THE FUNDAMENTAL CATEGORY
Abstract setting
Natural Transformations
morphisms of functors from $f : C \to D$ to $g : C \to D$
Natural Transformations

morphisms of functors from \( f : \mathcal{C} \to \mathcal{D} \) to \( g : \mathcal{C} \to \mathcal{D} \)

A natural transformation \( \eta : f \to g \) is a collection of morphisms \( (\eta_x)_{x \in \text{Ob}(\mathcal{C})} \) where \( \eta_x \in \mathcal{D}[f(x), g(x)] \) and such that for all \( \alpha \in \mathcal{C}[x, y] \) we have \( \eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x \) i.e. the following diagram commute
Natural Transformations

morphisms of functors from $f : \mathcal{C} \to \mathcal{D}$ to $g : \mathcal{C} \to \mathcal{D}$

A natural transformation $\eta : f \to g$ is a collection of morphisms $(\eta_x)_{x \in \text{Ob}(\mathcal{C})}$ where $\eta_x \in \mathcal{D}[f(x), g(x)]$ and such that for all $\alpha \in \mathcal{C}[x, y]$ we have $\eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x$ i.e. the following diagram commute.
Natural Transformations

morphisms of functors from $f : C \to D$ to $g : C \to D$

A natural transformation $\eta : f \to g$ is a collection of morphisms $(\eta_x)_{x \in \text{Ob}(C)}$ where $\eta_x \in D[f(x), g(x)]$ and such that for all $\alpha \in C[x, y]$ we have $\eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x$ i.e. the following diagram commute

\[
\begin{align*}
  x & \xrightarrow{\alpha} y \\
  f(x) & \xrightarrow{f(\alpha)} f(y) \\
  g(x) & \xrightarrow{g(\alpha)} g(y)
\end{align*}
\]

This description is summarized by the following diagram

\[
\begin{array}{c}
\circlearrowright \\
\downarrow^\eta \\
C \xrightarrow{f} D \\
\downarrow^g
\end{array}
\]
Congruences on small categories

A congruence on a small category $C$ is an equivalence relation $\sim$ over $\text{Mo}(C)$ such that:

- $\gamma \sim \gamma' \Rightarrow \partial^- \gamma = \partial^- \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$,
- $\gamma \sim \gamma', \delta \sim \delta' \Rightarrow \gamma \circ \delta \sim \gamma' \circ \delta'$,

In diagrams we have:

\[
\begin{array}{ccc}
\gamma & \sim \gamma' \\
\delta & \sim \delta' \\
\partial^- \gamma & = & \partial^- \gamma' \\
\partial^+ \delta & = & \partial^+ \delta' \\
\end{array}
\]

\[
\begin{array}{ccc}
x & \delta & \rightarrow \\
\rightarrow & \delta' & \rightarrow \\
\gamma & \sim \gamma' & \rightarrow \\
\rightarrow & \gamma' & \sim \gamma' \\
\end{array}
\]

Hence the $\sim$-equivalence class of $\gamma \circ \delta$ only depends on the $\sim$-equivalence classes of $\gamma$ and $\delta$ and we have a quotient category $C/\sim$ in which the composition is given by $\left[ \gamma \right] \circ \left[ \delta \right] = \left[ \gamma \circ \delta \right]$. The quotient map $q: \gamma \in \text{Mo}(C) \mapsto \left[ \gamma \right] \in \text{Mo}(C/\sim)$ induces a functor $q: C \rightarrow C/\sim$.
A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\text{Mo}(\mathcal{C})$ such that:

1. $\gamma \sim \gamma'$ implies $\partial^- \gamma = \partial^- \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$.
2. $\delta \sim \delta'$ and $\partial^- \gamma = \partial^+ \delta$ implies $\gamma \circ \delta \sim \gamma' \circ \delta'$.

Hence the $\sim$-equivalence class of $\gamma \circ \delta$ only depends on the $\sim$-equivalence classes of $\gamma$ and $\delta$, and we have a quotient category $\mathcal{C}/\sim$ in which the composition is given by $\left[\gamma\right] \circ \left[\delta\right] = \left[\gamma \circ \delta\right]$. The quotient map $q : \gamma \in \text{Mo}(\mathcal{C}) \mapsto \left[\gamma\right] \in \text{Mo}(\mathcal{C})/\sim$ induces a functor $q : \mathcal{C} \to \mathcal{C}/\sim$. 
Congruences on small categories

A congruence on a small category \( C \) is an equivalence relation \( \sim \) over \( \text{Mo}(C) \) such that:

- \( \gamma \sim \gamma' \) \( \Rightarrow \) \( \partial^- \gamma = \partial^- \gamma' \) and \( \partial^+ \gamma = \partial^+ \gamma' \)
Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\text{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \Rightarrow \partial^* \gamma = \partial^* \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$
- $\gamma \sim \gamma', \delta \sim \delta'$ and $\partial^* \gamma = \partial^+ \delta \Rightarrow \gamma \circ \delta \sim \gamma' \circ \delta'$
Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\text{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \Rightarrow \partial \gamma = \partial \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$
- $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\partial \gamma = \partial^+ \delta \Rightarrow \gamma \circ \delta \sim \gamma' \circ \delta'$

In diagrams we have

\[
\begin{array}{ccc}
  x & \overset{l}{\longrightarrow} & y \\
  \delta' & & \gamma' \\
  \delta & \overset{l}{\longrightarrow} & z
\end{array}
\Rightarrow
\begin{array}{ccc}
  x & \overset{l}{\longrightarrow} & z \\
  \gamma' \circ \delta' & & \gamma \circ \delta
\end{array}
\]
Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\text{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \Rightarrow \partial \gamma = \partial \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$
- $\gamma \sim \gamma', \delta \sim \delta'$ and $\partial \gamma = \partial^+ \delta \Rightarrow \gamma \circ \delta \sim \gamma' \circ \delta'$

In diagrams we have

$$
\begin{align*}
\begin{array}{c}
x \\ \delta' \\
\gamma'
\end{array} & \xymatrix@C=2em{ & \ar[l]_{\delta} y & x } & \begin{array}{c}
z \\ \gamma
\end{array} \\
\gamma' & \xymatrix@C=2em{ & \ar[l]_{\gamma'} z & y } & \begin{array}{c}
z \\ \gamma' \circ \delta'
\end{array}
\end{align*}
$$

Hence the $\sim$-equivalence class of $\gamma \circ \delta$ only depends on the $\sim$-equivalence classes of $\gamma$ and $\delta$ and we have a quotient category $\mathcal{C}/\sim$ in which the composition is given by

$$
[\gamma] \circ [\delta] = [\gamma \circ \delta]
$$
Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\text{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \Rightarrow \partial^- \gamma = \partial^- \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$
- $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\partial^- \gamma = \partial^+ \delta \Rightarrow \gamma \circ \delta \sim \gamma' \circ \delta'$

In diagrams we have

\[
\begin{aligned}
\begin{tikzpicture}
  \node (x) at (0,0) {x};
  \node (y) at (2,0) {y};
  \node (z) at (4,0) {z};
  \node (x') at (0,-1) {x'};
  \node (y') at (2,-1) {y'};
  \node (z') at (4,-1) {z'};
  \draw[->] (x) to node {$\delta$} (y);
  \draw[->] (y) to node {$\gamma$} (z);
  \draw[->] (x) to node[swap] {$\delta'$} (x');
  \draw[->] (y) to node[swap] {$\gamma'$} (y');
  \draw[->] (z) to node[swap] {$\text{wr}$} (z');
\end{tikzpicture}
\end{aligned}
\]

\[
\begin{aligned}
\begin{tikzpicture}
  \node (x) at (0,0) {x};
  \node (y) at (2,0) {y};
  \node (z) at (4,0) {z};
  \node (x') at (0,-1) {x'};
  \node (y') at (2,-1) {y'};
  \node (z') at (4,-1) {z'};
  \draw[->] (x) to node {$\gamma \circ \delta$} (y);
  \draw[->] (y) to node {$\gamma' \circ \delta'$} (z);
  \draw[->] (x) to node[swap] {$\text{wr}$} (x');
  \draw[->] (y) to node[swap] {$\text{wr}$} (y');
  \draw[->] (z) to node[swap] {$\text{wr}$} (z');
\end{tikzpicture}
\end{aligned}
\]

Hence the $\sim$-equivalence class of $\gamma \circ \delta$ only depends on the $\sim$-equivalence classes of $\gamma$ and $\delta$ and we have a quotient category $\mathcal{C}/\sim$ in which the composition is given by

\[
[\gamma] \circ [\delta] = [\gamma \circ \delta]
\]

The quotient map $q : \gamma \in \text{Mo}(\mathcal{C}) \mapsto [\gamma] \in \text{Mo}(\mathcal{C})/\sim$ induces a functor $q : \mathcal{C} \to \mathcal{C}/\sim$
Natural congruences on a functor $P : C \to \text{Cat}$
Natural congruences on a functor $P : \mathcal{C} \to \text{Cat}$

A natural congruence on a functor $P : \mathcal{C} \to \text{Cat}$ is a collection of congruences $\sim_X$ on $PX$, for $X$ ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \implies P(f)(\alpha) \sim_Y P(f)(\beta)$$
Natural congruences on a functor $P : C \to \text{Cat}$

A natural congruence on a functor $P : C \to \text{Cat}$ is a collection of congruences $\sim_X$ on $PX$, for $X$ ranging through the objects of $C$, such that for all morphisms $f : X \to Y$ of $C$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \implies P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\xrightarrow{\pi_1} : C \to \text{Cat}$ as follows:
Natural congruences on a functor $P : \mathcal{C} \to \mathbf{Cat}$

A natural congruence on a functor $P : \mathcal{C} \to \mathbf{Cat}$ is a collection of congruences $\sim_X$ on $PX$, for $X$ ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \implies P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\pi_1 : \mathcal{C} \to \mathbf{Cat}$ as follows:

- for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/_X$
Natural congruences on a functor $P : \mathcal{C} \to \text{Cat}$

A natural congruence on a functor $P : \mathcal{C} \to \text{Cat}$ is a collection of congruences $\sim_X$ on $PX$, for $X$ ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \implies P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\overrightarrow{\pi_1} : \mathcal{C} \to \text{Cat}$ as follows:

- for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/ \sim_X$
- for all $f : X \to Y$ in $\mathcal{C}$
Natural congruences on a functor $P : \mathcal{C} \to \mathbf{Cat}$

A natural congruence on a functor $P : \mathcal{C} \to \mathbf{Cat}$ is a collection of congruences $\sim_X$ on $PX$, for $X$ ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \implies P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\pi_1^\triangledown : \mathcal{C} \to \mathbf{Cat}$ as follows:

- for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/\sim_X$
- for all $f : X \to Y$ in $\mathcal{C}$

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow q_X & & \downarrow q_Y \\
\pi_1^\triangledown X & \xrightarrow{\pi_1^\triangledown f} & \pi_1^\triangledown Y \\
\end{array}
\]
A natural congruence on a functor $P : \mathcal{C} \to \mathbf{Cat}$ is a collection of congruences $\sim_X$ on $PX$, for $X$ ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \implies P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\overrightarrow{\pi_1} : \mathcal{C} \to \mathbf{Cat}$ as follows:
- for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/\sim_X$
- for all $f : X \to Y$ in $\mathcal{C}$

![Diagram](attachment://natural_congruences.png)

The collection of quotient functors $q_X$, for $X$ ranging through the objects of $\mathcal{C}$, provides a natural transformation from $P$ to $\overrightarrow{\pi_1}$.
The directed path functor
Object part

Let $X$ be a locally ordered space.
- The objects of $PX$ are the points of $X$.
- The homset $PX(\mathbf{a}, \mathbf{b})$ is $\bigcup_{r \geq 0} \{ \gamma \in L_{po}([0, r], X) | \gamma(0) = \mathbf{a} \text{ and } \gamma(r) = \mathbf{b} \}$
- For $\delta : [0, r] \to X$ and $\gamma : [0, r'] \to X$ with $\delta(r) = \gamma(0)$, define the concatenation $\gamma \cdot \delta : [0, r + r'] \to X$ by:

$$
\begin{align*}
\gamma \cdot \delta(t) &= \begin{cases} 
\delta(t) & \text{if } t \leq r \\
\gamma(t - r) & \text{if } t \geq r
\end{cases}
\end{align*}
$$
Let $X$ be a locally ordered space.
Object part

Let \( X \) be a locally ordered space.

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Let $X$ be a locally ordered space.
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$$
\bigcup_{r \geq 0} \{ \gamma \in Lpo([0, r], X) \mid \gamma(0) = a \text{ and } \gamma(r) = b \}
$$
Let $X$ be a locally ordered space.

- The objects of $PX$ are the points of $X$.
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$$\bigcup_{r \geq 0} \{ \gamma \in \text{Lpo}([0, r], X) \mid \gamma(0) = a \text{ and } \gamma(r) = b \}$$

- For $\delta : [0, r] \to X$ and $\gamma : [0, r'] \to X$ with $\delta(r) = \gamma(0)$, define the concatenation

$$\gamma \cdot \delta : [0, r + r'] \to X$$

$$t \mapsto \begin{cases} 
  \delta(t) & \text{if } t \leq r \\
  \gamma(t - r) & \text{if } t \geq r 
\end{cases}$$
Fundamental category

Directed path functor

Morphism part

The (Moore) path category construction gives rise to a functor $P$ from $L_{po}$ to $Cat$ since for all $f \in L_{po}(X,Y)$ and all paths $\gamma$ on $X$, the composite $f \circ \gamma$ is a path on $Y$. 

\[
P : L_{po} \rightarrow Cat
\]
Morphism part

The (Moore) path category construction gives rise to a functor \( P \) from \( Lpo \) to \( \text{Cat} \) since for all \( f \in Lpo(X, Y) \) and all paths \( \gamma \) on \( X \), the composite \( f \circ \gamma \) is a path on \( Y \).
The (Moore) path category construction gives rise to a functor $P$ from $\text{Lpo}$ to $\text{Cat}$ since for all $f \in \text{Lpo}(X, Y)$ and all paths $\gamma$ on $X$, the composite $f \circ \gamma$ is a path on $Y$.

$$
P : \text{Lpo} \rightarrow \text{Cat}
$$

$$
\begin{array}{cc}
X & \text{PX} \\
\downarrow f & \downarrow Pf \\
Y & \text{PY}
\end{array}
$$
The (Moore) path category construction gives rise to a functor $P$ from $Lpo$ to $Cat$ since for all $f \in Lpo(X, Y)$ and all paths $\gamma$ on $X$, the composite $f \circ \gamma$ is a path on $Y$.

\[
\begin{array}{ccc}
P : & Lpo & \rightarrow & Cat \\
& X & \rightarrow & PX \\
& f & \downarrow & Pf \\
& Y & \rightarrow & PY \\
\end{array}
\]

with

\[
\begin{array}{ccc}
Pf : & PX & \rightarrow & PY \\
p & \rightarrow & f(p) \\
\gamma & \downarrow & \rightarrow f \circ \gamma \\
q & \rightarrow & f(q) \\
\end{array}
\]
Natural congruences from directed homotopies
Equivalent directed paths on a local pospace $X$
Equivalent directed paths on a local pospace $X$

An elementary homotopy is a \textit{finite} concatenation of directed and anti-directed homotopies.
Equivalent directed paths on a local pospace $X$

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in Lpo([0, r], X)$, then $\gamma$ and $\gamma \circ \theta$ are dihomotopic.
Equivalent directed paths on a local pospace $X$

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Two directed paths $\gamma : [0, r'] \to X$ and $\delta : [0, r''] \to X$ on a local pospace are said to be equivalent (denoted by $\sim_X$) when there exists two reparametrizations $\theta : [0, r] \to [0, r']$ and $\psi : [0, r] \to [0, r'']$ such that there is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$. 

Equivalent directed paths on a local pospace $X$

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The relation $\sim_X$ is \textit{symmetric} because ...
Equivalent directed paths on a local pospace $X$

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in Lpo([0, r], X)$, then $\gamma$ and $\gamma \circ \theta$ are dihomotopic.

Two directed paths $\gamma : [0, r'] \to X$ and $\delta : [0, r''] \to X$ on a local pospace are said to be *equivalent* (denoted by $\sim_X$) when there exists two *reparametrizations* $\theta : [0, r] \to [0, r']$ and $\psi : [0, r] \to [0, r'']$ such that there is an *elementary homotopy* between $\gamma \circ \theta$ and $\delta \circ \psi$.

The relation $\sim_X$ is *symmetric* because if $h(s, t)$ is an elementary homotopy, then so is the mapping $(s, t) \mapsto h(-s, t)$.
Equivalent directed paths on a local pospace $X$

An elementary homotopy is a \textit{finite} concatenation of directed and anti-directed homotopies.

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The relation $\sim_X$ is \textit{symmetric} because if $h(s, t)$ is an elementary homotopy, then so is the mapping $(s, t) \mapsto h(-s, t)$.

The relation $\sim_X$ is \textit{transitive} because ...
Equivalent directed paths on a local pospace $X$

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The relation $\sim_X$ is \textbf{symmetric} because if $h(s, t)$ is an elementary homotopy, then so is the mapping $(s, t) \mapsto h(-s, t)$.

The relation $\sim_X$ is \textbf{transitive} because a concatenation of elementary homotopies is an elementary homotopy.
Equivalent directed paths on a local pospace $X$

An elementary homotopy is a \textit{finite} concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in \mathcal{L}po([0, r], X)$, then $\gamma$ and $\gamma \circ \theta$ are dihomotopic.

Two directed paths $\gamma : [0, r'] \to X$ and $\delta : [0, r''] \to X$ on a local pospace are said to be \textit{equivalent} (denoted by $\sim_X$) when there exists two \textit{reparametrizations} $\theta : [0, r] \to [0, r']$ and $\psi : [0, r] \to [0, r'']$ such that there is an \textit{elementary homotopy} between $\gamma \circ \theta$ and $\delta \circ \psi$.

The relation $\sim_X$ is \textit{symmetric} because if $h(s, t)$ is an elementary homotopy, then so is the mapping $(s, t) \mapsto h(-s, t)$.

The relation $\sim_X$ is \textit{transitive} because a concatenation of elementary homotopies is an elementary homotopy.

Given $x, y \in X$ and $r \in \mathbb{R}_+$, the relation $\sim_X$ is an equivalence relation on the set

$$\bigcup_{r \in \mathbb{R}_+} \{ \gamma \in \mathcal{L}po([0, r], X) \mid \gamma(0) = x; \gamma(r) = y \}$$
Juxtaposition of homotopies

horizontal composition

Let \( h: [0, r] \times [0, q] \to X \) and \( h': [0, r'] \times [0, q] \to X \) be homotopies from \( \gamma \) to \( \delta \) and from \( \gamma' \) to \( \delta' \) with \( \partial_- \gamma = \partial_+ \gamma' \).

The mapping \( h' \ast h: [0, r + r'] \times [0, q] \to X \) defined by

\[
h' \ast h(t, s) = \begin{cases} h(t, s) & \text{if } 0 \leq t \leq r \\ h'(t - r, s) & \text{if } r \leq t \leq r + r' \end{cases}
\]

is a homotopy from \( \gamma \) to \( \delta \).

If \( h \) and \( h' \) are (weakly) directed homotopies, then so is their juxtaposition \( h' \ast h \).
Juxtaposition of homotopies

horizontal composition

Let $h : [0, r] \times [0, q] \to X$ and $h' : [0, r'] \times [0, q] \to X$ be homotopies from $\gamma$ to $\delta$ and from $\gamma'$ to $\delta'$ with $\partial^+ \gamma = \partial^+ \gamma'$. If $h$ and $h'$ are ((weakly) directed) homotopies, then so is their juxtaposition $h' \cdot h$. 
Juxtaposition of homotopies

horizontal composition

Let \( h : [0, r] \times [0, q] \rightarrow X \) and \( h' : [0, r'] \times [0, q] \rightarrow X \) be homotopies from \( \gamma \) to \( \delta \) and from \( \gamma' \) to \( \delta' \) with \( \partial^+ \gamma = \partial^- \gamma' \).

The mapping \( h' \ast h : [0, r + r'] \times [0, q] \rightarrow X \) defined by

\[
(\gamma \ast h)(t, s) = \begin{cases} 
   h(t, s) & \text{if } 0 \leq t \leq r \\
   h'(t - r, s) & \text{if } r \leq t \leq r + r'
\end{cases}
\]

is a homotopy from \( \gamma \) to \( \delta \).
Juxtaposition of homotopies

horizontal composition

Let \( h : [0, r] \times [0, q] \to X \) and \( h' : [0, r'] \times [0, q] \to X \) be homotopies from \( \gamma \) to \( \delta \) and from \( \gamma' \) to \( \delta' \) with \( \partial^+ \gamma = \partial^+ \gamma' \).

The mapping \( h' * h : [0, r + r'] \times [0, q] \to X \) defined by

\[
h' * h(t, s) = \begin{cases} 
  h(t, s) & \text{if } 0 \leq t \leq r \\
  h'(t - r, s) & \text{if } r \leq t \leq r + r'
\end{cases}
\]

is a homotopy from \( \gamma \) to \( \delta \).

If \( h \) and \( h' \) are ((weakly) directed) homotopies, then so is their juxtaposition \( h' \cdot h \).
Suppose we have \( \delta, h, \xi, \gamma, g \) and \( \delta', h', \xi', \gamma', g' \) then it comes
\[
(g' \ast h') \cdot (g \ast h) = (g' \cdot g) \ast (h' \cdot h)
\]
Godement exchange law

Suppose we have

\[
\delta \quad h \quad \xi \quad \gamma \quad g
\]

then it comes

\[
(g' \ast h') \cdot (g \ast h) = (g' \cdot g) \ast (h' \cdot h)
\]
Godement exchange law

Suppose we have

\[ (g' \ast h') \cdot (g \ast h) = (g' \cdot g) \ast (h' \cdot h) \]
Applying Godement exchange law
Applying Godement exchange law
Applying Godement exchange law

\[ h \circ \text{id} = \text{id} \circ h \]
Applying Godement exchange law

\[ h_1 \cdot \text{id} = \text{id} \cdot h_2 \]
Equivalences are congruences
Equivalences are congruences

If:

\[ h \] is an elementary homotopy between \( \gamma \circ \theta \) and \( \delta \circ \psi \)

\[ h' \] is an elementary homotopy between \( \gamma' \circ \theta' \) and \( \delta' \circ \psi' \)

then \( h \cdot h' \) is an elementary homotopy from \( (\gamma \cdot \gamma') \circ (\theta \cdot \theta') \) to \( (\delta \cdot \delta') \circ (\psi \cdot \psi') \).

The relation \( \sim_X \) is a congruence on \( P(X) \).
Equivalences are congruences

If:
- \( h \) is an elementary homotopy between \( \gamma \circ \theta \) and \( \delta \circ \psi \)
Equivalences are congruences

If:
- $h$ is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$
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Equivalences are congruences

If:
- $h$ is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$
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- the endpoint of $\gamma$ is the starting point of $\gamma'$
Equivalences are congruences

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- \( h' \) is an elementary homotopy between \( \gamma' \circ \theta' \) and \( \delta' \circ \psi' \)
- the endpoint of \( \gamma \) is the starting point of \( \gamma' \)
then \( h \cdot h' \) is an elementary homotopy from ...
Equivalences are congruences

If:
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then $h \cdot h'$ is an elementary homotopy from $(\gamma \cdot \gamma') \circ (\theta \cdot \theta')$ to $(\delta \cdot \delta') \circ (\psi \cdot \psi')$. 
Equivalences are congruences

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The relation $\sim_X$ is a congruence on $P(X)$.
Naturality

If $h$ is a homotopy from $\gamma$ to $\gamma'$ on the topological space $X$ and $f: X \to Y$ is a continuous map, then $f \circ h$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the topological space $Y$.

If $h$ is a (weakly) directed homotopy from $\gamma$ to $\gamma'$ on the local pospace space $X$ and $f: X \to Y$ is a local pospace morphism, then $f \circ h$ is a (weakly) directed homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the local pospace space $Y$.

If $\gamma, \gamma': [0, r] \to X$ are ((weakly) di)homotopic, then so are $f \circ \gamma, f \circ \gamma': [0, r] \to Y$. 


If $h$ is a homotopy from $\gamma$ to $\gamma'$ on the topological space $X$ and $f : X \to Y$ is a continuous map, then $f \circ h$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the topological space $Y$. 

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If $\gamma, \gamma' : [0, r] \to X$ are ((weakly) di)homotopic, then so are $f \circ \gamma, f \circ \gamma' : [0, r] \to Y$. 
Conclusion
- The relations $\sim_X$ form a **natural congruence** on the directed path functor $P : \mathcal{Lpo} \to \text{Cat}$.
Conclusion

- The relations $\sim_X$ form a natural congruence on the directed path functor $P : Lpo \to Cat$.
- The fundamental category functor $\pi_1 : Lpo \to Cat$ is defined accordingly.
Conclusion

- The relations $\sim_X$ form a **natural congruence** on the directed path functor $P : Lpo \rightarrow \text{Cat}$.
- The **fundamental category** functor $\bar{\pi}_1 : Lpo \rightarrow \text{Cat}$ is defined accordingly.
- The fundamental groupoid functor $\Pi_1 : \text{Top} \rightarrow \text{Grd}$ is obtained by substituting “paths” and “homotopies” to “directed paths” and “elementary homotopies”.
Basic properties and computations
Fundamental category

Basic properties and computations

- The fundamental category of locally ordered real line is the corresponding partial order.

- For all local pospaces $X$ and $Y$ we have
  \[
  \pi_1(X \times Y) \cong \pi_1 X \times \pi_1 Y
  \]

- Given a pospace $X$, $\pi_1 X$ is loop-free i.e.
  \[
  \pi_1 X(x, y) \neq \emptyset \quad \text{and} \quad \pi_1 X(y, x) \neq \emptyset \implies x = y \quad \text{and} \quad \pi_1 X(x, x) = \{\text{id}_x\}
  \]

- The fundamental category of a local pospace has no nontrivial null homotopic directed paths i.e. any directed loop that is related to a constant path by an elementary homotopy is actually a constant.

- In particular the fundamental category of a local pospace has no isomorphism but its identities.
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- The fundamental category of locally ordered real line is the corresponding partial order.
- For all local pospaces $X$ and $Y$ we have
  \[ \vec{\pi}_1(X \times Y) \cong \vec{\pi}_1 X \times \vec{\pi}_1 Y \]
- Given a pospace $X$, $\vec{\pi}_1 X$ is loop-free i.e.
  \[ \vec{\pi}_1 X(x, y) \neq \emptyset \text{ and } \vec{\pi}_1 X(y, x) \neq \emptyset \implies x = y \text{ and } \vec{\pi}_1 X(x, x) = \{\text{id}_x\} \]
- The fundamental category of a local pospace has no nontrivial null homotopic directed paths i.e. any directed loop that is related to a constant path by an elementary homotopy is actually a constant.
- The fundamental category of locally ordered real line is the corresponding partial order.
- For all local pospaces $X$ and $Y$ we have

$$\pi_1^p(X \times Y) \cong \pi_1^p X \times \pi_1^p Y$$

- Given a pospace $X$, $\pi_1^p X$ is loop-free i.e.

$$\pi_1^p X(x, y) \neq \emptyset \text{ and } \pi_1^p X(y, x) \neq \emptyset \implies x = y \text{ and } \pi_1^p X(x, x) = \{\text{id}_x\}$$

- The fundamental category of a local pospace has no nontrivial null homotopic directed paths i.e. any directed loop that is related to a constant path by an elementary homotopy is actually a constant.
- In particular the fundamental category of a local pospace has no isomorphism but its identities.
The fundamental category of the locally ordered circle

- \( \pi_1(S^1(x, y)) = \{x\} \times \mathbb{N} \times \{y\} \)
- The identities are the tuples \((x, 0, x)\)
- The composition is given by
  \( (y, p, z) \circ (x, n, y) = (x, n + p, z) \) if \( \hat{xy} \cup \hat{yz} \neq S^1 \)
  \( (y, p, z) \circ (x, n, y) = (x, n + p + 1, z) \) if \( \hat{xy} \cup \hat{yz} = S^1 \)
The fundamental category of the locally ordered circle

- Given $x$, $y$, $\hat{xy}$ is the anticlockwise arc from $x$ to $y$. It is a singleton if $x = y$. 
The fundamental category of the locally ordered circle

- Given $x, y$, $xy$ is the anticlockwise arc from $x$ to $y$. It is a singleton if $x = y$.
- $\pi_1 S^1(x, y) = \{x\} \times \mathbb{N} \times \{y\}$
The fundamental category of the locally ordered circle

- Given $x, y$, $\widehat{xy}$ is the anticlockwise arc from $x$ to $y$. It is a singleton if $x = y$.
- $\pi_1^* S^1(x, y) = \{x\} \times \mathbb{N} \times \{y\}$
- The identities are the tuples $(x, 0, x)$
The fundamental category of the locally ordered circle

- Given $x$, $y$, $\widehat{xy}$ is the anticlockwise arc from $x$ to $y$. It is a singleton if $x = y$.
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The fundamental category of the locally ordered circle

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- $\pi_1(S^1)(x, y) = \{x\} \times \mathbb{N} \times \{y\}$
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  - $(y, p, z) \circ (x, n, y) = (x, n + p, z)$ if $\hat{xy} \cup \hat{yz} \neq S^1$
The fundamental category of the locally ordered circle

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  It is a singleton if \( x = y \).
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  - \((y, p, z) \circ (x, n, y) = (x, n + p + 1, z) \) if \( \hat{xy} \cup \hat{yz} = S^1 \)
Plane without a square

\[ x = \mathbb{R}^2 \setminus ]0, 1[^2 \]
Plane without a square

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Plane without a square

\[ x = \mathbb{R}_+^2 \setminus ]0, 1[^2 \]

If \( x \leq^2 y \), then \( \pi_1^X(x, y) \) only depends on the elements of the partition \( x \) and \( y \) belong to.
CATEGORY OF COMPONENTS
Motivation
Skeleta and equivalences of categories

- A skeleton of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose class of objects meets every isomorphism class of $\mathcal{C}$ exactly once.
- The skeleton of $\mathcal{C}$ is unique up to isomorphism, it is denoted by $\text{sk} \mathcal{C}$.
- Two categories are equivalent (i.e. there exists an equivalence of categories between them) iff their skeleta are isomorphic.
- The skeleton of the category of finite sets is the full subcategory whose objects are $\{0, \ldots, n-1\}$ for $n \in \mathbb{N}$.
- The skeleton of the fundamental groupoid of a path-connected space is the fundamental group of this space.
- Problem: The fundamental category of a local pospace has no isomorphisms but its identities, hence it is its own skeleton.
Skeleta and equivalences of categories

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Skeleta and equivalences of categories

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- The skeleton of the category of finite sets is the full subcategory whose objects are $\{0, \ldots, n - 1\}$ for $n \in \mathbb{N}$. 
- The fundamental category of a local pospace has no isomorphisms but its identities, hence it is its own skeleton.
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Skeleta and equivalences of categories

- A **skeleton** of $C$ is a full subcategory of $C$ whose class of objects meets every isomorphism class of $C$ exactly once.
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Loop-free and one-way categories
The categories \textit{LfCat} and \textit{OwCat}

\textbf{Category of components}  \hspace{1.5cm}  \textbf{Loop-free and one-way categories}

\begin{itemize}
  \item A category \( C \) is said to be one-way when all its endomorphisms are identities i.e. \( C(x, x) = \{ \text{id}_x \} \) for all \( x \).
  \item Every Grothendieck topos has a one-way site. C. MacLarty. Theor. Appl. of Cat. 16(5) pp 123-126 (2006).
  \item A one-way category \( C \) is said to be loop-free when for all \( x, y \), \( C(x, y) \neq \emptyset \) and \( C(y, x) \neq \emptyset \) implies \( x = y \).
  \item A loop-free category is its own skeleton.
  \item A category is one-way iff its skeleton is loop-free.
\end{itemize}
The categories $LfCat$ and $OwCat$

- A category $C$ is said to be one-way when all its endomorphisms are identities i.e. $C(x, x) = \{\text{id}_x\}$ for all $x$


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The categories \textit{LfCat} and \textit{OwCat}

- A category $\mathcal{C}$ is said to be \textit{one-way} when all its endomorphisms are identities i.e. $\mathcal{C}(x, x) = \{\text{id}_x\}$ for all $x$. Every Grothendieck topos has a one-way site. C. MacLarty. Theor. Appl. of Cat. 16(5) pp 123-126 (2006).

- A one-way category $\mathcal{C}$ is said to be \textit{loop-free} when for all $x, y$

$$\mathcal{C}(x, y) \neq \emptyset \text{ and } \mathcal{C}(y, x) \neq \emptyset \text{ implies } x = y$$

Complexes of groups and orbihedra \textit{in} Group theory from a geometrical viewpoint.
The categories $\text{LfCat}$ and $\text{OwCat}$

- A category $C$ is said to be one-way when all its endomorphisms are identities i.e. $C(x, x) = \{ \text{id}_x \}$ for all $x$.


- A one-way category $C$ is said to be loop-free when for all $x, y$

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Complexes of groups and orbihedra in Group theory from a geometrical viewpoint.

- A loop-free category is its own skeleton
The categories $LfCat$ and $OwCat$

- A category $C$ is said to be one-way when all its endomorphisms are identities i.e. $C(x, x) = \{\text{id}_x\}$ for all $x$
  

- A one-way category $C$ is said to be loop-free when for all $x, y$

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  Complexes of groups and orbihedra in Group theory from a geometrical viewpoint.

- A loop-free category is its own skeleton

- A category is one-way iff its skeleton is loop-free
Generalized congruences


- Given a binary relation $R$ on the set of morphisms of a category $C$, there is a unique category $C/R$ and a unique functor $q: C \to C/R$ such that for all functors $f: C \to D$, if $\alpha R \beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g: C/R \to D$ such that $f = g \circ q_C$.

- Examples

- Any congruence is a generalized congruence.

- $\langle N, +, 0 \rangle$ with $0 R n$ for some $n \in N$. 
Generalized congruences

- Given a binary relation $\mathcal{R}$ on the set of morphisms of a category $C$, there is a unique category $C/\mathcal{R}$ and a unique functor $q : C \rightarrow C/\mathcal{R}$ such that for all functors $f : C \rightarrow D$, if $\alpha \mathcal{R} \beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g : C/\mathcal{R} \rightarrow D$ such that $f = g \circ q$
Generalized congruences


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- Examples

![Diagram](image-url)
Generalized congruences


- Given a binary relation $\mathcal{R}$ on the set of morphisms of a category $\mathcal{C}$, there is a unique category $\mathcal{C}/\mathcal{R}$ and a unique functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{R}$ such that for all functors $f : \mathcal{C} \to \mathcal{D}$, if $\alpha \mathcal{R} \beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g : \mathcal{C}/\mathcal{R} \to \mathcal{D}$ such that $f = g \circ q$

- Examples
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Generalized congruences

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```
C/R
  ^
 / q
v
C ----> D
     ^
     g
```

- Examples
  - any congruence is a generalized congruence.
  - \( C \) freely generated by \( x \xrightarrow{\alpha} y \) with \( \text{id}_x R \text{id}_y \) (resp. with \( \alpha R \text{id}_x \)).
Generalized congruences


- Given a binary relation \( \mathcal{R} \) on the set of morphisms of a category \( C \), there is a unique category \( C/\mathcal{R} \) and a unique functor \( q : C \to C/\mathcal{R} \) such that for all functors \( f : C \to D \), if \( \alpha \mathcal{R} \beta \Rightarrow f(\alpha) = f(\beta) \), then there is a unique functor \( g : C/\mathcal{R} \to D \) such that \( f = g \circ q \)

\[
\begin{array}{c}
C/\mathcal{R} \\
\downarrow q \\
C \\
\downarrow f \\
\downarrow g \\
D
\end{array}
\]

- Examples
  - any congruence is a generalized congruence.
  - \( C \) freely generated by \( x \xrightarrow{\alpha} y \) with \( \text{id}_x \mathcal{R} \text{id}_y \) (resp. with \( \alpha \mathcal{R} \text{id}_x \)).
  - \((\mathbb{N}, +, 0)\) with \( 0 \mathcal{R} n \) for some \( n \in \mathbb{N} \).
Systems of weak isomorphisms
Goal

Let $C$ be a one-way category:

- Define a class $\Sigma$ of morphisms of $C$ so we can keep one representative in each class of $\Sigma$-related objects without loss of information
- To do so, we are in search for a class that behaves much like the one of isomorphisms
- From now on $C$ denotes a one-way category
Goal

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- From now on $\mathcal{C}$ denotes a one-way category.
Potential weak isomorphisms

Let $C$ be a one-way category
Potential weak isomorphisms

Let $C$ be a one-way category

- For all morphisms $\sigma$ and all objects $z$ define
Potential weak isomorphisms

Let $C$ be a one-way category

- For all morphisms $\sigma$ and all objects $z$ define
  - the $\sigma, z$-precomposition as $\gamma \in C(\partial^+ \sigma, z) \rightarrow \gamma \sigma \in C(\partial^- \sigma, z)$
Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

- For all morphisms $\sigma$ and all objects $z$ define
  - the $\sigma, z$-precomposition as $\gamma \in \mathcal{C}(\partial^+ \sigma, z) \rightarrow \gamma \circ \sigma \in \mathcal{C}(\partial \sigma, z)$
  - the $z, \sigma$-postcomposition as $\delta \in \mathcal{C}(z, \partial \sigma) \mapsto \sigma \circ \delta \in \mathcal{C}(z, \partial^+ \sigma)$
Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

- For all morphisms $\sigma$ and all objects $z$ define
  - the $\sigma, z$-precomposition as $\gamma \in \mathcal{C}(\partial^+ \sigma, z) \rightarrow \gamma \circ \sigma \in \mathcal{C}(\partial^+ \sigma, z)$
  - the $z, \sigma$-postcomposition as $\delta \in \mathcal{C}(z, \partial^+ \sigma) \mapsto \sigma \circ \delta \in \mathcal{C}(z, \partial^+ \sigma)$
- One may have $\mathcal{C}(\partial^+ \sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^+ \sigma) = \emptyset$
Potential weak isomorphisms

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- For all morphisms $\sigma$ and all objects $z$ define
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- One may have $C(\partial^+ \sigma, z) = \emptyset$ or $C(z, \partial^- \sigma) = \emptyset$
- Note that $\sigma$ is an isomorphism iff for all $z$ both precomposition and postcomposition are bijective.
Potential weak isomorphisms

Let $C$ be a one-way category

- For all morphisms $\sigma$ and all objects $z$ define
  - the $\sigma$, $z$-precomposition as $\gamma \in C(\partial^+\sigma, z) \rightarrow \gamma \circ \sigma \in C(\partial\sigma, z)$
  - the $z$, $\sigma$-postcomposition as $\delta \in C(z, \partial^+\sigma) \mapsto \sigma \circ \delta \in C(z, \partial\sigma)$

- One may have $C(\partial^+\sigma, z) = \emptyset$ or $C(z, \partial\sigma) = \emptyset$

- Note that $\sigma$ is an isomorphism iff for all $z$ both precomposition and postcomposition are bijective.

- The latter condition is weakened: $\sigma$ is said to preserve the future cones (resp. past cones) when for all $z$ if $C(\partial^+\sigma, z) \neq \emptyset$ (resp. $C(z, \partial\sigma) \neq \emptyset$) then the precomposition (resp. postcomposition) is bijective.
Potential weak isomorphisms

Let $C$ be a one-way category

- For all morphisms $\sigma$ and all objects $z$ define
  - the $\sigma, z$-precomposition as $\gamma \in C(\partial^+\sigma, z) \rightarrow \gamma \circ \sigma \in C(\partial^+\sigma, z)$
  - the $z, \sigma$-postcomposition as $\delta \in C(z, \partial^-\sigma) \mapsto \sigma \circ \delta \in C(z, \partial^+\sigma)$
- One may have $C(\partial^+\sigma, z) = \emptyset$ or $C(z, \partial^-\sigma) = \emptyset$
- Note that $\sigma$ is an isomorphism iff for all $z$ both precomposition and postcomposition are bijective.
- The latter condition is weakened: $\sigma$ is said to preserve the future cones (resp. past cones) when for all $z$ if $C(\partial^+\sigma, z) \neq \emptyset$ (resp. $C(z, \partial^-\sigma) \neq \emptyset$) then the precomposition (resp. postcomposition) is bijective.
- Then $\sigma$ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
Potential weak isomorphisms
Let $C$ be a one-way category

- For all morphisms $\sigma$ and all objects $z$ define
  - the $\sigma, z$-precomposition as $\gamma \in C(\partial^+ \sigma, z) \rightarrow \gamma \circ \sigma \in C(\partial^+ \sigma, z)$
  - the $z, \sigma$-postcomposition as $\delta \in C(z, \partial^- \sigma) \mapsto \sigma \circ \delta \in C(z, \partial^+ \sigma)$
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- Then $\sigma$ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $C(x, y)$ contains a potential weak isomorphism, then it is a singleton
  Requires the assumption that $C$ is one-way
An example of potential weak isomorphism
An example of potential weak isomorphism

Due to the lower dipath, the $\sigma, z$-precomposition is not bijective; yet $\sigma'$ is a potential weak isomorphism.
An unwanted example of potential weak isomorphism

Note that $\sigma''$ is a potential weak isomorphism though there exists a morphism from $\partial-\sigma''$ to $z$ but none from $\partial+\sigma''$ to $z$. 
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Note that $\sigma''$ is a potential weak isomorphism though there exists a morphism from $\partial \sigma''$ to $z$ but none from $\partial^+ \sigma''$ to $z$. 
Stability under pushout and pullback
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- A collection of morphisms $\Sigma$ is said to be **stable under pushout** when for all $\sigma \in \Sigma$, for all $\gamma$ with $\partial \gamma = \partial \sigma$, the pushout of $\sigma$ along $\gamma$ exists and belongs to $\Sigma$

$$\begin{array}{c}
\gamma' \\
\sigma \\
\gamma
\end{array} \quad \begin{array}{c}
\sigma' \\
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\end{array}$$
Stability under pushout and pullback

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Greatest inner collection stable under pushout and pullback

- Any collection $\Sigma$ of morphisms of a category $C$
  admits a greatest subcollection that is stable under pushout and pullback.
  
  Construction:
  - Start with $\Sigma_0 = \Sigma$.
  - For $n \in \mathbb{N}$ define $\Sigma_{n+1}$ as the collection of morphisms $\sigma \in \Sigma_n$ such that the pushout and the pullback of $\sigma$ along with all morphisms exist (when sources or targets match) and belong to $\Sigma_n$.

  $$\Sigma_0 \supseteq \cdots \supseteq \Sigma_1 \supseteq \cdots \supseteq \Sigma_n \supseteq \Sigma_{n+1} \supseteq \cdots$$

  The expected subcollection is the decreasing intersection $\Sigma_\infty := \bigcap_{n \in \mathbb{N}} \downarrow \Sigma_n$.

  The collection $\Sigma_\infty$ is stable under the action of $\text{Aut}(C)$. 

\[ \text{Category of components} \quad \text{Systems of weak isomorphisms} \]
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Systems of weak isomorphisms

- The class of isomorphisms of any category is stable under pushout and pullback.
- A system of weak isomorphisms is a collection of potential weak isomorphisms that is stable under pushout and pullback.
- The class of all isomorphisms of any category is a system of weak isomorphisms.
- If $\Sigma$ is a system of weak isomorphisms, then so is its closure under composition.
- Hence we suppose the systems of weak isomorphisms are closed under composition.
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Examples of systems of weak isomorphisms

- Given a partition $P$ of $\mathbb{R}$ into intervals, the following collection is a system of weak isomorphisms
  \[
  \{(x, y) \mid x \leq y; \exists I \in P, [x, y] \subseteq I\}
  \]

- In the preceding example, $\mathbb{R}$ can be replaced by any totally ordered set

- Let $\Sigma_i \subseteq C_i$ be a family of collections of morphisms, then
  \[
  \prod_i \Sigma_i
  \]
  is a swi of $\prod_i C_i$ iff each $\Sigma_i$ is a swi of $C_i$

- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.
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Pureness

- A collection $\Sigma$ of morphisms is said to be pure when $\gamma \circ \delta \in \Sigma \Rightarrow \gamma, \delta \in \Sigma$

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The locale of systems of weak isomorphisms
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- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

\[ x \land \left( \bigvee_{i} y_i \right) = \bigvee_{i} (x \land y_i) \]
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Components of a one-way category $C$

- From now on $C$ is a one-way category and $\Sigma$ is a system of weak isomorphisms on it.
- Recall that if $C(x,y)$ meets $\Sigma$, then $C(x,y)$ is a singleton, a fact that we represent on diagrams by:

$$x \xrightarrow[\Sigma]{\quad} y$$

- Given two objects $x$ and $y$ of $C$
  
  - there exists a $\Sigma$-zigzag between $x$ and $y$
  
  - there exists $z$ such that $x \xleftarrow[\Sigma]{\quad} z \xrightarrow[\Sigma]{\quad} y$
  
  - there exists $z$ such that $x \xrightarrow[\Sigma]{\quad} z \xleftarrow[\Sigma]{\quad} y$

- When any of the following property is satisfied $x$ and $y$ are said to be $\Sigma$-connected.

- $\Sigma$-connectedness is an equivalence relation on the objects of $C$

- The equivalence classes are called a $\Sigma$-components.
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Structure of the $\Sigma$-components

$\Sigma$ system of weak isomorphisms of $C$ one-way category
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A prelattice is a preordered set in which $x \land y$ and $x \lor y$ exist for all $x$ and $y$. However they are defined only up to isomorphism.
Structure of the $\Sigma$-components

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Let $K$ be a $\Sigma$-component of $C$ and $\mathcal{K}$ be the full subcategory of $C$ whose objects are the elements of $K$. The following properties are satisfied:
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1. The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preceq)$ where $x \preceq y$ stands for $C[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.
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3. If $d$ and $u$ are respectively a greatest lower bound and a least upper bound of the pair $\{x, y\}$, then Diagram 1 is both a pullback and a pushout in $C$, and all the arrows appearing on the diagram belong to $\Sigma$.

![Diagram 1](image1.png)

![Diagram 2](image2.png)

![Diagram 3](image3.png)
Structure of the $\Sigma$-components

$\Sigma$ system of weak isomorphisms of $C$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all $x$ and $y$. However they are defined only up to isomorphism.

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4. $C = \mathcal{K}$ iff $C$ is a prelattice, and $\Sigma$ is the greatest system of weak isomorphisms of $C$ i.e. all the morphisms in this case.

\[ \begin{array}{ccc} u & \xrightarrow{\quad} & u \\
\downarrow & & \downarrow \\
x & \quad & x \\
\downarrow & & \downarrow \\
d & \quad & d \\
\downarrow & & \downarrow \\
y & \quad & y \\
\end{array} \]

Diagram 1  \hspace{1cm} \text{Diagram 2}  \hspace{1cm} \text{Diagram 3}
Equivalent morphisms with respect to $\Sigma$

- Let $\delta \in C(x, y)$ and $\delta' \in C(x', y')$. Then write $\delta \sim \delta'$ when $x \sim x'$ and $y \sim y'$, and the inner hexagon of the next diagram commutes.

- Note that if $d \sim x \land x'$ and $u \sim y \lor y'$ then the outer hexagon also commutes, hence the relation $\sim$ is well-defined.

- If $\gamma \sim \delta$ then $\partial - \gamma \sim \partial - \delta$ and $\partial + \gamma \sim \partial + \delta$. 
Equivalent morphisms with respect to $\Sigma$

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Equivalent morphisms with respect to $\Sigma$

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- Note that if $d \cong x \wedge x'$ and $u \cong y \vee y'$ then the outer hexagon also commutes, hence the relation $\sim$ is well defined.

- If $\gamma \sim \delta$ then $\partial \gamma \sim \partial \delta$ and $\partial^+ \gamma \sim \partial^+ \delta$
The relation $\sim$ is an equivalence

- reflexive since $\Sigma$ contains all identities
- symmetric by definition
- transitive
The relation $\sim$ is an equivalence

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\[
\begin{array}{ccc}
\alpha & \rightarrow & \rightarrow \\
\downarrow & & \downarrow \\
\Sigma & \rightarrow & \rightarrow \\
\downarrow & & \downarrow \\
\Sigma & \rightarrow & \rightarrow \\
\end{array}
\]

- pullback commutes

\[
\begin{array}{ccc}
\Sigma & \rightarrow & \rightarrow \\
\uparrow & & \uparrow \\
\Sigma & \rightarrow & \rightarrow \\
\downarrow & & \downarrow \\
\Sigma & \rightarrow & \rightarrow \\
\end{array}
\]

- pushout commutes

\[
\begin{array}{ccc}
\Sigma & \rightarrow & \rightarrow \\
\downarrow & & \downarrow \\
\Sigma & \rightarrow & \rightarrow \\
\uparrow & & \uparrow \\
\Sigma & \rightarrow & \rightarrow \\
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![Diagram showing the relations and commutes]
The relation $\sim$ fits with composition
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- Suppose $\partial \gamma = \partial^+ \delta$, $\partial \gamma' = \partial^+ \delta'$ and $\gamma \sim \gamma'$ and $\delta \sim \delta'$. 
The relation \( \sim \) fits with composition

- Suppose \( \partial \gamma = \partial^+ \delta, \partial \gamma' = \partial^+ \delta' \) and \( \gamma \sim \gamma' \) and \( \delta \sim \delta' \).
- Then we have \( \gamma \circ \delta \sim \gamma' \circ \delta' \)
The category of components $\mathcal{C}/\Sigma$

- The objects are the $\Sigma$-components
- The morphisms are the $\sim$-equivalence classes
- If $\partial - \gamma \sim \partial + \delta$ then there exists $\gamma' \sim \gamma$, $\delta' \sim \delta$, and $\partial - \gamma' = \partial + \delta'$.
- We define $[\gamma] \circ [\delta] = [\gamma' \circ \delta']$.
- The category of components is $\mathcal{C}/\Sigma$ with $\Sigma$ being the greatest swi of $\mathcal{C}$.
The category of components $C/\Sigma$

- The quotient category $C/\Sigma$ (obtained by turning each morphism of $\Sigma$ into an identity) can be defined as follows:
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- The category of components is $C/\Sigma$ with $\Sigma$ being the greatest swi of $C$
Properties
Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism $\delta$ of $\mathcal{C}$ t.f.a.e. $\delta \in \Sigma - [\delta] \subseteq \Sigma$. $- [\delta]$ is an identity of $\mathcal{C}/\Sigma$. The quotient functor $Q: \mathcal{C} \to \mathcal{C}/\Sigma$ satisfies the following universal property: For all functors $F: \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{identities of } \mathcal{D}\}$ there exists a unique $G: \mathcal{C}/\Sigma \to \mathcal{D}$ s.t. $F = G \circ Q$. 

$Q \uparrow \downarrow$ $C$ $\uparrow \downarrow$ $\mathcal{D}$ $F \to \to$ $G \to \to$ $\to$ $\to$
Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism $\delta$ of $\mathcal{C}$ t.f.a.e.
- $\delta \in \Sigma$
Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism $\delta$ of $\mathcal{C}$ t.f.a.e.

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- $[\delta] \subseteq \Sigma$
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there exists a unique $G : \mathcal{C}/\Sigma \to \mathcal{D}$ s.t. $F = G \circ Q$

\[
\begin{array}{ccc}
\mathcal{C}/\Sigma & \xrightarrow{Q} & \mathcal{D} \\
\downarrow{G} & & \downarrow{F} \\
\mathcal{C} & \underset{\text{F}}{\xrightarrow{\rightarrow}} & \mathcal{D}
\end{array}
\]
The fundamental properties of $C/\Sigma$
with $\Sigma$ being a system of weak isomorphisms of a one-way category $C$
The fundamental properties of $\mathcal{C}/\Sigma$
with $\Sigma$ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $\mathcal{Q} : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
The fundamental properties of $C/\Sigma$

with $\Sigma$ being a system of weak isomorphisms of a one-way category $C$

- The quotient functor $Q : C \rightarrow C/\Sigma$ is surjective on morphisms
- The quotient category $C/\Sigma$ is loop-free
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- If $C(x, y) \neq \emptyset$ then the following map is a bijection.

$$\delta \in C(x, y) \mapsto Q(\delta) \in C/\Sigma(Q(x), Q(y))$$
The fundamental properties of \( C/\Sigma \)

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  \]
- If \( C/\Sigma(Q(x), Q(y)) \neq \emptyset \) then there exist \( x' \) and \( y' \) such that \( \Sigma(x', x), \Sigma(y, y'), C(x', y), \) and \( C(x, y') \) are nonempty.
The fundamental properties of $\mathcal{C}/\Sigma$

with $\Sigma$ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

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\[
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\]

- If $\mathcal{C}/\Sigma(Q(x), Q(y)) \neq \emptyset$ then there exist $x'$ and $y'$ such that $\Sigma(x', x)$, $\Sigma(y, y')$, $\mathcal{C}(x', y)$, and $\mathcal{C}(x, y')$ are nonempty.

- The quotient functor $Q$ preserves and reflects potential weak isomorphisms
The fundamental properties of $C/\Sigma$

with $\Sigma$ being a system of weak isomorphisms of a one-way category $C$

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- If $C$ is finite then so is the quotient $C/\Sigma$
The fundamental properties of $C/\Sigma$

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- If $C$ is finite then so is the quotient $C/\Sigma$
- $C$ is a preorder iff $C/\Sigma$ is a poset
Describing the localization of $\mathcal{C}$ by $\Sigma$

with $\Sigma$ being a system of weak isomorphisms of a one-way category $\mathcal{C}$
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- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$
Describing the localization of $\mathcal{C}$ by $\Sigma$

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- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$
- The morphisms are the equivalence classes of ordered pairs of coinitial morphisms $(\gamma, \sigma)$ with $\sigma \in \Sigma$, 

\[ \text{pushout} \]

\[ \text{Q(}\gamma'\circ\gamma''\text{)} = \text{Q(}\gamma'\text{)} \circ \text{Q(}\gamma''\text{)} = \text{Q(}\gamma'\text{)} \circ \text{Q(}\gamma\text{)} \]

hence the composite $(\gamma'\circ\gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pushout nor on the representatives $(\gamma, \sigma)$ and $(\gamma', \sigma')$. 

\[ \sigma'' \downarrow \downarrow \gamma'' \]

\[ \sigma' \downarrow \downarrow \gamma' \]
Describing the localization of $\mathcal{C}$ by $\Sigma$

with $\Sigma$ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$
- The morphisms are the equivalence classes of ordered pairs of coinitial morphisms $(\gamma, \sigma)$ with $\sigma \in \Sigma$,
  - Two pairs $(\gamma, \sigma)$ and $(\gamma', \sigma')$ being equivalent when $\partial \sigma = \partial \sigma'$, $\partial \gamma = \partial \gamma'$, and $Q(\gamma) = Q(\gamma')$
Describing the localization of $\mathcal{C}$ by $\Sigma$

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  - Two pairs $(\gamma, \sigma)$ and $(\gamma', \sigma')$ being equivalent when $\partial \sigma = \partial \sigma'$, $\partial \gamma = \partial \gamma'$, and $Q(\gamma) = Q(\gamma')$
  - In the diagram below we have $Q(\gamma' \circ \gamma'') = Q(\gamma') \circ Q(\gamma'') = Q(\gamma') \circ Q(\gamma)$ hence the composite $(\gamma' \circ \gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pushout nor on the representatives $(\gamma, \sigma)$ and $(\gamma', \sigma')$.

```
  σ''  γ''
  /    /  \
 σ     |   \ pushout
  |  σ'  σ''
  |    |   \
  |  γ   γ'
  |    |   \
  |    |   \
  |    |   \
  |    γ     γ'
```

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The canonical comparison $P : C[\Sigma^{-1}] \to C/\Sigma$

with $\Sigma$ being a system of weak isomorphisms of a one-way category $C$
The canonical comparison \( P : C[\Sigma^{-1}] \to C/\Sigma \)
with \( \Sigma \) being a system of weak isomorphisms of a one-way category \( C \)

- Define \( I \) by \( I(\gamma) := (\gamma, \text{id}_{\partial \gamma}) \) and the identity on objects
The canonical comparison $P : C[\Sigma^{-1}] \to C/\Sigma$
with $\Sigma$ being a system of weak isomorphisms of a one-way category $C$

- Define $I$ by $I(\gamma) := (\gamma, \text{id}_{\mathcal{D}} \cdot \gamma)$ and the identity on objects
- Given a functor $F : C \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$ define
The canonical comparison \( P : C[\Sigma^{-1}] \to C/\Sigma \)
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- Define \( I \) by \( I(\gamma) := (\gamma, \text{id}_{\text{D}-\gamma}) \) and the identity on objects
- Given a functor \( F : C \to D \) s.t. \( F(\Sigma) \subseteq \{\text{isomorphisms of } D\} \) define
  - \( G(x) := F(x) \) for all objects \( x \) of \( C[\Sigma^{-1}] \) and
The canonical comparison $P : C[\Sigma^{-1}] \to C/\Sigma$
with $\Sigma$ being a system of weak isomorphisms of a one-way category $C$

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The canonical comparison \( P : C[\Sigma^{-1}] \to C/\Sigma \)
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- The functor \( I : C \to C[\Sigma^{-1}] \) then satisfies the universal property: for all functors \( F : C \to D \) there exists a unique \( G : C \to C[\Sigma^{-1}] \) s.t. \( F = G \circ I \)
The canonical comparison $P : C[\Sigma^{-1}] \to C/\Sigma$

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- In particular there is a unique functor $P$ s.t. $Q = P \circ I$ with $Q : C \to C/\Sigma$ and we have

The functor $P$ is an equivalence of categories
The canonical comparison \( P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma \)
with \( \Sigma \) being a system of weak isomorphisms of a one-way category \( \mathcal{C} \)

- Define \( I \) by \( I(\gamma) := (\gamma, \text{id}_{\mathcal{D}} \gamma) \) and the identity on objects

- Given a functor \( F : \mathcal{C} \to \mathcal{D} \) s.t. \( F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\} \) define
  - \( G(x) := F(x) \) for all objects \( x \) of \( \mathcal{C}[\Sigma^{-1}] \) and
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- In particular there is a unique functor \( P \) s.t. \( Q = P \circ I \) with \( Q : \mathcal{C} \to \mathcal{C}/\Sigma \) and we have

  The functor \( P \) is an equivalence of categories

- The skeleton of \( \mathcal{C}[\Sigma^{-1}] \) is \( \mathcal{C}/\Sigma \) and \( \mathcal{C}[\Sigma^{-1}] \) is one-way.
Embedding $\mathcal{C}/\Sigma$ into $\mathcal{C}$
Embedding $C/\Sigma$ into $C$

- Let $\phi : \Sigma$-components of $C \to \text{Ob}(C)$ such that

- for all $\Sigma$-components $K$, $K'$, if there exists $x \in K$ and $x' \in K'$ such that $C(x, x') \neq \emptyset$, then $C(\phi(K), \phi(K')) \neq \emptyset$.

- In this case, $C/\Sigma$ is isomorphic with the full subcategory of $C$ whose set of objects is $\text{im}(\phi)$.

- The mapping $\phi$ is called an admissible choice (of canonical objects).

- Write $\phi \preceq \phi'$ when $C(\phi(K), \phi'(K)) \neq \emptyset$ for all $\Sigma$-components $K$.

- The collection of admissible choices then forms a (pre)lattice.

- If $C/\Sigma$ is finite, then there exists an admissible choice.

- If $C/\Sigma$ is infinite, the existence of an admissible choice is an open question.
Embedding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let $\phi : \Sigma$-components of $\mathcal{C} \to \text{Ob}(\mathcal{C})$ such that
  - for all $\Sigma$-components $K, K'$, if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
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Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

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  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\text{im}(\phi)$.
  - the mapping $\phi$ is called an admissible choice (of canonical objects)

- Write $\phi \preceq \phi'$ when $\mathcal{C}(\phi(K), \phi'(K')) \neq \emptyset$ for all $\Sigma$-components $K$
  - The collection of admissible choice then forms a (pre)lattice
  - If $\mathcal{C}/\Sigma$ is finite then there exists an admissible choice
Embedding $C/\Sigma$ into $C$

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  - If $C/\Sigma$ is finite then there exists an admissible choice
  - If $C/\Sigma$ is infinite the existence of an admissible choice is a open question
Examples
Plane without a square

\( x = \mathbb{R}_+^2 \setminus ]0, 1[^2 \)

Plane without a square

\[ x = \mathbb{R}^2 \setminus [0, 1]^2 \]
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Let \( x, y \) such that \( x \leq y \), then \( \pi_1 X(x, y) \) only depends on which elements of the partition \( x \) and \( y \) belong to

\[
\begin{array}{c|c|c|c|c}
\rightarrow & A & B & C & D \\
\hline
A & \sigma & \beta & \gamma & \beta' \circ \beta \\
& & & \alpha' \circ \alpha & \\
B & \sigma & & & \\
C & \sigma & & & \\
D & & & \sigma & \\
\end{array}
\]
Plane without a square

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\hline
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\hline
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\hline
C & & & \gamma' & \\
\hline
D & & & \sigma & \\
\end{array}
\]
Two rectangles
Two rectangles
Two rectangles
Two rectangles
Two rectangles
Two rectangles
Two rectangles
Two rectangles
Swiss Flag
Swiss Flag
Swiss Flag
Swiss Flag
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Swiss Flag
<table>
<thead>
<tr>
<th>Category of components</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Achronal overlapping square</td>
<td><img src="chart.png" alt="Diagram of Achronal overlapping square" /></td>
</tr>
</tbody>
</table>
Achronal overlapping square
Achronal overlapping square
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Achronal overlapping square
Diagonal overlapping squares
Diagonal overlapping squares
Diagonal overlapping squares
Diagonal overlapping squares
Diagonal overlapping squares
Diagonal overlapping squares
The floating cube

Non potential weak isomorphisms
The floating cube

Non potential weak isomorphisms
The floating cube

A “vee” that does not extend to a pushout
The floating cube
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Some pushouts squares
The floating cube

Some pushouts squares
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- Since the pushout of \( f \) (resp. \( g \)) along \( g \) (resp. \( f \)) does not exist, \( f, g \notin \Sigma \)
The floating cube

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- The commutative square \( f, g, f', \) and \( g' \) is a pullback:
The floating cube

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- The commutative square $f, g, f', g'$ is a pullback:
  - Therefore $f', g' \not\in \Sigma$ (anyway they do not preserve the future cones)
<table>
<thead>
<tr>
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<tr>
<td>The floating cube</td>
<td>boundaries of the components</td>
</tr>
</tbody>
</table>
The floating cube

boundaries of the components
Finite connected loop-free categories
Commutative monoid

of nonempty finite connected loop-free categories
Commutative monoid
of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.
  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
Commutative monoid
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- The Cartesian product of categories $A \times B$ is non-empty iff so are $A$ and $B$.
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- The collection of isomorphism classes of nonempty finite connected loop-free categories is thus a commutative monoid $\mathcal{M}$
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- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$
- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
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The commutative monoid $\mathcal{M}$ is free.
Criteria for primality

- The monoid $M$ is graded by the following morphisms:
  - $\#\text{Ob}: C \in M \mapsto \text{card}(\text{Ob}(C)) \in (\mathbb{N} \{0\}, \times, 1)$
  - $\#\text{Mo}: C \in M \mapsto \text{card}(\text{Mo}(C)) \in (\mathbb{N} \{0\}, \times, 1)$
  - $\#\text{Mo}(C) \geq 2 \times \#\text{Ob}(C) - 1$, for all $C \in M$

- In particular if $\#\text{Ob}(C)$ or $\#\text{Mo}(C)$ is prime, then so is $C$.

The converse is false.

- Any element of $M$ freely generated by a graph, is prime.
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- For all d-spaces $X$ and $Y$, $\pi_1(X \times Y) \cong \pi_1 X \times \pi_1 Y$
- Hence $\mathcal{N'} := \{X \in \mathcal{H}_f|G| \mid \pi_1 X$ is nonempty, connected, and loop-free} is a pure submonoid of $\mathcal{H}_f|G|$
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- Hence $\mathcal{N'} := \{X \in \mathcal{H}_f \uparrow G \uparrow | \pi_1X \text{ is nonempty, connected, and loop-free} \}$ is a pure submonoid of $\mathcal{H}_f \uparrow G \uparrow$
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- For all d-spaces $X$ and $Y$, $\pi_1(X \times Y) \cong \pi_1 X \times \pi_1 Y$
- Hence $\mathcal{N}' := \{X \in \mathcal{H}_f \downarrow G \mid \pi_1 X$ is nonempty, connected, and loop-free\} is a pure submonoid of $\mathcal{H}_f \downarrow G$
- Then $\mathcal{N} := \{X \in \mathcal{N}' \mid \pi_0(\pi_1 X)$ is finite\} is a pure submonoid of $\mathcal{N}'$
- Therefore it is free commutative and we would like to know which prime elements are preserved by $X \in \mathcal{N} \mapsto \pi_0(\pi_1 X) \in \mathcal{M}$
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- For all d-spaces $X$ and $Y$, $\pi_1(X \times Y) \cong \pi_1 X \times \pi_1 Y$.
- Hence $\mathcal{N}' := \{X \in \mathcal{H}_f | G \mid | \pi_1 X \text{ is nonempty, connected, and loop-free}\}$ is a pure submonoid of $\mathcal{H}_f | G \mid$.
- Then $\mathcal{N} := \{X \in \mathcal{N}' \mid \pi_0(\pi_1 X) \text{ is finite}\}$ is a pure submonoid of $\mathcal{N}'$.
- Therefore it is free commutative and we would like to know which prime elements are preserved by $X \in \mathcal{N} \mapsto \pi_0(\pi_1 X) \in \mathcal{M}$.
- Conjecture

\[
\text{If } P \in \mathcal{N} \text{ is prime and } \pi_1(P) \text{ is not a lattice, then } \pi_0(\pi_1(P)) \text{ is prime.}
\]