

Directed Algebraic Topology and Concurrency

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Extracting the “largest” graph from a d-space

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The following functor is right adjoint to graph realization in $d\mathit{Top}$

$$\begin{array}{ccc}
 d\mathit{Top} & & \mathit{Grph} \\
 \\
 (X, dX) & & dX^{[0,1]} \begin{array}{c} \xrightarrow{\gamma \mapsto \gamma(0)} \\ \xrightarrow{\gamma \mapsto \gamma(1)} \end{array} \Rightarrow X \\
 \downarrow f & & \downarrow f \\
 (Y, dY) & & dY^{[0,1]} \begin{array}{c} \xrightarrow{\gamma \mapsto \gamma(0)} \\ \xrightarrow{\gamma \mapsto \gamma(1)} \end{array} \Rightarrow Y
 \end{array}$$

$f \circ -$

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- Given $X, Y \in \mathcal{Top}$, a mapping $f : X \rightarrow Y$ is said to be *k-continuous* when for all *k-probe* on X , the mapping $f \circ t$ is continuous.

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- The *k*-spaces and the continuous functions between them form the category CG .

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That construction is functorial and provides the following inclusion functors with a **right** adjoint.

- $Top \leftrightarrow CG$
- $wHaus \leftrightarrow CGw\mathcal{H}$
- $Haus \leftrightarrow CG\mathcal{H}$

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- $X \in \mathcal{CG}\mathcal{H}$ iff it is a colimit in *Haus* of compact Hausdorff spaces.

$$\begin{array}{ccc}
 CG & \xrightleftharpoons[k]{\cong} & Top \\
 \uparrow w & & \uparrow w \\
 \downarrow \Psi & & \downarrow \Psi \\
 CGw\mathcal{H} & \xrightleftharpoons[k]{\cong} & wHaus \\
 \uparrow h & & \uparrow h \\
 \downarrow \Psi & & \downarrow \Psi \\
 CG\mathcal{H} & \xrightleftharpoons[k]{\cong} & Haus
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$$(- \times E): \mathcal{C} \longrightarrow \mathcal{C}$$

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$$\begin{array}{ccccc}
 & & X \times E & & \\
 & \swarrow & \downarrow f \times \text{id}_E & \searrow & \\
 X & & Y \times E & & E \\
 \downarrow f & \swarrow & & \searrow & \downarrow \text{id}_E \\
 Y & & & & E
 \end{array}$$

with $f \times \text{id}_E$ defined by right hand side diagram
(the unlabelled arrows being the projection morphism)

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- The categories \mathcal{CG} , $\mathcal{CG}w\mathcal{H}$, and $\mathcal{CG}\mathcal{H}$ are cartesian closed. For $X, E \in \mathcal{CG}$ ($\mathcal{CG}w\mathcal{H}$, $\mathcal{CG}\mathcal{H}$), X^E is the k -ification of the compact-open topology on $\mathcal{Top}(E, X)$, and $X \times E$ is the k -ification of the product $X \times_{\mathcal{Top}} E$.

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- The categories \mathcal{Set} and \mathcal{Grph} are cartesian closed (they are actually toposes).

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S. Krishnan, 2006

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A **stream** is a Hausdorff space X together with a mapping that associates each open subset U of X with a preorder \preceq_U on U such that for all open covering \mathcal{V} of U , the preorder \preceq_U is the least preorder containing \preceq_V for all $V \in \mathcal{V}$.

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By restricting the class of topological spaces allowed in the definition of streams to ((weak) Hausdorff) compactly generated spaces, one obtains Cartesian closed categories.

Directed paths on a stream

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For all U open subsets of an interval of \mathbb{R} , write $u \preceq_U v$ when $u \leq v$ and $[u, v] \subseteq U$. Then any interval of \mathbb{R} is a stream. In particular $[0, r]$ is a stream.

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A **directed path** on a stream X is a stream morphism from some stream $[0, r]$ to X .

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That adjunction induces an isomorphism between the full subcategories $\text{Str}_d = \{SX \mid X \in \text{dTop}\}$ and $\text{dTop}_f = \{DX \mid X \in \text{Str}\}$ of Str and dTop .

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Characterization of $dTop_f$ and Str_d

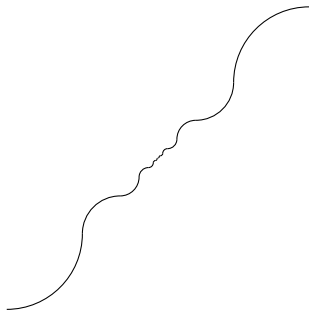
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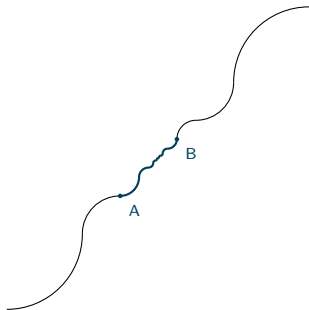
A d-space D belongs to $dTop_f$ when for every pseudo-directed path on X is directed. Such d-spaces are said to be **filled**.

Infinite rounded staircase

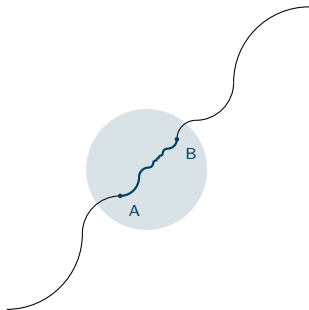
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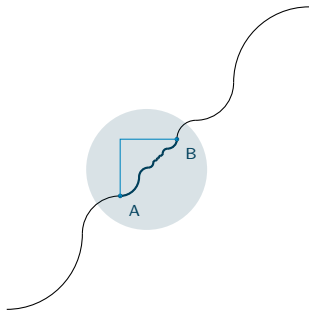
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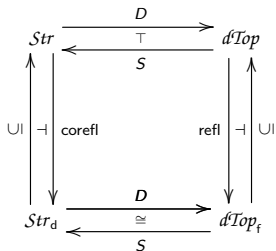
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- The collection of filled directions on X is thus a complete lattice.
- Hence a Galois connection $\{\text{filled directions on } X\} \rightleftarrows \{\text{directions on } X\}$.
- The inclusion functor $d\mathcal{Top}_f \hookrightarrow d\mathcal{Top}$ has a left adjoint

Relating streams and d-spaces

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- The full subcategory $dTop_f$ of filled d-spaces is complete and cocomplete.

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For all **filled** d-spaces X , $\overrightarrow{\pi}_1 X \cong \overrightarrow{\pi}_1 SX$

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- For $n \in \mathbb{N}$, $k \in \{0, \dots, n-1\}$, $\varepsilon \in \{+, -\}$, we have the face inclusion map

$$\begin{array}{ccc} [0, 1]^n & \rightarrow & [0, 1]^{n+1} \\ (t_0, \dots, t_{n-1}) & \mapsto & (t_0, \dots, t_{k-1}, \varepsilon, t_k, \dots, t_{n-1}) \end{array}$$

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- Then \square^+ is the subcategory of *Set* generated by all the face inclusions

A description of the \square^+ category

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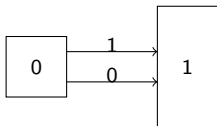
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- if $w : a \rightarrow b$ and $w' : b \rightarrow c$ then $w'w$ is obtained by replacing the k^{th} occurrence of x in w' by the k^{th} letter of w .

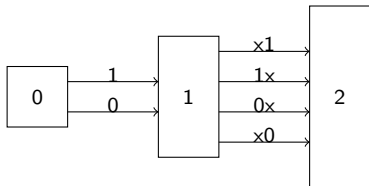
Presentation of \square^+ with faces as generators

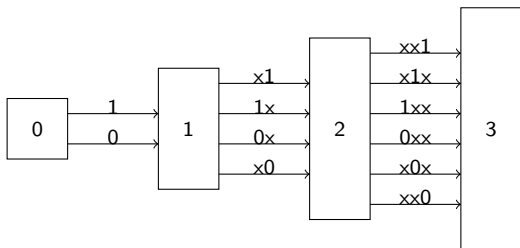
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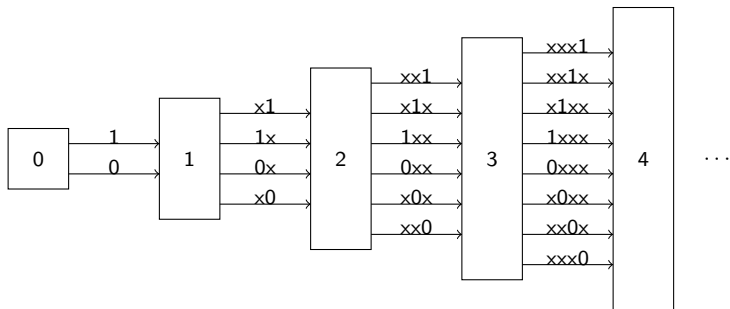
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Presentation of \square^+ : the (co)precubical relations

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The category \square^+ is generated by the morphisms $\delta_{k,n}^\varepsilon$ for $n \in \mathbb{N}$, $k \in \{0, \dots, n-1\}$, and $\varepsilon \in \{+, -\}$ together with the following relations for all $n \in \mathbb{N}$, $i \in [n]$, $j \in [n+1]$ and $\alpha, \beta \in \{0, 1\}$

$$\delta_{j,n+1}^\beta \circ \delta_{i,n}^\alpha = \begin{cases} \delta_{i,n+1}^\alpha \circ \delta_{j-1,n}^\beta & \text{if } i < j \\ \delta_{i+1,n+1}^\alpha \circ \delta_{j,n}^\beta & \text{if } i \geq j \end{cases}$$

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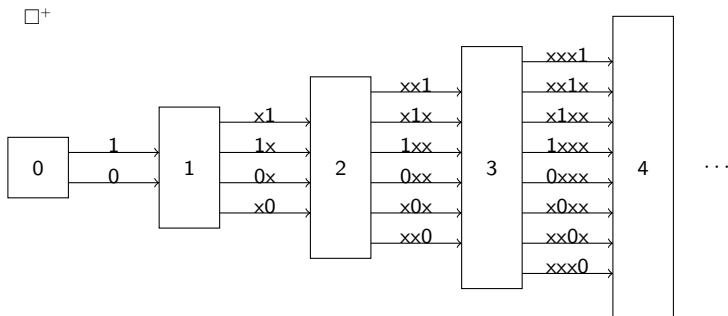
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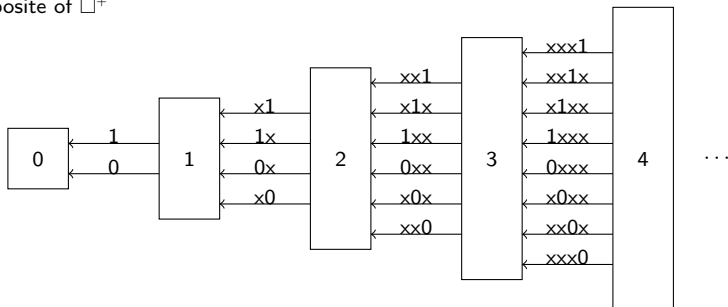
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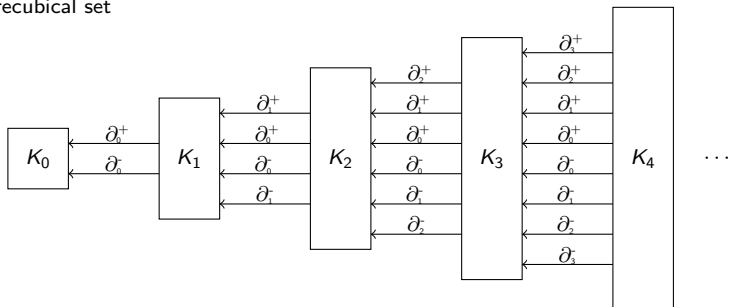
opposite of \square^+



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a precubical set

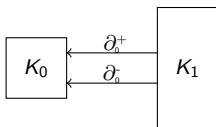


Precubical sets as higher dimensional sets

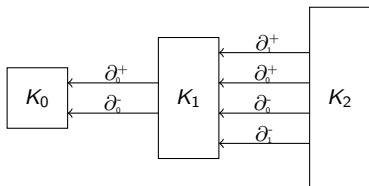
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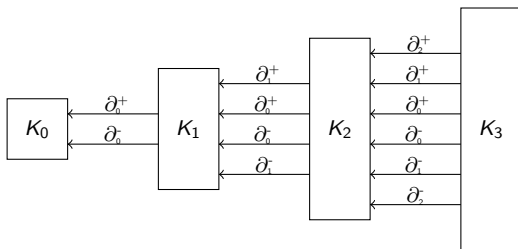
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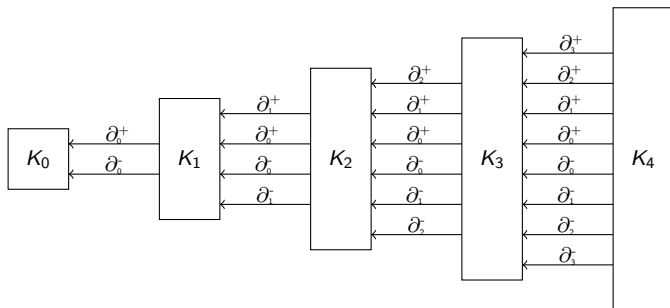
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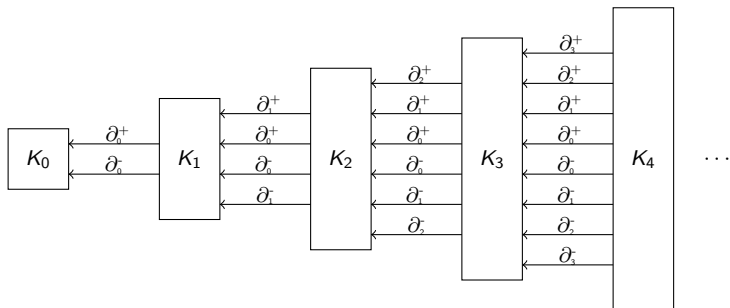
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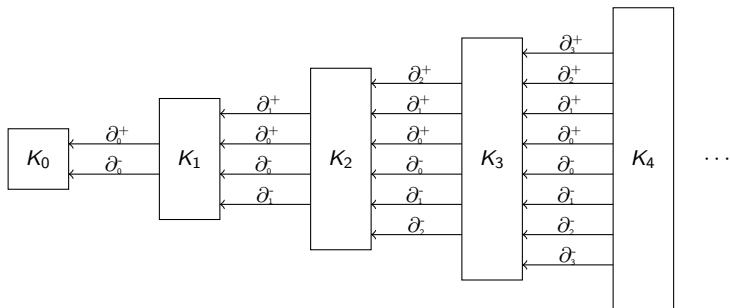
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Precubical sets as higher dimensional sets



satisfying the [precubical relations](#)

Higher dimensional automata

labelled precubical sets

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Modeling Concurrency with Geometry. Pratt, V. PoPL 1991.

Bisimulations for Higher Dimensional Automata. van Glabbeek, R.J. Manuscript 1991.

<http://theory.stanford.edu/~rvg/hda>

Higher dimensional automata revisited. Pratt, V. Mathematical Structures in Computer Science 10(4):525-548, 2000.

Erratum to "On the Expressiveness of Higher Dimensional Automata". van Glabbeek, R.J. Theoretical Computer Science 368(1-2):168-194. 2006.

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Objects

$$\begin{array}{c}
 \vdots \\
 K_{n+1} \\
 \begin{array}{c} \partial_k^- \downarrow \downarrow \partial_k^+ \\ \downarrow \downarrow \\ K_n \end{array} \\
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 & & \partial_k^+ & & \partial_k'^+ \\
 \vdots & & \vdots & & \vdots
 \end{array}$$

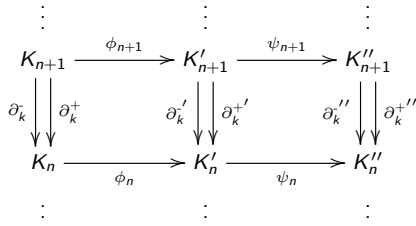
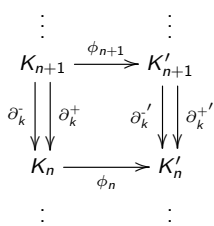
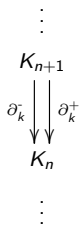
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Objects

Morphisms

Composition



Cartesian product of precubical sets

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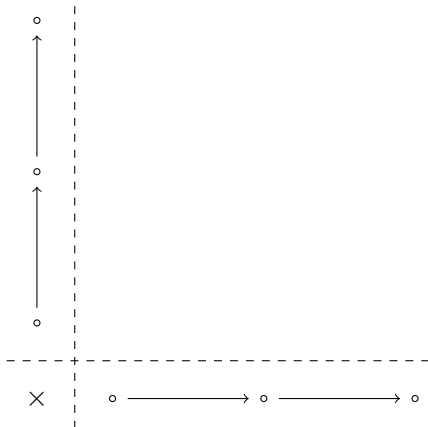
$$\left(\begin{array}{c} \vdots \\ K_{n+1} \\ \partial_k^+ \downarrow \downarrow \partial_k^- \\ K_n \\ \vdots \end{array} \right) \times \left(\begin{array}{c} \vdots \\ K'_{n+1} \\ \partial_k^{+'} \downarrow \downarrow \partial_k^{-'} \\ K'_n \\ \vdots \end{array} \right) \cong$$

Cartesian product of precubical sets

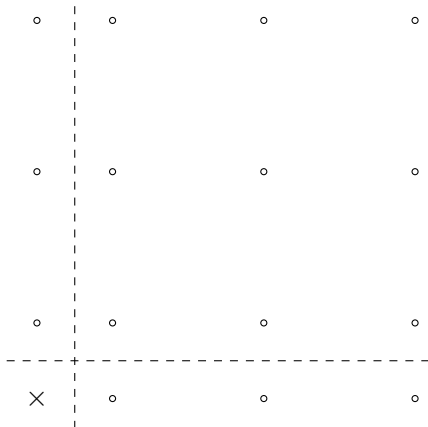
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The Cartesian product in $pcSet$ is deduced from the Cartesian product in Set

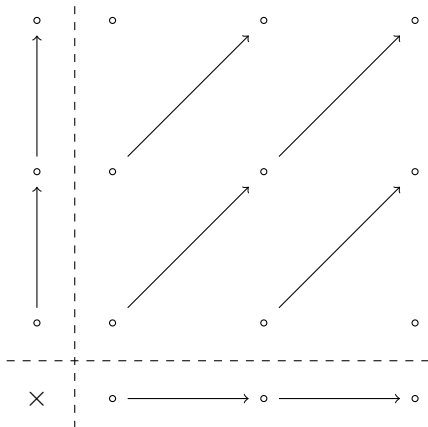
Example of Cartesian product



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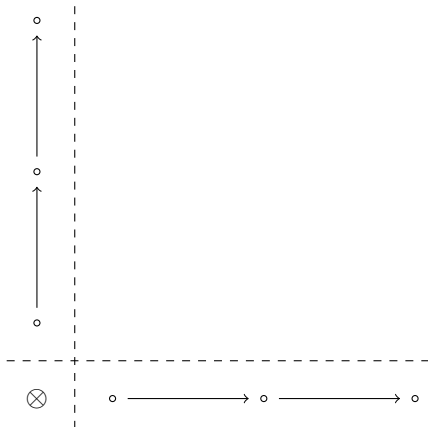
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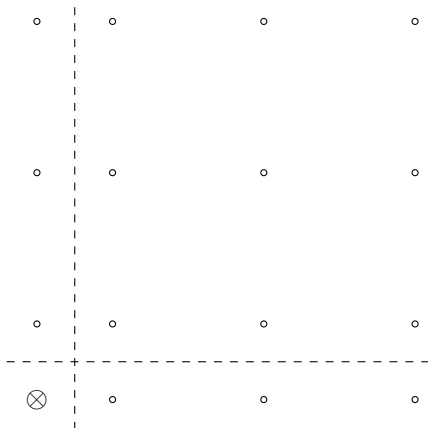
Remark that $\square^+(-, 1) = \{\cdot \rightarrow \cdot\}$ and defines the standard n -cube as $\square_n^+ := \square^+(-, n)$. Then

$$\square_n^+ = \bigotimes_{i=1}^n \square_1^+$$

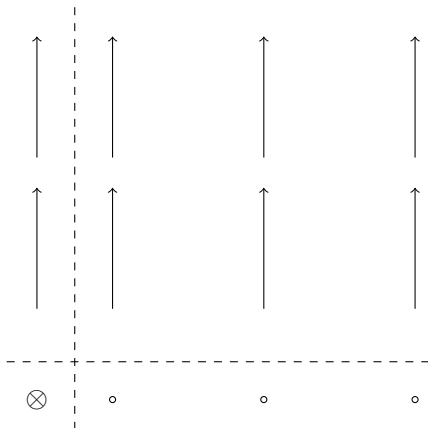
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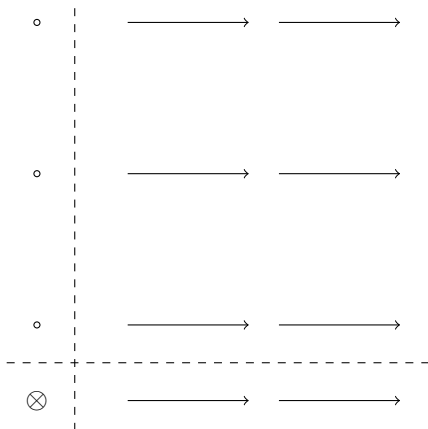
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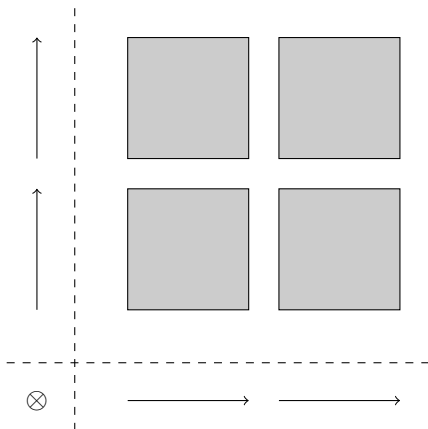
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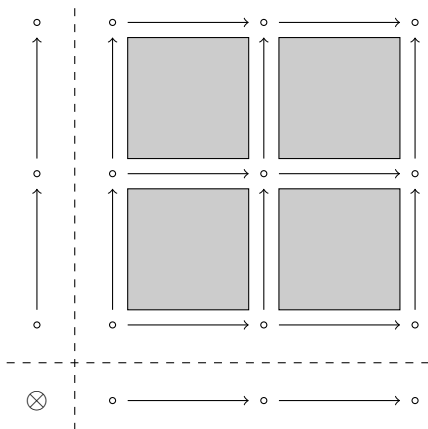
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That construction defines a functor $|-| : pcSet \rightarrow \mathcal{C}$.

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- in particular for $x, y \in K_{n+1}$, the equality $\partial_k^\varepsilon x = \partial_{k'}^{\varepsilon'} y$ makes the k^{th} ε -face of $\{x\} \times [0, 1]^{n+1}$ be identified with the k'^{th} ε' -face of $\{y\} \times [0, 1]^{n+1}$.
This is the way cubes are “glued” with each other.

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 - at least one of the realizations $|K|$ and $|K'|$ is exponentiable

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Examples of realizations

- $C(n) = \{n\}$ in *Set*, *Top*, *Haus*, or \mathcal{K}_c : connected components
- $C(n) = \square_n^+$ in *pcSet*: $|K| = K$
- $C(n)$ the underlying graph of \square_n^+ in *Grph*:
the underlying graph of K . NB: $|\square_n^+|_{\text{Grph}} \not\cong |\square_1^+|^n_{\text{Grph}}$
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- **Cubical sets** are presheaves over \square i.e. $pcSet = Set^{\square^{op}}$

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 - every variable occurs at most once in w , and
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- if $w : a \rightarrow b$ and $w' : b \rightarrow c$ then $w'w$ is obtained by replacing, for $k \in \{0, \dots, b-1\}$, the occurrence of x_k in w' (if any) by the k^{th} letter of w .

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That construction defines a functor $|-| : \mathcal{c}Set \rightarrow \mathcal{C}$.

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$$\begin{array}{ccc}
 & |K| & \\
 \phi_{\sigma_{k,n}(x)} \nearrow & & \nwarrow \phi_x \\
 \{\sigma_{k,n}(x)\} \times [0, 1]^{n+1} & \xrightarrow{\text{proj}} & \{x\} \times [0, 1]^n
 \end{array}$$

Hence $\phi_{\sigma_{k,n}(x)}(t_0, \dots, t_n)$ does not depend on t_k .

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$$\left(\begin{array}{c} \vdots \\ K_{n+1} \\ \partial_k^+ \uparrow \sigma_k \downarrow \partial_{\bar{k}} \\ K_n \\ \vdots \end{array} \right) \times \left(\begin{array}{c} \vdots \\ K'_{n+1} \\ \partial_k^{+'} \uparrow \sigma_k' \downarrow \partial_{\bar{k}}' \\ K'_n \\ \vdots \end{array} \right) \cong$$

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The Cartesian product in $pcSet$ is deduced from the Cartesian product in Set

Cartesian product of two segments in $cSet$

Compute the product $\square_1 \times \square_1$

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The “segment” is $\square(-, 1)$ and the standard n -cube is $\square_n := \square(-, n)$. We have

$$\square_n = \bigotimes_{i=1}^n \square_1$$

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Theorem

Nonabelian Algebraic Topology, Brown, R., Higgins, P. J., and Sivera R., EMS, 2011.
Proposition 11.1.17, p.372

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For any “topological space” X , the counit at X

$$\varepsilon_X : |\mathit{Sing} X| \rightarrow X$$

of the adjunction $\mathit{Top} \begin{array}{c} \xrightarrow{\mathit{Sing}} \\ \xleftarrow{|\cdot|} \end{array} \mathit{cSet}$ is a **weak homotopy equivalence**.

(“topological space” maybe mean compactly generated space here.)

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- Denote by $G : \mathit{Cat} \rightarrow \mathit{Grd}$ the left adjoint to the inclusion functor $\mathit{Grd} \hookrightarrow \mathit{Cat}$

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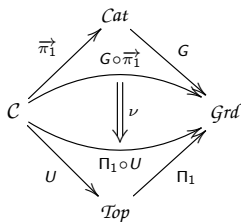
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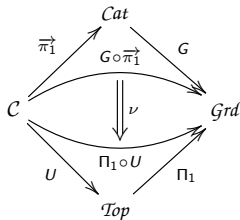
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