Directed Algebraic Topology and Concurrency

Emmanuel Haucourt

emmanuel.haucourt@polytechnique.edu

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Extracting the “largest” graph from a d-space

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The following functor is right adjoint to graph realization in $\mathcal{dTop}$

$\mathcal{dTop}$ $\qquad\qquad$ $\mathcal{Grph}$

$$
\begin{tikzcd}
(X, dX) \arrow[d, "f"'] \\
(Y, dY)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
dX^{[0,1]} \arrow[rr, shift left, "\gamma\mapsto\gamma(0)"] \arrow[rr, shift right, "\gamma\mapsto\gamma(1)"'] \arrow[d, "f\circ -"'] & & X \arrow[d, "f"] \\
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- The k-spaces and the continuous functions between them form the category $\mathcal{CG}$.

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That construction is functorial and provides the following inclusion functors with a right adjoint.

- $\mathcal{Top} \hookleftarrow \mathcal{CG}$
- $\mathcal{wHaus} \hookleftarrow \mathcal{CGwH}$
- $\mathcal{Haus} \hookleftarrow \mathcal{CGH}$

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$$\begin{tikzcd}
\mathcal{CG} \arrow[r, shift left] \arrow[d, shift right, dashed, "w"'] & \mathcal{T}op \arrow[l, shift left, dashed, "k"] \arrow[d, shift right, dashed, "w"'] \\
\mathcal{CG}w\mathcal{H} \arrow[r, shift left] \arrow[u, shift right] \arrow[d, shift right, dashed, "h"'] & w\mathcal{H}aus \arrow[l, shift left, dashed, "k"] \arrow[u, shift right] \arrow[d, shift right, dashed, "h"'] \\
\mathcal{CGH} \arrow[r, shift left] \arrow[u, shift right] & \mathcal{H}aus \arrow[l, shift left, dashed, "k"] \arrow[u, shift right]
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with $f \times \mathrm{id}_E$ defined by right hand side diagram
(the unlabelled arrows being the projection morphism)

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- The categories $\mathcal{Set}$ and $\mathcal{Grph}$ are cartesian closed (they are actually toposes).

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A stream is a Hausdorff space X together with a mapping that associates each open subset U of X with a preorder $\preccurlyeq_U$ on U such that for all open covering $\mathcal{V}$ of U, the preorder $\preccurlyeq_U$ is the least preorder containing $\preccurlyeq_V$ for all $V \in \mathcal{V}$.

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A stream morphism from X to Y is a mapping from the underlying set of X to that of Y such that for all $x \in X$, there exist U and V, open neighbourhoods of x and $f(x)$ such that f induces a preorder morphism from U to V.

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By restricting the class of topological spaces allowed in the definition of streams to ((weak) Hausdorff) compactly generated spaces, one obtains Cartesian closed categories.

Directed paths on a stream

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For all U open subsets of an interval of $\mathbb{R}$, write $u \preccurlyeq_U v$ when $u \leqslant v$ and $[u, v] \subseteq U$. Then any interval of $\mathbb{R}$ is a stream. In particular $[0, r]$ is a stream.

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A directed path on a stream X is a stream morphism from some stream $[0, r]$ to X.

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That adjunction induces an isomorphism between the full subcategories $\mathcal{S}tr_{\mathrm{d}} = \{SX \mid X \in d\mathcal{T}op\}$ and $d\mathcal{T}op_{\mathrm{f}} = \{DX \mid X \in \mathcal{S}tr\}$ of $\mathcal{S}tr$ and $d\mathcal{T}op$.

Characterization of $d\mathcal{Top}_{\mathrm{f}}$ and $\mathcal{Str}_{\mathrm{d}}$

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A pseudo-directed path on a d-space X is a continuous path γ such that for all subpaths γ' of γ and all neighbourhood U of $\mathrm{im}(\gamma')$, there exists a directed path on U from $\partial^{-}\gamma'$ to $\partial^{+}\gamma'$.

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A d-space D belongs to $d\mathcal{T}op_{\mathrm{f}}$ when for every pseudo-directed path on X is directed. Such d-spaces are said to be filled.

Infinite rounded staircase

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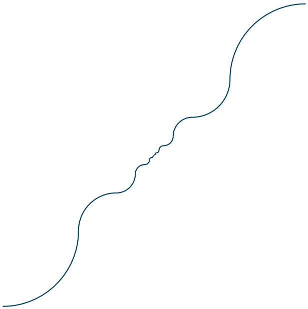

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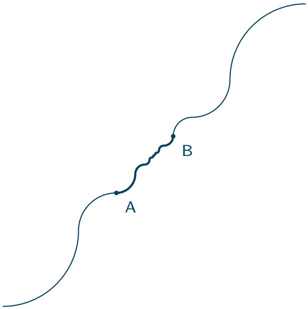

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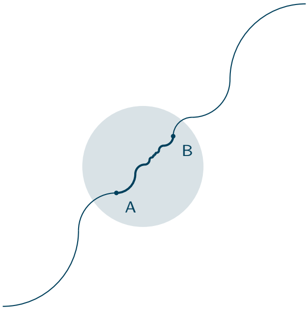

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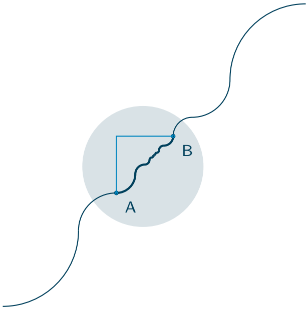

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- The collection of filled directions on X is thus a complete lattice.
- Hence a Galois connection $\{\text{filled directions on } X\} \rightleftarrows \{\text{directions on } X\}$.
- The inclusion functor $\mathit{dTop}_{\mathrm{f}} \hookrightarrow \mathit{dTop}$ has a left adjoint

Relating streams and d-spaces

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$$
\begin{tikzcd}[column sep=huge, row sep=huge]
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\mathcal{S}tr_{\mathsf{d}} \arrow[u, shift left, "\cup\mid"] \arrow[r, shift left, "D"] \arrow[r, phantom, "\cong"] & d\mathcal{T}op_{\mathsf{f}} \arrow[l, shift left, "S"] \arrow[u, shift right, "\cup\mid"']
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Conjecture: given $V_1, \ldots, V_n$ vector fields on $\mathcal{M}$,

$$F\big(d_{V_1}\mathcal{M} \vee \cdots \vee d_{V_n}\mathcal{M}\big) = F\big(d_{V_1,\ldots,V_n}\mathcal{M}\big)$$

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- The full subcategory $d\mathcal{Top}_{\mathrm{f}}$ of filled d-spaces is complete and cocomplete.

Fundamental categories

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- For $n \in \mathbb{N}$, $k \in \{0, \ldots, n-1\}$, $\varepsilon \in \{+, -\}$, we have the face inclusion map

$$\begin{array}{rcl} [0,1]^n & \to & [0,1]^{n+1} \\ (t_0, \ldots, t_{n-1}) & \mapsto & (t_0, \ldots, t_{k-1}, \varepsilon, t_k, \ldots, t_{n-1}) \end{array}$$

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- Then $\square^+$ is the subcategory of $\mathcal{S}et$ generated by all the face inclusions

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- if $w : a \to b$ and $w' : b \to c$ then $w'w$ is obtained by replacing the k^{th} occurrence of x in w' by the k^{th} letter of w.

Presentation of $\square^+$ with faces as generators

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$\boxed{0}$

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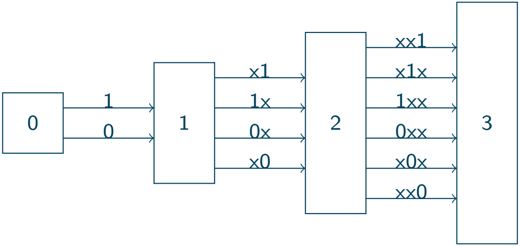
$$\begin{tikzcd} 0 \arrow[r, shift left, "1"] \arrow[r, shift right, "0"] & 1 \end{tikzcd}$$

Presentation of $\square^+$ with faces as generators

$$
\begin{tikzcd}
0 \arrow[r, shift left=1, "1"] \arrow[r, shift right=1, "0"'] & 1 \arrow[r, shift left=3, "x1"] \arrow[r, shift left=1, "1x"] \arrow[r, shift right=1, "0x"] \arrow[r, shift right=3, "x0"'] & 2
\end{tikzcd}
$$

Presentation of $\square^+$ with faces as generators



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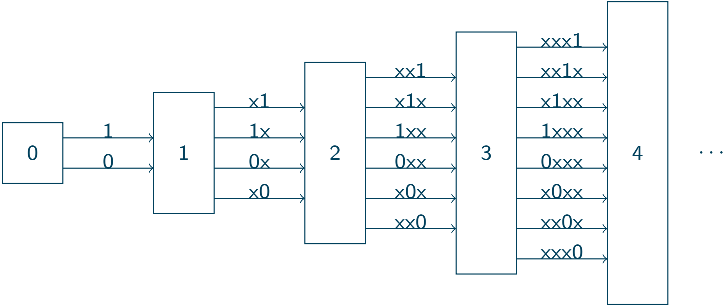

Presentation of $\square^+$: the (co)precubical relations

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The category $\square^+$ is generated by the morphisms $\delta^{\varepsilon}_{k,n}$ for $n \in \mathbb{N}$, $k \in \{0, \dots, n-1\}$, and $\varepsilon \in \{+, -\}$ together with the following relations for all $n \in \mathbb{N}$, $i \in [n]$, $j \in [n+1]$ and $\alpha, \beta \in \{0, 1\}$

$$\delta^{\beta}_{j,n+1} \circ \delta^{\alpha}_{i,n} = \begin{cases} \delta^{\alpha}_{i,n+1} \circ \delta^{\beta}_{j-1,n} & \text{if } i < j \\ \delta^{\alpha}_{i+1,n+1} \circ \delta^{\beta}_{j,n} & \text{if } i \geqslant j \end{cases}$$

Precubical sets are presheaves over $\square^+$

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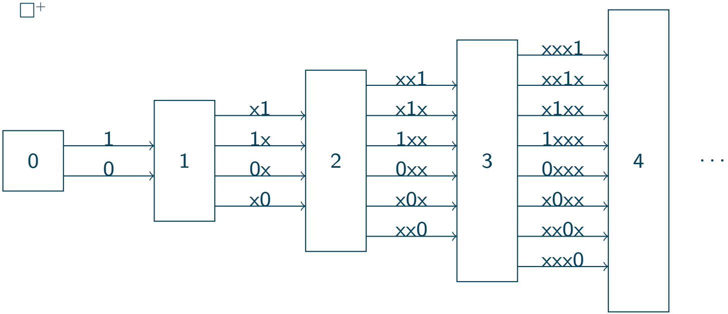

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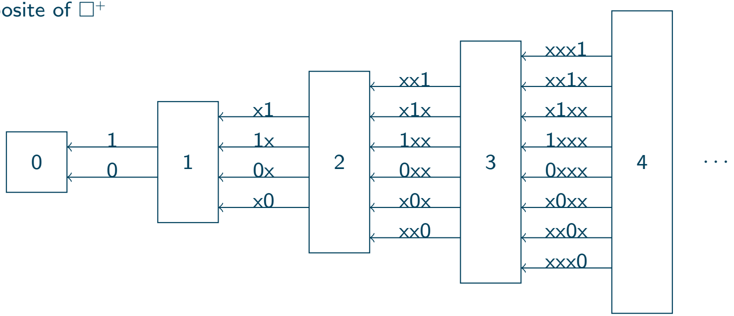
opposite of $\square^{+}$



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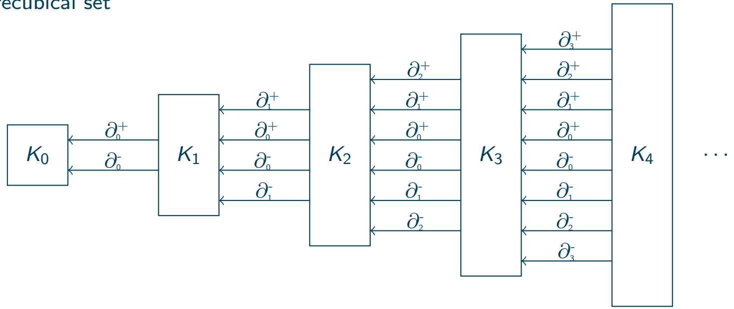
a precubical set



Precubical sets as higher dimensional sets

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$\boxed{K_0}$

Precubical sets as higher dimensional sets

$$\begin{tikzcd} K_0 & K_1 \arrow[l, shift right, "\partial_0^+"'] \arrow[l, shift left, "\partial_0^-"] \end{tikzcd}$$

Precubical sets as higher dimensional sets

$$
\begin{tikzcd}
K_0 & K_1 \arrow[l, shift right=2, "\partial_0^+"'] \arrow[l, shift left=2, "\partial_0^-"'] & K_2 \arrow[l, shift right=6, "\partial_1^+"'] \arrow[l, shift right=2, "\partial_0^+"'] \arrow[l, shift left=2, "\partial_0^-"'] \arrow[l, shift left=6, "\partial_1^-"']
\end{tikzcd}
$$

Precubical sets as higher dimensional sets

$$
\begin{tikzcd}
K_0 & K_1 \arrow[l, shift right=1.5ex, "\partial_0^+"'] \arrow[l, shift left=1.5ex, "\partial_0^-"] & K_2 \arrow[l, shift right=4.5ex, "\partial_1^+"'] \arrow[l, shift right=1.5ex, "\partial_0^+"'] \arrow[l, shift left=1.5ex, "\partial_0^-"] \arrow[l, shift left=4.5ex, "\partial_1^-"] & K_3 \arrow[l, shift right=7.5ex, "\partial_2^+"'] \arrow[l, shift right=4.5ex, "\partial_1^+"'] \arrow[l, shift right=1.5ex, "\partial_0^+"'] \arrow[l, shift left=1.5ex, "\partial_0^-"] \arrow[l, shift left=4.5ex, "\partial_1^-"] \arrow[l, shift left=7.5ex, "\partial_2^-"]
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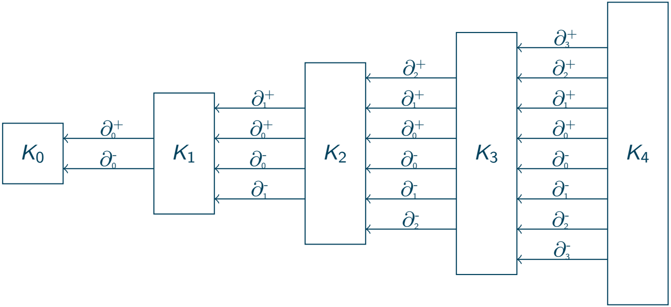

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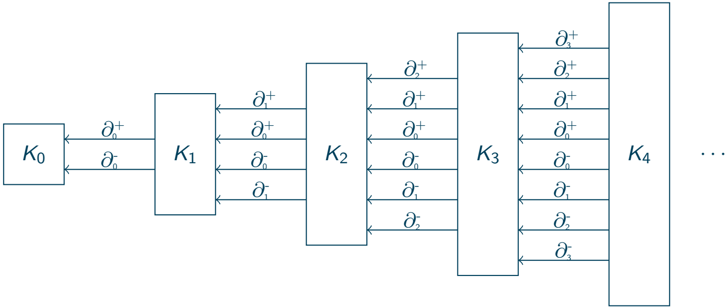

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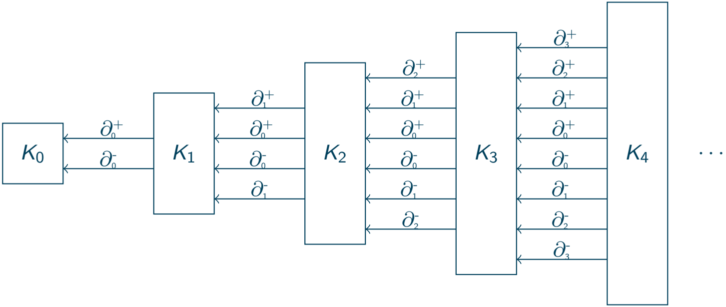

satisfying the precubical relations

Higher dimensional automata

labelled precubical sets

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Modeling Concurrency with Geometry. Pratt, V. PoPL 1991.

Bisimulations for Higher Dimensional Automata. van Glabbeek, R.J. Manuscript 1991.
`http://theory.stanford.edu/~rvg/hda`

Higher dimensional automata revisited. Pratt, V. Mathematical Structures in Computer Science 10(4):525-548, 2000.

Erratum to "On the Expressiveness of Higher Dimensional Automata". van Glabbeek, R.J. Theoretical Computer Science 368(1-2):168-194. 2006.

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Objects

$$\begin{tikzcd}
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\end{tikzcd}
$$

Composition

$$
\begin{tikzcd}
\vdots & \vdots & \vdots \\
K_{n+1} \arrow[r, "\phi_{n+1}"] \arrow[d, shift right, "\partial^-_k"'] \arrow[d, shift left, "\partial^+_k"] & K'_{n+1} \arrow[r, "\psi_{n+1}"] \arrow[d, shift right, "\partial^{-}_k{}'"'] \arrow[d, shift left, "\partial^{+}_k{}'"] & K''_{n+1} \arrow[d, shift right, "\partial^{-}_k{}''"'] \arrow[d, shift left, "\partial^{+}_k{}''"] \\
K_n \arrow[r, "\phi_n"'] & K'_n \arrow[r, "\psi_n"'] & K''_n \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

Cartesian product of precubical sets

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$$
\left(\begin{array}{c} \vdots \\ K_{n+1} \\ \partial_k^+ \Big\downarrow \Big\downarrow \partial_k^- \\ K_n \\ \vdots \end{array}\right) \times \left(\begin{array}{c} \vdots \\ K'_{n+1} \\ {\partial_k^+}' \Big\downarrow \Big\downarrow {\partial_k^-}' \\ K'_n \\ \vdots \end{array}\right) \cong
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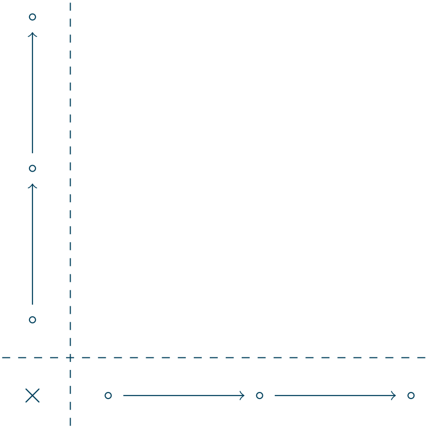
The Cartesian product in pcSet is deduced form the Cartesian product in $\mathcal{S}et$

Example of Cartesian product



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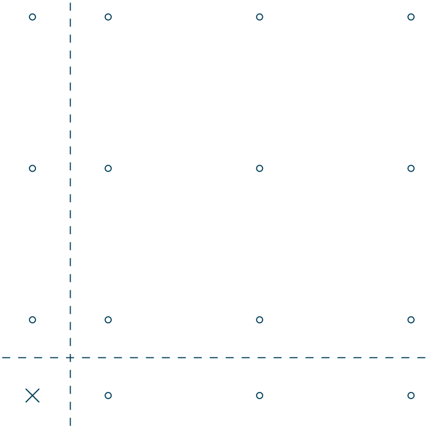

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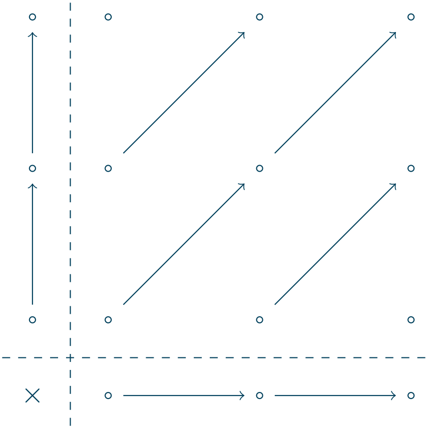

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Remark that $\square^+(-,1) = \{\cdot \to \cdot\}$ and defines the standard n-cube as $\square_n^+ := \square^+(-,n)$. Then

$$\square_n^+ = \bigotimes_{i=1}^{n} \square_1^+$$

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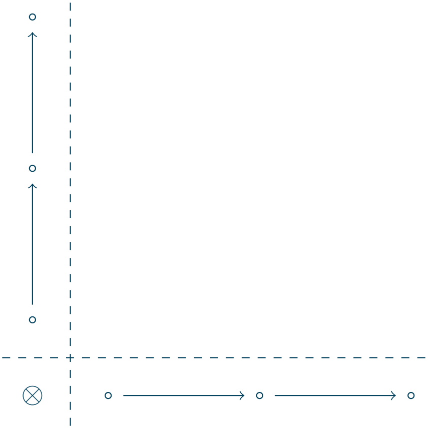

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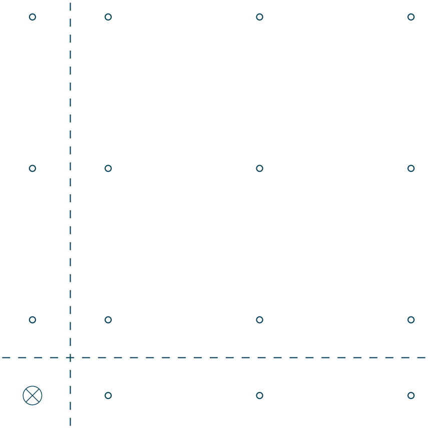

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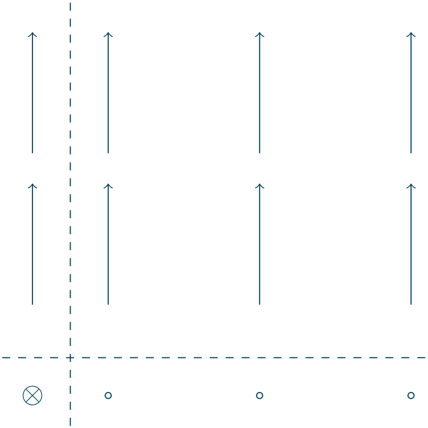

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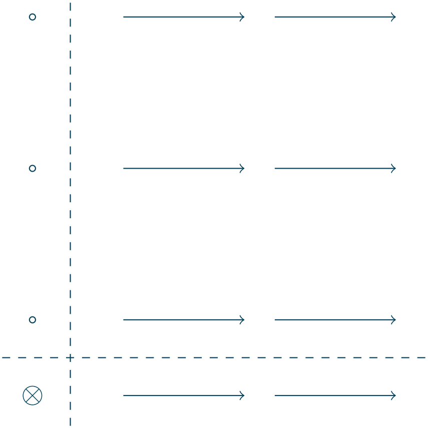

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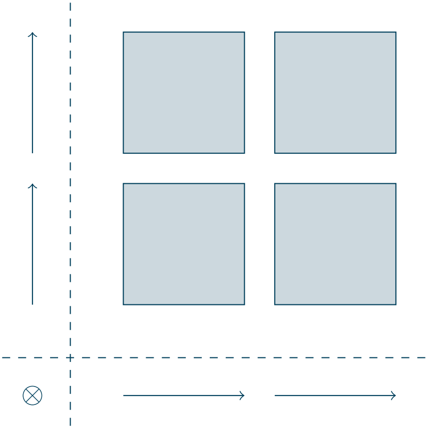

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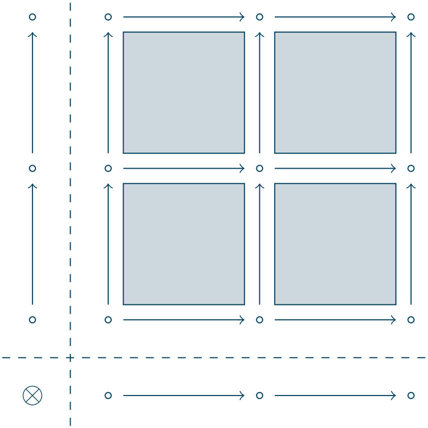

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Consider the diagram made of the arrows

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That construction defines a functor $\rceil \_ \lfloor : \mathit{pcSet} \to \mathcal{C}$.

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- in particular for $x, y \in K_{n+1}$, the equality $\partial_k^\varepsilon x = \partial_{k'}^{\varepsilon'} y$ makes
  the k^{th} ε-face of $\{x\} \times [0,1]^{n+1}$ be identified with
  the k'^{th} ε'-face of $\{y\} \times [0,1]^{n+1}$.
  This is the way cubes are "glued" with each other.

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  - $|K \otimes K'| \cong |K| \times |K'|$ in $\mathcal{Top}$ and $\mathcal{Haus}$ for all K and K' with K or K' finite. The finiteness of K or K' implies that the corresponding realization is compact, hence exponentiable

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- Then we have $|\_| \dashv Sing$

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- $\upharpoonleft K \downharpoonright_{d\mathcal{Top}} \quad \cong \quad \upharpoonleft K \downharpoonright_{d\mathcal{Top}_{\mathsf{f}}}$
- There exists a precubical set K such that $\upharpoonleft K \downharpoonright \quad \cong \quad \vec{\mathbb{S}}^3$

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- Cubical sets are presheaves over $\square$ i.e. $pc\mathcal{S}et = \mathcal{S}et^{\square^{\mathrm{op}}}$

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- if $w : a \to b$ and $w' : b \to c$ then $w'w$ is obtained by replacing, for $k \in \{0, \ldots, b-1\}$, the occurrence of x_k in w' (if any) by the k^{th} letter of w.

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That construction defines a functor $\rceil\_\lfloor : c\mathcal{S}et \to \mathcal{C}$.

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$$
\begin{tikzcd}
 & {|K|} & \\
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\end{tikzcd}
$$

Hence $\phi_{\sigma_{k,n}(x)}(t_0, \ldots, t_n)$ does not depend on t_k.

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- $\rceil K \lfloor_{\mathcal{Str}} \quad \cong \quad \rceil K \lfloor_{\mathcal{Str}_{\mathsf{d}}}$
- $\rceil K \lfloor_{\mathcal{dTop}}$ and $\rceil K \lfloor_{\mathcal{dTop}_{\mathsf{f}}}$ may differ.

Cartesian product of cubical sets

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$$\left(\begin{array}{c}\vdots \\ K_{n+1} \\ \partial_k^+ \Big\downarrow \; \Big\uparrow \sigma_k \; \Big\downarrow \partial_k^- \\ K_n \\ \vdots\end{array}\right) \times \left(\begin{array}{c}\vdots \\ K'_{n+1} \\ \partial_k^{+\prime} \Big\downarrow \; \Big\uparrow \sigma'_k \; \Big\downarrow \partial_k^{-\prime} \\ K'_n \\ \vdots\end{array}\right) \cong$$

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The Cartesian product in pcSet is deduced form the Cartesian product in $\mathcal{S}et$

Cartesian product of two segments in $c\mathcal{S}et$

Compute the product $\square_1 \times \square_1$

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For $x \otimes y \in K_i \times K'_j$ with $i + j = n$ the k^{th} face map, with $0 \leqslant k < n$, is given by

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The "segment" is $\square(-, 1)$ and the standard n-cube is $\square_n := \square(-, n)$. We have

$$\square_n = \bigotimes_{i=1}^{n} \square_1$$

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Theorem

Nonabelian Algebraic Topology, Brown, R., Higgins, P. J., and Sivera R., EMS, 2011. Proposition 11.1.17, p.372

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For any "topological space" X, the counit at X

$$\varepsilon_X : |Sing\ X| \to X$$

of the adjunction $\text{“}\mathcal{Top}\text{”} \underset{|\_|}{\overset{\text{Sing}}{\rightleftarrows}} c\mathcal{S}et$ is a weak homotopy equivalence.

("topological space" maybe mean compactly generated space here.)

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we would like to known when it is an isomorphism.

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