DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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ISOTHETIC REGIONS

Boolean structure

One-dimensional regions

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Yet some infinite graphs may not enjoy the property e.g. when G is a graph with a single vertex and infinitely many arrows.

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- The collection of n-dimensional block coverings is denoted by $\mathsf{Cov}_n G$, it is preordered by

$$C \preccurlyeq C' \quad \equiv \quad \forall b \in C \; \exists b' \in C', \; b \subseteq b'$$

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- $\alpha_n(X) = \emptyset$ if and only if $X = \emptyset$.

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We have a Galois connection (γ_n, α_n) between $\mathsf{Cov}_n G$ and $\mathsf{Pow}(|G|^n)$ with $\gamma_n(D) = \bigcup D$ for all $D \in \mathsf{Cov}_n G$.

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Proof: any block is contained in a maximal block (by the Hausdorff maximal principle).

$$\bigcup_i{}^{\uparrow}\left(B_1^{(i)} \times \cdots \times B_n^{(i)}\right) \quad = \quad \left(\bigcup_i{}^{\uparrow} B_1^{(i)}\right) \times \cdots \times \left(\bigcup_i{}^{\uparrow} B_n^{(i)}\right)$$

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$$\mathrm{Cov}_{nf}\ G \underset{\alpha_n}{\overset{\gamma_n}{\rightleftarrows}} \mathcal{R}_n G$$

A subset $X \subseteq |G|^n$ is an isothetic region iff the collection of maximal subblocks of X is finite and covers X.

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Its maximal blocks are found among that of B^c therefore they have the form

$$D_1 \times \cdots \times D_{k-1} \times C_k \times D_{k+1} \times \cdots \times D_n$$

with $k \in \{1, \ldots, n\}$, C_k ranging through the connected components of B_k^c and D_j, for $j \neq k$, ranging through the connected components of $|G|$.

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Moreover if $\mathcal{B}$ and $\mathcal{B}'$ are block coverings of X and X' containing all their maximal blocks, then the union of the collections of maximal blocks of $B \cap B'$ for $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$ is a block covering of $X \cap X'$ containing all its maximal blocks.

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where M_B is a maximal block of B^c.

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where M_B is a maximal block of B^c.
- The maximal blocks of X^c thus form a finite block covering of X^c.

A result from directed topology

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For all directed paths γ on $\upharpoonleft G \downharpoonright^n$ and all $X \in \mathcal{R}_n G$, the inverse image of X by γ has finitely many connected components.

Additional operators

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The interior of a set is the difference between its closure and its boundary. It follows that the interior operator also preserves isothetic regions.

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$$\mathrm{frw}(A, B) = \{\partial^{+}\delta \mid \delta \text{ directed path on } X;\ \partial^{-}\delta \in A;\ \mathrm{im}(\delta) \subseteq A \cup B\}$$

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Theorem: if A, B, and X are isothetic regions, then so are $\mathrm{frw}(A, B)$, $\mathrm{cone}^{\mathrm{f}} A$, $\overline{A}^{\mathrm{f}}$, and their duals.

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- The same holds for past stable subsets.

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- The terminal points form a future stable isothetic region of $[\![P]\!]$
- A point $p \in [\![P]\!]$ is said to be deadlock when its future cone neither contains directed loops (i.e. it is loop-free) nor terminal points.

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Let $[\![P]\!]$ be the geometric model of the program.

- The reachable space of $[\![P]\!]$ is the future cone of the initial point
- A point $p \in \upharpoonleft G_i \downharpoonright$ is said to be terminal when $[\![\gamma]\!]$ is empty for all directed paths on $\upharpoonleft G_i \downharpoonright$ starting at p.
- A point $p \in [\![P]\!]$ is said to be terminal when so are all its projections
- The terminal points form a future stable isothetic region of $[\![P]\!]$
- A point $p \in [\![P]\!]$ is said to be deadlock when its future cone neither contains directed loops (i.e. it is loop-free) nor terminal points.
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- The deadlock attractor of the program is the past attractor of its deadlock region.

Deadlock attractor of the Swiss Cross

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proc:
p = P(a).P(b).V(b).V(a)
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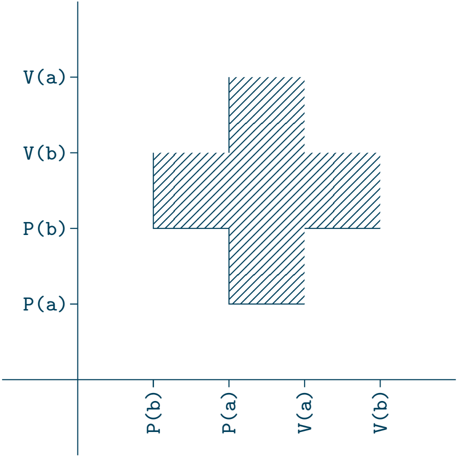

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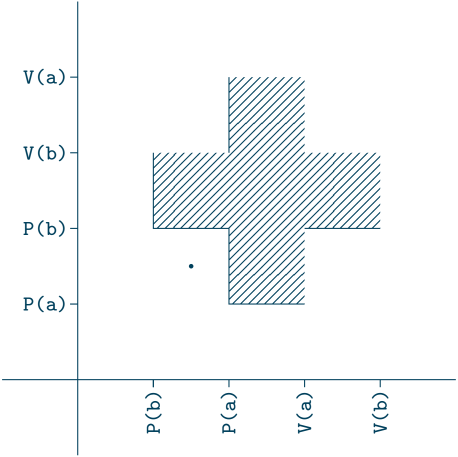

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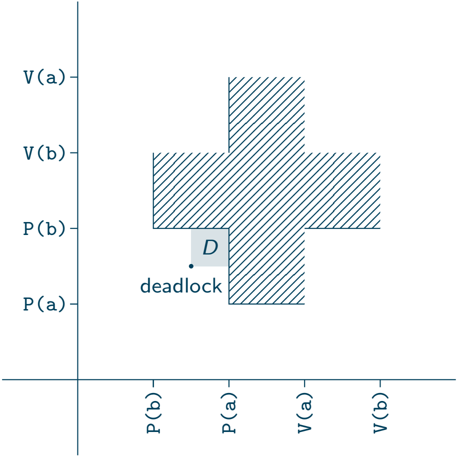

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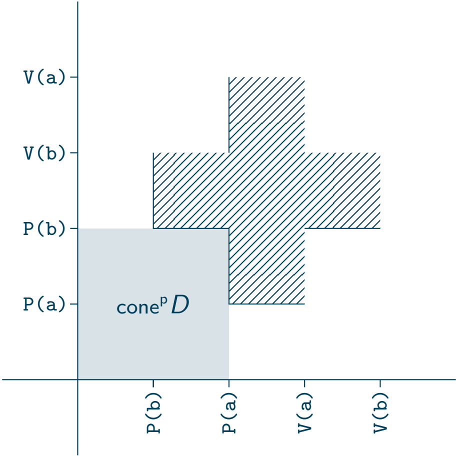

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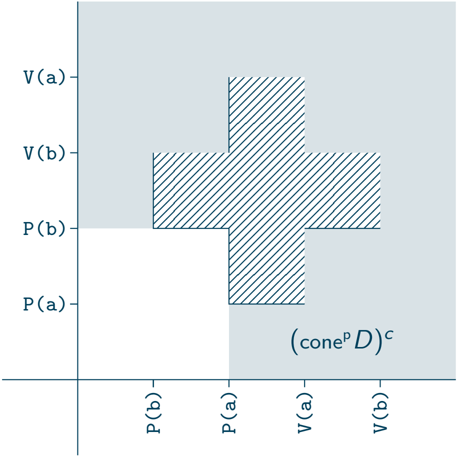

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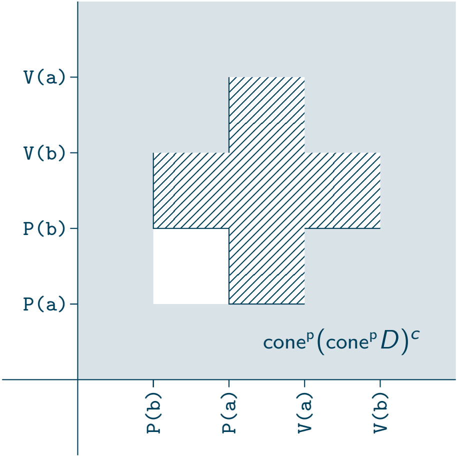

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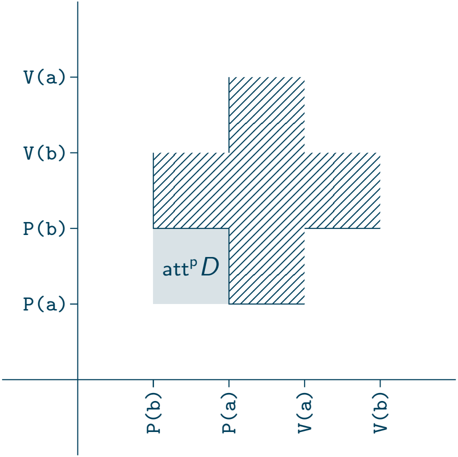
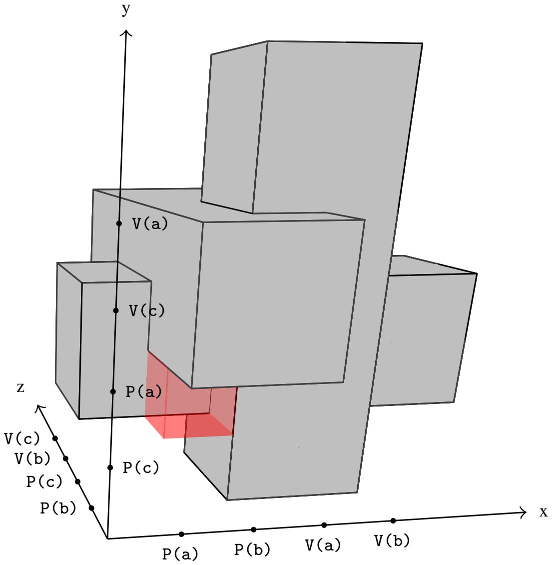


Three dining philosophers



FACTORING ISOTHETIC REGIONS

Free commutative monoids

Commutative monoids

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  - In particular, if $f : X \to Y$ is a set map, then

$$M(f)(\phi) = \sum_{x \in X} \phi(x)f(x)$$

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- If M contains nontrivial units, then one can consider the quotient monoid $M/\sim$ where $x \sim y$ stands for: there exists a unit u s.t. $y = ux$

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$\mathbb{Z}_6, \times, 1$	$\emptyset$	$\{0, 2, 3, 4\}$	$\{1, 5\}$

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  number of prime factors of $N(u)$

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On Direct Product Decomposition of Partially Ordered Sets. Junji Hashimoto

Annals of Mathematics **2**(54), pp 315-318 (1951)

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  - $(\mathbb{N}, +, 0)$ and its finite products in the category of commutative monoids.
    Indeed $(\mathbb{N}, +, 0)^n \cong M(\{1, \ldots, n\})$

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  - $\{\text{primes of } M\} = \{\text{irreducibles of } M\}$ and generates M
  - M is graded and $\{\text{irreducibles of } M\} \subseteq \{\text{primes of } M\}$
- Standard examples:
  - $(\mathbb{N} \setminus \{0\}, \times, 1)$
  - $(\mathbb{N}, +, 0)$ and its finite products in the category of commutative monoids.
    Indeed $(\mathbb{N}, +, 0)^n \cong M(\{1, \ldots, n\})$
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Characterization of the free commutative monoids

Unique factorization

- The following are equivalent:
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  - Note that two free commutative monoids are isomorphic in $\mathcal{Cmon}$ iff their set of prime elements have the same cardinality
    e.g. $(\mathbb{N} \setminus \{0\}, \times, 1) \cong (\mathbb{Z}[X] \setminus \{0\}, \times, 1)$ in $\mathcal{Cmon}$

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In particular $\mathcal{M}_2 \cong (\mathbb{N}, +, 0)$ and $\mathcal{M}_3 \cong (\mathbb{N} \setminus \{0\}, \times, 1)$

Monoids of homogeneous languages

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$$\sigma \otimes \sigma'(k) := \begin{cases} \sigma(k) & \text{if} \quad 1 \leqslant k \leqslant n \\ (\sigma'(k-n)) + n & \text{if} \quad n+1 \leqslant k \leqslant n+n' \end{cases}$$

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- A Godement exchange law is satisfied, which ensures that $\sim$ is actually a congruence:

$$(\sigma \cdot H) \cdot (\sigma' \cdot H') = (\sigma \otimes \sigma') \cdot (H \cdot H')$$

i.e. $H \sim K$ and $H' \sim K'$ implies $HH' \sim KK'$

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- For any homogeneous language H and $\sigma \in \mathfrak{S}_{\dim(H)}$, $\mathrm{card}(H) = \mathrm{card}(\sigma \cdot H)$ so we can define the cardinality of any element of $\mathcal{H}(\mathbb{A})$

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- The primality of H does not imply that of Card(H)
  e.g. $H = \{a, b, c, d\}$ is prime though card(H) $= 4$

The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Theory

The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

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Given $w \in \mathbb{A}^n$ and $I \subseteq \{1, \ldots, n\}$, we write $w_{|_I}$ for the subword of w consisting of letters with indices in I.

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Given a homogeneous language H of dimension n, we write

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Denoting I^c for $\{1, \ldots, n\} \setminus I$, we have

$$[H] \quad = \quad [H_{|_I}] \cdot [H_{|_{I^c}}]$$

in $\mathcal{H}_f(\mathbb{A})$ if and only if for all words $u, v \in H$ there exists a word $w \in H$ such that

$$w_{|_I} \quad = \quad u_{|_I} \qquad \text{and} \qquad w_{|_{I^c}} \quad = \quad v_{|_{I^c}}$$

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For $I \subseteq \{1,\ldots,n\}$ let $\pi_{|_I}$ be the "projection" that sends $w \in H$ to $w_{|_I} \in \mathbb{A}^{\text{card}(I)}$.

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2. if $\pi_{|_{I^c}}(\pi^{-1}_{|_I}(u))$ does not depend on $u \in H_{|_I}$, then we have the factorization

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   and we are done

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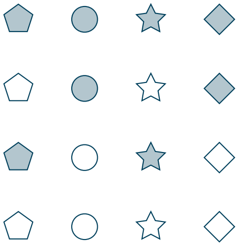
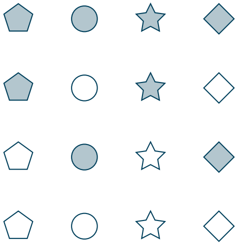
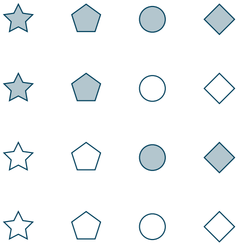
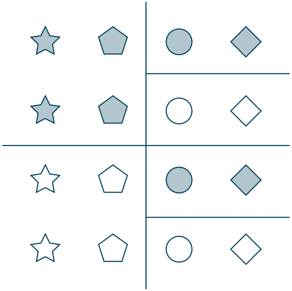
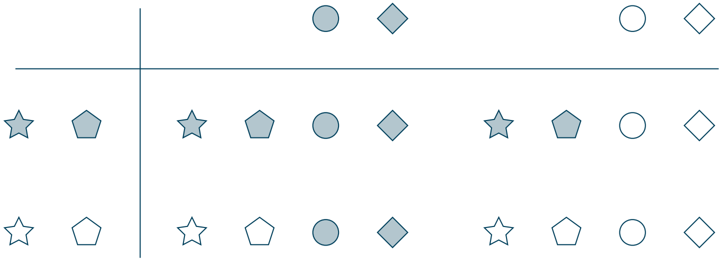
3. otherwise check whether there are still subsets of $\{1, \dots, n\}$ to check:
   3.1. yes: go to step 1
   3.2. no: $[H]$ is prime











Homogeneous languages and isothetic regions

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- The induced morphism γ does not preserve the prime elements e.g. consider a covering of $[0,1]^2$ with 3 disctinct rectangles

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- therefore $\mathrm{im}(\alpha)$ is commutative free

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> if $\mathcal{F} = \alpha(X^c)$ then we obtain the prime factorization of X

Parallelizing a program

```
sem:  1 a b
sem:  2 c
```

```
proc:
  p = P(a);P(c);V(c);V(a)

  q = P(b);P(c);V(c);V(b)
```

```
init:  p q p q
```

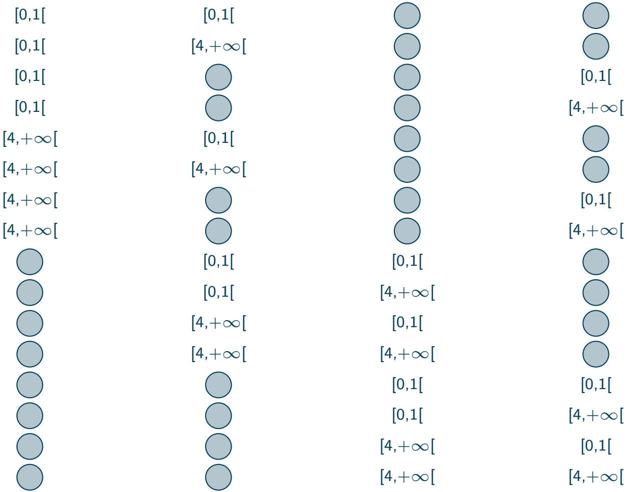
Factoring the space of states

brute force

[0,1[	[0,1[	[0,+∞[	[0,+∞[
[0,1[	[4,+∞[	[0,+∞[	[0,+∞[
[0,1[	[0,+∞[	[0,+∞[	[0,1[
[0,1[	[0,+∞[	[0,+∞[	[4,+∞[
[4,+∞[	[0,1[	[0,+∞[	[0,+∞[
[4,+∞[	[4,+∞[	[0,+∞[	[0,+∞[
[4,+∞[	[0,+∞[	[0,+∞[	[0,1[
[4,+∞[	[0,+∞[	[0,+∞[	[4,+∞[
[0,+∞[	[0,1[	[0,1[	[0,+∞[
[0,+∞[	[0,1[	[4,+∞[	[0,+∞[
[0,+∞[	[4,+∞[	[0,1[	[0,+∞[
[0,+∞[	[4,+∞[	[4,+∞[	[0,+∞[
[0,+∞[	[0,+∞[	[0,1[	[0,1[
[0,+∞[	[0,+∞[	[0,1[	[4,+∞[
[0,+∞[	[0,+∞[	[4,+∞[	[0,1[
[0,+∞[	[0,+∞[	[4,+∞[	[4,+∞[

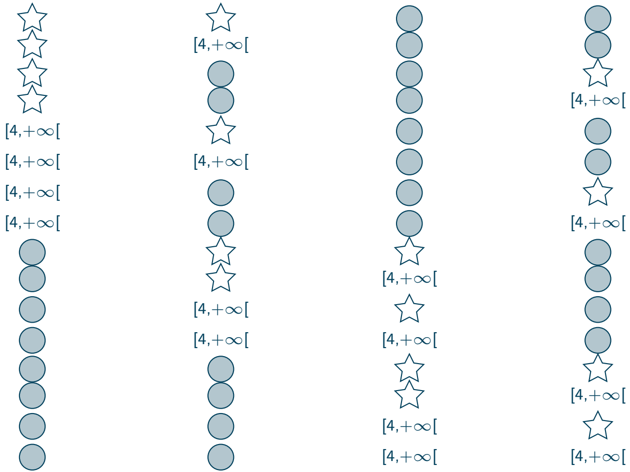
Factoring the space of states

brute force



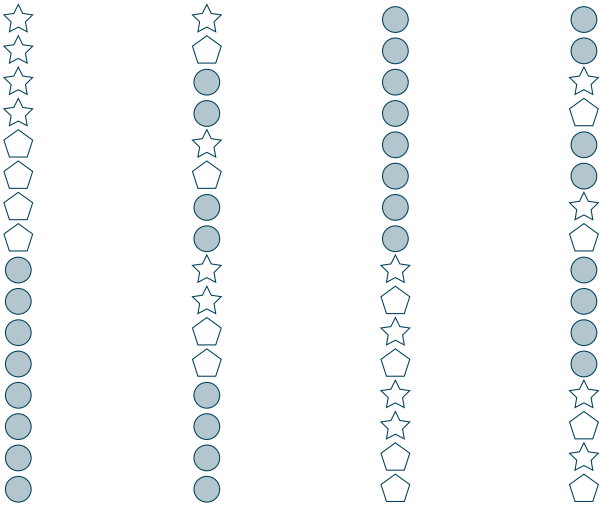
Factoring the space of states

brute force



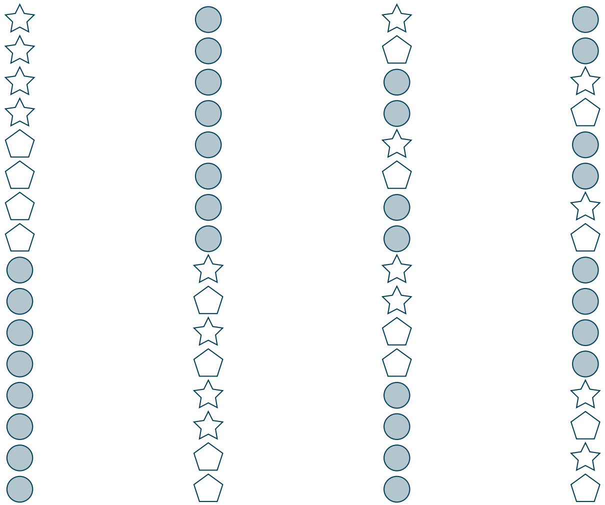
Factoring the space of states

brute force



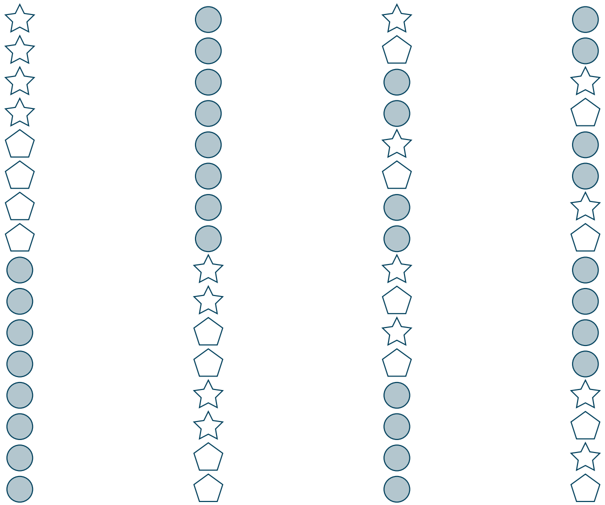
Factoring the space of states

brute force



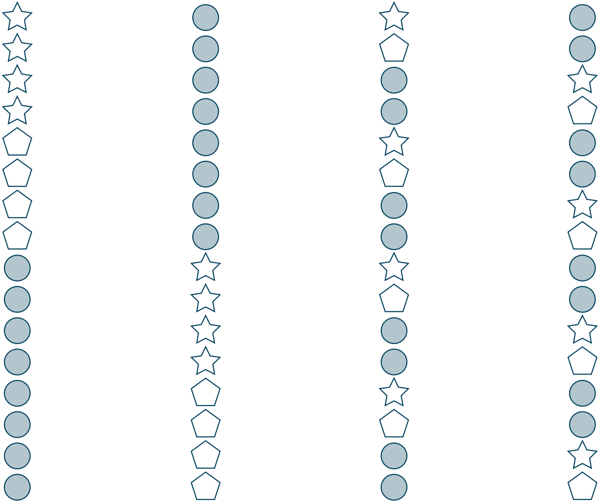
Factoring the space of states

brute force



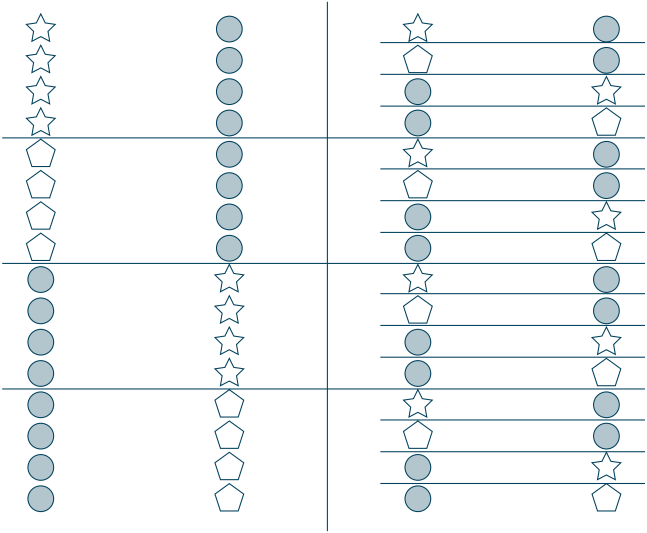
Factoring the space of states

brute force



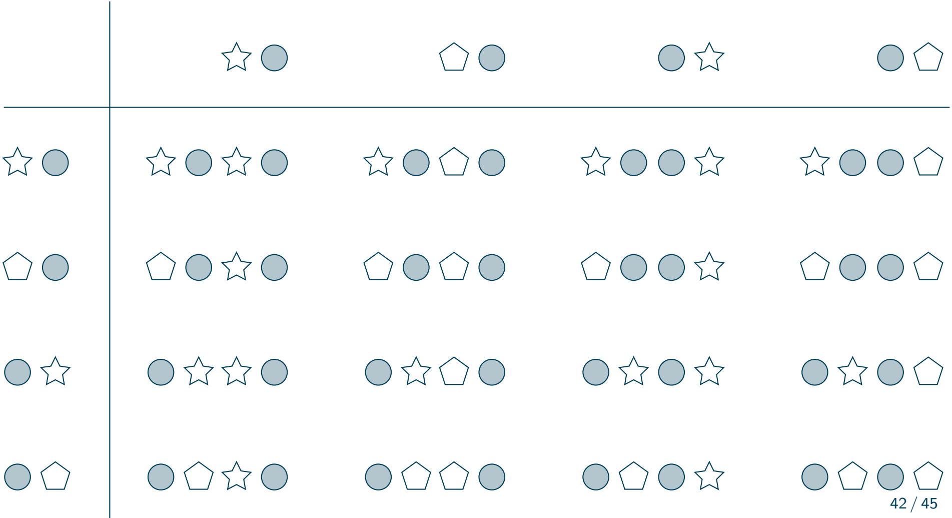
Factoring the space of states

brute force



Factoring the space of states

brute force



Factoring the space of states

subtle

[2,3[	[2,3[	[2,3[	[0,+∞[
[2,3[	[2,3[	[0,+∞[	[2,3[
[1,4[	[0,+∞[	[1,4[	[0,+∞[
[2,3[	[0,+∞[	[2,3[	[2,3[
[0,+∞[	[1,4[	[0,+∞[	[1,4[
[0,+∞[	[2,3[	[2,3[	[2,3[

Factoring the space of states

subtle

[2,3[	[2,3[	[2,3[	[0,+∞[
[2,3[	[2,3[	[0,+∞[	[2,3[
[1,4[	[0,+∞[	[1,4[	[0,+∞[
[2,3[	[0,+∞[	[2,3[	[2,3[
[0,+∞[	[1,4[	[0,+∞[	[1,4[
[0,+∞[	[2,3[	[2,3[	[2,3[

Factoring the space of states

subtle

[1,4[	[0,+∞[	[1,4[	[0,+∞[
[0,+∞[	[1,4[	[0,+∞[	[1,4[

Factoring the space of states

subtle

Parallelizing a program

`sem: 1 a b` `sem: 2 c`	`sem: 1 a b` `sem: 2 c`
`proc:` `p = P(a);P(c);V(c);V(a)`	`proc:` `q = P(b);P(c);V(c);V(b)`
`init: 2p`	`init: 2q`

Parallelizing a program

`sem: 1 a`	`sem: 1 b`
`proc:` `p = P(a);V(a)`	`proc:` `q = P(b);V(b)`
`init: 2p`	`init: 2q`