ISOTHETIC REGIONS
Boolean structure
One-dimensional regions

Let $G$ be a finite graph, the collection $R_1^G$ of all finite unions of connected subsets of $|G|$ forms a Boolean subalgebra of $\text{Pow}(|G|)$. Moreover $R_1^G \cong \text{Pow}(V) \times (R_1^{[0,1[})$ with $A$ (resp. $V$) being the set of arrows (resp. vertices) of $G$, and $R_1^{[0,1[}$ being the Boolean algebra of finite unions of subintervals of $[0,1[$. The elements of $R_1^G$ are seen as one-dimensional blocks.

Proof: If $X$ is a connected subset of $|G|$ then for all arrows $\alpha \in G$, $X \cap (\{\alpha\} \times [0,1[)$ has at most two connected components.

The finiteness condition is not necessary e.g. yet some infinite graphs may not enjoy the property e.g. when $G$ is a graph with a single vertex and infinitely many arrows.
Let $G$ be a finite graph, the collection $\mathcal{R}_1 G$ of all finite unions of connected subsets of $|G|$ forms a Boolean subalgebra of $\text{Pow}(|G|)$.

The elements of $\mathcal{R}_1 G$ are seen as one-dimensional blocks.

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Moreover

$$\mathcal{R}_1 G \cong \text{Pow}(V) \times (\mathcal{R}_1]0,1[)^{\text{card}A}$$

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Yet some infinite graphs may not enjoy the property e.g. when $G$ is a graph with a single vertex and infinitely many arrows.
Higher dimensional blocks

- A block of dimension $n \in \mathbb{N}$, or $n$-block, is the product of $n$ connected subsets of the metric graph $|G|$.

- A collection of blocks is called a block covering of $X \subseteq |G|$ when the union of its elements is $X$.

- The collection of $n$-dimensional block coverings is denoted by $\text{Cov}_n G$, it is preordered by $C \preceq C'$ if $\forall b \in C \exists b' \in C'$, $b \subseteq b'$.
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C \preceq C' \iff \forall b \in C \exists b' \in C', \ b \subseteq b'
\]
Maximal blocks

- A block contained in $X$ is said to be a block of $X$. Such a block is said to be maximal when no block of $X$ strictly contains it.

- The maximal connected block covering of $X \subseteq |G|^n$ is the set of all its maximal connected blocks, it is denoted by $\alpha_n(X)$.

- $\alpha_n(X) = \emptyset$ if and only if $X = \emptyset$. 
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A Galois connection

We have a Galois connection \((\gamma_n, \alpha_n)\) between \(\text{Cov}_n G\) and \(\text{Pow}(|G|^n)\) with \(\gamma_n(D) = SD\) for all \(D \in \text{Cov}_n G\).

In particular, \(\gamma_n \circ \alpha_n = \text{id}\) and \(\text{id} \leq \alpha_n \circ \gamma_n\).

That Galois connection induces an isomorphism of Boolean algebras between \(\text{Pow}(|G|^n)\) and the image of \(\alpha_n\), i.e., the collection of maximal connected block coverings.

Proof: any connected block is contained in a maximal connected block (by the Hausdorff maximal principle).
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We have a Galois connection \((\gamma_n, \alpha_n)\) between \(\text{Cov}_n G\) and \(\text{Pow}(|G|^n)\) with \(\gamma_n(D) = \bigcup D\) for all \(D \in \text{Cov}_n G\).

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\text{Cov}_n G \xrightarrow{\gamma_n} \text{Pow}(|G|^n) \\
\xleftarrow{\alpha_n}
\end{array}
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In particular \(\gamma_n \circ \alpha_n = id\) and \(id \not\approx \alpha_n \circ \gamma_n\). That Galois connection induces an isomorphism of Boolean algebras between \(\text{Pow}(|G|^n)\) and the image of \(\alpha_n\) i.e. the collection of maximal connected block coverings.
A Galois connection

We have a Galois connection $\left( \gamma_n, \alpha_n \right)$ between $\text{Cov}_n G$ and $\text{Pow}(\lvert G \rvert^n)$ with $\gamma_n(D) = \bigcup D$ for all $D \in \text{Cov}_n G$.

\[
\begin{align*}
\text{Cov}_n G & \overset{\gamma_n}{\longrightarrow} \text{Pow}(\lvert G \rvert^n) \\
\text{Pow}(\lvert G \rvert^n) & \overset{\alpha_n}{\longleftarrow} \text{Cov}_n G
\end{align*}
\]

In particular $\gamma_n \circ \alpha_n = \text{id}$ and $\text{id} \preceq \alpha_n \circ \gamma_n$. That Galois connection induces an isomorphism of Boolean algebras between $\text{Pow}(\lvert G \rvert^n)$ and the image of $\alpha_n$, i.e. the collection of maximal connected block coverings.

**Proof:** any connected block is contained in a maximal connected block (by the Hausdorff maximal principle).

\[
\bigcup_i^{\uparrow} \left( B_1^{(i)} \times \cdots \times B_n^{(i)} \right) = \left( \bigcup_i^{\uparrow} B_1^{(i)} \right) \times \cdots \times \left( \bigcup_i^{\uparrow} B_n^{(i)} \right)
\]
Isothetic regions

- An isothetic region of dimension $n$ is a subset of $|G|^n$ that admits a finite block covering.
- The geometric model of a conservative program is an isothetic region.
- The collection of isothetic regions of dimension $n$ is denoted by $R_n^G$.
- The collection of finite block covering of dimension $n$ is denoted by $\text{Cov}_{nf}^G$. 
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The previous Galois connection restricted to isothetic regions
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Suppose that the graph $G$ is finite. The collection of $n$-dimensional isothetic regions $\mathcal{R}_n G$ forms a Boolean subalgebra of $\text{Pow}(|G|^n)$.
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Suppose that the graph $G$ is finite. The collection of $n$-dimensional isothetic regions $\mathcal{R}_n G$ forms a Boolean subalgebra of $\text{Pow}(|G|^n)$ and the previous Galois connection restricts to a Galois connection between $\text{Cov}_n G$ and $\mathcal{R}_n G$. 
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\[
\text{Cov}_{nf} G \xrightarrow{\gamma_n} \mathcal{R}_n G \xleftarrow{\alpha_n}
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The previous Galois connection restricted to isothetic regions

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$$\text{Cov}_{nf} G \xrightarrow{\gamma_n} \mathcal{R}_nG \xleftarrow{\alpha_n}$$

A subset $X \subseteq |G|^n$ is an isothetic region iff the collection of maximal subblocks of $X$ is finite and covers $X$. 

The complement of a block is an isothetic region
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If $X$ is 1-dimensional then its maximal blocks are its connected components.
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If $X$ is 1-dimensional then its maximal blocks are its connected components. The complement of a block $B = B_1 \times \cdots \times B_n$ can be written as

$$B^c = \bigcup_{k=1}^{n} |G| \times \cdots \times B_k^c \times \cdots \times |G|$$
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B^c = \bigcup_{k=1}^{n} |G| \times \cdots \times B_k^c \times \cdots \times |G|
\]

Its maximal blocks are found among that of \( B^c \) therefore they have the form

\[
D_1 \times \cdots \times D_{k-1} \times C_k \times D_{k+1} \times \cdots \times D_n
\]

with \( k \in \{1, \ldots, n\} \), \( C_k \) ranging through the connected components of \( B_k^c \) and \( D_j \), for \( j \neq k \), ranging through the connected components of \( |G| \).
Intersection of two isothetic regions

The intersection of the blocks $B$ and $B'$ is given by $B \cap B' = (B_1 \cap B_1') \times \cdots \times (B_n \cap B_n')$.

The maximal blocks of $B \cap B'$ are therefore of the form $C_1 \times \cdots \times C_n$ with each $C_k$ ranging through the connected components of $(B_k \cap B_k')$.

It follows from De Morgan's laws that the intersection of two regions is still a region. Moreover if $B$ and $B'$ are block coverings of $X$ and $X'$ containing all their maximal blocks, then the union of the collections of maximal blocks of $B \cap B'$ for $B \in B$ and $B' \in B'$ is a block covering of $X \cap X'$ containing all its maximal blocks.
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It follows from De Morgan’s laws that the intersection of two regions is still a region.

Moreover if $B$ and $B'$ are block coverings of $X$ and $X'$ containing all their maximal blocks, then the union of the collections of maximal blocks of $B \cap B'$ for $B \in B$ and $B' \in B'$ is a block covering of $X \cap X'$ containing all its maximal blocks.
Concluding the proof

If $F$ is any finite block covering of $X$, then

- The collection of maximal blocks of $X$ is finite and covers $X$.
- The maximal blocks of $X$ are obtained as certain finite intersection of the form $\bigcap \{M_B | B \in F\}$ where $M_B$ is a maximal block of $B$.
- The maximal blocks of $X$ thus form a finite block covering of $X$. 

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Isotthetic regions

Boolean structure

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Concluding the proof

If $\mathcal{F}$ is any finite block covering of $X$, then

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$$\bigcap\{M_B \mid B \in \mathcal{F}\}$$

where $M_B$ is a maximal block of $B^c$. 
Concluding the proof

If \( \mathcal{F} \) is any finite block covering of \( X \), then

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where \( M_B \) is a maximal block of \( B^c \).
- The maximal blocks of \( X^c \) thus form a finite block covering of \( X^c \).
A result from directed topology

For all directed paths $\gamma$ on $\mathcal{G}$ and all $X \in \mathbb{R}^n_G$, the inverse image of $X$ by $\gamma$ has finitely many connected components.
A result from directed topology

For all directed paths $\gamma$ on $|G|^n$ and all $X \in \mathcal{R}_n G$, the inverse image of $X$ by $\gamma$ has finitely many connected components.
Additional operators
Closure, interior, and boundary of an isothetic region

The closure operator preserves finite products, therefore it preserves blocks. The closure operator preserves finite unions hence it preserves isothetic regions. The boundary of a set is the intersection of its closure and the closure of its complement, hence it also preserves isothetic regions. The interior of a set is the difference between its closure and its boundary. It follows that the interior operator also preserves isothetic regions.
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The boundary of a set is the intersection of its closure and the closure of its complement, hence it also preserves isothetic regions.

The interior of a set is the difference between its closure and its boundary. It follows that the interior operator also preserves isothetic regions.
The forward and the backward operators

Let $A, B$ be subsets of a local pospace $X$.

- The forward and the backward operators are defined as

$$frw(A, B) = \{ \partial^+ \delta | \delta \text{ directed path on } X; \partial^- \delta \in A; \text{im}(\delta) \subseteq A \cup B \}$$

$$bck(A, B) = \{ \partial^- \delta | \delta \text{ directed path on } X; \partial^+ \delta \in A; \text{im}(\delta) \subseteq A \cup B \}$$

- The future cone of $A$ in $X$ is cone $f_A := frw(A, X)$ and the past cone of $A$ in $X$ is cone $p_A := bck(A, X)$.

- The future closure of $A$ in $X$ is $A_f := frw(A, A)$ and the past closure of $A$ in $X$ is $A_p := bck(A, A)$.

The closure $A$ being understood in $X$.

Theorem: if $A, B, X$ are isothetic regions, then so are $frw(A, B)$, cone $f_A$, $A_f$, and their duals.
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- The future cone of $A$ in $X$ is cone $f_A := \text{frw}(A, X)$ and the past cone of $A$ in $X$ is cone $p_A := \text{bck}(A, X)$.

- The future closure of $A$ in $X$ is $A_f := \text{frw}(A, A)$ and the past closure of $A$ in $X$ is $A_p := \text{bck}(A, A)$.

Theorem: if $A, B$, and $X$ are isothetic regions, then so are $\text{frw}(A, B)$, $f_A$, $A_f$, and their duals.
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The forward and the backward operators

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  \text{bck}(A, B) = \{ \partial^- \delta \mid \delta \text{ directed path on } X; \partial^+ \delta \in A; \text{im}(\delta) \subseteq A \cup B \}
  \]
- The future cone of $A$ in $X$ is $\text{cone}^f A := \text{frw}(A, X)$ and
  the past cone of $A$ in $X$ is $\text{cone}^p A := \text{bck}(A, X)$. 
The forward and the backward operators

Let $A, B$ be subsets of a local pospace $X$.

- The forward and the backward operators are defined as

$$\text{frw}(A, B) = \{ \partial^+ \delta \mid \delta \text{ directed path on } X; \partial^+ \delta \in A; \text{im}(\delta) \subseteq A \cup B\}$$

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- The future cone of $A$ in $X$ is $\text{cone}^f A := \text{frw}(A, X)$ and the past cone of $A$ in $X$ is $\text{cone}^p A := \text{bck}(A, X)$.

- The future closure of $A$ in $X$ is $\overline{A}^f := \text{frw}(A, \overline{A})$ and the past closure of $A$ in $X$ is $\overline{A}^p := \text{bck}(A, \overline{A})$. The closure $\overline{A}$ being understood in $X$. 

Theorem: if $A, B, \text{ and } X$ are isothetic regions, then so are $\text{frw}(A, B), \text{ cone}^f A, \overline{A}^f$, and their duals.
The forward and the backward operators

Let $A, B$ be subsets of a local pospace $X$.

- The **forward** and the **backward** operators are defined as

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\text{frw}(A, B) = \{ \partial^+ \delta \mid \delta \text{ directed path on } X; \partial^+ \delta \in A; \text{im}(\delta) \subseteq A \cup B \} \\
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- The future cone of $A$ in $X$ is $\text{cone}^f A := \text{frw}(A, X)$ and the past cone of $A$ in $X$ is $\text{cone}^p A := \text{bck}(A, X)$.

- The future closure of $A$ in $X$ is $\overline{A}^f := \text{frw}(A, \overline{A})$ and the past closure of $A$ in $X$ is $\overline{A}^p := \text{bck}(A, \overline{A})$. The closure $\overline{A}$ being understood in $X$.

**Theorem:** if $A$, $B$, and $X$ are isothetic regions, then so are $\text{frw}(A, B)$, $\text{cone}^f A$, $\overline{A}^f$, and their duals.
Future/past stable subsets of $X$
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Let $A$ be a subset of a local pospace $X$. 
Future/past stable subsets of $X$

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- $\text{cone}^f \text{cone}^f A = \text{cone}^f A$ and $\text{cone}^p \text{cone}^p A = \text{cone}^p A$
Future/past stable subsets of $X$

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- The same holds for past stable subsets.
Past/future attractors

Let $A$ be a subset of a local pospace $X$.

$$\text{cone}_p A = \{ p \in X \text{ from which } A \text{ can be reached} \} = bck(A, X)$$

$$\text{escape}_f A = \{ p \in X \text{ from which } A \text{ cannot be reached} \} = (\text{cone}_p A)^c$$

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$$\text{att}^p A \ = \ \text{escape}^f (\text{escape}^f A)$$
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $J_P^K$ be the geometric model of the program.

- The reachable space of $J_P^K$ is the future cone of the initial point.
- A point $p \in \mathcal{G}^i$ is said to be terminal when $J^\gamma_K$ is empty for all directed paths on $\mathcal{G}^i$ starting at $p$.
- A point $p \in J_P^K$ is said to be terminal when so are all its projections.
- The terminal points form a future stable isothetic region of $J_P^K$.
- A point $p \in J_P^K$ is said to be deadlock when its future cone neither contains directed loops (i.e. it is loop-free) nor terminal points.
- The deadlock points form a future stable isothetic region of $J_P^K$.
- The deadlock attractor of the program is the past attractor of its deadlock region.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.
The deadlock attractor of a conservative program

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- A point $p \in \llbracket P \rrbracket$ is said to be terminal when so are all its projections

- The terminal points form a future stable isothetic region of $\llbracket P \rrbracket$.
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The deadlock attractor of the program is the past attractor of its deadlock region.
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $[P]$ be the geometric model of the program.

- The reachable space of $[P]$ is the future cone of the initial point.
- A point $p \in G_i$ is said to be terminal when $[\gamma]$ is empty for all directed paths on $G_i$ starting at $p$.
- A point $p \in [P]$ is said to be terminal when so are all its projections.
- The terminal points form a . . .
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

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The deadlock attractor of a conservative program
The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\mathbb{R}[P]$ be the geometric model of the program.

- The reachable space of $\mathbb{R}[P]$ is the future cone of the initial point
- A point $p \in \downarrow G_i\uparrow$ is said to be terminal when $\mathbb{R}[\gamma]$ is empty for all directed paths on $\downarrow G_i\uparrow$ starting at $p$.
- A point $p \in \mathbb{R}[P]$ is said to be terminal when so are all its projections
- The terminal points form a future stable isothetic region of $\mathbb{R}[P]$.
- A point $p \in \mathbb{R}[P]$ is said to be deadlock when its future cone neither contains directed loops (i.e. it is loop-free) nor terminal points.
- The deadlock points form a . . .
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Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

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The deadlock attractor of a conservative program

Let $G_1, \ldots, G_n$ be the running processes of a conservative program $P$. Let $\llbracket P \rrbracket$ be the geometric model of the program.

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The deadlock attractor of a conservative program

Let \( G_1, \ldots, G_n \) be the running processes of a conservative program \( P \).
Let \( \llbracket P \rrbracket \) be the geometric model of the program.

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Deadlock attractor of the Swiss Cross

sem 1 a b
proc:
q = P(b).P(a).V(a).V(b)
init:  p q
Deadlock attractor of the Swiss Cross

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Deadlock attractor of the Swiss Cross

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Three dining philosophers
FACTORING ISOTHETIC REGIONS
Free commutative monoids
Commutative monoids

- A commutative monoid is said to be free when it is isomorphic with some $M^X$.

- For all set $X$, the collection $M^X$ of multisets over $X$, i.e., maps $\phi : X \to \mathbb{N}$ s.t. $\{x \in X | \phi(x) \neq 0\}$ is finite forms a commutative monoid with pointwise addition.

- A multiset $\phi$ can be written as $\sum_{x \in X} \phi(x)x$.

- In particular, if $f : X \to Y$ is a set map, then $M(f)(\phi) = \sum_{x \in X} \phi(x)f(x)$.
Commutative monoids

- \((M, \ast, \varepsilon)\) such that for all \(a, b, c \in M\),
  - \((ab)c = a(bc)\)
  - \(\varepsilon a = a = a\varepsilon\)
  - \(ab = ba\)
Commutative monoids

- \((M, \ast, \varepsilon)\) such that for all \(a, b, c \in M\),
  - \((ab)c = a(bc)\)
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- For all set \(X\) the collection \(MX\) of multisets over \(X\) i.e. maps \(\phi: X \rightarrow \mathbb{N}\) s.t. \(\{x \in X \mid \phi(x) \neq 0\}\) is finite forms a commutative monoid with pointwise addition
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- Functor \(M : \text{Set} \to \text{Cmon}\)
Commutative monoids

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    \sum_{x \in X} \phi(x)x
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- Functor \(M : \text{Set} \rightarrow \text{Cmon}\)
  - A multiset \(\phi\) can be written as
    \[
    \sum_{x \in X} \phi(x)x
    \]
  - In particular, if \(f : X \rightarrow Y\) is a set map, then
    \[
    M(f)(\phi) = \sum_{x \in X} \phi(x)f(x)
    \]
Prime vs irreducible

**Prime:**
- $p$ is nonunit and $p | ab$ implies $p | a$ or $p | b$.

**Irreducible:**
- $i$ is nonunit and $x | i$ implies $x ∼ i$ or $x$ is unit.

If $M$ contains nontrivial units, then one can consider the quotient monoid $M/∼$ where $x ∼ y$ stands for:
- there exists a unit $u$ s.t. $y = ux$. 

- $d$ divides $x$, denoted by $d | x$, when there exists $x' = dx′$ and $u$ is a unit: exists $u′$ s.t. $uu′ = ε$.
Prime vs irreducible

- $d$ divides $x$, denoted by $d|x$, when there exists $x'$ such that $x = dx'$
Prime vs irreducible

- $d$ divides $x$, denoted by $d|x$, when there exists $x'$ such that $x = dx'$
- $u$ unit: exists $u'$ s.t. $uu' = \epsilon$ then write $x \sim y$ when $y = ux$ for some unit $u$
Prime vs irreducible

- $d$ divides $x$, denoted by $d | x$, when there exists $x'$ such that $x = dx'$
- $u$ unit: exists $u'$ s.t. $uu' = \varepsilon$ then write $x \sim y$ when $y = ux$ for some unit $u$
- $i$ irreducible: $i$ nonunit and $x | i$ implies $x \sim i$ or $x$ unit
Prime vs irreducible

- \( d \) divides \( x \), denoted by \( d | x \), when there exists \( x' \) such that \( x = dx' \)
- \( u \) unit: exists \( u' \) s.t. \( uu' = \varepsilon \) then write \( x \sim y \) when \( y = ux \) for some unit \( u \)
- \( i \) irreducible: \( i \) nonunit and \( x | i \) implies \( x \sim i \) or \( x \) unit
- \( p \) prime: \( p \) nonunit and \( p | ab \) implies \( p | a \) or \( p | b \)
Prime vs irreducible

- $d$ divides $x$, denoted by $d | x$, when there exists $x'$ such that $x = dx'$
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- $i$ irreducible: $i$ nonunit and $x | i$ implies $x \sim i$ or $x$ unit
- $p$ prime: $p$ nonunit and $p | ab$ implies $p | a$ or $p | b$
- If $M$ contains nontrivial units, then one can consider the quotient monoid $M / \sim$ where $x \sim y$ stands for: there exists a unit $u$ s.t. $y = ux$
### Examples

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Graded commutative monoid
Graded commutative monoid

- \((M, \ast, \varepsilon)\) graded: there is a morphism \(g : (M, \ast, \varepsilon) \rightarrow (\mathbb{N}, +, 0)\) s.t. 
  \(g^{-1}(\{0\}) = \{\text{units of } M\}\)
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\[ M = \{ a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0 \}, \times, 1 \]
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\[ M = \left( \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0 \}, \times, 1 \right) \]

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Factoring isothetic regions

Free commutative monoids

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  $N(a + b\sqrt{10}) \mod 10 \in \{0, 1, 4, 5, 6, 9\}$
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- 2, 3, and 4 \( \pm \sqrt{10} \) are irreducible but not prime
  since \( 2 \cdot 3 = (4 + \sqrt{10}) \cdot (4 - \sqrt{10}) \)
Irreducible that are not prime

\[ M = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0\}, \times, 1 \]

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- 2, 3, and \(4 \pm \sqrt{10}\) are irreducible but not prime
  since \(2 \cdot 3 = (4 + \sqrt{10}) \cdot (4 - \sqrt{10})\)
- \(\{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \setminus \{0\}\) is graded by the number of prime factors of \(N(u)\)
\[ \mathbb{N}[X] \text{ polynomials with coefficients in } \mathbb{N} \]

*On Direct Product Decomposition of Partially Ordered Sets.* Junji Hashimoto
Annals of Mathematics 2(54), pp 315-318 (1951)
\[ \mathbb{N}[X] \] polynomials with coefficients in \( \mathbb{N} \)

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\[ X^5 + X^4 + X^3 + X^2 + X + 1 = \]

\[ (X + 1)(X^4 + 1) = (X^3 + 1)(X^2 + 1) \] in \( \mathbb{N}[X] \) and \( \mathbb{Z}[X] \) respectively

\( X + 1, X^3 + 1, X^2 + 1 \) are irreducible but not prime.

\( \mathbb{N}[X] \) is graded by the degree.
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\[
X^5 + X^4 + X^3 + X^2 + X + 1 = (X + 1)(X^4 + X^2 + 1)
\]

Therefore \(X + 1, X^2 + X + 1, X^3 + 1, \) and \(X^4 + X^2 + 1\) are not irreducible but not prime.

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(X + 1)(X^4 + X^2 + 1) = (X^3 + 1)(X^2 + X + 1) & \text{in } \mathbb{N}[X]
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Characterization of the free commutative monoids

Unique factorization

The following are equivalent:

- $M$ is free commutative
- any element of $M$ can be written as a product of irreducibles in a unique way up to reordering
- \{primes of $M$\} = \{irreducibles of $M$\} and generates $M$
- $M$ is graded and \{irreducibles of $M$\} ⊆ \{primes of $M$\}

Standard examples:

- $(\mathbb{N} \setminus \{0\}, \times, 1)$
- $(\mathbb{N}, +, 0)$ and its finite products in the category of commutative monoids.
  Indeed $(\mathbb{N}, +, 0)$ isomorphic to $M(\{1, \ldots, n\})$
- $(\mathbb{Z}[\{X\} \setminus \{0\}, \times, 1)$ (if $F$ is a factorial ring, then so is $F[\{X\}$)


Note that two free commutative monoids are isomorphic in $\text{Cmon}$ iff their set of prime elements have the same cardinality e.g. $(\mathbb{N} \setminus \{0\}, \times, 1) \sim = (\mathbb{Z}[\{X\} \setminus \{0\}, \times, 1)$ in $\text{Cmon}$
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  - Note that two free commutative monoids are isomorphic in $\mathcal{C}_{\text{mon}}$ iff
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In particular $\mathcal{M}_2 \cong (\mathbb{N}, +, 0)$ and $\mathcal{M}_3 \cong (\mathbb{N} \setminus \{0\}, \times, 1)$
Monoids of homogeneous languages
### Factoring isothetic regions

#### Monoids of homogeneous languages

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26 / 43
Let $\varepsilon$ denote the empty word. A language is a set of words on $A$. Let $D$ and $D'$ be languages.

Define $D \cdot D' := \{ w \cdot w' | w \in D; w' \in D' \}$. One has $\emptyset \cdot D = D \cdot \emptyset = \emptyset$ and $\{\varepsilon\} \cdot D = D \cdot \{\varepsilon\} = D$.

The monoid of nonempty languages is $D(A)$, which is commutative if $\text{Card}(A) \leq 1$. Note that $D(\emptyset) \sim = \{\varepsilon\}$, however $D(\{a\})$ is not freely commutative.
The noncommutative monoid of languages

- $A^*$ (non commutative) monoid of words on the alphabet $A$.
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Factoring isothetic regions

Monoids of homogeneous languages

The noncommutative monoid of homogeneous languages

- Define $\dim(H)$ as the length common to all the words of $H$. It is well defined since $H$ is nonempty.
- $H \cdot H' = \{w \cdot w' | w \in H; w' \in H'\}$ is homogeneous iff so are $H$ and $H'$.
- $D_h(A) \subseteq D(A)$ the pure submonoid of homogeneous languages.
- $H \in D_h(A) \rightarrow \dim(H) \in (\mathbb{N}, +, 0)$ is a morphism of monoid.
- $\dim(H) = 0$ iff $H = \{\varepsilon\}$.
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- $\mathcal{D}_h\{a\} \simeq \ldots$
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- \( \dim(H) = 0 \) iff \( H = \{ \varepsilon \} \).
- \( \mathcal{D}_h(A) \) is commutative iff \( \text{Card}(A) \leq 1 \).
- \( \mathcal{D}_h(\{a\}) \cong (\mathbb{N}, +, 0) \).
Action of the symmetric groups
on the left of the homogeneous languages
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on the left of the homogeneous languages

- The $n^{th}$ symmetric group $\mathfrak{S}_n$ acts on the left of the set of words of length $n$
  i.e. mappings from $\{1, \ldots, n\}$ to $\mathfrak{A}$, by $\sigma \cdot \omega := \omega \circ \sigma^{-1}$
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- If $\sigma \in \mathfrak{S}_n$ and $\sigma' \in \mathfrak{S}_{n'}$ then define $\sigma \otimes \sigma' \in \mathfrak{S}_{n+n'}$ as:

$$
\sigma \otimes \sigma'(k) := \begin{cases} 
\sigma(k) & \text{if } 1 \leq k \leq n \\
(\sigma'(k-n)) + n & \text{if } n+1 \leq k \leq n+n'
\end{cases}
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- A Godement exchange law is satisfied, which ensures that $\sim$ is actually a congruence:

\[(\sigma \cdot H) \cdot (\sigma' \cdot H') = (\sigma \otimes \sigma') \cdot (H \cdot H')\]

i.e. $H \sim K$ and $H' \sim K'$ implies $HH' \sim KK'$
The commutative monoid of homogeneous languages

- The commutative monoid of homogeneous languages is

\[ H(A) = (\mathcal{D}_h(A), \cdot, \{\varepsilon\}) / \sim \]

- The monoid \( H(A) \) is graded by \( H \in H(A) \mapsto \dim(H) \in (\mathbb{N}, +, 0) \)

- The commutative monoid \( H(A) \) is free

- For any homogeneous language \( H \) and \( \sigma \in S \), \( \dim(H) = \dim(\sigma \cdot H) \) so we can define the cardinality of any element of \( H(A) \).
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- For any homogeneous language $H$ and $\sigma \in \mathcal{S}_{\dim(H)}$, $\text{card}(H) = \text{card}(\sigma \cdot H)$ so we can define the cardinality of any element of $\mathcal{H}(A)$
The commutative monoid of finite homogeneous languages
The commutative monoid of finite homogeneous languages

- \( M' \subseteq M \) is said to be pure when for all \( x, y \in M \), \( xy \in M' \) implies \( x, y \in M' \)
Factoring isothetic regions

Monoids of homogeneous languages

The commutative monoid of finite homogeneous languages

- $M' \subseteq M$ is said to be pure when for all $x, y \in M$, $xy \in M'$ implies $x, y \in M'$
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  e.g. $H = \{ab, ac\} = \{a\} \cdot \{b, c\}$ though card(H) = 2
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- The primality of $H$ does not imply that of $\text{Card}(H)$
  e.g. $H = \{a, b, c, d\}$ is prime though $\text{card}(H) = 4$
The brute force algorithm for factoring in \( \mathcal{H}_f(\mathbb{A}) \)

Theory
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Theory

Given $w \in \mathbb{A}^n$ and $I \subseteq \{1, \ldots, n\}$, we write $w_{\mid I}$ for the subword of $w$ consisting of letters with indices in $I$. 
The brute force algorithm for factoring in $\mathcal{H}_f(A)$

Theory

Given $w \in A^n$ and $I \subseteq \{1, \ldots, n\}$, we write $w_I$ for the subword of $w$ consisting of letters with indices in $I$.

Given a homogeneous language $H$ of dimension $n$, we write

$$H_I = \{ w_I \mid w \in H \}$$
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Theory

Given $w \in \mathbb{A}^n$ and $I \subseteq \{1, \ldots, n\}$, we write $w_{|I}$ for the subword of $w$ consisting of letters with indices in $I$.

Given a homogeneous language $H$ of dimension $n$, we write

$$H_{|I} = \{w_{|I} \mid w \in H\}$$

Denoting $I^c$ for $\{1, \ldots, n\} \setminus I$, we have

$$[H] = [H_{|I}] \cdot [H_{|I^c}]$$

in $\mathcal{H}_f(\mathbb{A})$ if and only if for all words $u, v \in H$ there exists a word $w \in H$ such that

$$w_{|I} = u_{|I} \quad \text{and} \quad w_{|I^c} = v_{|I^c}$$
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Practice

For $I \subseteq \{1, \ldots, n\}$ let $\pi_I$ be the "projection" that sends $w \in H$ to $w_I \in \mathbb{A}^{\text{card}(I)}$. 
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Practice

For $I \subseteq \{1, \ldots, n\}$ let $\pi_I$ be the “projection” that sends $w \in H$ to $w|_I \in \mathbb{A}^{\text{card}(I)}$.

1. choose $I \subseteq \{1, \ldots, n\}$ of cardinality $k \leq n/2$
The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Practice

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1. choose $I \subseteq \{1, \ldots, n\}$ of cardinality $k \leq n/2$

2. if $\pi_{I^c}(\pi_I^{-1}(u))$ does not depend on $u \in H|_I$, then we have the factorization

$$[H] = [H|_I] \cdot [H|_{I^c}]$$

and we are done.
The brute force algorithm for factoring in $\mathcal{H}_f(A)$

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For $I \subseteq \{1, \ldots, n\}$ let $\pi_{|I}$ be the “projection” that sends $w \in H$ to $w_{|I} \in A^{\text{card}(I)}$.

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2. if $\pi_{|I^c}(\pi_{|I}^{-1}(u))$ does not depend on $u \in H_{|I}$, then we have the factorization

\[ [H] = [H_{|I}] \cdot [H_{|I^c}] \]

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3. otherwise check whether there are still subsets of $\{1, \ldots, n\}$ to check:

3.1. yes: go to step 1

3.2. no: $[H]$ is prime
Homogeneous languages and isothetic regions
Factoring a program

sem:  1 a b
sem:  2 c

proc:
  p = P(a);P(c);V(c);V(a)

  q = P(b);P(c);V(c);V(b)

init:  p q p q
### Factoring the space of states

#### brute force

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Factoring the space of states

brute force
Factoring the space of states

brute force
Factoring the space of states

brute force
Factoring isothetic regions

Homogeneous languages and isothetic regions

Factoring the space of states

brute force
Factoring the space of states

brute force
Factoring isothetic regions

Homogeneous languages and isothetic regions

Factoring the space of states

brute force
Factoring the space of states

brute force

![Diagram showing factoring the space of states through brute force]

- Homogeneous languages and isothetic regions
Factoring the space of states

brute force
Factoring the space of states

brute force

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Factoring a program

sem: 1 a b
sem: 2 c

proc:
    p = P(a); P(c); V(c); V(a)
    q = P(b); P(c); V(c); V(b)

init: p q p q
Factoring a program

sem: 1 a b
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proc:
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  q = P(b); P(c); V(c); V(b)

init:  p p q q
Factoring a program

<table>
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Factoring a program

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The preorder $\preceq$ over $\mathcal{H}(A)$

inherited from a preorder $\preceq$ over $A$
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- Let $\preceq^n$ by the product preorder on the words of length $n$
The preorder $\preceq$ over $\mathcal{H}(\mathbb{A})$

inherited from a preorder $\preceq$ over $\mathbb{A}$

- Let $\preceq^n$ by the product preorder on the words of length $n$
- Given $H, H' \in \mathcal{D}_h(\mathbb{A})$ of the same dimension $n$, write $H \preceq H'$ when for all $\omega \in H$ there exists $\omega' \in H'$ such that $\omega \preceq^n \omega'$
The preorder $\preccurlyeq$ over $\mathcal{H}(\mathbb{A})$

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- $X \preceq Y$ and $X' \preceq Y'$ implies $X \cdot X' \preceq Y \cdot Y'$
  i.e. $(\mathcal{H}(A), \preceq)$ is a preordered commutative monoid
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- If $\preceq$ is the equality relation, then $X \preceq Y$ amounts to $H_X \subseteq H_Y$ for some representatives $H_X$ and $H_Y$ of $X$ and $Y$. 
Homogeneous languages

over the alphabets $|G|$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with $G$ being a finite graph
Homogeneous languages

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Homogeneous languages
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Homogeneous languages

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- $\mathcal{A} = \mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\}$ is the collection of nonempty finite unions of connected subsets of $\mathcal{G}$:
Homogeneous languages

over the alphabets $\mathcal{L}_G$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with $G$ being a finite graph

- $\mathbb{A} = \mathcal{L}_G$ is the geometric realization of a finite graph:
  - a point of $\mathcal{L}_G^n$ can be seen as a word of length $n$ on $\mathbb{A}$
  - a nonempty subset of $\mathcal{L}_G^n$ is thus a homogeneous language on $\mathbb{A}$
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- $\mathbb{A} = \mathcal{R}_1 G \setminus \{\emptyset\}$ is the collection of nonempty finite unions of connected subsets of $\mathcal{L}_G$:
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    i.e. a word of length $n$ on $\mathbb{A}$
  - a nonempty family of $n$-blocks is thus an homogeneous language on $\mathbb{A}$ (of dimension $n$)
  - the concatenation of words on $\mathbb{A}$ corresponds to the cartesian product of blocks
The canonical morphism of monoids $\gamma : \mathcal{H}(R_1 G \setminus \{\emptyset\}) \to \mathcal{H}(1G \downarrow)$
The canonical morphism of monoids $\gamma : \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\}) \to \mathcal{H}(\mathcal{R} G)$

- Let $\gamma$ be the map sending an homogeneous language on $\mathcal{R}_1 G \setminus \{\emptyset\}$ to the union of its elements.
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The canonical morphism of monoids \( \gamma : \mathcal{H}(\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\}) \to \mathcal{H}(\mathcal{G} \perp) \)

- Let \( \gamma \) be the map sending an homogeneous language on \( \mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\} \) to the union of its elements
- \( \gamma \) is a morphism of monoids from \( \mathcal{D}_h(\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\}) \) to \( \mathcal{D}_h(\mathcal{G} \perp) \)
- \( \gamma \) is compatible with the action of the symmetric groups in the sense that
  \[ H' = \sigma \cdot H \Rightarrow \bigcup H' = \sigma \cdot \bigcup H \]
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  - $\gamma$ induces a morphism of monoids from $\mathcal{H}(R_1 G \setminus \{\emptyset\})$ to $\mathcal{H}(\uparrow G \downarrow)$
- The induced morphism $\gamma$ does not preserve the prime elements e.g. consider a covering of $[0, 1]^2$ with 3 distinct rectangles
The canonical morphism of monoids $\alpha : \mathcal{H}(\mathbb{G}) \to \mathcal{H}(\mathcal{R}_1 \mathbb{G} \setminus \{\emptyset\})$
The canonical morphism of monoids \( \alpha : \mathcal{H}(|G|) \rightarrow \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\}) \)

- Define \( \alpha(X) \) as the collection of maximal blocks of \( X \):
The canonical morphism of monoids \( \alpha : \mathcal{H}(\mathcal{1}|G|) \to \mathcal{H}(\mathcal{R}_1G \setminus \{\emptyset\}) \)

- Define \( \alpha(X) \) as the collection of maximal blocks of \( X \):
  - given \( X \subseteq \mathcal{1}|G|^n \) and \( Y \subseteq \mathcal{1}|G|^m \), the collection of maximal blocks of \( X \times Y \) is \( \{ C \times D \mid C \text{ and } D \text{ are maximal blocks of } X \text{ and } Y \} \)
The canonical morphism of monoids $\alpha : \mathcal{H}(1G) \to \mathcal{H}(R_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:  
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  - the unique maximal block of the unique nonempty subset of $|G|^0$ is $\varepsilon$
  - $\alpha$ is a morphism of monoids from $D_h(|G|)$ to $D_h(R_1 G \setminus \{\emptyset\})$
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  - $\alpha$ is a morphism of monoids from $\mathcal{D}_h(\mathcal{G})$ to $\mathcal{D}_h(\mathcal{R}_1 \mathcal{G} \setminus \{\emptyset\})$
  - if $C$ is a maximal block of $X \subseteq \mathcal{G}^n$ then $\sigma \cdot C$ is a maximal block of $\sigma \cdot X$. 
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- the morphisms $\gamma$ and $\alpha$ induce isomorphisms of ordered monoids between $\text{im}(\alpha)$ and $\mathcal{H}(\mathcal{G})$, the order relation being inherited from inclusion over $\mathcal{R}_1 G \setminus \{\emptyset\}$ and equality over $\mathcal{G}$. 
The canonical morphism of monoids $\alpha : \mathcal{H}(|G|) \to \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of $X$:
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  - $\text{im}(\alpha)$ is a submonoid of $\mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$
- the morphisms $\gamma$ and $\alpha$ induce isomorphisms of ordered monoids between $\text{im}(\alpha)$ and $\mathcal{H}(|G|)$, the order relation being inherited from inclusion over $\mathcal{R}_1 G \setminus \{\emptyset\}$ and equality over $|G|$.
- therefore $\text{im}(\alpha)$ is commutative free
The free commutative monoids of isothetic regions
The free commutative monoids of isothetic regions

- By definition, an isothetic region is a finite union of blocks of $X \subseteq |G|^n$. 
Factoring isothetic regions

Homogeneous languages and isothetic regions

The free commutative monoids of isothetic regions

- By definition, an isothetic region is a finite union of blocks of $X \subseteq \uparrow G \downarrow ^n$.
- We have seen that an isothetic region has finitely many maximal blocks.
The free commutative monoids of isothetic regions

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The free commutative monoids of isothetic regions

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- We have seen that an isothetic region has finitely many maximal blocks.
- For $X, Y \in \mathcal{H}(G)$, $\alpha(X \cdot Y)$ is finite iff $\alpha(X)$ and $\alpha(Y)$ are so:
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  - this commutative monoid is thus free and isomorphic to the monoid of isothetic regions, the latter being defined as

$$\gamma(\{X \in \text{im}(\alpha) \mid \text{card}(X) \text{ is finite}\})$$
The free commutative monoids of isothetic regions

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  \[
  \gamma(\{X \in \text{im}(\alpha) \mid \text{card}(X) \text{ is finite}\})
  \]

- The monoid of isothetic regions is thus free commutative.
Factoring isothetic regions

Homogeneous languages and isothetic regions

A better factoring algorithm

by Nicolas Ninin

Let $X \subseteq |G|^n$ be an isothetic region and $F$ be a finite block covering of $X$.

- For each block $(\omega_1, \ldots, \omega_n)$ that belongs to $F$ define the subset $B_{\omega} = \{ k \in \{1, \ldots, n\} | \omega_k \neq |G| \}$.

- The finest partition of $\{1, \ldots, n\}$ that is coarser than the collection $\{B_{\omega} | \omega \in F\}$ induces a factorization of $X$.

If $F = \alpha(X)$ then we obtain the prime factorization of $X$. 

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A better factoring algorithm

by Nicolas Ninin

Let $X \subseteq |G|^n$ be an isothetic region and $\mathcal{F}$ be a finite block covering of $X^c$
A better factoring algorithm
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$$\{B_\omega \mid \omega \in \mathcal{F}\}$$

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  \[
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  \]

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If $\mathcal{F} = \alpha(X^c)$ then we obtain the prime factorization of $X$.
Factoring a program

sem: 1 a b
sem: 2 c

proc:
  p = P(a);P(c);V(c);V(a)
  q = P(b);P(c);V(c);V(b)

init: p q p q
Factoring the space of states

subtle

\[
\begin{align*}
[1,4] &\quad [0, +\infty] &\quad [1,4] &\quad [0, +\infty] \\
[0, +\infty] &\quad [1,4] &\quad [0, +\infty] &\quad [1,4] \\
[0, +\infty] &\quad [2,3] &\quad [2,3] &\quad [2,3]
\end{align*}
\]
Factoring the space of states

subtle
Factoring the space of states

subtle
Factoring isothetic regions

Homogeneous languages and isothetic regions

Factoring the space of states

subtle