INDEPENDENCE
Two programs $P$ and $Q$ are said to be compatible when their initial valuations and their arity maps coincide on the intersection of their domains of definition. In that case we define the parallel composition $P \parallel Q$.

By extension we define the parallel composition of $P_1, \ldots, P_N$ when the programs are pairwise compatible.
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By extension we define the parallel composition of $P_1, \ldots, P_N$ when the programs are pairwise compatible.
Syntactical independence
Independence

Syntactical independence

Two programs are said to be syntactically independent when the set of resources they use are disjoint:

- they have no variables in common,
- they have no semaphores in common, and
- they have no barriers in common.

Syntactically independent programs are compatible.

Syntactical independence can be decided statically, it is compositional, but it is too restrictive.
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Model independence
Suppose the programs $P_1, \ldots, P_N$ are conservative. The programs $P_1, \ldots, P_N$ are said to be model independent when

$$J_{P_1} \cdots J_{P_N} = J_{P_1} \times \cdots \times J_{P_N}$$

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Model Independence

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Model independence can be decided statically.
Observational independence
Compatible permutations

Assume we have a partition \( \{1, \ldots, n\} = S_1 \sqcup \cdots \sqcup S_N \).

Two multi-instructions \( \mu \) and \( \mu' \) (\( \text{dom}(\mu), \text{dom}(\mu') \subseteq \{1, \ldots, n\} \)) can be swapped when

\[
\forall j \in \{1, \ldots, N\} \ | \ S_j \cap \text{dom}(\mu) \neq \emptyset \quad \land \quad \forall j \in \{1, \ldots, N\} \ | \ S_j \cap \text{dom}(\mu') \neq \emptyset
\]

A permutation \( \pi \) of the set \( \{0, \ldots, q-1\} \) is said to be compatible with the sequence of multi-instructions \( \mu_0, \ldots, \mu_{q-1} \) when it is order preserving on all pairs \( \{k, k'\} \) such that \( \mu_k \) and \( \mu_{k'} \) cannot be swapped.

The permutation \( \pi \) is said to be compatible with the directed path \( \gamma \) when it is compatible with its associated sequence of multi-instructions.
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Observational independence

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\[ \gamma_{12345} \mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \]
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The identifiers of the running processes of $P_1|\cdots|P_N$ are the elements of $\{1, \ldots, n\}$ with

$$n = \sum_{j=1}^{N} n_j,$$
and for $j \in \{1, \ldots, N\}$

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  there exists an execution trace $\gamma'$ whose associated sequence of multi-instructions is $\pi \cdot (\mu_0 \cdots \mu_{q-1})$, which has the same action on the system state than $\gamma$, that is to say

$$\sigma \cdot (\mu_0 \cdots \mu_{q-1}) = \sigma \cdot (\mu_{\pi^{-1}(0)} \cdots \mu_{\pi^{-1}(q-1)}) .$$
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Observational independence cannot be decided statically, moreover it is too loose.
Comparison
<table>
<thead>
<tr>
<th>Independence</th>
<th>Comparison</th>
</tr>
</thead>
<tbody>
<tr>
<td>syntactic independence</td>
<td>model independence</td>
</tr>
<tr>
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</tr>
</tbody>
</table>

Main theorem
Main theorem

syntactic independence \(\Downarrow\) model independence \(\Downarrow\) observational independence
DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

Emmanuel Haucourt
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MPRI : Concurrency (2.3.1)
– Lecture 4 –

2023 – 2024
ISOTHETIC REGIONS
Boolean structure
Let $G$ be a finite graph, the collection $R_1^G$ of all finite unions of connected subsets of $|G|$ forms a Boolean subalgebra of $\text{Pow}(|G|)$. Moreover $R_1^G \sim \text{Pow}(V) \times (R_1[^0,1[)$ with $A$ (resp. $V$) being the set of arrows (resp. vertices) of $G$, and $R_1[^0,1[)$ being the Boolean algebra of finite unions of subintervals of $]0,1[$. The elements of $R_1^G$ are seen as one-dimensional blocks.

Proof: If $X$ is a connected subset of $|G|$ then for all arrows $\alpha \in G$, $X \cap (\{\alpha\} \times ]0,1[)$ has at most two connected components. The finiteness condition is not necessary e.g. yet some infinite graphs may not enjoy the property e.g. when $G$ is a graph with a single vertex and infinitely many arrows.
One-dimensional regions

Let $G$ be a finite graph, the collection $\mathcal{R}_1 G$ of all finite unions of connected subsets of $|G|$ forms a Boolean subalgebra of $\text{Pow}(|G|)$.

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$$\cdots \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdots$$
One-dimensional regions

Let \( G \) be a \textit{finite} graph, the collection \( \mathcal{R}_1 G \) of all finite unions of connected subsets of \( |G| \) forms a \textit{Boolean subalgebra} of \( \text{Pow}(|G|) \).

Moreover

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Higher dimensional blocks
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- A block of dimension $n \in \mathbb{N}$, or $n$-block, is the product of $n$ connected subsets of the metric graph $|G|$. 
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- A collection of blocks is called a block covering of $X \subseteq |G|^n$ when the union of its elements is $X$.
- The collection of $n$-dimensional block coverings is denoted by $\text{Cov}_n G$, it is preordered by

$$C \preceq C' \iff \forall b \in C \exists b' \in C', b \subseteq b'$$
Maximal blocks

- A block contained in $X$ is said to be a block of $X$. Such a block is said to be maximal when no block of $X$ strictly contains it.
- The maximal connected block covering of $X \subseteq |G|$ is the set of all its maximal connected blocks, it is denoted by $\alpha_n(X)$.
- $\alpha_n(X) = \{\emptyset\}$ if and only if $X = \emptyset$. 
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A Galois connection

We have a Galois connection \((\gamma_n, \alpha_n)\) between \(\text{Cov}_n G\) and \(\text{Pow}(|G|^n)\) with \(\gamma_n(D) = SD\) for all \(D \in \text{Cov}_n G\).

In particular \(\gamma_n \circ \alpha_n = \text{id}\) and \(\text{id} \leq \alpha_n \circ \gamma_n\).

That Galois connection induces an isomorphism of Boolean algebras between \(\text{Pow}(|G|^n)\) and the image of \(\alpha_n\) i.e. the collection of maximal connected block coverings.

Proof: any connected block is contained in a maximal connected block (by the Hausdorff maximal principle).
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We have a Galois connection $(\gamma_n, \alpha_n)$ between $\text{Cov}_n G$ and $\text{Pow}(\lvert G \rvert^n)$ with $\gamma_n(D) = \bigcup D$ for all $D \in \text{Cov}_n G$.

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In particular $\gamma_n \circ \alpha_n = id$ and $id \preceq \alpha_n \circ \gamma_n$. That Galois connection induces an isomorphism of Boolean algebras between $\text{Pow}(\lvert G \rvert^n)$ and the image of $\alpha_n$ i.e. the collection of maximal connected block coverings.
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\text{Cov}_n G \\
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Proof: any connected block is contained in a maximal connected block (by the Hausdorff maximal principle).

\[
\bigcup_i \uparrow \left( B_1^{(i)} \times \cdots \times B_n^{(i)} \right) = \left( \bigcup_i \uparrow B_1^{(i)} \right) \times \cdots \times \left( \bigcup_i \uparrow B_n^{(i)} \right)
\]
Isothetic regions

- An isothetic region of dimension $n$ is a subset of $|G|^n$ that admits a finite block covering.
- The geometric model of a conservative program is an isothetic region.
- The collection of isothetic regions of dimension $n$ is denoted by $R^G_n$.
- The collection of finite block covering of dimension $n$ is denoted by $\text{Cov}^{nf}G^n$. 
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The previous Galois connection restricted to isothetic regions
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Suppose that the graph $G$ is finite. The collection of $n$-dimensional isothetic regions $\mathcal{R}_nG$ forms a Boolean subalgebra of $\text{Pow}(|G|^n)$.
The previous Galois connection restricted to isothetic regions

Suppose that the graph $G$ is finite. The collection of $n$-dimensional isothetic regions $\mathcal{R}_nG$ forms a Boolean subalgebra of $\text{Pow}(|G|^n)$ and the previous Galois connection restricts to a Galois connection between $\text{Cov}_{nf} G$ and $\mathcal{R}_nG$, where $\alpha_n$ is an isomorphism of Boolean algebras between $\mathcal{R}_nG$ and the collection of finite maximal block coverings.
The previous Galois connection restricted to isothetic regions

Suppose that the graph $G$ is finite. The collection of $n$-dimensional isothetic regions $\mathcal{R}_n G$ forms a Boolean subalgebra of $\text{Pow}(\lvert G \rvert^n)$ and the previous Galois connection restricts to a Galois connection between $\text{Cov}_{nf} G$ and $\mathcal{R}_n G$, which induces an isomorphism of Boolean algebras between $\mathcal{R}_n G$ and the image of $\alpha_n$ i.e. the collection of finite maximal block coverings.

$\text{Cov}_{nf} G \xrightarrow{\gamma_n} \mathcal{R}_n G \xleftarrow{\alpha_n}$
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$$\text{Cov}_{nf} G \xrightarrow{\gamma_n} \mathcal{R}_n G$$

A subset $X \subseteq |G|^n$ is an isothetic region iff the collection of maximal subblocks of $X$ is finite and covers $X$. 
The complement of a block is an isothetic region
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If $X$ is 1-dimensional then its maximal blocks are its connected components.
The complement of a block is an isothetic region

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$$B^c = \bigcup_{k=1}^n |G| \times \cdots \times B_k^c \times \cdots \times |G|$$
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Its maximal blocks are found among that of $B^c$ therefore they have the form

$$D_1 \times \cdots \times D_{k-1} \times C_k \times D_{k+1} \times \cdots \times D_n$$

with $k \in \{1, \ldots, n\}$, $C_k$ ranging through the connected components of $B_k^c$ and $D_j$, for $j \neq k$, ranging through the connected components of $|G|$.
Intersection of two isothetic regions

The intersection of the blocks \( B \) and \( B' \) is given by
\[
B \cap B' = (B_1 \cap B'_1) \times \cdots \times (B_n \cap B'_n)
\]

The maximal blocks of \( B \cap B' \) are therefore of the form
\[
C_1 \times \cdots \times C_n
\]
with each \( C_k \) ranging through the connected components of \( (B_k \cap B'_k) \).

It follows from De Morgan's laws that the intersection of two regions is still a region.

Moreover if \( B \) and \( B' \) are block coverings of \( X \) and \( X' \) containing all their maximal blocks, then the collection of maximal blocks of \( B \cap B' \) for \( B \in B \) and \( B' \in B' \) is a block covering of \( X \cap X' \) containing all its maximal blocks.
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It follows from De Morgan's laws that the intersection of two regions is still a region.

Moreover if $B$ and $B'$ are block coverings of $X$ and $X'$ containing all their maximal blocks, then the collection of maximal blocks of $B \cap B'$ for $B \in B$ and $B' \in B'$ is a block covering of $X \cap X'$ containing all its maximal blocks.
Concluding the proof

If $F$ is any finite block covering of $X$, then $X^c = \bigcap_{B \in F} B^c$.

- The collection of maximal blocks of $B^c$ is finite and covers $B^c$.
- The maximal blocks of $X^c$ are obtained as certain finite intersection of the form $\bigcap \{ M_B | B \in F \}$ where $M_B$ is a maximal block of $B^c$.
- The maximal blocks of $X^c$ thus form a finite block covering of $X^c$. 
Concluding the proof

If $\mathcal{F}$ is any finite block covering of $X$, then

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- The maximal blocks of $X^c$ thus form a finite block covering of $X^c$. 
A result from directed topology
A result from directed topology

For all directed paths $\gamma$ on $|G|^n$ and all $X \in \mathcal{R}_n G$, the inverse image of $X$ by $\gamma$ has finitely many connected components.
Additional operators
Closure, interior, and boundary of an isothetic region

The closure operator preserves finite products, therefore it preserves blocks.

The closure operator preserves finite unions hence it preserves isothetic regions.

The boundary of a set is the intersection of its closure and the closure of its complement, hence it also preserves isothetic regions.

The interior of a set is the difference between its closure and its boundary. It follows that the interior operator also preserves isothetic regions.
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The interior of a set is the difference between its closure and its boundary. It follows that the interior operator also preserves isothetic regions.
The forward and the backward operators

Let $A$, $B$ be subsets of a local pospace $X$.

- The forward and the backward operators are defined as
  
  $\text{frw}(A, B) = \{ \partial^+ \delta | \delta \text{ directed path on } X; \partial^- \delta \in A; \text{im}(\delta) \subseteq A \cup B \}$

  $\text{bck}(A, B) = \{ \partial^- \delta | \delta \text{ directed path on } X; \partial^+ \delta \in A; \text{im}(\delta) \subseteq A \cup B \}$

- The future cone of $A$ in $X$ is $\text{cone} f A := \text{frw}(A, X)$ and the past cone of $A$ in $X$ is $\text{cone} p A := \text{bck}(A, X)$.

- The future closure of $A$ in $X$ is $A_f := \text{frw}(A, A)$ and the past closure of $A$ in $X$ is $A_p := \text{bck}(A, A)$.

Theorem: if $A$, $B$, and $X$ are isothetic regions, then so are $\text{frw}(A, B)$, $\text{cone} f A$, $A_f$, and their duals.
The forward and the backward operators

Let $A, B$ be subsets of a local pospace $X$. 
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- The **future cone** of $A$ in $X$ is $cone^f A := frw(A, X)$ and the **past cone** of $A$ in $X$ is $cone^p A := bck(A, X)$.

- The **future closure** of $A$ in $X$ is $\overline{A}^f := frw(A, \overline{A})$ and the **past closure** of $A$ in $X$ is $\overline{A}^p := bck(A, \overline{A})$. The closure $\overline{A}$ being understood in $X$.

Theorem: if $A, B$, and $X$ are isothetic regions, then so are $frw(A, B)$, $cone^f A$, $A^f$, and their duals.
The forward and the backward operators

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**Theorem:** if $A$, $B$, and $X$ are isothetic regions, then so are $\text{frw}(A, B)$, $\text{cone}^f A$, $\overline{A}^f$, and their duals.
Future/past stable subsets of $X$

- Let $A$ be a subset of a local pospace $X$.
  - Cone $f(A)$ and cone $p(A)$ are future stable (resp. past) when $f(A) = A$ (resp. $p(A) = A$).
  - $A$ is future stable iff $X \setminus A$ is past stable.

- The collection of future stable subsets of $X$ is a complete lattice, the greatest lower (resp. least upper) bound of a family being given by its intersection (resp. union).

- The same holds for past stable subsets.
Future/past stable subsets of $X$

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- $\text{cone}^f \text{cone}^f A = \text{cone}^f A$ and $\text{cone}^p \text{cone}^p A = \text{cone}^p A$
- $A$ is said to be future (resp. past) stable (in $X$) when $\text{cone}^f A = A$ (resp. $\text{cone}^p A = A$)

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- The same holds for past stable subsets.
Past/future attractors

Let $A$ be a subset of a local pospace $X$.

- **escape** $f_A = \{ p \in X \text{ from which } A \text{ is avoided} \} = \{ p \in X \text{ from which } A \text{ cannot be reached} \}

- **att** $p_A = \{ p \in X \text{ from which } A \text{ cannot be avoided} \}

- **cone** $p_A = \{ p \in X \text{ from which } A \text{ can be reached} \} = \text{bck}(A, X) = cone p_A$

- **Past/future attractors**
Let $A$ be a subset of a local pospace $X$. 
Isothetic regions  

Additional operators

Past/future attractors

Let $A$ be a subset of a local pospace $X$.

$$\text{cone}^p A = \{ p \in X \text{ from which } A \text{ can be reached} \} = \cdots$$
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\[
\text{escape}^f A = (\text{cone}^p A)^c
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Past/future attractors

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\[
\text{att}^p A = \text{escape}^f (\text{escape}^f A)
\]
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $J_{KP}$ be the geometric model of the program.

- The reachable space of $J_{KP}$ is the future cone of the initial point.
- A point $p \in \mathcal{R}(G_i) \mathcal{H}$ is said to be terminal when $J_{\gamma K}$ is empty for all directed paths on $\mathcal{R}(G_i) \mathcal{H}$ starting at $p$.
- A point $p \in J_{KP}$ is said to be terminal when so are all its projections.
- The terminal points form a future stable isothetic region of $J_{KP}$.
- A point $p \in J_{KP}$ is said to be deadlock when its future cone neither contains directed loops (i.e. it is loop-free) nor terminal points.
- The deadlock points form a future stable isothetic region of $J_{KP}$.
- The deadlock attractor of the program is the past attractor of its deadlock region.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

- The reachable space of $\llbracket P \rrbracket$ is the future cone of the initial point.
Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$.
Let $\mathcal{P}$ be the geometric model of the program.

- The reachable space of $\mathcal{P}$ is the future cone of the initial point
- A point $p \in \mathcal{G}_i$ is said to be terminal when $\gamma$ is empty for all directed paths on $\mathcal{G}_i$ starting at $p$.

The deadlock attractor of a conservative program
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

- The reachable space of $\llbracket P \rrbracket$ is the future cone of the initial point
- A point $p \in \llbracket G_i \rrbracket$ is said to be terminal when $\llbracket \gamma \rrbracket$ is empty for all directed paths on $\llbracket G_i \rrbracket$ starting at $p$.
- A point $p \in \llbracket P \rrbracket$ is said to be terminal when so are all its projections

- The deadlock points form a future stable isothetic region of $\llbracket P \rrbracket$.
- The deadlock attractor of the program is the past attractor of its deadlock region.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

- The reachable space of $\llbracket P \rrbracket$ is the future cone of the initial point.
- A point $p \in \downarrow G_i \uparrow$ is said to be terminal when $\llbracket \gamma \rrbracket$ is empty for all directed paths on $\downarrow G_i \uparrow$ starting at $p$.
- A point $p \in \llbracket P \rrbracket$ is said to be terminal when so are all its projections.
- The terminal points form a...
The deadlock attractor of a conservative program

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- The deadlock points form a future stable isothetic region of $\llbracket P \rrbracket$
- The deadlock attractor of the program is the past attractor of its deadlock region.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\mathcal{J}[P]$ be the geometric model of the program.

- The reachable space of $\mathcal{J}[P]$ is the future cone of the initial point
- A point $p \in \mathcal{J}[G_i]$ is said to be terminal when $\mathcal{J}[\gamma]$ is empty for all directed paths on $\mathcal{J}[G_i]$ starting at $p$.
- A point $p \in \mathcal{J}[P]$ is said to be terminal when so are all its projections
- The terminal points form a future stable isothetic region of $\mathcal{J}[P]$
- A point $p \in \mathcal{J}[P]$ is said to be deadlock when its future cone neither contains directed loops (i.e. it is loop-free) nor terminal points.

The deadlock attractor of the program is the past attractor of its deadlock region.
The deadlock attractor of a conservative program

Let \( G_1, \ldots, G_n \) be the running processes of a conservative program \( P \). Let \( \llbracket P \rrbracket \) be the geometric model of the program.

- The reachable space of \( \llbracket P \rrbracket \) is the future cone of the initial point
- A point \( p \in \downarrow G_i \) is said to be terminal when \( \llbracket \gamma \rrbracket \) is empty for all directed paths on \( \downarrow G_i \) starting at \( p \).
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- The deadlock points form a . . .
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

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The deadlock attractor of a conservative program

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- The reachable space of $\llbracket P \rrbracket$ is the future cone of the initial point.
- A point $p \in \downarrow G_i \downarrow$ is said to be terminal when $\llbracket \gamma \rrbracket$ is empty for all directed paths on $\downarrow G_i \downarrow$ starting at $p$.
- A point $p \in \llbracket P \rrbracket$ is said to be terminal when so are all its projections.
- The terminal points form a future stable isothetic region of $\llbracket P \rrbracket$.
- A point $p \in \llbracket P \rrbracket$ is said to be deadlock when its future cone neither contains directed loops (i.e. it is loop-free) nor terminal points.
- The deadlock points form a future stable isothetic region of $\llbracket P \rrbracket$.
- The deadlock attractor of the program is the . . .
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

- The reachable space of $\llbracket P \rrbracket$ is the future cone of the initial point.
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- The deadlock points form a future stable isothetic region of $\llbracket P \rrbracket$.
- The deadlock attractor of the program is the past attractor of its deadlock region.
Deadlock attractor of the Swiss Cross

sem 1 a b
proc:
q = P(b).P(a).V(a).V(b)
init: p q
Deadlock attractor of the Swiss Cross

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\[
\begin{align*}
\text{sem} &\ 1\ a\ b \\
\text{proc:} & \\
p & = P(a) . P(b) . V(b) . V(a) \\
q & = P(b) . P(a) . V(a) . V(b) \\
\text{init:} &\ p \ q
\end{align*}
\]
Three dining philosophers
FACTORING ISOTHETIC REGIONS
Free commutative monoids
Commutative monoids
Commutative monoids

- $(M, *, \varepsilon)$ such that for all $a, b, c \in M$,
  - $(ab)c = a(bc)$
  - $\varepsilon a = a = a\varepsilon$
  - $ab = ba$
Commutative monoids

- \((M, \ast, \varepsilon)\) such that for all \(a, b, c \in M\),
  - \((ab)c = a(bc)\)
  - \(\varepsilon a = a = a\varepsilon\)
  - \(ab = ba\)
- For all set \(X\) the collection \(MX\) of multisets over \(X\)
  i.e. maps \(\phi : X \to \mathbb{N}\) s.t. \(\{x \in X \mid \phi(x) \neq 0\}\) is finite
  forms a commutative monoid with pointwise addition
Commutative monoids

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- Functor \(M : Set \rightarrow \mathcal{Cmon}\)
Commutative monoids

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- Functor \(M : Set \to Cmon\)
  - A multiset \(\phi\) can be written as
    \[
    \sum_{x \in X} \phi(x)x
    \]
Commutative monoids

- \((M, *, \varepsilon)\) such that for all \(a, b, c \in M\),
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- For all set \(X\) the collection \(MX\) of multisets over \(X\)
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- A commutative monoid is said to be free when
  it is isomorphic with some \(MX\)

- Functor \(M : \text{Set} \rightarrow \text{Cmon}\)
  - A multiset \(\phi\) can be written as
    \[
    \sum_{x \in X} \phi(x)x
    \]
  - In particular, if \(f : X \rightarrow Y\) is a set map, then
    \[
    M(f)(\phi) = \sum_{x \in X} \phi(x)f(x)
    \]
Prime vs irreducible
Prime vs irreducible

- $d$ divides $x$, denoted by $d | x$, when there exists $x'$ such that $x = dx'$
Prime vs irreducible

- $d$ divides $x$, denoted by $d | x$, when there exists $x'$ such that $x = dx'$
- $u$ unit: exists $u'$ s.t. $uu' = \varepsilon$ then write $x \sim y$ when $y = ux$ for some unit $u$
Prime vs irreducible

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- $i$ irreducible: $i$ nonunit and $x|i$ implies $x \sim i$ or $x$ unit
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- $p$ prime: $p$ nonunit and $p|ab$ implies $p|a$ or $p|b$
- If $M$ contains nontrivial units, then one can consider the quotient monoid $M/\sim$ where $x \sim y$ stands for: there exists a unit $u$ s.t. $y = ux$
## Examples

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\( \mathbb{N} \) = \{0, 1, 2, 3, 4, \ldots\}  
\( \mathbb{R}_+ \) = \{0, 1, 2, 3, \ldots\}  
\( \mathbb{R} \) = \mathbb{R}_+ \cup \{0, \infty\}
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<td>{1, 5}</td>
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Graded commutative monoid

- (M, *, ε) graded: there is a morphism g: (M, *, ε) → (N, +, 0) s.t.
g − 1({0}) = {units of M}

- If M is graded then
- {irreducibles of M} generates M
- {primes of M} ⊆ {irreducibles of M}
Graded commutative monoid

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- If \(M\) is graded then
  - \{irreducibles of \(M\)\} generates \(M\)
  - \{primes of \(M\)\} \subseteq \{irreducibles of \(M\)\}
Irreducible that are not prime

\[ M = \left\{ a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0 \right\}, \times, 1 \]
Irreducible that are not prime

\( M = (\{a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0\}, \times, 1) \)

\( - \quad N : M \to (\mathbb{Z} \setminus \{0\}, \times, 1); \quad N(a + b\sqrt{10}) = a^2 - 10b^2 \)
Irreducible that are not prime

$M = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0\}, \times, 1$

- $N : M \to (\mathbb{Z} \setminus \{0\}, \times, 1)$; $N(a + b\sqrt{10}) = a^2 - 10b^2$
  $N(uv) = N(u)N(v)$
Irreducible that are not prime

\( M = (\{a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0\}, \times, 1) \)

- \( N : M \to (\mathbb{Z} \setminus \{0\}, \times, 1); \ N(a + b\sqrt{10}) = a^2 - 10b^2 \)
  \( N(uv) = N(u)N(v) \)
  \( u \) unit iff \( N(u) \in \{\pm 1\} \) [hint: \( u^{-1} = N(u)\bar{u} \) with \( \bar{u} = a - b\sqrt{10} \) if \( u = a + b\sqrt{10} \)]
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  \( N(a + b\sqrt{10}) \text{ mod } 10 \in \{0, 1, 4, 5, 6, 9\} \)
Irreducible that are not prime

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- \( N(a + b\sqrt{10}) \mod 10 \in \{0, 1, 4, 5, 6, 9\} \)
- therefore \( N(a + b\sqrt{10}) \not\in \{\pm 2, \pm 3\} \)
Irreducible that are not prime

\$M = (\{ a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0 \}, \times, 1)\$

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\( N(a + b\sqrt{10}) \mod 10 \in \{0, 1, 4, 5, 6, 9\} \)

therefore \( N(a + b\sqrt{10}) \not\in \{\pm 2, \pm 3\} \)

<table>
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<th>( uv )</th>
<th>( N(uv) )</th>
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<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>( \pm 1, \pm 2, \pm 4 )</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>( \pm 1, \pm 3, \pm 9 )</td>
</tr>
<tr>
<td>( 4 \pm \sqrt{10} )</td>
<td>6</td>
<td>( \pm 1, \pm 2, \pm 3, \pm 6 )</td>
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Irreducible that are not prime

\[ M = (\{a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0\}, \times, 1) \]

- \( N : M \to (\mathbb{Z} \setminus \{0\}, \times, 1); N(a + b\sqrt{10}) = a^2 - 10b^2 \)
  \[ N(uv) = N(u)N(v) \]
  \( N(u) \) unit iff \( N(u) \in \{\pm 1\} \) [hint: \( u^{-1} = N(u)\bar{u} \) with \( \bar{u} = a - b\sqrt{10} \) if \( u = a + b\sqrt{10} \)]
  \( N(a + b\sqrt{10}) \mod 10 \in \{0, 1, 4, 5, 6, 9\} \)
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<tr>
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<td>±1, ±2, ±4</td>
</tr>
<tr>
<td>3</td>
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<td>±1, ±3, ±9</td>
</tr>
<tr>
<td>4 ± \sqrt{10}</td>
<td>6</td>
<td>±1, ±2, ±3, ±6</td>
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- 2, 3, and \( 4 \pm \sqrt{10} \) are irreducible but not prime
  since \( 2 \cdot 3 = (4 + \sqrt{10}) \cdot (4 - \sqrt{10}) \)
Irreducible that are not prime

\[ M = \left\{ a + b\sqrt{10} \mid a, b \in \mathbb{Z}; \ a \neq 0 \text{ or } b \neq 0 \right\}, \times, 1 \]

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\[ N(uv) = N(u)N(v) \]

\( u \) unit iff \( N(u) \in \{\pm1\} \) [hint: \( u^{-1} = N(u)\bar{u} \) with \( \bar{u} = a - b\sqrt{10} \) if \( u = a + b\sqrt{10} \)]

\[ N(a + b\sqrt{10}) \mod 10 \in \{0, 1, 4, 5, 6, 9\} \]

therefore \( N(a + b\sqrt{10}) \not\in \{\pm2, \pm3\} \)

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<th>uv</th>
<th>N(uv)</th>
<th>N(u)</th>
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<tr>
<td>2</td>
<td>4</td>
<td>( \pm1, \pm2, \pm4 )</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
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</tr>
<tr>
<td>4 \pm \sqrt{10}</td>
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</tr>
</tbody>
</table>

- 2, 3, and \( 4 \pm \sqrt{10} \) are irreducible but not prime
  since \( 2 \cdot 3 = (4 + \sqrt{10}) \cdot (4 - \sqrt{10}) \)

- \( \{ a + b\sqrt{10} \mid a, b \in \mathbb{Z} \setminus \{0\} \) is graded by the number of prime factors of \( N(u) \)
Factoring isothetic regions
Free commutative monoids

\[ \mathbb{N}[X] \text{ polynomials with coefficients in } \mathbb{N} \]

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Annals of Mathematics 2(54), pp 315-318 (1951)
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\[
X^5 + X^4 + X^3 + X^2 + X + 1 = \left\{ (X + 1)(X^4 + X^2 + 1) \right\}
\]
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\[
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\end{array} \right.
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\( \mathbb{N}[X] \) polynomials with coefficients in \( \mathbb{N} \)

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X^5 + X^4 + X^3 + X^2 + X + 1 = \\
\begin{cases}
(X + 1)(X^4 + X^2 + 1) = (X^3 + 1)(X^2 + X + 1) & \text{in } \mathbb{N}[X] \\
(X + 1)(X^2 + X + 1)(X^2 - X + 1) & \text{in } \mathbb{Z}[X]
\end{cases}
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- therefore \( X + 1, X^2 + X + 1, X^3 + 1, \) and \( X^4 + X^2 + 1 \)
  
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\end{align*}
\]

- therefore $X + 1$, $X^2 + X + 1$, $X^3 + 1$, and $X^4 + X^2 + 1$

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- $\mathbb{N}[X] \setminus \{0\}$ is graded by the degree
Characterization of the free commutative monoids

Unique factorization

- The following are equivalent:
  - $M$ is free commutative
  - any element of $M$ can be written as a product of irreducibles in a unique way up to reordering
  - $\{\text{primes of } M\} = \{\text{irreducibles of } M\}$ and generates $M$
  - $M$ is graded and $\{\text{irreducibles of } M\} \subseteq \{\text{primes of } M\}$

- Standard examples:
  - $(\mathbb{N} \setminus \{0\}, \times, 1)$
  - $(\mathbb{N}, +, 0)$ and its finite products in the category of commutative monoids.
  - Indeed $(\mathbb{N}, +, 0) \sim M(\{1, \ldots, n\})$
  - $(\mathbb{Z}[X] \setminus \{0\}, \times, 1)$ (if $F$ is a factorial ring, then so is $F[X]$)


- Note that two free commutative monoids are isomorphic in $\mathbb{C}_{\text{mon}}$ iff their set of prime elements have the same cardinality e.g. $(\mathbb{N} \setminus \{0\}, \times, 1) \sim (\mathbb{Z}[X] \setminus \{0\}, \times, 1)$ in $\mathbb{C}_{\text{mon}}$
Characterization of the free commutative monoids

Unique factorization

- The following are equivalent:

- The monoid is free commutative.
- Any element of the monoid can be written as a product of irreducibles in a unique way up to reordering.
- The set of primes of the monoid is equal to the set of irreducibles of the monoid and generates the monoid.
- The monoid is graded and the set of irreducibles is contained in the set of primes.

Standard examples:
- \((\mathbb{N} \setminus \{0\}, \times, 1)\)
- \((\mathbb{N}, +, 0)\) and its finite products in the category of commutative monoids. Indeed, \((\mathbb{N}, +, 0)\) is isomorphic to \(\mathbb{M}(\{1, \ldots, n\})\).
- \((\mathbb{Z}[X] \setminus \{0\}, \times, 1)\) (if \(F\) is a factorial ring, then so is \(F[X]\)).


Note that two free commutative monoids are isomorphic in \(C_{\text{mon}}\) if and only if their set of prime elements have the same cardinality. For example, \((\mathbb{N} \setminus \{0\}, \times, 1) \cong (\mathbb{Z}[X] \setminus \{0\}, \times, 1)\) in \(C_{\text{mon}}\).
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Connected sum of manifolds

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In particular $\mathcal{M}_2 \cong (\mathbb{N}, +, 0)$ and $\mathcal{M}_3 \cong (\mathbb{N} \setminus \{0\}, \times, 1)$
Monoids of homogeneous languages
Factoring isothetic regions

Monoids of homogeneous languages
### Factoring Isothetic Regions

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### Monoids of Homogeneous Languages

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The noncommutative monoid of languages

Let $\varepsilon$ denote the empty word. A language is a set of words on $A$. Let $D$ and $D'$ be languages. Define $D \cdot D' := \{ w \cdot w' | w \in D; w' \in D' \}$. One has $\emptyset \cdot D = D \cdot \emptyset = \emptyset$ and $\{ \varepsilon \} \cdot D = D \cdot \{ \varepsilon \} = D$. The monoid of nonempty languages is $D(A)$, which is commutative iff $\operatorname{Card}(A) \leq 1$. Note that $D(\emptyset) \sim = \{ \varepsilon \}$, but $D(\{ a \})$ is not freely commutative.
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Factoring isothetic regions
Monoids of homogeneous languages

The noncommutative monoid of homogeneous languages

- \( H \in D(A) \) is homogeneous when all the words in \( H \) have the same length.
- Define \( \dim(H) \) as the length common to all the words of \( H \).
  It is well defined since \( H \) is nonempty.
- \( H \cdot H' = \{ w \cdot w' | w \in H; w' \in H' \} \) is homogeneous iff so are \( H \) and \( H' \).
- \( D_h(A) \subseteq D(A) \) the pure submonoid of homogeneous languages.
- \( H \in D_h(A) \mapsto \dim(H) \in (\mathbb{N}, +, 0) \) is a morphism of monoid.
- \( \dim(H) = 0 \) iff \( H = \{ \epsilon \} \).
- \( D_h(A) \) is commutative iff \( \text{Card}(A) \leq 1 \).
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- \( \mathcal{D}_h(\{a\}) \cong (\mathbb{N}, +, 0) \).
Action of the symmetric groups on the left of the homogeneous languages

- The $n$th symmetric group $S_n$ acts on the left of the set of words of length $n$, i.e. mappings from $\{1, \ldots, n\}$ to $A$, by

\[
\sigma \cdot \omega := \omega \circ \sigma^{-1}
\]

- Then $S_n$ acts on the left of the homogeneous languages of dimension $n$.

Write $H \sim H'$ when $\dim(H) = \dim(H')$ and $H' = \sigma \cdot H$ for some $\sigma \in S_{\dim(H)}$.

- If $\sigma \in S_n$ and $\sigma' \in S_{n'}$ then define $\sigma \otimes \sigma' \in S_{n + n'}$ as:

\[
\sigma \otimes \sigma' (k) := \begin{cases} 
\sigma(k) & \text{if } 1 \leq k \leq n \\
\sigma'(k - n) + n & \text{if } n + 1 \leq k \leq n + n'
\end{cases}
\]

- A Godement exchange law is satisfied, which ensures that $\sim$ is actually a congruence:

\[
(\sigma \cdot H) \cdot (\sigma' \cdot H') = (\sigma \otimes \sigma') \cdot (H \cdot H')
\]

i.e. $H \sim K$ and $H' \sim K'$ implies $HH' \sim KK'$. 

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Action of the symmetric groups
on the left of the homogeneous languages

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Factoring isothetic regions
Monoids of homogeneous languages

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The commutative monoid of homogeneous languages
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- The commutative monoid of homogeneous languages is $\mathcal{H}(A) = (\mathcal{D}_h(A), \cdot, \{\varepsilon\})/\sim$
The commutative monoid of homogeneous languages

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The commutative monoid $\mathcal{H}(A)$ is free
The commutative monoid of homogeneous languages

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\[
\begin{array}{c}
\text{The commutative monoid } H(A) \text{ is free}
\end{array}
\]

- For any homogeneous language \( H \) and \( \sigma \in \mathcal{G}_{\dim(H)} \), \( \text{card}(H) = \text{card}(\sigma \cdot H) \) so we can define the cardinality of any element of \( H(A) \)
The commutative monoid of finite homogeneous languages
The commutative monoid of finite homogeneous languages

- \( M' \subseteq M \) is said to be pure when for all \( x, y \in M \), \( xy \in M' \) implies \( x, y \in M' \)
The commutative monoid of finite homogeneous languages

- $M' \subseteq M$ is said to be pure when for all $x, y \in M$, $xy \in M'$ implies $x, y \in M'$
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  - e.g. $H = \{ab, ac\} = \{a\} \cdot \{b, c\}$ though card($H$) = 2
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  - e.g. $H = \{ab, ac\} = \{a\} \cdot \{b, c\}$ though card($H$) = 2
- The primality of $H$ does not imply that of Card($H$)
  - e.g. $H = \{a, b, c, d\}$ is prime though card($H$) = 4
The brute force algorithm for factoring in $\mathcal{H}_f(A)$
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

**Theory**

Given $w \in \mathbb{A}^n$ and $I \subseteq \{1, \ldots, n\}$, we write $w|_I$ for the subword of $w$ consisting of letters with indices in $I$. 
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

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Given a homogeneous language $H$ of dimension $n$, we write

$$H_{|I} = \{ w_{|I} \mid w \in H \}$$
The brute force algorithm for factoring in $\mathcal{H}_f(A)$

Theory

Given $w \in A^n$ and $I \subseteq \{1, \ldots, n\}$, we write $w|_I$ for the subword of $w$ consisting of letters with indices in $I$.

Given a homogeneous language $H$ of dimension $n$, we write

$$H|_I = \{ w|_I | w \in H \}$$

Denoting $I^c$ for $\{1, \ldots, n\} \setminus I$, we have

$$[H] = [H|_I] \cdot [H|_{I^c}]$$

in $\mathcal{H}_f(A)$ if and only if for all words $u, v \in H$ there exists a word $w \in H$ such that

$$w|_I = u|_I \quad \text{and} \quad w|_{I^c} = v|_{I^c}$$
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

1. Choose $I \subseteq \{1, \ldots, n\}$ of cardinality $k \leq n/2$.
2. If $\pi_{\mid I^c}(\pi_{\mid I}^{-1}(u))$ does not depend on $u \in \mathcal{H}_{\mid I}$, then we have the factorization $\mathcal{H} = \mathcal{H}_{\mid I} \cdot \mathcal{H}_{\mid I^c}$ and we are done.
3. Otherwise, check whether there are still subsets of $\{1, \ldots, n\}$ to check:
   3.1. Yes: go to step 1.
   3.2. No: $\mathcal{H}$ is prime.
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Practice

For $I \subseteq \{1, \ldots, n\}$ let $\pi_I$ be the “projection” that sends $w \in H$ to $w_I \in \mathbb{A}^{|I|}$. 
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

For $l \subseteq \{1, \ldots, n\}$ let $\pi_{||}$ be the “projection” that sends $w \in H$ to $w_{\mid l} \in \mathbb{A}^{\text{card}(l)}$.

1. choose $l \subseteq \{1, \ldots, n\}$ of cardinality $k \leq n/2$
The brute force algorithm for factoring in $\mathcal{H}_f(A)$

Practice

For $I \subseteq \{1, \ldots, n\}$ let $\pi|_I$ be the “projection” that sends $w \in H$ to $w|_I \in A^{\text{card}(I)}$.

1. choose $I \subseteq \{1, \ldots, n\}$ of cardinality $k \leq n/2$
2. if $\pi|_{I^c}(\pi^{-1}|_I(u))$ does not depend on $u \in H|_I$, then we have the factorization

$$[H] = [H|_I] \cdot [H|_{I^c}]$$

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The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

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   3.1. yes: go to step 1
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Homogeneous languages and isothetic regions
Factoring a program

sem: 1 a b
sem: 2 c

proc:
  p = P(a);P(c);V(c);V(a)

  q = P(b);P(c);V(c);V(b)

init: p q p q
Factoring the space of states
brute force

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Factoring the space of states

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Factoring isothetic regions

Homogeneous languages and isothetic regions
Factoring the space of states

brute force

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Factoring isothetic regions

Homogeneous languages and isothetic regions
Factoring a program

sem: 1 a b
sem: 2 c

proc:
  p = P(a);P(c);V(c);V(a)

  q = P(b);P(c);V(c);V(b)

init: p q p q
Factoring a program

sem: 1 a b
sem: 2 c

proc:
  p = P(a);P(c);V(c);V(a)
  q = P(b);P(c);V(c);V(b)

init: p p q q
## Factoring a program

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| init: 2p  | init: 2q  |
## Factoring a program

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The preorder \( \preceq \) over \( \mathcal{H}(A) \)

inherited from a preorder \( \preceq \) over \( A \)
The preorder \( \preceq \) over \( \mathcal{H}(A) \)

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- Let \( \preceq^n \) be the product preorder on the words of length \( n \)
The preorder \( \preceq \) over \( \mathcal{H}(\mathbb{A}) \)

inherited from a preorder \( \preceq \) over \( \mathbb{A} \)

- Let \( \preceq^n \) be the product preorder on the words of length \( n \)

- Given \( H, H' \in \mathcal{D}_h(\mathbb{A}) \) of the same dimension \( n \), write \( H \preceq H' \) when for all \( \omega \in H \) there exists \( \omega' \in H' \) such that \( \omega \preceq^n \omega' \)
The preorder $\preceq$ over $\mathcal{H}(\mathcal{A})$

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The preorder $\preceq$ over $\mathcal{H}(A)$

Inherited from a preorder $\preceq$ over $A$

- Let $\preceq^n$ be the product preorder on the words of length $n$
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- $X \preceq Y$ and $X' \preceq Y'$ implies $X \cdot X' \preceq Y \cdot Y'$
  i.e. $(\mathcal{H}(A), \preceq)$ is a preordered commutative monoid
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i.e. $(\mathcal{H}(\mathbb{A}), \preceq)$ is a preordered commutative monoid
- If $\preceq$ is actually a partial order on $\mathbb{A}$, then so is $\preceq$ on $\mathcal{H}(\mathbb{A})$
- If $\preceq$ is the equality relation, then $X \preceq Y$ amounts to $H_X \subseteq H_Y$ for some representatives $H_X$ and $H_Y$ of $X$ and $Y$. 
Homogeneous languages

over the alphabets $|G|$ and $\mathcal{R}_1 \ G \ \{\emptyset\}$ with $G$ being a finite graph
Homogeneous languages

over the alphabets $\mathcal{G}$ and $\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\}$ with $G$ being a finite graph

- $A = |G|$ is the geometric realization of a finite graph:
Homogeneous languages
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- $\mathbb{A} = \mathcal{R}_1 G \setminus \{\emptyset\}$ is the collection of nonempty finite unions of connected subsets of $|G|$:
  - an $n$-block is an $n$-fold product of nonempty elements of $\mathcal{R}_1 G$
    i.e. a word of length $n$ on $\mathbb{A}$
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    i.e. a word of length $n$ on $\mathbb{A}$
  - a nonempty family of $n$-blocks is thus an homogeneous language on $\mathbb{A}$ (of dimension $n$)
  - the concatenation of words on $\mathbb{A}$ corresponds to the cartesian product of blocks
The canonical morphism of monoids $\gamma : \mathcal{H}(R_1 G \setminus \{\emptyset\}) \to \mathcal{H}(1G\downarrow)$
The canonical morphism of monoids $\gamma : \mathcal{H}(R_1G \setminus \{\emptyset\}) \rightarrow \mathcal{H}(1G\downarrow)$

- Let $\gamma$ be the map sending an homogeneous language on $R_1G \setminus \{\emptyset\}$ to the union of its elements
The canonical morphism of monoids $\gamma : \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\}) \rightarrow \mathcal{H}(\mathcal{G})$

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- $\gamma$ is a morphism of monoids from $\mathcal{D}_h(\mathcal{R}_1 G \setminus \{\emptyset\})$ to $\mathcal{D}_h(\mathcal{G})$
The canonical morphism of monoids $\gamma: \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\}) \to \mathcal{H}(\mathcal{I} \mathcal{G})$

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- $\gamma$ induces a morphism of monoids from $\mathcal{H}(R_1 G \setminus \{\emptyset\})$ to $\mathcal{H}(\uparrow G \downarrow)$
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  - $\gamma$ induces a morphism of monoids from $\mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$ to $\mathcal{H}(\uparrow G \downarrow)$

- The induced morphism $\gamma$ does not preserve the prime elements e.g. consider a covering of $[0, 1]^2$ with 3 distinct rectangles
The canonical morphism of monoids \( \alpha : \mathcal{H}(\mathcal{G}) \to \mathcal{H}(\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\}) \)
The canonical morphism of monoids $\alpha : \mathcal{H}(\mathcal{L} \mathcal{I} \mathcal{G}) \rightarrow \mathcal{H}(\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$: 
The canonical morphism of monoids $\alpha : \mathcal{H}(|G|) \to \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
  - given $X \subseteq |G|^n$ and $Y \subseteq |G|^m$, the collection of maximal blocks of $X \times Y$ is
    $\{C \times D \mid C$ and $D$ are maximal blocks of $X$ and $Y\}$
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  - the unique maximal block of the unique nonempty subset of $|G|^0$ is $\varepsilon$
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  - if $C$ is a maximal block of $X \subseteq \lvert G \rvert^n$ then $\sigma \cdot C$ is a maximal block of $\sigma \cdot X$. 
The canonical morphism of monoids $\alpha : \mathcal{H}(\mathcal{L}G) \rightarrow \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
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  - the unique maximal block of the unique nonempty subset of $\mathcal{L}G^0$ is $\varepsilon$
  - $\alpha$ is a morphism of monoids from $\mathcal{D}_h(\mathcal{L}G)$ to $\mathcal{D}_h(\mathcal{R}_1 G \setminus \{\emptyset\})$
  - if $C$ is a maximal block of $X \subseteq \mathcal{L}G^n$ then $\sigma \cdot C$ is a maximal block of $\sigma \cdot X$.
  - $\alpha$ induces a morphism of monoids from $\mathcal{H}(\mathcal{L}G)$ to $\mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$
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  - if $C$ is a maximal block of $X \subseteq |G|^n$ then $\sigma \cdot C$ is a maximal block of $\sigma \cdot X$.
  - $\alpha$ induces a morphism of monoids from $\mathcal{H}(|G|)$ to $\mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$
  - $\text{im}(\alpha)$ is a submonoid of $\mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$
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- the morphisms $\gamma$ and $\alpha$ induce isomorphisms of ordered monoids between $\text{im}(\alpha)$ and $\mathcal{H}(\mathcal{G})$, the order relation being inherited from inclusion over $\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\}$ and equality over $\mathcal{G}$.
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- the morphisms $\gamma$ and $\alpha$ induce isomorphisms of ordered monoids between $\text{im}(\alpha)$ and $\mathcal{H}(|G|)$, the order relation being inherited from inclusion over $\mathcal{R}_1 G \setminus \{\emptyset\}$ and equality over $|G|$.

- therefore $\text{im}(\alpha)$ is commutative free
The free commutative monoids of isothetic regions

- By definition, an isothetic region is a finite union of blocks of $X \subseteq \overset{\rightarrow}{G} \overset{\rightarrow}{n}$.
- We have seen that an isothetic region has finitely many maximal blocks.
- For $X, Y \in H(\overset{\rightarrow}{G} \overset{\rightarrow}{n})$, $\alpha(X \cdot Y)$ is finite iff $\alpha(X)$ and $\alpha(Y)$ are so:
- then $\{X \in \text{im}(\alpha) | \text{card}(X) \text{ is finite}\}$ is a pure submonoid of $\text{im}(\alpha)$
- this commutative monoid is thus free and isomorphic to the monoid of isothetic regions, the latter being defined as $\gamma(\{X \in \text{im}(\alpha) | \text{card}(X) \text{ is finite}\})$
- The monoid of isothetic regions is thus free commutative.
The free commutative monoids of isothetic regions

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The free commutative monoids of isothetic regions

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A better factoring algorithm

by Nicolas Ninin

Let $X \subseteq |G|^n$ be an isothetic region and $F$ be a finite block covering of $X$.

- For each block $(\omega_1, \ldots, \omega_n)$ that belongs to $F$, define the subset $B_\omega = \{ k \in \{1, \ldots, n\} \mid \omega_k \neq |G| \}$.

- The finest partition of $\{1, \ldots, n\}$ that is coarser than the collection $\{B_\omega \mid \omega \in F\}$ induces a factorization of $X$.

If $F = \alpha(X)$ then we obtain the prime factorization of $X$. 

\[\frac{429}{50}\]
A better factoring algorithm
by Nicolas Ninin

Let $X \subseteq |G|^n$ be an isothetic region and $\mathcal{F}$ be a finite block covering of $X^c$
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Factoring a program

sem: 1 a b
sem: 2 c

proc:
  p = P(a); P(c); V(c); V(a)

  q = P(b); P(c); V(c); V(b)

init: p q p q
Factoring the space of states

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**Factoring the space of states**

subtle

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