INDEPENDENCE
Two programs $P$ and $Q$ are said to be compatible when their initial valuations and their arity maps coincide on the intersection of their domains of definition. In that case we define the parallel composition $P \mid Q$. By extension we define the parallel composition of $P_1, \ldots, P_N$ when the programs are pairwise compatible.
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By extension we define the parallel composition of $P_1, \ldots, P_N$ when the programs are pairwise compatible.
Syntactical independence
Independence

Syntactical independence

Two programs are said to be syntactically independent when the set of resources they use are disjoint:

- they have no variables in common,
- they have no semaphores in common, and
- they have no barriers in common.

Syntactically independent programs are compatible.

Syntactical independence can be decided statically, it is compositional, but it is too restrictive.
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Syntactical independence can be decided **statically**, it is **compositional**, but it is too **restrictive**.
Model independence
Model Independence

Independence

Model independence

Suppose the programs $P_1$, ..., $P_N$ are conservative. The programs $P_1$, ..., $P_N$ are said to be model independent when

$$J_{P_1} | \cdots | J_{P_N} = J_{P_1} \times \cdots \times J_{P_N}$$

Model independence can be decided statically.
Model Independence

Suppose the programs $P_1, \ldots, P_N$ are conservative.
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Model independence can be decided statically.
Observational independence
Compatible permutations
Compatible permutations

Assume we have a partition

\[ \{1, \ldots, n\} = S_1 \sqcup \cdots \sqcup S_N \]
Compatible permutations

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Two multi-instructions \( \mu \) and \( \mu' \) (\( \text{dom}(\mu), \text{dom}(\mu') \subseteq \{1, \ldots, n\} \)) should not be swapped when

\[
\{ j \in \{1, \ldots, N\} \mid S_j \cap \text{dom}(\mu) \neq \emptyset \} \cap \{ j \in \{1, \ldots, N\} \mid S_j \cap \text{dom}(\mu') \neq \emptyset \} \neq \emptyset
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A permutation $\pi$ of the set $\{0, \ldots, q - 1\}$ is said to be compatible with the sequence of multi-instructions $\mu_0, \ldots, \mu_{q-1}$ when it does not swap multi-instructions that should not be (it is order preserving on all pairs $\{k, k'\}$ such that $\mu_k$ and $\mu_{k'}$ should not be swapped).
Compatible permutations

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The permutation $\pi$ is said to be compatible with the directed path $\gamma$ when it is compatible with its associated sequence of multi-instructions.
<table>
<thead>
<tr>
<th>Independence</th>
<th>Observational independence</th>
</tr>
</thead>
</table>

Assume that $S_1 = \{1, 3, 5\}$ and $S_2 = \{2, 4\}$. 

\[
\begin{array}{cccccc}
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \\
\mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\
\end{array}
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Observational independence
related to partial order reduction (?)
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Suppose that the programs \( P_1, \ldots, P_N \) are compatible and that \( P_j \) has \( n_j \) running processes.
Observational independence
related to partial order reduction (?)

Suppose that the programs $P_1, \ldots, P_N$ are compatible and that $P_j$ has $n_j$ running processes.

The identifiers of the running processes of $P_1|\cdots|P_N$ are the elements of \{1, \ldots, n\} with

$$n = \sum_{j=1}^{N} n_j,$$

and for $j \in \{1, \ldots, N\}$

$$s_j = \sum_{k=1}^{j} n_k.$$
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The programs $P_1, \ldots, P_N$ are said to be observationally independent when:

- for all execution traces $\gamma$
- for all permutations $\pi$ compatible with the sequence of multi-instructions $(\mu_0 \cdots \mu_{q-1})$ associated with $\gamma$, there exists an execution trace $\gamma'$ whose associated sequence of multi-instructions is $\pi \cdot (\mu_0 \cdots \mu_{q-1})$, which has the same action on the system state than $\gamma$, that is to say

$$\sigma \cdot (\mu_0 \cdots \mu_{q-1}) = \sigma \cdot (\mu_{\pi^{-1}(0)} \cdots \mu_{\pi^{-1}(q-1)}) \cdot$$
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Observational independence cannot be decided statically, moreover it is too loose.
Comparison
Main theorem
Main theorem

syntactic independence
↓
model independence
↓
observational independence
ISOTHETIC REGIONS
Boolean structure
Let $G$ be a finite graph, the collection $R_1^G$ of all finite unions of connected subsets of $|G|$ forms a Boolean subalgebra of $\text{Pow}(|G|)$. Moreover $R_1^G \cong \text{Pow}(V) \times (R_1^{[0,1[})$ with $A$ (resp. $V$) being the set of arrows (resp. vertices) of $G$, and $R_1^{[0,1[}$ being the Boolean algebra of finite unions of subintervals of $[0, 1[$. The elements of $R_1^G$ are seen as one-dimensional blocks.

Proof: If $X$ is a connected subset of $|G|$ then for all arrows $\alpha \in G$, $X \cap (\{\alpha\} \times [0, 1[)$ has at most two connected components.

Yet some infinite graphs may not enjoy the property e.g. when $G$ is a graph with a single vertex and infinitely many arrows.
One-dimensional regions

Let $G$ be a finite graph, the collection $\mathcal{R}_1 G$ of all finite unions of connected subsets of $|G|$ forms a Boolean subalgebra of $\text{Pow}(|G|)$.
One-dimensional regions

Let $G$ be a finite graph, the collection $\mathcal{R}_1 G$ of all finite unions of connected subsets of $|G|$ forms a Boolean subalgebra of $\text{Pow}(|G|)$.

Moreover

$$\mathcal{R}_1 G \cong \text{Pow}(V) \times (\mathcal{R}_1]0, 1[)^{\text{card}A}$$

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The elements of $\mathcal{R}_1G$ are seen as one-dimensional blocks.
One-dimensional regions

Let \( G \) be a finite graph, the collection \( \mathcal{R}_1 G \) of all finite unions of connected subsets of \(|G|\) forms a Boolean subalgebra of \( \text{Pow}(|G|) \).

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**Proof**: If \( X \) is a connected subset of \(|G|\) then for all arrows \( \alpha \in G \), \( X \cap (\{\alpha\} \times ]0, 1[) \) has at most two connected components.
One-dimensional regions

Let $G$ be a finite graph, the collection $\mathcal{R}_1 G$ of all finite unions of connected subsets of $|G|$ forms a Boolean subalgebra of $\text{Pow}(|G|)$.

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Proof: If $X$ is a connected subset of $|G|$ then for all arrows $\alpha \in G$, $X \cap (\{\alpha}\times]0,1[)$ has at most two connected components.

The finiteness condition is not necessary e.g.

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⋅⋅⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→⋅→cdot
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One-dimensional regions

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Moreover

$$\mathcal{R}_1 G \cong \text{Pow}(V) \times (\mathcal{R}_1]0,1[)^{\text{card}A}$$

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The elements of $\mathcal{R}_1 G$ are seen as one-dimensional blocks.

Proof: If $X$ is a connected subset of $|G|$ then for all arrows $\alpha \in G$, $X \cap (\{\alpha\} \times ]0,1[)$ has at most two connected components.

The finiteness condition is not necessary e.g.

$$\cdots \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdots$$

Yet some infinite graphs may not enjoy the property e.g. when $G$ is a graph with a single vertex and infinitely many arrows.
Higher dimensional blocks

- A block of dimension $n \in \mathbb{N}$, or $n$-block, is the product of $n$ connected subsets of the metric graph $|G|$.
- A collection of blocks is called a block covering of $X \subseteq |G|^n$ when the union of its elements is $X$.
- The collection of $n$-dimensional block coverings is denoted by $\text{Cov}_n G$, it is preordered by $C \preceq C'$ if $\forall b \in C \exists b' \in C'$, $b \subseteq b'$. 


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Higher dimensional blocks

- A block of dimension $n \in \mathbb{N}$, or $n$-block, is the product of $n$ connected subsets of the metric graph $|G|$.  

- A collection of blocks is called a block covering of $X \subseteq |G|^n$ when the union of its elements is $X$.  

- The collection of $n$-dimensional block coverings is denoted by $\text{Cov}_n G$, it is preordered by  

$$C \preceq C' \iff \forall b \in C \exists b' \in C',\; b \subseteq b'$$
Maximal blocks

- A block contained in $X$ is said to be a block of $X$. Such a block is said to be maximal when no block of $X$ strictly contains it.

- The maximal connected block covering of $X \subseteq |G|_n$ is the set of all its maximal connected blocks, it is denoted by $\alpha_n(X)$.

- $\alpha_n(X) = \{\emptyset\}$ if and only if $X = \emptyset$. 

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A Galois connection

We have a Galois connection \((\gamma_n, \alpha_n)\) between \(\text{Cov}_n \mathcal{G}\) and \(\text{Pow}(|\mathcal{G}|^n)\) with 
\[
\gamma_n(D) = SD
\]
for all \(D \in \text{Cov}_n \mathcal{G}\).

In particular \(\gamma_n \circ \alpha_n = \text{id}\) and \(\text{id} \preceq \alpha_n \circ \gamma_n\).

That Galois connection induces an isomorphism of Boolean algebras between 
\(\text{Pow}(|\mathcal{G}|^n)\) and the image of \(\alpha_n\), i.e. the collection of maximal connected block coverings.

Proof: any connected block is contained in a maximal connected block (by the Hausdorff maximal principle).
A Galois connection

We have a Galois connection \((\gamma_n, \alpha_n)\) between \(\text{Cov}_n G\) and \(\text{Pow}(|G|^n)\) with \(\gamma_n(D) = \bigcup D\) for all \(D \in \text{Cov}_n G\).

\[
\begin{array}{c}
\text{Cov}_n G & \xrightarrow{\gamma_n} & \text{Pow}(|G|^n) \\
\xleftarrow{\alpha_n} & & \\
\end{array}
\]
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\[
\text{Cov}_n G \overset{\gamma_n}{\longrightarrow} \text{Pow}(|G|^n) \overset{\alpha_n}{\longleftarrow}
\]

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**Proof:** any *connected* block is contained in a maximal *connected* block (by the Hausdorff maximal principle).

\[
\bigcup_i \uparrow \left( B_1^{(i)} \times \cdots \times B_n^{(i)} \right) = \left( \bigcup_i \uparrow B_1^{(i)} \right) \times \cdots \times \left( \bigcup_i \uparrow B_n^{(i)} \right)
\]
Isothetic regions

- An isothetic region of dimension $n$ is a subset of $|G|^n$ that admits a finite block covering.
- The geometric model of a conservative program is an isothetic region.
- The collection of isothetic regions of dimension $n$ is denoted by $R_n G$.
- The collection of finite block covering of dimension $n$ is denoted by $\text{Cov}_{nf} G$.
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The previous Galois connection restricted to isothetic regions
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restricted to isothetic regions

Suppose that the graph $G$ is finite. The collection of $n$-dimensional isothetic regions $\mathcal{R}_nG$ forms a Boolean subalgebra of $\text{Pow}(|G|)$.
The previous Galois connection restricted to isothetic regions

Suppose that the graph $G$ is finite. The collection of $n$-dimensional isothetic regions $\mathcal{R}_n G$ forms a Boolean subalgebra of $\text{Pow}(|G|^n)$ and the previous Galois connection restricts to a Galois connection between $\text{Cov}_{nf} G$ and $\mathcal{R}_n G$. 
The previous Galois connection restricted to isothetic regions

Suppose that the graph $G$ is finite. The collection of $n$-dimensional isothetic regions $\mathcal{R}_n G$ forms a Boolean subalgebra of $\text{Pow}(|G|^n)$ and the previous Galois connection restricts to a Galois connection between $\text{Cov}_{nf} G$ and $\mathcal{R}_n G$, which induces an isomorphism of Boolean algebras between $\mathcal{R}_n G$ and the image of $\alpha_n$ i.e. the collection of finite maximal block coverings.

\[
\text{Cov}_{nf} G \xrightarrow{\gamma_n} \mathcal{R}_n G \xleftarrow{\alpha_n}
\]
The previous Galois connection
restricted to isothetic regions

Suppose that the graph $G$ is finite. The collection of $n$-dimensional isothetic regions $\mathcal{R}_n G$ forms a Boolean subalgebra of $\text{Pow}(\mathcal{G}^n)$ and the previous Galois connection restricts to a Galois connection between $\text{Cov}_{nf} G$ and $\mathcal{R}_n G$, which induces an isomorphism of Boolean algebras between $\mathcal{R}_n G$ and the image of $\alpha_n$ i.e. the collection of finite maximal block coverings.

$$\begin{array}{ccc}
\text{Cov}_{nf} G & \xrightarrow{\gamma_n} & \mathcal{R}_n G \\
\mathcal{R}_n G & \xleftarrow{\alpha_n} & \text{Cov}_{nf} G
\end{array}$$

A subset $X \subseteq \mathcal{G}^n$ is an isothetic region iff the collection of maximal subblocks of $X$ is finite and covers $X$. 
The complement of a block is an isothetic region
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If $X$ is 1-dimensional then its maximal blocks are its connected components.
The complement of a block is an isothetic region

If \( X \) is 1-dimensional then its maximal blocks are its connected components. The complement of a block \( B = B_1 \times \cdots \times B_n \) can be written as

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B^c = \bigcup_{k=1}^{n} |G| \times \cdots \times B_k^c \times \cdots \times |G|
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$$B^c = \bigcup_{k=1}^{n} |G| \times \cdots \times B_k^c \times \cdots \times |G|$$

Its maximal blocks are found among that of $B^c$ therefore they have the form

$$D_1 \times \cdots \times D_{k-1} \times C_k \times D_{k+1} \times \cdots \times D_n$$

with $k \in \{1, \ldots, n\}$, $C_k$ ranging through the connected components of $B_k^c$ and $D_j$, for $j \neq k$, ranging through the connected components of $|G|$.
Intersection of two isothetic regions

The intersection of the blocks $B$ and $B'$ is given by

$$B \cap B' = (B_1 \cap B'_1) \times \cdots \times (B_n \cap B'_n)$$

The maximal blocks of $B \cap B'$ are therefore of the form $C_1 \times \cdots \times C_n$ with each $C_k$ ranging through the connected components of $(B_k \cap B'_k)$.

It follows from De Morgan's laws that the intersection of two regions is still a region.

Moreover if $B$ and $B'$ are block coverings of $X$ and $X'$ containing all their maximal blocks, then the collection of maximal blocks of $B \cap B'$ for $B \in B$ and $B' \in B'$ is a block covering of $X \cap X'$ containing all its maximal blocks.
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Concluding the proof
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If $\mathcal{F}$ is any finite block covering of $X$, then

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If $\mathcal{F}$ is any finite block covering of $X$, then

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- The collection of maximal blocks of $B^c$ is finite and covers $B^c$.
Concluding the proof

If $\mathcal{F}$ is any finite block covering of $X$, then

$$X^c = \bigcap_{B \in \mathcal{F}} B^c$$

- The collection of maximal blocks of $B^c$ is finite and covers $B^c$.  
- The maximal blocks of $X^c$ are obtained as certain finite intersection of the form

$$\bigcap \{M_B \mid B \in \mathcal{F}\}$$

where $M_B$ is a maximal block of $B^c$. 
Concluding the proof

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- The collection of maximal blocks of $B^c$ is finite and covers $B^c$.
- The maximal blocks of $X^c$ are obtained as certain finite intersection of the form

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where $M_B$ is a maximal block of $B^c$.
- The maximal blocks of $X^c$ thus form a finite block covering of $X^c$. 
A result from directed topology
A result from directed topology

For all directed paths $\gamma$ on $|G|^n$ and all $X \in R_nG$, the inverse image of $X$ by $\gamma$ has finitely many connected components.
Additional operators
Closure, interior, and boundary of an isothetic region

The closure operator preserves finite products, therefore it preserves blocks. The closure operator preserves finite unions hence it preserves isothetic regions. The boundary of a set is the intersection of its closure and the closure of its complement, hence it also preserves isothetic regions. The interior of a set is the difference between its closure and its boundary. It follows that the interior operator also preserves isothetic regions.
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The interior of a set is the difference between its closure and its boundary. It follows that the interior operator also preserves isothetic regions.
The forward and the backward operators

Let $A, B$ be subsets of a local pospace $X$.

- The forward and the backward operators are defined as
  
  \[ \text{frw}(A, B) = \{ \partial^+ \delta \mid \delta \text{ directed path on } X; \partial^- \delta \in A; \text{im}(\delta) \subseteq A \cup B \} \]
  
  \[ \text{bck}(A, B) = \{ \partial^- \delta \mid \delta \text{ directed path on } X; \partial^+ \delta \in A; \text{im}(\delta) \subseteq A \cup B \} \]

- The future cone of $A$ in $X$ is cone $f_A := \text{frw}(A, X)$ and
  
  - The past cone of $A$ in $X$ is cone $p_A := \text{bck}(A, X)$.
  
- The future closure of $A$ in $X$ is $A_f := \text{frw}(A, A)$ and
  
  - The past closure of $A$ in $X$ is $A_p := \text{bck}(A, A)$.

Theorem: if $A, B,$ and $X$ are isothetic regions, then so are $\text{frw}(A, B)$, cone $f_A$, $A_f$, and their duals.
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- The future cone of $A$ in $X$ is $\text{cone}^f A := \text{frw}(A, X)$ and the past cone of $A$ in $X$ is $\text{cone}^p A := \text{bck}(A, X)$.
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- The future closure of \(A\) in \(X\) is \(\bar{A}^f := \text{frw}(A, \bar{A})\) and the past closure of \(A\) in \(X\) is \(\bar{A}^p := \text{bck}(A, \bar{A})\).

The closure \(\bar{A}\) being understood in \(X\).
The forward and the backward operators

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**Theorem:** if $A$, $B$, and $X$ are isothetic regions, then so are $\text{frw}(A, B)$, $\text{cone}^f A$, $\bar{A}^f$, and their duals.
Future/past stable subsets of $X$

- Let $A$ be a subset of a local pospace $X$.
- $\text{cone}_f A = \text{cone}_f A$ and $\text{cone}_p A = \text{cone}_p A - A$ is said to be future (resp. past) stable (in $X$) when $\text{cone}_f A = A$ (resp. $\text{cone}_p A = A$).
- $A$ is future stable iff $X \setminus A$ is past stable.
- The collection of future stable subsets of $X$ is a complete lattice, the greatest lower (resp. least upper) bound of a family being given by its intersection (resp. union).
- The same holds for past stable subsets.
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- The collection of future stable subsets of $X$ is a complete lattice, the greatest lower (resp. least upper) bound of a family being given by its intersection (resp. union).
- The same holds for past stable subsets.
Past/future attractors

Let $A$ be a subset of a local pospace $X$. Then:

- The cone $p_A$ of $A$ is the set of all points $p$ in $X$ from which $A$ can be reached.
- The escape set $\text{escape}_f A$ is the set of all points $p$ in $X$ from which $A$ is avoided.
- The apsis set $\text{att}_p A$ is the set of all points $p$ in $X$ from which $A$ cannot be avoided.

$$\text{cone}_p A = bck(\text{escape}_f A, X) = \text{escape}_f (\text{escape}_f A)$$
Past/future attractors

Let $A$ be a subset of a local pospace $X$. 
Past/future attractors

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$$\text{cone}^p A = \{ p \in X \text{ from which } A \text{ can be reached} \} = \cdots$$
Past/future attractors

Let $A$ be a subset of a local pospace $X$.

\[
\text{cone}^p A = \{ p \in X \text{ from which } A \text{ can be reached} \} = \text{bck}(A, X) = \text{cone}^p A
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Past/future attractors

Let $A$ be a subset of a local pospace $X$.

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\[
\text{escape}^f A = (\text{cone}^p A)^c
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Past/future attractors

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$$
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Past/future attractors

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$$\text{escape}^f A = (\text{cone}^p A)^c$$

$$\text{att}^p A = \{ p \in X \text{ from which } A \text{ cannot be avoided} \}$$

$$\text{att}^p A = \text{escape}^f (\text{escape}^f A)$$
Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$.

- The reachable space of $J_{GP_K}$ is the future cone of the initial point.

- A point $p \in \uparrow G_i \downarrow$ is said to be terminal when $J_{\gamma K}$ is empty for all directed paths on $\uparrow G_i \downarrow$ starting at $p$.

- A point $p \in J_{GP_K}$ is said to be terminal when so are all its projections.

- The terminal points form a future stable isothetic region of $J_{GP_K}$.

- A point $p \in J_{GP_K}$ is said to be deadlock when its future cone neither contains directed loops (i.e. it is loop-free) nor terminal points.

- The deadlock points form a future stable isothetic region of $J_{GP_K}$.

- The deadlock attractor of the program is the past attractor of its deadlock region.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

- The reachable space of $\llbracket P \rrbracket$ is the future cone of the initial point.
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- A point $p \in \llbracket P \rrbracket$ is said to be terminal when so are all its projections.

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- The terminal points form a . . .
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- The deadlock points form a future stable isothetic region of $\llbracket P \rrbracket$.
- The deadlock attractor of the program is the past attractor of its deadlock region.
The deadlock attractor of a conservative program

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The deadlock attractor of the program is the past attractor of its deadlock region.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\langle P \rangle$ be the geometric model of the program.

- The reachable space of $\langle P \rangle$ is the future cone of the initial point.
- A point $p \in \downarrow G_i \downarrow$ is said to be terminal when $\langle \gamma \rangle$ is empty for all directed paths on $\downarrow G_i \downarrow$ starting at $p$.
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The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $[P]$ be the geometric model of the program.

- The reachable space of $[P]$ is the future cone of the initial point.
- A point $p \in G_i$ is said to be terminal when $\gamma$ is empty for all directed paths on $\downarrow G_i$ starting at $p$.
- A point $p \in [P]$ is said to be terminal when so are all its projections.
- The terminal points form a future stable isothetic region of $[P]$.
- A point $p \in [P]$ is said to be deadlock when its future cone neither contains directed loops (i.e. it is loop-free) nor terminal points.
- The deadlock points form a future stable isothetic region of $[P]$.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let \( [P] \) be the geometric model of the program.

- The reachable space of \( [P] \) is the future cone of the initial point
- A point $p \in |G_i|$ is said to be **terminal** when \( [\gamma] \) is empty for all directed paths on $|G_i|$ starting at $p$.
- A point $p \in [P]$ is said to be **terminal** when so are all its projections
- The terminal points form a future stable isothetic region of \( [P] \)
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- The **deadlock attractor** of the program is the \ldots
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

- The reachable space of $\llbracket P \rrbracket$ is the future cone of the initial point.
- A point $p \in |G_i|$ is said to be terminal when $\llbracket \gamma \rrbracket$ is empty for all directed paths on $|G_i|$ starting at $p$.
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- The deadlock attractor of the program is the past attractor of its deadlock region.
Deadlock attractor of the Swiss Cross

sem 1 a b
proc:
q = P(b).P(a).V(a).V(b)
init:  p q
Deadlock attractor of the Swiss Cross

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Deadlock attractor of the Swiss Cross

\[
\text{sem } 1 \ a \ b \\
\text{proc:} \\
p = P(a) . P(b) . V(b) . V(a) \\
q = P(b) . P(a) . V(a) . V(b) \\
\text{init: } p \ q
\]
Three dining philosophers
FACTORYING ISOTHETIC REGIONS
Free commutative monoids
Commutative monoids

For all set $X$, the collection $\mathcal{M}_X$ of multisets over $X$, i.e. maps $\phi: X \rightarrow \mathbb{N}$ such that $\{x \in X | \phi(x) \neq 0\}$ is finite, forms a commutative monoid with pointwise addition.

A commutative monoid is said to be free when it is isomorphic with some $\mathcal{M}_X$.

A multiset $\phi$ can be written as $\sum_{x \in X} \phi(x) x$.

In particular, if $f: X \rightarrow Y$ is a set map, then $\mathcal{M}(f)(\phi) = \sum_{x \in X} \phi(x) f(x)$.
Commutative monoids

- \((M, *, \varepsilon)\) such that for all \(a, b, c \in M\),
  - \((ab)c = a(bc)\)
  - \(\varepsilon a = a = a\varepsilon\)
  - \(ab = ba\)
Commutative monoids

- \((M, *, \varepsilon)\) such that for all \(a, b, c \in M\),
  - \((ab)c = a(bc)\)
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- For all set \(X\) the collection \(MX\) of multisets over \(X\)
  i.e. maps \(\phi : X \rightarrow \mathbb{N}\) s.t. \(\{x \in X \mid \phi(x) \neq 0\}\) is finite
  forms a commutative monoid with pointwise addition
Commutative monoids

- \( (M, \ast, \varepsilon) \) such that for all \( a, b, c \in M \),
  - \( (ab)c = a(bc) \)
  - \( \varepsilon a = a = a\varepsilon \)
  - \( ab = ba \)
- For all set \( X \) the collection \( M^X \) of multisets over \( X \)
  i.e. maps \( \phi : X \to \mathbb{N} \) s.t. \( \{x \in X \mid \phi(x) \neq 0\} \) is finite
  forms a commutative monoid with pointwise addition
- A commutative monoid is said to be free when
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Commutative monoids

- \((M, *, \varepsilon)\) such that for all \(a, b, c \in M\),
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- Functor \(M : Set \to Cmon\)
Commutative monoids

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- Functor \(M : \text{Set} \to \text{Cmon}\)
  - A multiset \(\phi\) can be written as
    \[\sum_{x \in X} \phi(x)x\]
- In particular, if \(f : X \to Y\) is a set map, then
  \[M(f)(\phi) = \sum_{x \in X} \phi(x)f(x)\]
Prime vs irreducible
Prime vs irreducible

- $d$ divides $x$, denoted by $d|x$, when there exists $x'$ such that $x = dx'$
Prime vs irreducible

- $d$ divides $x$, denoted by $d|x$, when there exists $x'$ such that $x = dx'$
- $u$ unit: exists $u'$ s.t. $uu' = \varepsilon$ then write $x \sim y$ when $y = ux$ for some unit $u$
Prime vs irreducible

- *d divides* $x$, denoted by $d | x$, when there exists $x'$ such that $x = dx'$
- *$u$ unit*: exists $u'$ s.t. $uu' = \varepsilon$ then write $x \sim y$ when $y = ux$ for some unit $u$
- *$i$ irreducible*: $i$ nonunit and $x | i$ implies $x \sim i$ or $x$ unit
Prime vs irreducible

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- $i$ irreducible: $i$ nonunit and $x | i$ implies $x \sim i$ or $x$ unit

- $p$ prime: $p$ nonunit and $p | ab$ implies $p | a$ or $p | b$
Prime vs irreducible

- \(d\) divides \(x\), denoted by \(d|x\), when there exists \(x'\) such that \(x = dx'\)
- \(u\) unit: exists \(u'\) s.t. \(uu' = \varepsilon\) then write \(x \sim y\) when \(y = ux\) for some unit \(u\)
- \(i\) irreducible: \(i\) nonunit and \(x|i\) implies \(x \sim i\) or \(x\) unit
- \(p\) prime: \(p\) nonunit and \(p|ab\) implies \(p|a\) or \(p|b\)
- If \(M\) contains nontrivial units, then one can consider the quotient monoid \(M/\sim\) where \(x \sim y\) stands for: there exists a unit \(u\) s.t. \(y = ux\)
### Examples

<table>
<thead>
<tr>
<th>monoid</th>
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- $N$: the set of natural numbers excluding zero.
- $\times$: multiplication.
- $1$: the multiplicative identity.
- $\{\}$: empty set.
- $\mathbb{N}$: the set of natural numbers.
- $\{0\}$: the set containing zero.
- $\times$: multiplication.
- $1$: the multiplicative identity.
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Graded commutative monoid
Graded commutative monoid

- \((M, *, \varepsilon)\) graded: there is a morphism \(g : (M, *, \varepsilon) \to (\mathbb{N}, +, 0)\)
  s.t. \(g^{-1}(\{0\}) = \{\text{units of } M\}\)
Graded commutative monoid

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- If \(M\) is graded then
Graded commutative monoid

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- If \(M\) is graded then
  - \{irreducibles of \(M\}\} generates \(M\)
Graded commutative monoid

- \((M, *, \varepsilon)\) graded: there is a morphism \(g : (M, *, \varepsilon) \to (\mathbb{N}, +, 0)\) s.t. \(g^{-1}(\{0\}) = \{\text{units of } M\}\)

- If \(M\) is graded then
  - \(\{\text{irreducibles of } M\}\) generates \(M\)
  - \(\{\text{primes of } M\} \subseteq \{\text{irreducibles of } M\}\)
Irreducible that are not prime

\( M = (\{ a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0 \}, \times, 1) \)
Irreducible that are not prime

\[ M = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0\}, \times, 1\]

- \( N : M \rightarrow (\mathbb{Z} \setminus \{0\}, \times, 1) \); \( N(a + b\sqrt{10}) = a^2 - 10b^2 \)
Irreducible that are not prime

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- 2, 3, and \( 4 \pm \sqrt{10} \) are irreducible but not prime
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- 2, 3, and 4 \( \pm \sqrt{10} \) are irreducible but not prime
  since \( 2 \cdot 3 = (4 + \sqrt{10}) \cdot (4 - \sqrt{10}) \)
- \( \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \setminus \{0\} \) is graded by the number of prime factors of \( N(u) \)
\( \mathbb{N}[X] \) polynomials with coefficients in \( \mathbb{N} \)

*On Direct Product Decomposition of Partially Ordered Sets.* Junji Hashimoto
Annals of Mathematics 2(54), pp 315-318 (1951)
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\{ (X + 1)(X^4 + X^2 + 1) \}
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- \( \mathbb{N}[X] \setminus \{0\} \) is graded by the degree
Characterization of the free commutative monoids

Unique factorization

The following are equivalent:

- \( M \) is free commutative
- any element of \( M \) can be written as a product of irreducibles in a unique way up to reordering
- \( \{ \text{primes of } M \} = \{ \text{irreducibles of } M \} \) and generates \( M \)

- \( M \) is graded and \( \{ \text{irreducibles of } M \} \subseteq \{ \text{primes of } M \} \)

Standard examples:

- \((\mathbb{N} \setminus \{0\}, \times, 1)\)
- \((\mathbb{N}, +, 0)\) and its finite products in the category of commutative monoids.

Indeed \((\mathbb{N}, +, 0)\) isomorphic to \( \mathbb{Z}[X] \) if \( F \) is a factorial ring, then so is \( F[X] \).


Note that two free commutative monoids are isomorphic in \( \text{Cmon} \) iff their set of prime elements have the same cardinality e.g. \((\mathbb{N} \setminus \{0\}, \times, 1) \cong (\mathbb{Z}[X] \setminus \{0\}, \times, 1) \) in \( \text{Cmon} \).
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In particular $\mathcal{M}_2 \cong (\mathbb{N}, +, 0)$ and $\mathcal{M}_3 \cong (\mathbb{N} \setminus \{0\}, \times, 1)$
Monoids of homogeneous languages
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Factoring isothetic regions

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- **Factoring isothetic regions**
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The noncommutative monoid of languages
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- $A^*$ (non commutative) monoid of words on the alphabet $A$.
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  - The monoid of nonempty languages is $D(A)$
  - $D(A)$ is commutative iff $\text{Card}(A) \leq 1$. Note that $D(\emptyset) \cong \{\varepsilon\}$
  - however $D(\{a\})$ is not freely commutative
The noncommutative monoid of homogeneous languages
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- \( \mathcal{D}_h\{a\} \cong \ldots \)
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on the left of the homogeneous languages
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- The $n^{th}$ symmetric group $\mathfrak{S}_n$ acts on the left of the set of words of length $n$
i.e. mappings from $\{1, \ldots, n\}$ to $\mathcal{A}$, by $\sigma \cdot \omega := \omega \circ \sigma^{-1}$
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- If $\sigma \in \mathfrak{S}_n$ and $\sigma' \in \mathfrak{S}_{n'}$ then define $\sigma \otimes \sigma' \in \mathfrak{S}_{n+n'}$ as:

$$\sigma \otimes \sigma'(k) := \begin{cases} 
\sigma(k) & \text{if } 1 \leq k \leq n \\
(\sigma'(k - n)) + n & \text{if } n + 1 \leq k \leq n + n'
\end{cases}$$
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- A Godement exchange law is satisfied, which ensures that $\sim$ is actually a congruence:

\[(\sigma \cdot H) \cdot (\sigma' \cdot H') = (\sigma \otimes \sigma') \cdot (H \cdot H')\]

i.e. $H \sim K$ and $H' \sim K'$ implies $HH' \sim KK'$
The commutative monoid of homogeneous languages

\[
\text{H}(A) = (\text{Dh}(A), \cdot, \{\varepsilon\}) / \sim
\]

The monoid \( \text{H}(A) \) is graded by \( H \in \text{H}(A) \mapsto \dim(H) \in (\mathbb{N}, +, 0) \)

The commutative monoid \( \text{H}(A) \) is free

For any homogeneous language \( H \) and \( \sigma \in \text{S}, \dim(H), \text{card}(H) = \text{card}(\sigma \cdot H) \) so we can define the cardinality of any element of \( \text{H}(A) \).
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The commutative monoid of finite homogeneous languages

- A pure submonoid of a free commutative monoid is free.
- The submonoid $H_f(A) \subseteq H(A)$ of finite languages is pure, therefore it is free.
- $H \in H_f(A) \mapsto \text{Card}(H) \in (\mathbb{N} \setminus \{0\}, \times, 1)$ is a morphism of monoid.

- The primality of $\text{Card}(H)$ does not imply that of $H$.
  - e.g. $H = \{ab, ac\} = \{a\} \cdot \{b, c\}$ though card($H$) = 2.
  - The primality of $H$ does not imply that of Card($H$).
    - e.g. $H = \{a, b, c, d\}$ is prime though card($H$) = 4.
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- $M' \subseteq M$ is said to be pure when for all $x, y \in M$, $xy \in M'$ implies $x, y \in M'$
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The brute force algorithm for factoring in $\mathcal{H}_f(A)$

Theory
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

**Theory**

Given $w \in \mathbb{A}^n$ and $I \subseteq \{1, \ldots, n\}$, we write $w|_I$ for the subword of $w$ consisting of letters with indices in $I$. 


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Given a homogeneous language $H$ of dimension $n$, we write

$$H_{|I} = \{ w_{|I} \mid w \in H \}$$
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Denoting $I^c$ for $\{1, \ldots, n\} \setminus I$, we have

$$[H] = [H|_I] \cdot [H|_{I^c}]$$

in $\mathcal{H}_f(\mathbb{A})$ if and only if for all words $u, v \in H$ there exists a word $w \in H$ such that

$$w|_I = u|_I \quad \text{and} \quad w|_{I^c} = v|_{I^c}$$
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Practice
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Practice

For $I \subseteq \{1, \ldots, n\}$ let $\pi|_I$ be the “projection” that sends $w \in H$ to $w|_I \in \mathbb{A}^{\text{card}(I)}$. 
The brute force algorithm for factoring in $\mathcal{H}_f(A)$

For $I \subseteq \{1, \ldots, n\}$ let $\pi_I$ be the “projection” that sends $w \in H$ to $w_I \in A^{\text{card}(I)}$.

1. choose $I \subseteq \{1, \ldots, n\}$ of cardinality $k \leq n/2$
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Practice

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1. choose $I \subseteq \{1, \ldots, n\}$ of cardinality $k \leq n/2$
2. if $\pi_{|I^c} (\pi_{|I}^{-1}(u))$ does not depend on $u \in H_{|I}$, then we have the factorization

$$[H] = [H_{|I}] \cdot [H_{|I^c}]$$

and we are done
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3. otherwise check whether there are still subsets of $\{1, \ldots, n\}$ to check:
   3.1. yes: go to step 1
   3.2. no: $[H]$ is prime
Homogeneous languages and isothetic regions
Factoring a program

\text{sem:} \quad 1 \ a \ b \\
\text{sem:} \quad 2 \ c

\underline{\text{proc:}} \\
\text{p} = P(a); P(c); V(c); V(a) \\
\text{q} = P(b); P(c); V(c); V(b)

\underline{\text{init:}} \quad p \ q \ p \ q
## Factoring the space of states

**brute force**

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brute force
Factoring isothetic regions

Homogeneous languages and isothetic regions

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Factoring the space of states

brute force
Factoring the space of states

brute force

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Factoring a program

sem: 1 a b
sem: 2 c

proc:  
  p = P(a);P(c);V(c);V(a)
  q = P(b);P(c);V(c);V(b)

init: p q p q
Factoring a program

```
sem:  1 a b
sem:  2 c

proc:
p = P(a); P(c); V(c); V(a)
q = P(b); P(c); V(c); V(b)

init:  p p q q
```
## Factoring a program

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<td>q = P(b);P(c);V(c);V(b)</td>
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## Factoring a program

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The preorder $\preceq$ over $\mathcal{H}(A)$ inherited from a preorder $\preceq$ over $A$
The preorder \( \preceq \) over \( \mathcal{H}(\mathcal{A}) \)

inherited from a preorder \( \preceq \) over \( \mathcal{A} \)

- Let \( \preceq^n \) be the product preorder on the words of length \( n \)
Factoring isothetic regions

Homogeneous languages and isothetic regions

The preorder \(\preceq\) over \(\mathcal{H}(\mathbb{A})\)

inherited from a preorder \(\preceq\) over \(\mathbb{A}\)

- Let \(\preceq^n\) be the product preorder on the words of length \(n\)
- Given \(H, H' \in \mathcal{D}_h(\mathbb{A})\) of the same dimension \(n\), write \(H \preceq H'\) when for all \(\omega \in H\) there exists \(\omega' \in H'\) such that \(\omega \preceq^n \omega'\)

- If \(\preceq\) is actually a partial order on \(\mathbb{A}\), then so is \(\preceq\) on \(\mathcal{H}(\mathbb{A})\)
- If \(\preceq\) is the equality relation, then \(X \preceq Y\) amounts to \(H_X \subseteq H_Y\) for some representatives \(H_X\) and \(H_Y\) of \(X\) and \(Y\).
The preorder $\preceq$ over $\mathcal{H}(A)$

inherited from a preorder $\preceq$ over $A$

- Let $\preceq^n$ be the product preorder on the words of length $n$
- Given $H, H' \in \mathcal{D}_h(A)$ of the same dimension $n$, write $H \preceq H'$ when for all $\omega \in H$ there exists $\omega' \in H'$ such that $\omega \preceq^n \omega'$
- Given $X, Y \in \mathcal{H}(A)$ of the same dimension $n$ write $X \preceq Y$ when there exist $H \in X$ and $K \in Y$ such that $H \preceq K$
The preorder $\preccurlyeq$ over $\mathcal{H}(\mathbb{A})$

inherited from a preorder $\preccurlyeq$ over $\mathbb{A}$

- Let $\preccurlyeq^n$ be the product preorder on the words of length $n$
- Given $H, H' \in \mathcal{D}_h(\mathbb{A})$ of the same dimension $n$, write $H \preccurlyeq H'$ when for all $\omega \in H$ there exists $\omega' \in H'$ such that $\omega \preccurlyeq^n \omega'$
- Given $X, Y \in \mathcal{H}(\mathbb{A})$ of the same dimension $n$ write $X \preccurlyeq Y$ when there exist $H \in X$ and $K \in Y$ such that $H \preccurlyeq K$
- $X \preccurlyeq Y$ and $X' \preccurlyeq Y'$ implies $X \cdot X' \preccurlyeq Y \cdot Y'$
  i.e. $(\mathcal{H}(\mathbb{A}), \preccurlyeq)$ is a preordered commutative monoid
The preorder $\preceq$ over $\mathcal{H}(\mathbb{A})$

inherited from a preorder $\preceq$ over $\mathbb{A}$

- Let $\preceq^n$ be the product preorder on the words of length $n$
- Given $H, H' \in D_h(\mathbb{A})$ of the same dimension $n$, write $H \preceq H'$ when for all $\omega \in H$ there exists $\omega' \in H'$ such that $\omega \preceq^n \omega'$
- Given $X, Y \in \mathcal{H}(\mathbb{A})$ of the same dimension $n$ write $X \preceq Y$ when there exist $H \in X$ and $K \in Y$ such that $H \preceq K$
- $X \preceq Y$ and $X' \preceq Y'$ implies $X \cdot X' \preceq Y \cdot Y'$
  i.e. $(\mathcal{H}(\mathbb{A}), \preceq)$ is a preordered commutative monoid
- If $\preceq$ is actually a partial order on $\mathbb{A}$, then so is $\preceq$ on $\mathcal{H}(\mathbb{A})$
The preorder \( \preceq \) over \( \mathcal{H}(A) \)

inherited from a preorder \( \preceq \) over \( A \)

- Let \( \preceq^n \) be the product preorder on the words of length \( n \)
- Given \( H, H' \in D_h(A) \) of the same dimension \( n \), write \( H \preceq H' \) when for all \( \omega \in H \) there exists \( \omega' \in H' \) such that \( \omega \preceq^n \omega' \)
- Given \( X, Y \in \mathcal{H}(A) \) of the same dimension \( n \) write \( X \preceq Y \) when there exist \( H \in X \) and \( K \in Y \) such that \( H \preceq K \)
- \( X \preceq Y \) and \( X' \preceq Y' \) implies \( X \cdot X' \preceq Y \cdot Y' \)
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Homogeneous languages
over the alphabets $|G|$ and $R_1G \setminus \{\emptyset\}$ with $G$ being a finite graph
Homogeneous languages
over the alphabets $|G|$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with $G$ being a finite graph

- $A = |G|$ is the geometric realization of a finite graph:
Homogeneous languages
over the alphabets $|G|$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with $G$ being a finite graph

- $\mathbb{A} = |G|$ is the geometric realization of a finite graph:
  - a point of $|G|^n$ can be seen as a word of length $n$ on $\mathbb{A}$
Homogeneous languages
over the alphabets $|G|$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with $G$ being a finite graph

- $\mathbb{A} = |G|$ is the geometric realization of a finite graph:
  - a point of $|G|^n$ can be seen as a word of length $n$ on $\mathbb{A}$
  - a nonempty subset of $|G|^n$ is thus a homogeneous language on $\mathbb{A}$
Homogeneous languages over the alphabets $\mathcal{A} = \mathcal{G}$ and $\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\}$ with $\mathcal{G}$ being a finite graph

- $\mathcal{A} = \mathcal{G}$ is the geometric realization of a finite graph:
  - a point of $\mathcal{G}^n$ can be seen as a word of length $n$ on $\mathcal{A}$
  - a nonempty subset of $\mathcal{G}^n$ is thus a homogeneous language on $\mathcal{A}$
  - the product of the monoid $\mathcal{D}_h(\mathcal{A})$ corresponds to the cartesian product of isothetic regions
Homogeneous languages

over the alphabets $|G|$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with $G$ being a finite graph

- $\mathbb{A} = |G|$ is the geometric realization of a finite graph:
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- $\mathbb{A} = \mathcal{R}_1 G \setminus \{\emptyset\}$ is the collection of nonempty finite unions of connected subsets of $|G|$:
Homogeneous languages

over the alphabets $\mathcal{V}G$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with $G$ being a finite graph

- $A = \mathcal{V}G$ is the geometric realization of a finite graph:
  - a point of $\mathcal{V}G^n$ can be seen as a word of length $n$ on $A$
  - a nonempty subset of $\mathcal{V}G^n$ is thus a homogeneous language on $A$
  - the product of the monoid $D_h(A)$ corresponds to the cartesian product of isothetic regions

- $A = \mathcal{R}_1 G \setminus \{\emptyset\}$ is the collection of nonempty finite unions of connected subsets of $\mathcal{V}G$:
  - an $n$-block is an $n$-fold product of nonempty elements of $\mathcal{R}_1 G$
    i.e. a word of length $n$ on $A$
Homogeneous languages
over the alphabets $|G|$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with $G$ being a finite graph

- $\mathcal{A} = |G|$ is the geometric realization of a finite graph:
  - a point of $|G|^n$ can be seen as a word of length $n$ on $\mathcal{A}$
  - a nonempty subset of $|G|^n$ is thus a homogeneous language on $\mathcal{A}$
  - the product of the monoid $D_h(\mathcal{A})$ corresponds to the cartesian product of isothetic regions

- $\mathcal{A} = \mathcal{R}_1 G \setminus \{\emptyset\}$ is the collection of nonempty finite unions of connected subsets of $|G|$:
  - an $n$-block is an $n$-fold product of nonempty elements of $\mathcal{R}_1 G$
    i.e. a word of length $n$ on $\mathcal{A}$
  - a nonempty family of $n$-blocks is thus an homogeneous language on $\mathcal{A}$ (of dimension $n$)
Homogeneous languages
over the alphabets $|G|$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with $G$ being a finite graph

- $\mathbb{A} = |G|$ is the geometric realization of a finite graph:
  - a point of $|G|^n$ can be seen as a word of length $n$ on $\mathbb{A}$
  - a nonempty subset of $|G|^n$ is thus a homogeneous language on $\mathbb{A}$
  - the product of the monoid $\mathcal{D}_h(\mathbb{A})$ corresponds to the cartesian product of isothetic regions

- $\mathbb{A} = \mathcal{R}_1 G \setminus \{\emptyset\}$ is the collection of nonempty finite unions of connected subsets of $|G|$:
  - an $n$-block is an $n$-fold product of nonempty elements of $\mathcal{R}_1 G$
    i.e. a word of length $n$ on $\mathbb{A}$
  - a nonempty family of $n$-blocks is thus an homogeneous language on $\mathbb{A}$ (of dimension $n$)
  - the concatenation of words on $\mathbb{A}$ corresponds to the cartesian product of blocks
The canonical morphism of monoids $\gamma : H(R_1 G \setminus \{\emptyset\}) \rightarrow H(|G|)$
The canonical morphism of monoids $\gamma : \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\}) \to \mathcal{H}(\mathcal{H}(G))$

- Let $\gamma$ be the map sending an homogeneous language on $\mathcal{R}_1 G \setminus \{\emptyset\}$ to the union of its elements.
The canonical morphism of monoids \( \gamma : \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\}) \rightarrow \mathcal{H}(\downharpoonright G \downharpoonleft) \)

- Let \( \gamma \) be the map sending an homogeneous language on \( \mathcal{R}_1 G \setminus \{\emptyset\} \) to the union of its elements.
- \( \gamma \) is a morphism of monoids from \( \mathcal{D}_h(\mathcal{R}_1 G \setminus \{\emptyset\}) \) to \( \mathcal{D}_h(\downharpoonright G \downharpoonleft) \).
The canonical morphism of monoids \( \gamma : \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\}) \rightarrow \mathcal{H}(\downarrow G \uparrow) \)

- Let \( \gamma \) be the map sending an homogeneous language on \( \mathcal{R}_1 G \setminus \{\emptyset\} \) to the union of its elements
- \( \gamma \) is a morphism of monoids from \( \mathcal{D}_h(\mathcal{R}_1 G \setminus \{\emptyset\}) \) to \( \mathcal{D}_h(\downarrow G \uparrow) \)
- \( \gamma \) is compatible with the action of the symmetric groups in the sense that
  \[ H' = \sigma \cdot H \Rightarrow \bigcup H' = \sigma \cdot (\bigcup H) \]
The canonical morphism of monoids $\gamma : \mathcal{H}(R_1 G \setminus \{\emptyset\}) \rightarrow \mathcal{H}(\uparrow G \downarrow)$

- Let $\gamma$ be the map sending an homogeneous language on $R_1 G \setminus \{\emptyset\}$ to the union of its elements
- $\gamma$ is a morphism of monoids from $D_h(R_1 G \setminus \{\emptyset\})$ to $D_h(\uparrow G \downarrow)$
- $\gamma$ is compatible with the action of the symmetric groups in the sense that $H' = \sigma \cdot H \Rightarrow \bigcup H' = \sigma \cdot (\bigcup H)$
- $\gamma$ induces a morphism of monoids from $\mathcal{H}(R_1 G \setminus \{\emptyset\})$ to $\mathcal{H}(\uparrow G \downarrow)$
The canonical morphism of monoids $\gamma : \mathcal{H}(R_1 G \setminus \{\emptyset\}) \rightarrow \mathcal{H}(\uparrow G \downarrow)$

- Let $\gamma$ be the map sending an homogeneous language on $R_1 G \setminus \{\emptyset\}$ to the union of its elements
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- The induced morphism $\gamma$ does not preserve the prime elements e.g. consider a covering of $[0,1]^2$ with 3 distinct rectangles
The canonical morphism of monoids $\alpha : \mathcal{H}(\mathbb{R}^1 G) \rightarrow \mathcal{H}(\mathbb{R}^1 G \setminus \{\emptyset\})$
The canonical morphism of monoids $\alpha : H(⇑G⇂) \rightarrow H(R_1G \setminus \{∅\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
The canonical morphism of monoids $\alpha : \mathcal{H}(|G|) \rightarrow \mathcal{H}(R_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:  
  - given $X \subseteq |G|^n$ and $Y \subseteq |G|^m$, the collection of maximal blocks of $X \times Y$ is \{ $C \times D$ | $C$ and $D$ are maximal blocks of $X$ and $Y$ \}
The canonical morphism of monoids $\alpha : \mathcal{H}(\mathcal{L}G) \rightarrow \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
  - given $X \subseteq \mathcal{L}G^n$ and $Y \subseteq \mathcal{L}G^m$, the collection of maximal blocks of $X \times Y$ is
    \[ \{ C \times D \mid C \text{ and } D \text{ are maximal blocks of } X \text{ and } Y \} \]
  - the unique maximal block of the unique nonempty subset of $\mathcal{L}G^0$ is $\varepsilon$
The canonical morphism of monoids $\alpha : \mathcal{H}(\mathcal{G}) \to \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
  - given $X \subseteq \mathcal{G}^n$ and $Y \subseteq \mathcal{G}^m$, the collection of maximal blocks of $X \times Y$ is $\{C \times D \mid C$ and $D$ are maximal blocks of $X$ and $Y\}$
  - the unique maximal block of the unique nonempty subset of $\mathcal{G}^0$ is $\varepsilon$
  - $\alpha$ is a morphism of monoids from $\mathcal{D}_h(\mathcal{G})$ to $\mathcal{D}_h(\mathcal{R}_1 G \setminus \{\emptyset\})$
The canonical morphism of monoids $\alpha : \mathcal{H}(\lvert G \rvert) \to \mathcal{H}(\mathcal{R}_1G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
  - given $X \subseteq \lvert G \rvert^n$ and $Y \subseteq \lvert G \rvert^m$, the collection of maximal blocks of $X \times Y$ is $\{C \times D \mid C$ and $D$ are maximal blocks of $X$ and $Y\}$
  - the unique maximal block of the unique nonempty subset of $\lvert G \rvert^0$ is $\varepsilon$
  - $\alpha$ is a morphism of monoids from $\mathcal{D}_h(\lvert G \rvert)$ to $\mathcal{D}_h(\mathcal{R}_1G \setminus \{\emptyset\})$
  - if $C$ is a maximal block of $X \subseteq \lvert G \rvert^n$ then $\sigma \cdot C$ is a maximal block of $\sigma \cdot X$. 
The canonical morphism of monoids $\alpha : \mathcal{H}(\uparrow G) \to \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
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  - $\alpha$ is a morphism of monoids from $\mathcal{D}_h(\uparrow G)$ to $\mathcal{D}_h(\mathcal{R}_1 G \setminus \{\emptyset\})$
  - if $C$ is a maximal block of $X \subseteq \uparrow G^n$ then $\sigma \cdot C$ is a maximal block of $\sigma \cdot X$.
  - $\alpha$ induces a morphism of monoids from $\mathcal{H}(\uparrow G)$ to $\mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$
The canonical morphism of monoids $\alpha : \mathcal{H}(\mathcal{G}) \rightarrow \mathcal{H}(\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
  - given $X \subseteq |G|^n$ and $Y \subseteq |G|^m$, the collection of maximal blocks of $X \times Y$ is
    \[ \{ C \times D \mid C \text{ and } D \text{ are maximal blocks of } X \text{ and } Y \} \]
  - the unique maximal block of the unique nonempty subset of $|G|^0$ is $\varepsilon$
  - $\alpha$ is a morphism of monoids from $\mathcal{D}_h(|G|)$ to $\mathcal{D}_h(\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\})$
  - if $C$ is a maximal block of $X \subseteq |G|^n$ then $\sigma \cdot C$ is a maximal block of $\sigma \cdot X$.
  - $\alpha$ induces a morphism of monoids from $\mathcal{H}(\mathcal{G})$ to $\mathcal{H}(\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\})$
  - $\text{im}(\alpha)$ is a submonoid of $\mathcal{H}(\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\})$
The canonical morphism of monoids $\alpha : H(\lvert G \rvert) \to H(R_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
  - given $X \subseteq \lvert G \rvert^n$ and $Y \subseteq \lvert G \rvert^m$, the collection of maximal blocks of $X \times Y$ is
    $\{ C \times D \mid C \text{ and } D \text{ are maximal blocks of } X \text{ and } Y \}$
  - the unique maximal block of the unique nonempty subset of $\lvert G \rvert^0$ is $\varepsilon$
- $\alpha$ is a morphism of monoids from $D_h(\lvert G \rvert)$ to $D_h(R_1 G \setminus \{\emptyset\})$
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- $\text{im}(\alpha)$ is a submonoid of $H(R_1 G \setminus \{\emptyset\})$
- the morphisms $\gamma$ and $\alpha$ induce isomorphisms of ordered monoids between $\text{im}(\alpha)$ and $H(\lvert G \rvert)$, the order relation being inherited from inclusion over $R_1 G \setminus \{\emptyset\}$ and equality over $\lvert G \rvert$. 
The canonical morphism of monoids $\alpha : \mathcal{H}(|G|) \to \mathcal{H}(R_1 G \setminus \{\emptyset\})$

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- the morphisms $\gamma$ and $\alpha$ induce isomorphisms of ordered monoids between $\text{im}(\alpha)$ and $\mathcal{H}(|G|)$, the order relation being inherited from inclusion over $R_1 G \setminus \{\emptyset\}$ and equality over $|G|$.
- therefore $\text{im}(\alpha)$ is commutative free
The free commutative monoids of isothetic regions
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The free commutative monoids of isothetic regions

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- We have seen that an isothetic region has finitely many maximal blocks.
The free commutative monoids of isothetic regions

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The free commutative monoids of isothetic regions

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  - this commutative monoid is thus free and isomorphic to the monoid of isothetic regions, the latter being defined as

$$\gamma(\{X \in \text{im}(\alpha) \mid \text{card}(X) \text{ is finite}\})$$
The free commutative monoids of isothetic regions

- By definition, an isothetic region is a finite union of blocks of \( X \subseteq \uparrow G \downarrow^n \).
- We have seen that an isothetic region has finitely many maximal blocks.
- For \( X, Y \in \mathcal{H}(\uparrow G \downarrow) \), \( \alpha(X \cdot Y) \) is finite iff \( \alpha(X) \) and \( \alpha(Y) \) are so:
  - then \( \{ X \in \text{im}(\alpha) \mid \text{card}(X) \text{ is finite} \} \) is a pure submonoid of \( \text{im}(\alpha) \)
  - this commutative monoid is thus free and isomorphic to the monoid of isothetic regions, the latter being defined as

\[
\gamma(\{ X \in \text{im}(\alpha) \mid \text{card}(X) \text{ is finite} \})
\]

- The monoid of isothetic regions is thus free commutative.
A better factoring algorithm
by Nicolas Ninin

Let $X \subseteq |G|_n$ be an isothetic region and $F$ be a finite block covering of $X$.

- For each block $(\omega_1, \ldots, \omega_n)$ that belongs to $F$, define the subset $B_{\omega} = \{k \in \{1, \ldots, n\} | \omega_k \neq |G|\}$.

- The finest partition of $\{1, \ldots, n\}$ that is coarser than the collection $\{B_{\omega} | \omega \in F\}$ induces a factorization of $X$.

If $F = \alpha(X_c)$ then we obtain the prime factorization of $X$.
A better factoring algorithm
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Let $X \subseteq |G|^n$ be an isothetic region and $\mathcal{F}$ be a finite block covering of $X^c$
A better factoring algorithm
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If $\mathcal{F} = \alpha(X^c)$ then we obtain the prime factorization of $X$. 
Factoring a program

sem: 1 a b
sem: 2 c

proc:
    p = P(a); P(c); V(c); V(a)
    q = P(b); P(c); V(c); V(b)

init: p q p q
Factoring the space of states

subtle

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Factoring the space of states

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