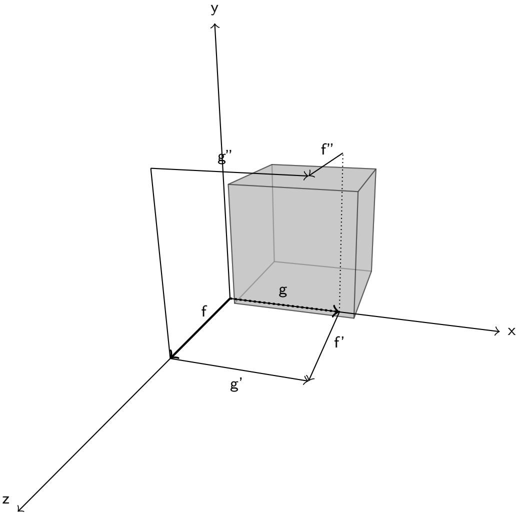

The floating cube

Some pushouts squares

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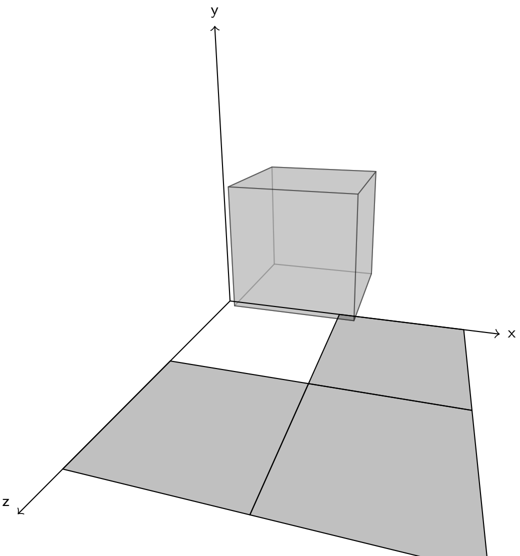

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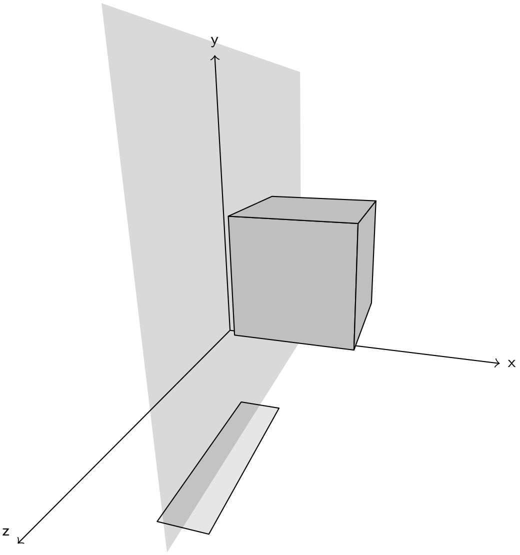

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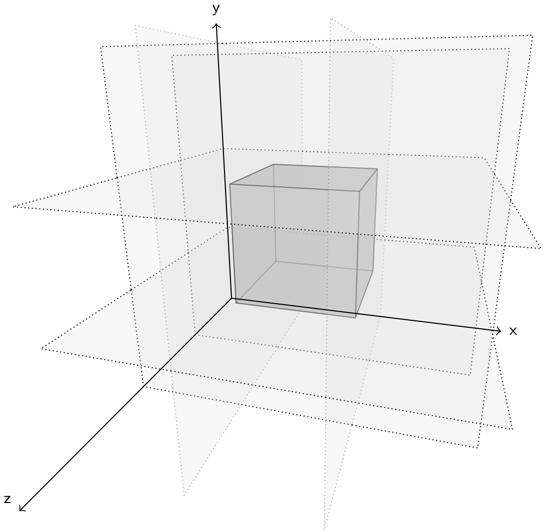
  - Therefore $f', g' \notin \Sigma$ (anyway they do not preserve the future cones)

The floating cube

boundaries of the components

The floating cube

boundaries of the components



Homotopical category and homotopical functor

Homotopy Limit Functors on Model Categories and Homotopical Categories

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- The element of $\mathcal{W}$ are called the weak equivalences
- A functor between homotopical categories is said to be homotopical if it preserves weak equivalences.

Homotopical category and homotopical functor

Homotopy Limit Functors on Model Categories and Homotopical Categories

Dwyer, W. G., Hirschhorn, P. S., Kan, D. M., and Smith, J. H. Amer. Math. Soc. (2004)

- A category is said to be homotopical when it comes with a collection of morphisms $\mathcal{W}$ that
  - contains all the identities
  - has the two out of six property i.e. if $\gamma \circ \beta$ and $\beta \circ \alpha$ both exist and belong to $\mathcal{W}$, then α, β, γ, and $\gamma \circ \beta \circ \alpha$ belong to $\mathcal{W}$
- The second condition may be replaced by the conjonction of the following ones:
  - has the weak invertibility property i.e. if there exists γ and δ such that $\gamma \circ \omega$ and $\omega \circ \delta$ both exist and belong to $\mathcal{W}$, then $\omega \in \mathcal{W}$
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- The corresponding homotopy category is defined as the localization $\mathcal{C}[\mathcal{W}^{-1}]$

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- If the collection of isomorphisms of $\mathcal{C}$ is pure then so is its greatest p.o. and p.b. stable collection of morphisms.

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- In fact there is a unique model category structure on $\mathcal{C}at$ whose weak equivalences are the equivalences of category.

Finding a homotopical category structure on $\mathcal{O}w\mathcal{C}at$

Finding a homotopical category structure on *OwCat*

- The class of isomorphisms of a category $\mathcal{C}$ is stable under the action of $\mathrm{Aut}(\mathcal{C})$ therefore any one-way category should be equipped with its greatest system of weak isomorphisms

Finding a homotopical category structure on $\mathcal{O}w\mathcal{C}at$

- The class of isomorphisms of a category $\mathcal{C}$ is stable under the action of $\mathrm{Aut}(\mathcal{C})$ therefore any one-way category should be equipped with its greatest system of weak isomorphisms
- The class of isomorphisms of a category $\mathcal{C}$ is preserved by any functor, so the only functors between one-way categories that should be considered are the ones that preserve the greatest systems of weak isomorphisms.
  These functors are therefore the homotopical ones.

A guiding result

for defining the weak equivalences

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Given a system of weak isomorphisms Σ of a one-way category $\mathcal{C}$ we have:

> the functor $g : \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}[\Sigma'^{-1}]$ is an equivalence of category
> iff the quotient functor $q : \mathcal{C}/\Sigma \to \mathcal{D}/\Sigma'$ is an isomorphism

$$
\begin{tikzcd}
\mathcal{C}[\Sigma^{-1}] \arrow[rrr, "h"] \arrow[dd, "P_\Sigma"'] & & & \mathcal{D}[\Sigma'^{-1}] \arrow[dd, "P_{\Sigma'}"] \\
& \mathcal{C} \arrow[r, "f"] \arrow[ul, "l_\Sigma"'] \arrow[dl, "Q_\Sigma"] & \mathcal{D} \arrow[ur, "l_{\Sigma'}"] \arrow[dr, "Q_{\Sigma'}"'] & \\
\mathcal{C}/\Sigma \arrow[rrr, "g"'] & & & \mathcal{D}/\Sigma'
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When Σ anf Σ' are the greatest systems of weak isomorphisms of $\mathcal{C}$ and $\mathcal{D}$, then we define:

$$\overrightarrow{\pi_0}(f) = g \qquad \text{and} \qquad Ho(f) = h$$

The functors $\overrightarrow{\pi_0}$ and Ho

The functors $\overrightarrow{\pi_0}$ and *Ho*

- In particular we have a functor $\overrightarrow{\pi_0} : \mathcal{O}w\mathcal{C}at_h \to \mathcal{L}f\mathcal{C}at$ with $\overrightarrow{\pi_0}\mathcal{C}$ being the quotient of $\mathcal{C}$ by its greatest system of weak isomorphisms

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\mathcal{C} \arrow[r, "f"] \arrow[d, "q_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "q_{\mathcal{D}}"] \\
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- We also have the functor $Ho : \mathcal{O}w\mathcal{C}at_h \to \mathcal{O}w\mathcal{C}at$ defined by

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "i_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "i_{\mathcal{D}}"] \\ Ho\mathcal{C} \arrow[r, "Ho(f)"'] & Ho\mathcal{D} \end{tikzcd}$$

The homotopical category OwCat_h

The homotopical category $\mathcal{O}w\mathcal{C}at_h$

- Then $\mathcal{O}w\mathcal{C}at_h$ is the subcategory of $\mathcal{O}w\mathcal{C}at$ whose morphisms are the functors that preserve the greatest systems of weak isomorphisms

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$$\mathcal{W} := \left\{f \mid \overrightarrow{\pi_0} f \text{ is an isomorphism}\right\} = \left\{f \mid Ho(f) \text{ is an equivalence of category}\right\}$$

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- The pushout in $\mathcal{O}w\mathcal{C}at_h$ of two copies of $\{0\} \hookrightarrow \{0 < 1\}$ is $\{0, 1\}^2$