DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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MPRI : Concurrency (2.3)

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THE FUNDAMENTAL CATEGORY

Abstract setting

Natural Transformations

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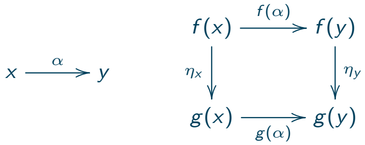

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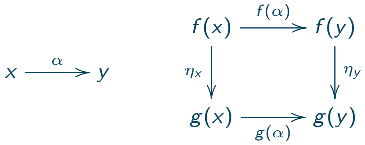
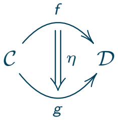


This description is summarized by the following diagram



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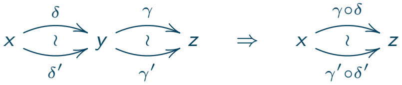

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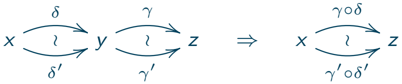

Hence the $\sim$-equivalence class of $\gamma \circ \delta$ does not depend on the $\sim$-equivalence classes of γ and δ and we have a quotient category $\mathcal{C}/\sim$ in which the composition is given by

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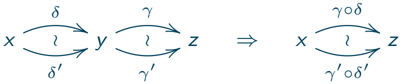

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The quotient map $q : \gamma \in \mathrm{Mo}(\mathcal{C}) \mapsto [\gamma] \in \mathrm{Mo}(\mathcal{C})/\sim$ induces a functor $q : \mathcal{C} \to \mathcal{C}/\sim$

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The collection of quotient functors q_X, for X ranging through the objects of $\mathcal{C}$, provides a natural transformation from P to $\overrightarrow{\pi_1}$.

The directed path functor

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- For $\delta : [0,r] \to X$ and $\gamma : [0,r'] \to X$ with $\delta(r) = \gamma(0)$, define the concatenation

$$\gamma \cdot \delta : [0, r + r'] \longrightarrow X$$

$$t \longmapsto \begin{cases} \delta(t) & \text{if } t \leqslant r \\ \gamma(t-r) & \text{if } t \geqslant r \end{cases}$$

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Natural congruences from directed homotopies

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Given $x, y \in X$ and $r \in \mathbb{R}_+$, the relation $\sim_X$ is an equivalence relation on the set

$$\bigcup_{r \in \mathbb{R}_+} \{\gamma \in \mathcal{L}po([0, r], X) \mid \gamma(0) = x;\ \gamma(r) = y\}$$

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The mapping $h' * h : [0, r + r'] \times [0, q] \to X$ defined by

$$h' * h(t, s) \quad = \quad \begin{cases} h(t, s) & \text{if } 0 \leqslant t \leqslant r \\ h'(t - r, s) & \text{if } r \leqslant t \leqslant r + r' \end{cases}$$

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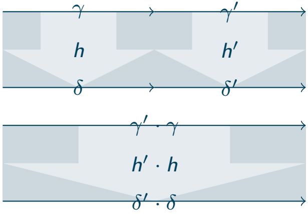

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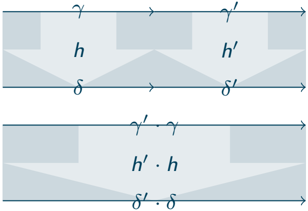

If h and h' are ((weakly) directed) homotopies, then so is their juxtaposition $h' \cdot h$.

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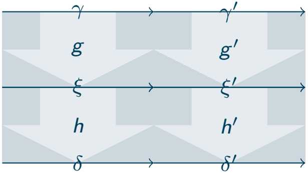

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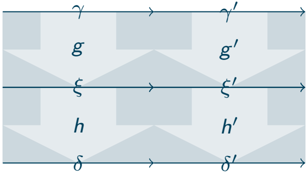

then it comes

$$(g' * h') \cdot (g * h) = (g' \cdot g) * (h' \cdot h)$$

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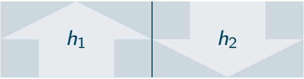

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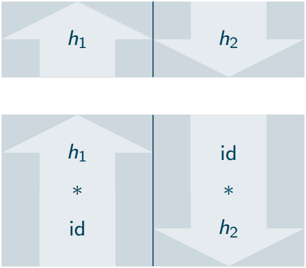

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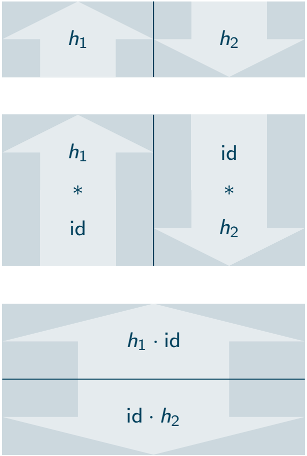

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The relation $\sim_X$ is a congruence on $P(X)$

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If $\gamma, \gamma' : [0, r] \to X$ are ((weakly) di)homotopic, then so are $f \circ \gamma, f \circ \gamma' : [0, r] \to Y$.

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- The fundamental groupoid functor $\Pi_1 : \mathcal{Top} \to \mathcal{Grd}$ is obtained by substituting "paths" and "homotopies" to "directed paths" and "elementary homotopies".

Basic properties and computations

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- In particular the fundamental category of a local pospace has no isomorphism but its identities.

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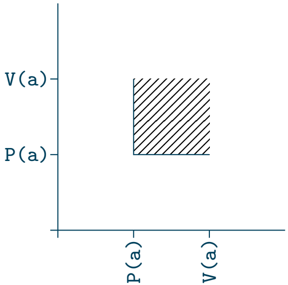

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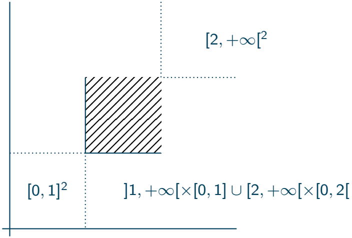

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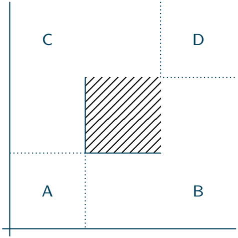

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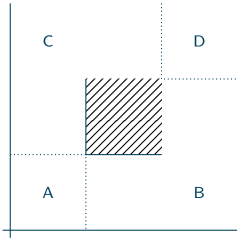

If $x \leqslant^2 y$, then $\vec{\pi}_1 X(x,y)$ only depends on the elements of the partition x and y belong to.

$\to$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

CATEGORY OF COMPONENTS

Motivation

Skeleta and equivalences of categories

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- The skeleton of the fundamental groupoid of a path-connected space is the fundamental group of this space.
- Problem: The fundamental category of a local pospace has no isomorphisms but its identities, hence it is its own skeleton.

Loop-free and one-way categories

The categories *LfCat* and *OwCat*

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- A loop-free category is its own skeleton
- A category is one-way iff its skeleton is loop-free

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$$\begin{array}{ccc} \mathcal{C}/\mathcal{R} & & \\ {\scriptstyle q}\uparrow & \searrow^{g} & \\ \mathcal{C} & \xrightarrow[f]{} & \mathcal{D} \end{array}$$

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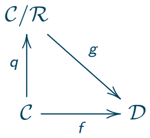

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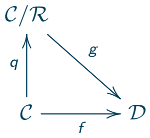

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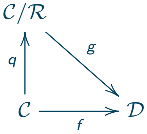

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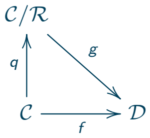

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  - $(\mathbb{N}, +, 0)$ with $0 \mathcal{R} n$ for some $n \in \mathbb{N}$.

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Systems of weak isomorphisms

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- One may have $\mathcal{C}(\partial^{+}\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^{-}\sigma) = \emptyset$

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  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^+\sigma, z) \;\to\; \gamma \circ \sigma \in \mathcal{C}(\partial^- \sigma, z)$
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- One may have $\mathcal{C}(\partial^+\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^- \sigma) = \emptyset$
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Potential weak isomorphisms

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- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $\mathcal{C}(x, y)$ contains a potential weak isomorphism, then it is a singleton
  Requires the assumption that $\mathcal{C}$ is one-way

An example of potential weak isomorphism

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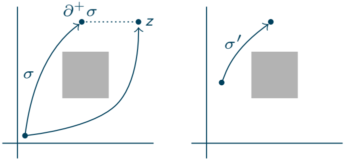

Due to the lower dipath, the σ, z-precomposition is not bijective; yet σ' is a potential weak isomorphism.

An unwanted example of potential weak isomorphism

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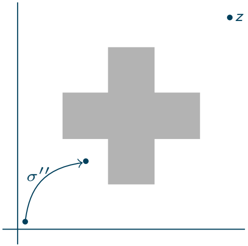

Note that σ'' is a potential weak isomorphism though there exists a morphism from $\partial^{-}\sigma''$ to z but none from $\partial^{+}\sigma''$ to z.

Stability under pushout and pullback

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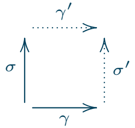
- A collection of morphisms Σ is said to be stable under pushout when for all $\sigma \in \Sigma$, for all γ with $\partial^- \gamma = \partial^- \sigma$, the pushout of σ along γ exists and belongs to Σ



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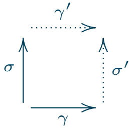

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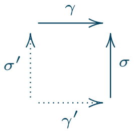

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- The collection Σ_∞ is stable under the action of $\mathrm{Aut}(\mathcal{C})$

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- Hence we suppose the systems of weak isomorphisms are closed under composition

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- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.

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- Given a one-way category $\mathcal{C}$ we have:

All the systems of weak isomorphisms of $\mathcal{C}$ are pure

The locale of systems of weak isomorphisms

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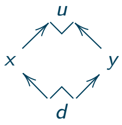

Diagram 1

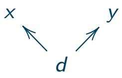

Diagram 2

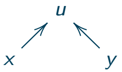

Diagram 3

Structure of the Σ-components

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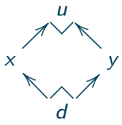
4. $\mathcal{C} = \mathcal{K}$ iff $\mathcal{C}$ is a prelattice, and Σ is the greatest system of weak isomorphisms of $\mathcal{C}$ i.e. all the morphisms in this case.



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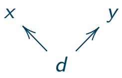

Diagram 2

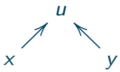

Diagram 3

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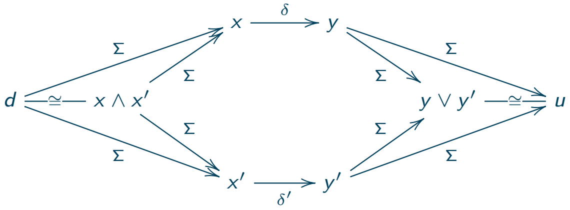

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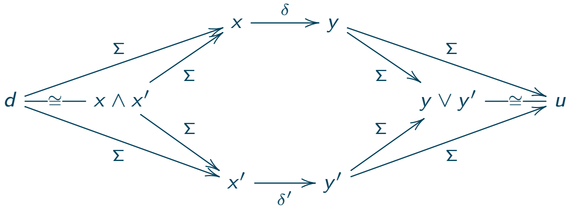

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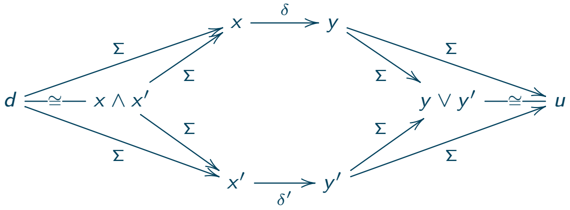

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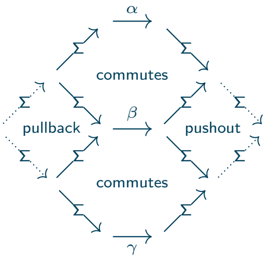

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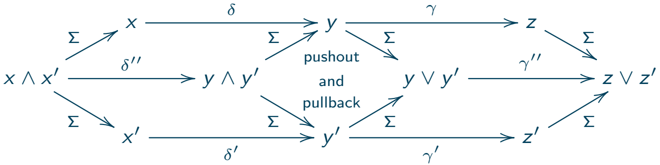

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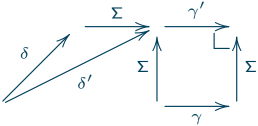

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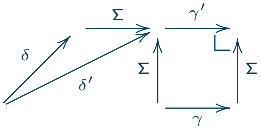

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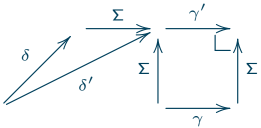

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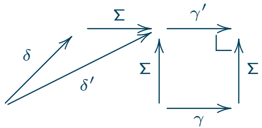

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Properties

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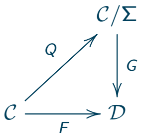

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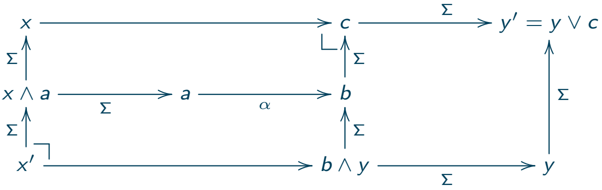

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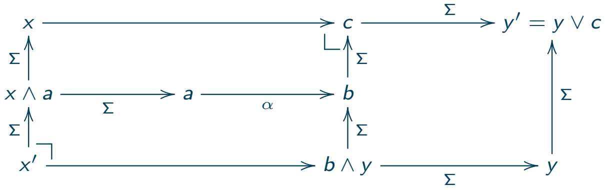

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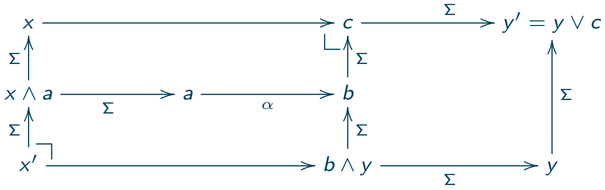

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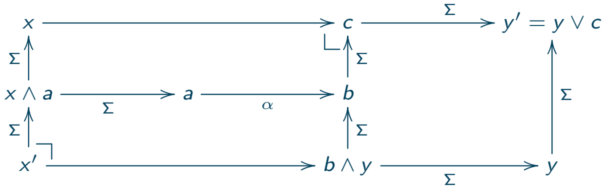

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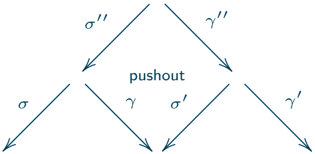

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  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.

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  - If $\mathcal{C}/\Sigma$ is finite then there exists an admissible choice
  - If $\mathcal{C}/\Sigma$ is infinite the existence of an admissible choice is a open question

Examples

Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$

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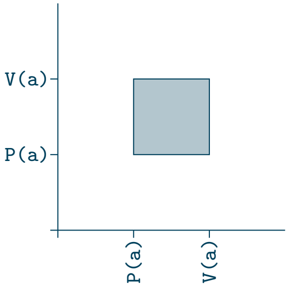

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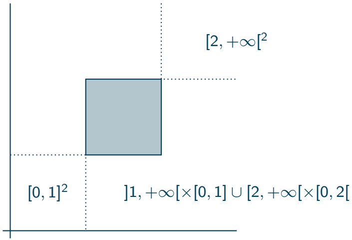

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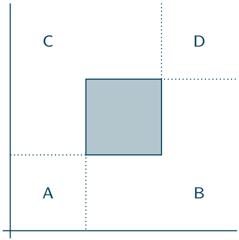

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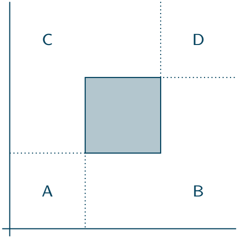

Let x, y such that $x \leqslant^2 y$, then $\overrightarrow{\pi_1}X(x, y)$ only depends on which elements of the partition x and y belong to

$\to$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

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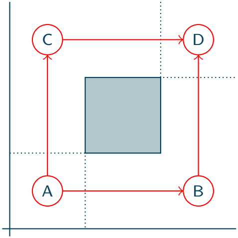

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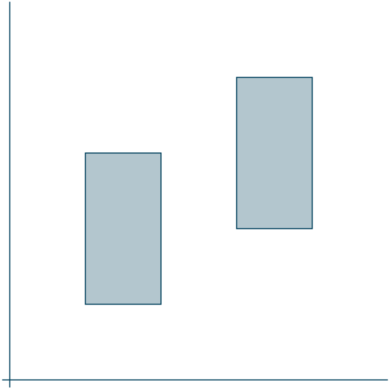
C			σ	γ'
D				σ

Two rectangles

Two rectangles



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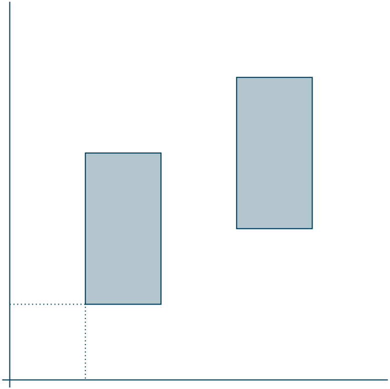

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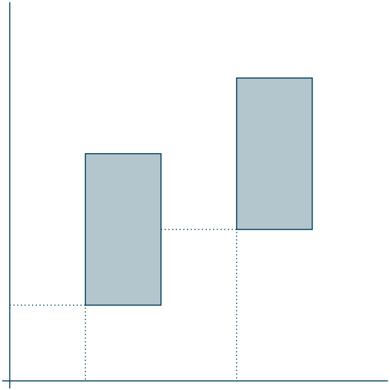

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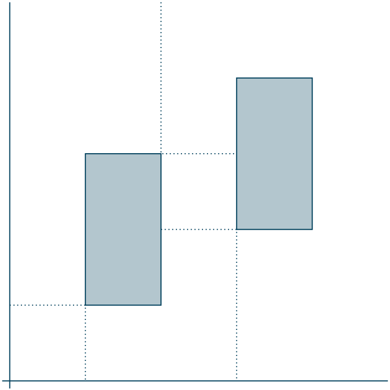

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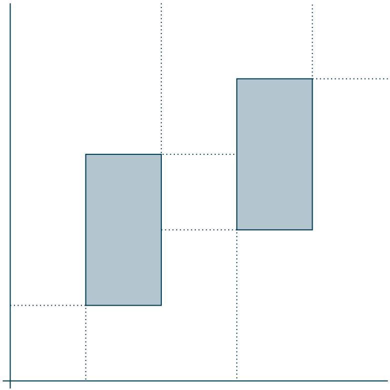

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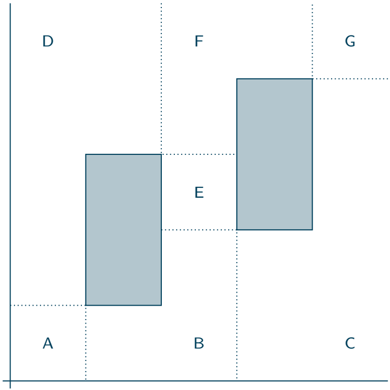

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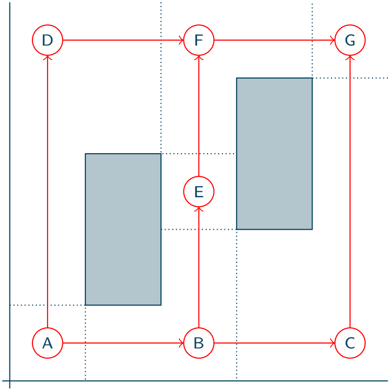

Swiss Flag

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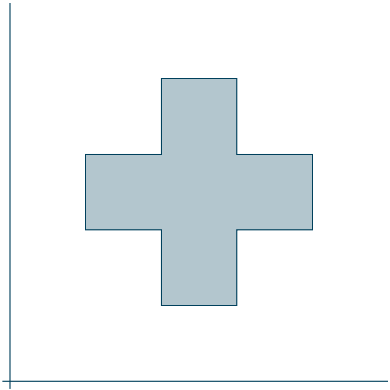

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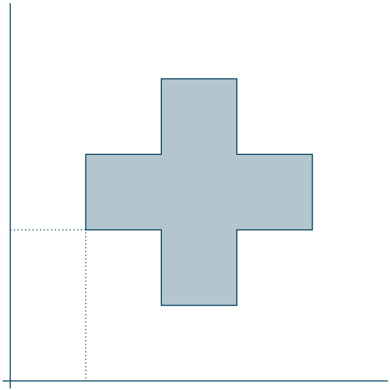

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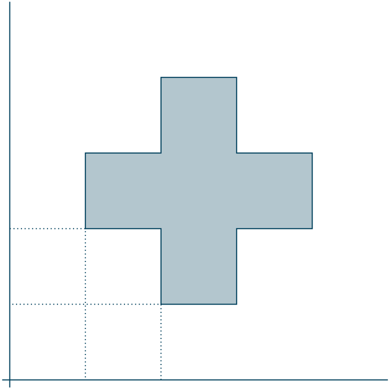

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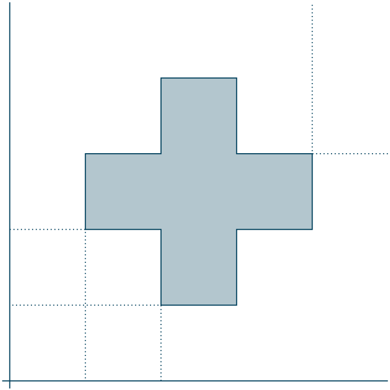

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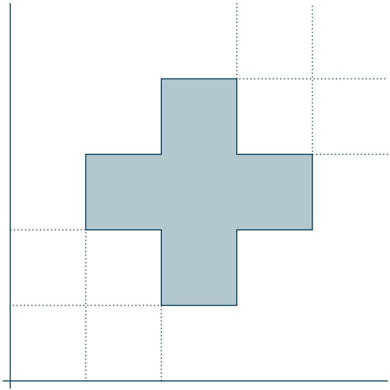

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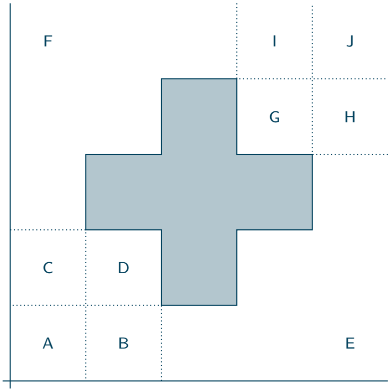

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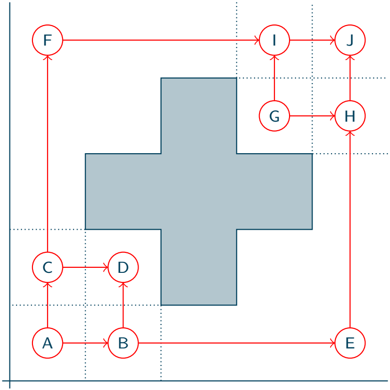

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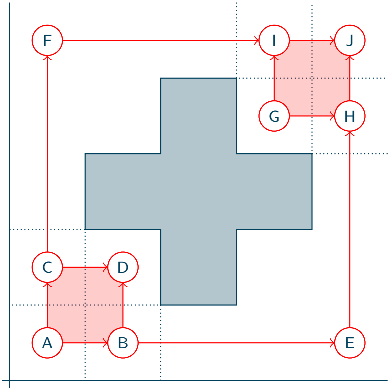

Achronal overlaping square

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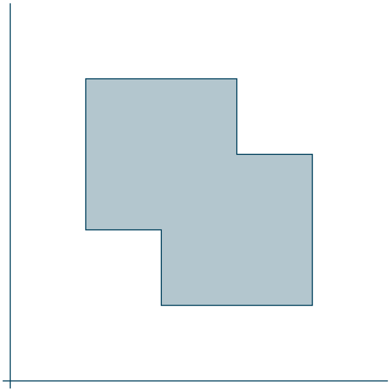

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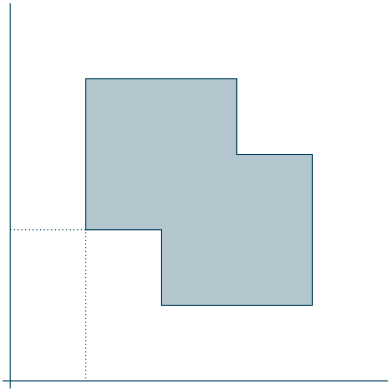

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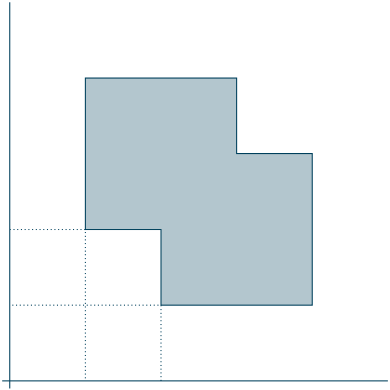

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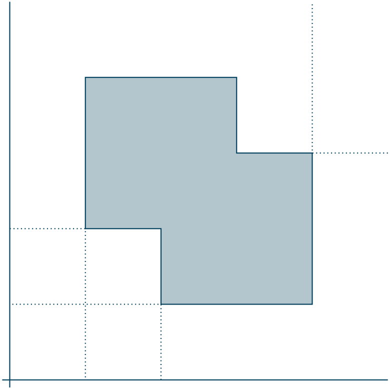

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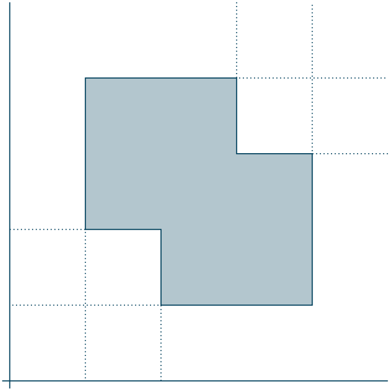

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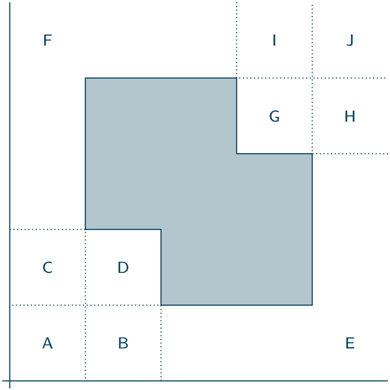

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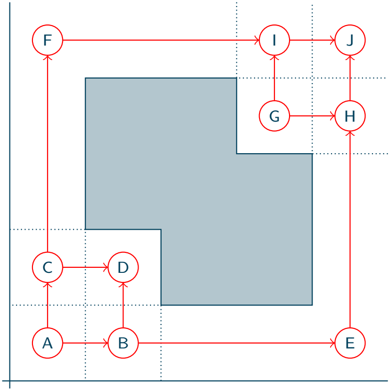

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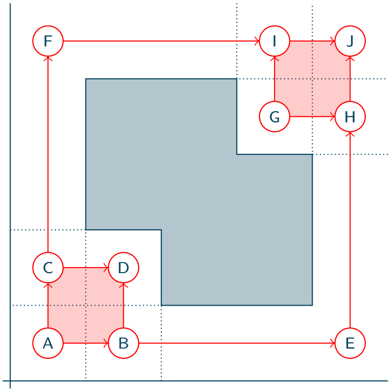

Diagonal overlaping squares

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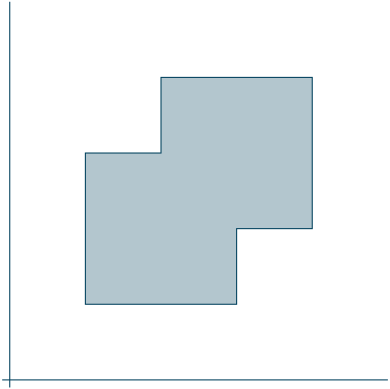

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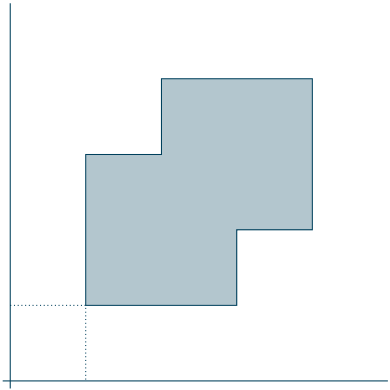

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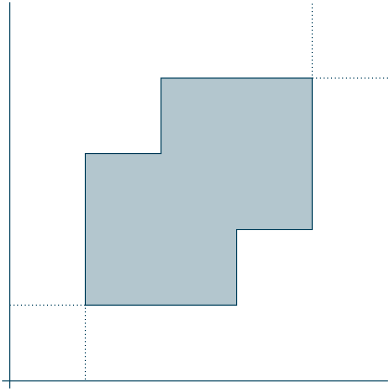

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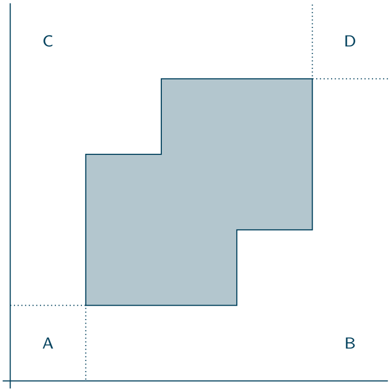

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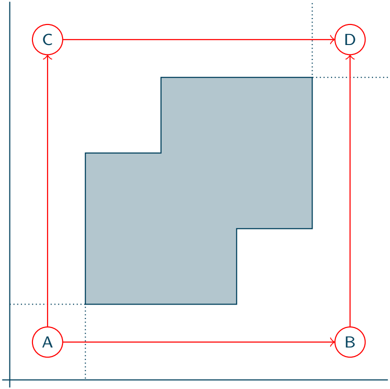

The floating cube

Non potential weak isomorphisms

The floating cube

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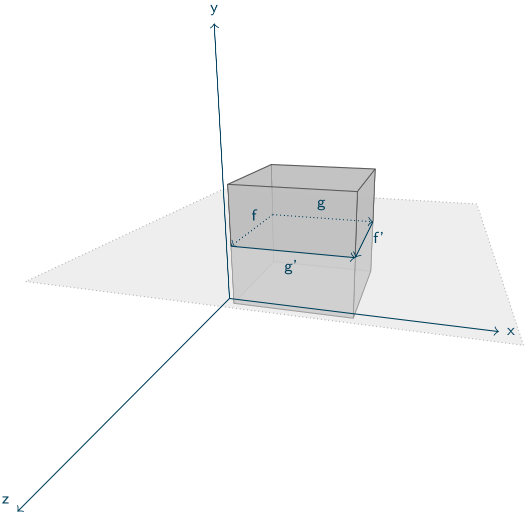

The floating cube

A "vee" that does not extend to a pushout

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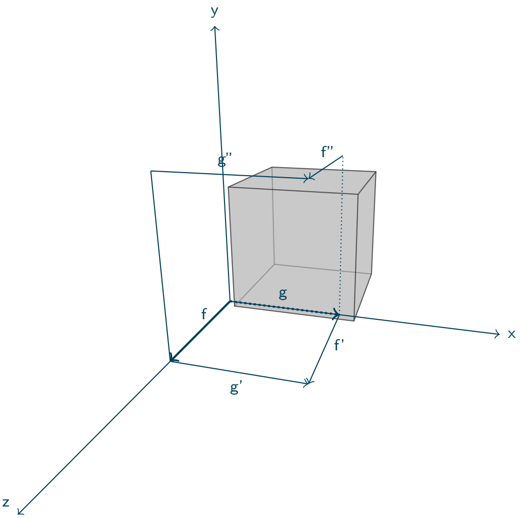
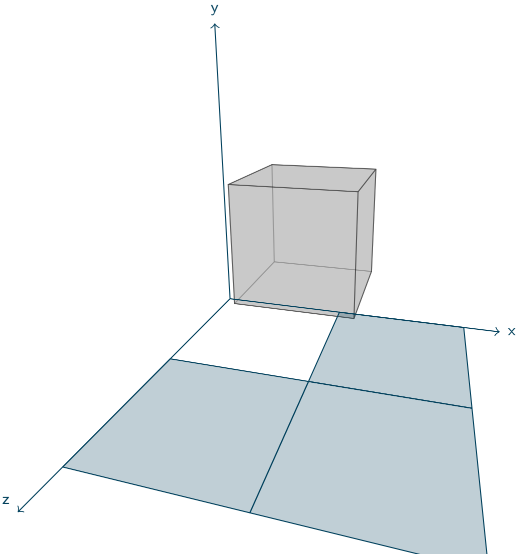


The floating cube

Some pushouts squares

The floating cube

Some pushouts squares



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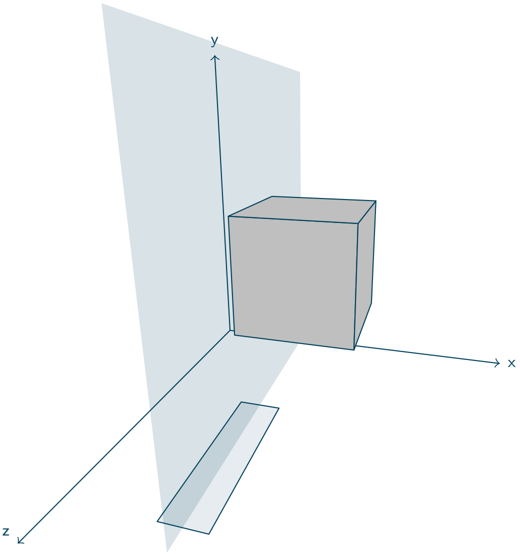

The floating cube

The floating cube

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  - Therefore $f', g' \notin \Sigma$ (anyway they do not preserve the future cones)

The floating cube

boundaries of the components

The floating cube

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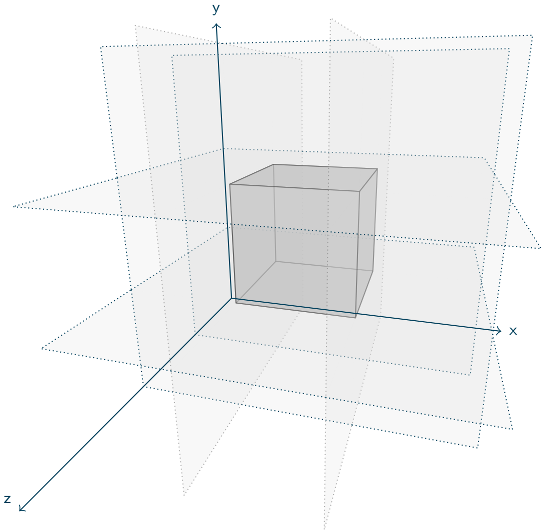

Finite connected loop-free categories

Commutative monoid

of nonempty finite connected loop-free categories

Commutative monoid

of nonempty finite connected loop-free categories

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- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
- $1 \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times 1$
- $\mathcal{A} \times \mathcal{B} \cong \mathcal{B} \times \mathcal{A}$
- The collection of isomorphism classes of nonempty finite connected loop-free categories is thus a commutative monoid $\mathcal{M}$

Commutative monoid

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  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$
- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
- $1 \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times 1$
- $\mathcal{A} \times \mathcal{B} \cong \mathcal{B} \times \mathcal{A}$
- The collection of isomorphism classes of nonempty finite connected loop-free categories is thus a commutative monoid $\mathcal{M}$

The commutative monoid $\mathcal{M}$ is free.

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- Any element of $\mathcal{M}$ freely generated by a graph, is prime

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- Conjecture

> If $P \in \mathcal{N}$ is prime and $\overrightarrow{\pi_1}(P)$ is not a lattice, then $\overrightarrow{\pi_0}(\overrightarrow{\pi_1}(P))$ is prime