Directed Algebraic Topology and Concurrency

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MPRI : Concurrency (2.3)

Wednesday, the 4^{th} of January 2017

Partially ordered spaces

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A partially ordered space (or pospace) is a topological space X together with a partial order $\sqsubseteq$ on (the underlying set of) X such that

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- The collection of compact subsets of a metric space equipped with the Hausdorff distance is a metric space.

$$
\begin{aligned}
d_H(K, K') &= \sup \left\{ d(x, K'), d(x', K) \mid x \in K; x' \in K' \right\} \\
d(x, K) &= \inf \left\{ d(x, k) \mid k \in K \right\}
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The induced topological space ordered by inclusion is a pospace.

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  The induced topological space ordered by inclusion is a pospace.
- Problem: there is no pospace on the circle whose collection of directed paths is

$$\left\{\rho(t) \cdot e^{i\theta(t)} \mid \rho, \theta : [0, r] \to \mathbb{R}_+ \text{ increasing}\right\}$$

Ordered atlas

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- for all $U, V \in \mathcal{U}$ for all $x \in U \cap V$ there exists $W \in \mathcal{U}$ such that $x \in W \subseteq U \cap V$ and denoting by $\sqsubseteq_{U|_W}$ the relation induced by $\sqsubseteq_U$ on the underlying set of W, the restrictions of $\sqsubseteq_U$ and $\sqsubseteq_V$ to W match $\sqsubseteq_W$.

$$\sqsubseteq_{U|_W} \quad = \quad \sqsubseteq_W \quad = \quad \sqsubseteq_{V|_W}$$

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Any subset of X inherits an ordered atlas from $\mathcal{U}$.

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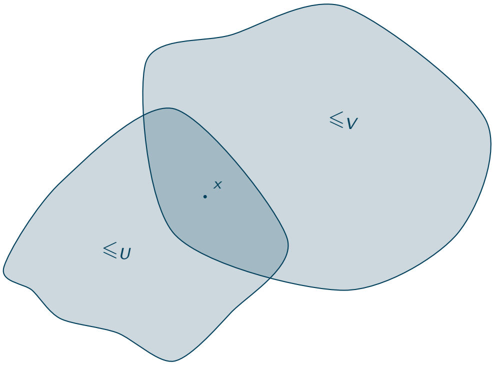

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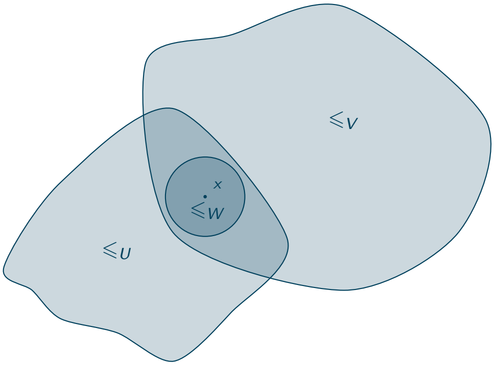

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A local pospace is a Hausdorff space togethter with an equivalence class of ordered atlas.

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- $\{(U, \sqsubseteq'_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq'_U y$ is any extension of $\sqsubseteq_U$.

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- $\{(A, \leqslant) \mid A$ open arc$\}$ where $\leqslant$ is the order induced by $\mathbb{R}$ and the restriction of the exponential map to an open subinterval of $\{t \in \mathbb{R} \mid e^{it} \in A\}$ of length at most 2π,

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- $\{(U, \sqsubseteq_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $x \sqsubseteq_U y$ means that the anticlockwise compact arc from x to y is included in U,
- $\{(U, \sqsubseteq'_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $\sqsubseteq'_U$ is any extension of the partial order $\sqsubseteq_U$.

Local pospace morphisms

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An atlas morphism from $\mathcal{U}$ to $\mathcal{V}$ is a map f (between the underlying sets of $\mathcal{U}$ and $\mathcal{V}$) such that for all $x \in \mathrm{dom}(f)$ there exists an ordered chart $U \in \mathcal{U}$ and an ordered chart $V \in \mathcal{V}$ such that $x \in U$ and f induces a pospace morphism from U to V (implicitly $f(U) \subseteq V$).

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A local pospace morphism defined over a locally ordered compact interval is called a directed path.

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A local pospace has no vortex.

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The locally ordered metric graph construction is functorial.

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such that for all sets X the following map is a bijection

$$\mathcal{S}et[X, A \times B] \longrightarrow \mathcal{S}et[X, A] \times \mathcal{S}et[X, B]$$

$$h \longmapsto (\pi_A \circ h, \pi_B \circ h)$$

Cartesian product

in a category $\mathcal{C}$

The object c is the Cartesian product (in $\mathcal{C}$) of a and b when there exist two morphisms $\pi_a : c \to a$ and $\pi_b : c \to b$ such that for all objects x of $\mathcal{C}$ the following map is a bijection

$$\begin{array}{ccc} \mathcal{C}[x,c] & \longrightarrow & \mathcal{C}[x,a] \times \mathcal{C}[x,b] \\ h & \longmapsto & (\ \pi_a \circ h\ ,\ \pi_b \circ h\ ) \end{array}$$

When such an object c exists we write $c = a \times b$

Cartesian product in the category of graphs ($\mathcal{Grph}$)

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$$\left(\begin{tikzcd} A \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\ V \end{tikzcd}\right) \times \left(\begin{tikzcd} A' \arrow[d, shift right, "\mathsf{t}'"'] \arrow[d, shift left, "\mathsf{s}'"] \\ V' \end{tikzcd}\right) \cong \left(\begin{tikzcd} A \times A' \arrow[d, shift right, "\mathsf{t}\times\mathsf{t}'"'] \arrow[d, shift left, "\mathsf{s}\times\mathsf{s}'"] \\ V \times V' \end{tikzcd}\right)$$

The Cartesian product in $\mathcal{Grph}$ is deduced form the Cartesian product in $\mathcal{Set}$

Examples of Cartesian products

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- Categories of models of algebraic theories.

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Infinite products of directed circle does not exist in $\mathcal{L}po$.

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where $p = (p_1, \ldots, p_n)$, $p_i \in V_i \sqcup A_i$, and $\dim p = \#\big\{i \in \{1, \ldots, n\} \mid p_i \in A_i\big\}$

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The collection of canonical blocks forms the canonical partition of $\lceil G_1 \rfloor \times \cdots \times \lceil G_n \rfloor$.

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Discretization and lifting

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- Given a directed path γ on the local pospace $\lceil G_1 \rfloor \times \cdots \times \lceil G_n \rfloor$ we have a finite partition $I_0 < \cdots < I_N$ of $\mathrm{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point p^k such that $\gamma(I_k) \subseteq B_{p^k}$.

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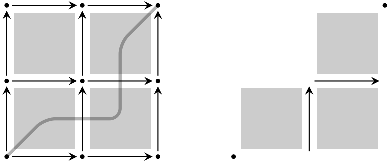
- Given a directed path δ on $(G_1, \ldots, G_n)$ there exists a directed path γ on $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$ whose discretization is δ, such a directed path γ is said to be a lifting of δ.

Example of discretization



Admissible directed paths and execution traces

on $\lceil G_1 \rfloor \times \cdots \times \lceil G_n \rfloor$

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The action of a directed path γ on $\lceil G_1 \rfloor \times \cdots \times \lceil G_n \rfloor$ on the right of a state σ is that of its discretization of $D(\gamma)$.

Example

```
var x = 0
var y = 0
var z = 0
sync 1 b
sem 1 a
```

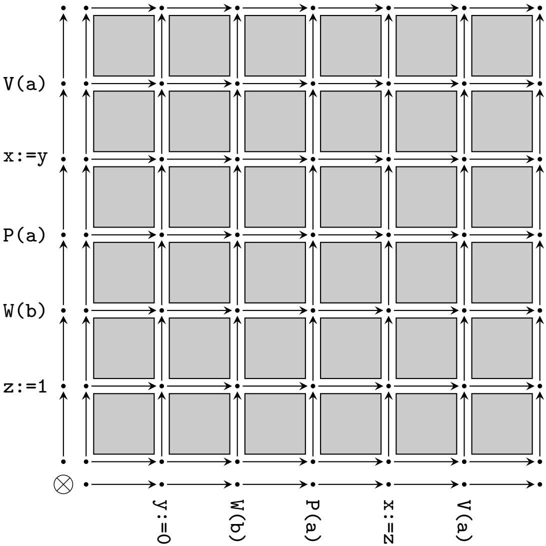
```
proc p = y:=0 ; W(b) ; P(a) ; x:=z ; V(a)

proc q = z:=1 ; W(b) ; P(a) ; x:=y ; V(a)
```

```
init p q
```

Discretization of an execution trace

`sem: 1 a` `sync: 1 b`



Discretization of an execution trace

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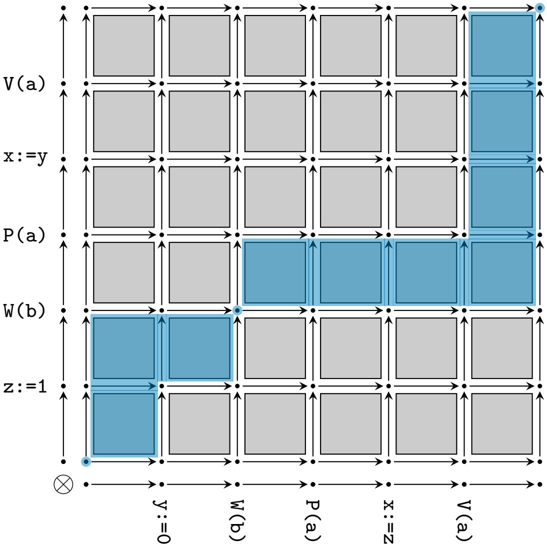
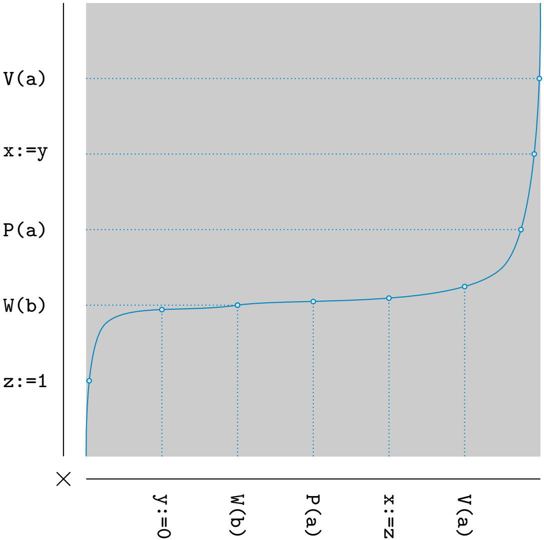


Discretization of an execution trace

`sem: 1 a        sync: 1 b`



Potential function on $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$

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If the program under consideration is conservative, then we have the potential function

$$F : \upharpoonright G_1 \downharpoonleft \times \cdots \times \upharpoonright G_n \downharpoonleft \times \mathcal{S} \to \{\text{multisets over } \{1, \ldots, n\}\}$$

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The function F is constant on each canonical block.

Geometric model

Geometric model

The forbidden region is

$$\bigsqcup_{\substack{\text{forbidden points } p\\ \text{of } (G_1,\ldots,G_n)}} B_p$$

Geometric model

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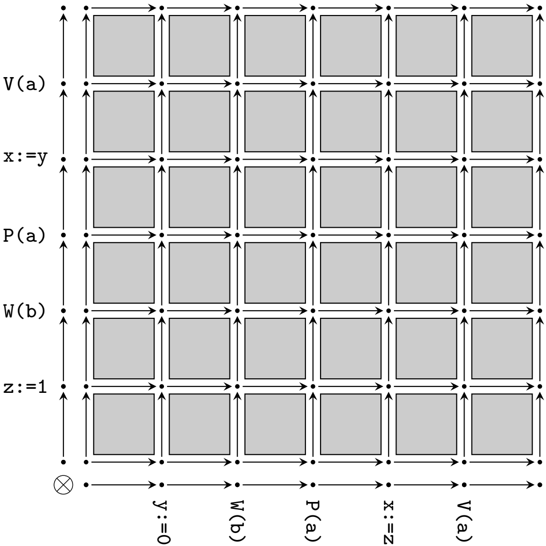
The directed continuous model is

$$|G_1| \times \cdots \times |G_n| \setminus \{\text{forbidden region}\}$$

From discrete to continuous

sem: 1 a sync: 1 b



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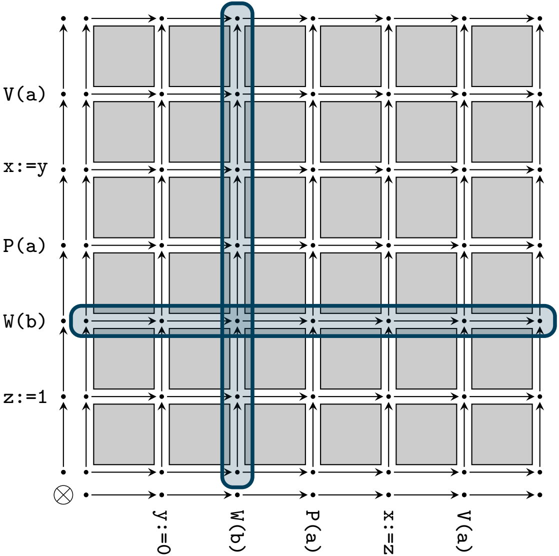

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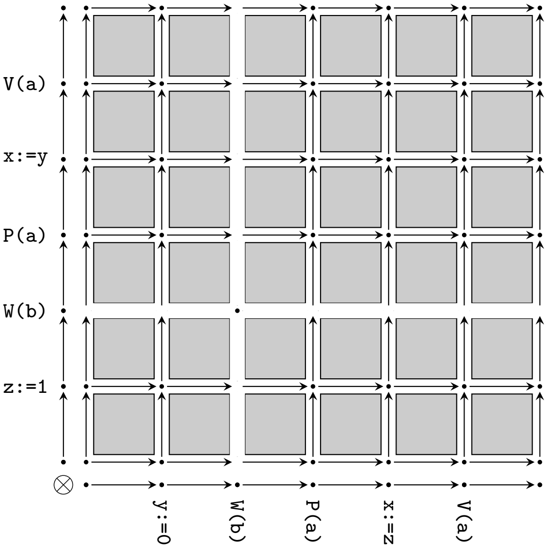
sem: 1 a sync: 1 b



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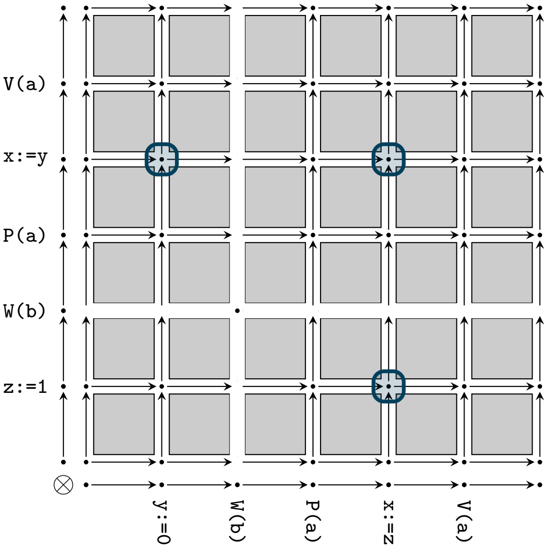

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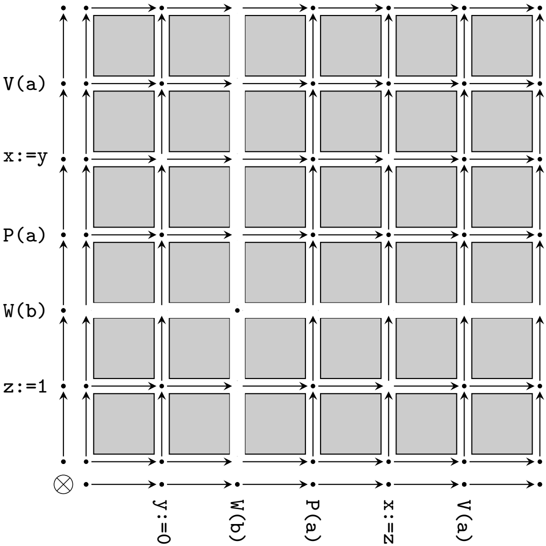
`sem: 1 a        sync: 1 b`



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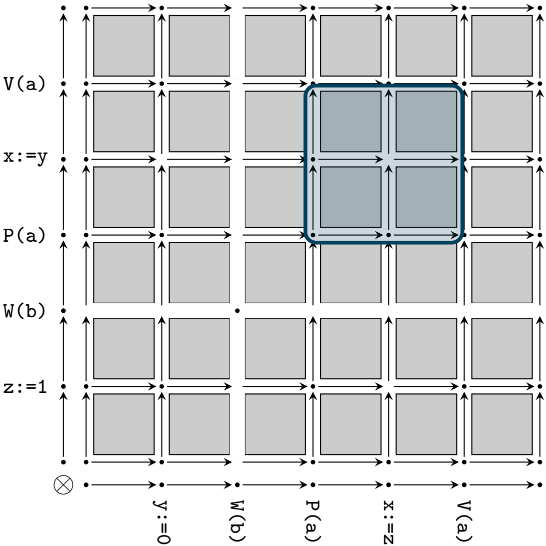
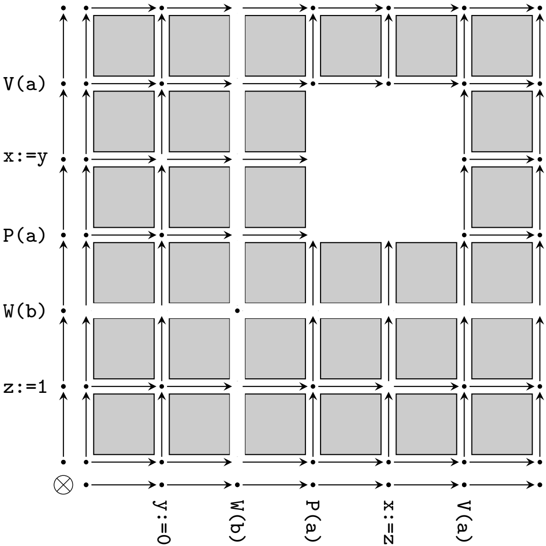
sem: 1 a sync: 1 b



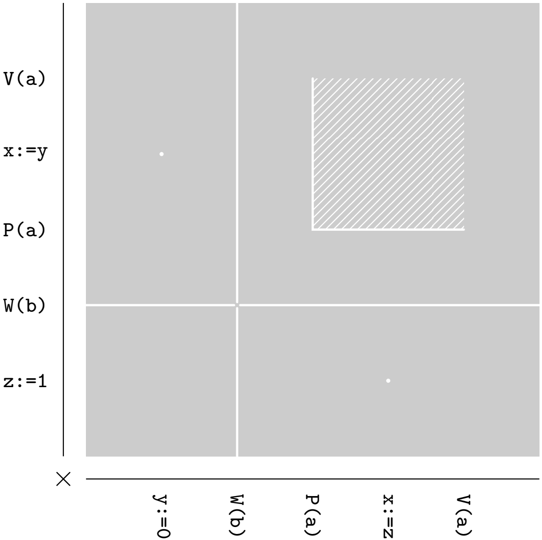
From discrete to continuous

sem: 1 a sync: 1 b



From discrete to continuous

`sem: 1 a` `sync: 1 b`



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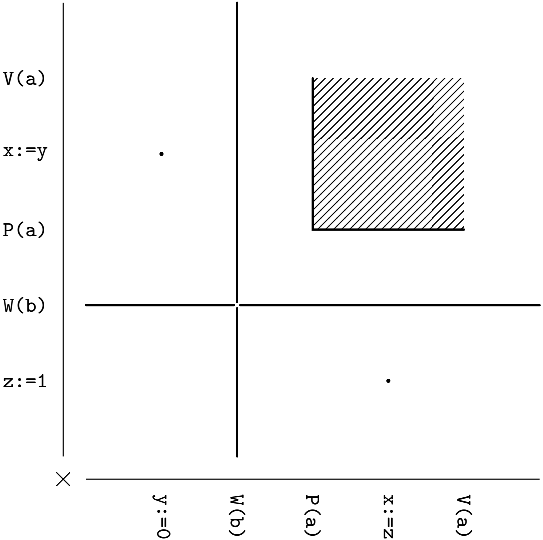

Square

Square

```
sem 1 a
proc:  p = P(a);V(a)
init:  2p
```

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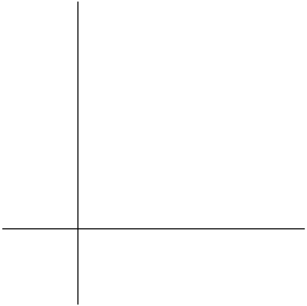

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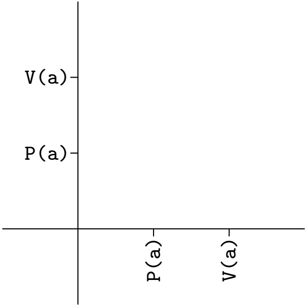

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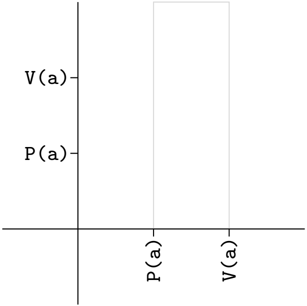

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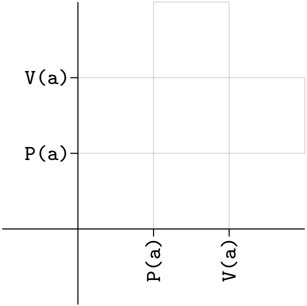

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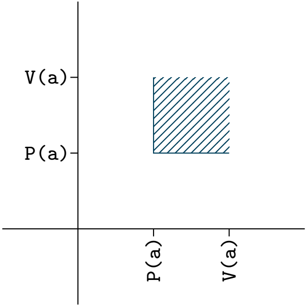

Swiss Cross

Swiss Cross

```
sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init:  p q
```

Swiss Cross

```
sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
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```

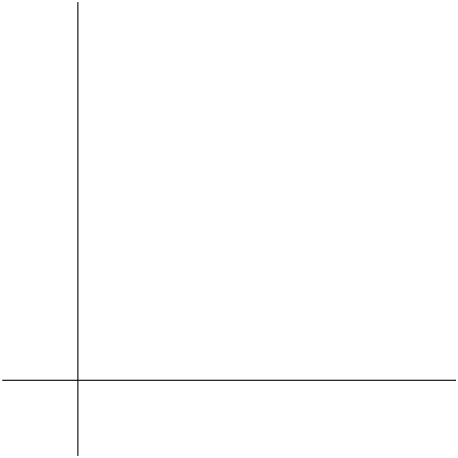

Swiss Cross

```
sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
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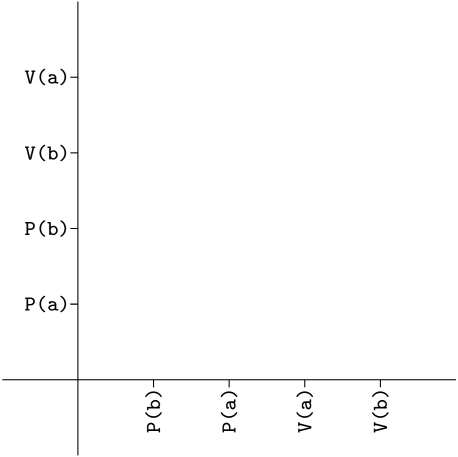

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proc:
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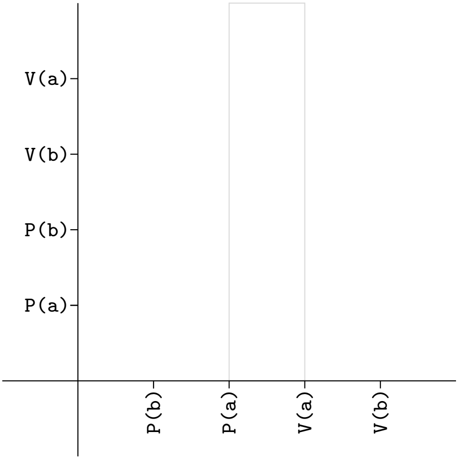

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p = P(a);P(b);V(b);V(a)
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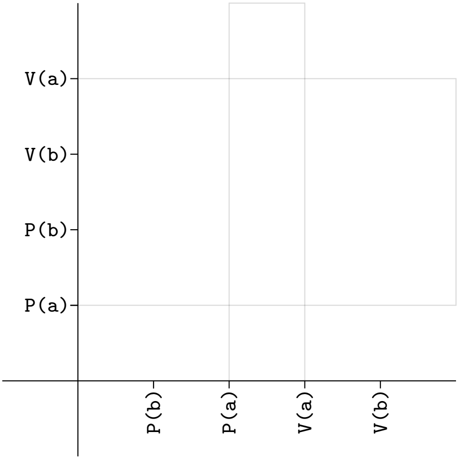

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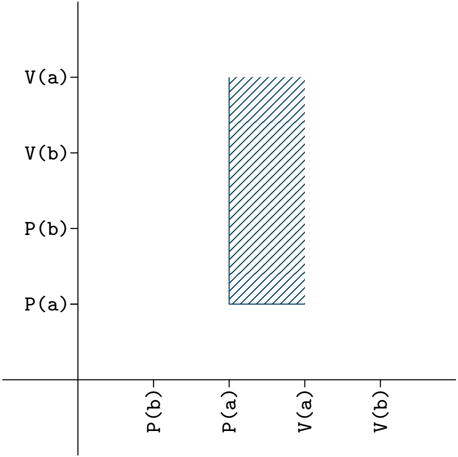

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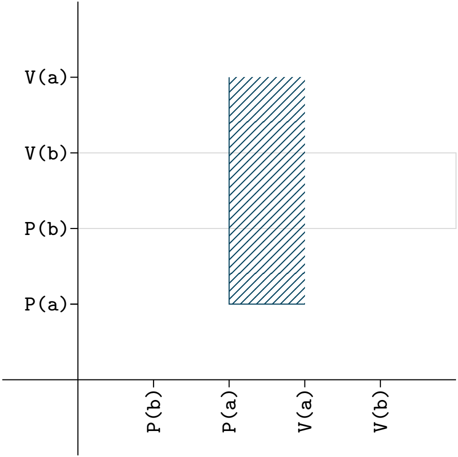

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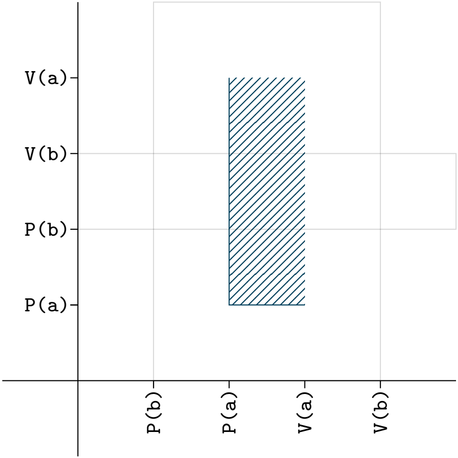
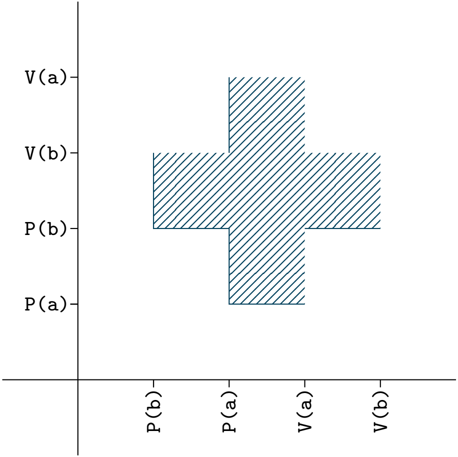


Swiss Cross

```
sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init:  p q
```



Binary synchronization

Binary synchronization

```
sync 1 a
proc:  p = W(a)
init:  2p
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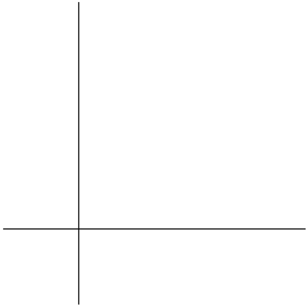

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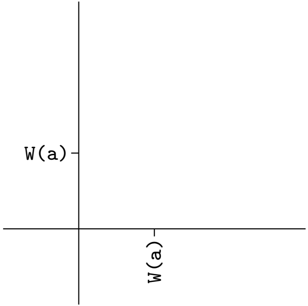

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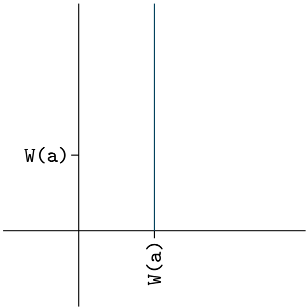

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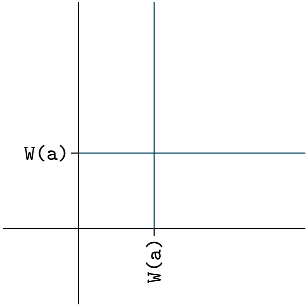

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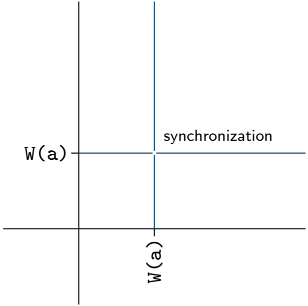

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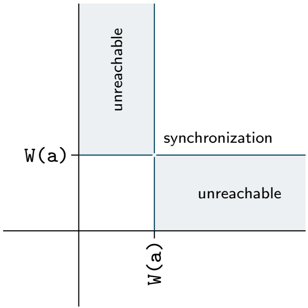

Producer/Consumer

nonlooping

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nonlooping

```
sync 1 a
proc:
  p = x:=x+1 ; W(a)
  c = W(a)   ; x:=x-1
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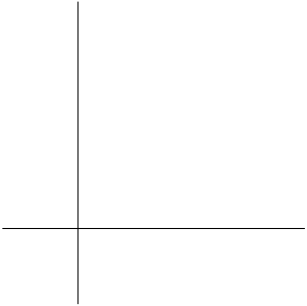

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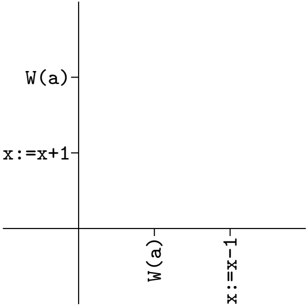

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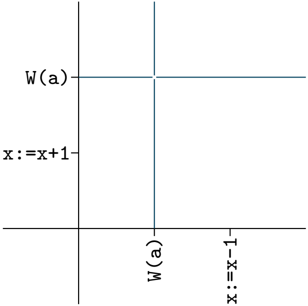

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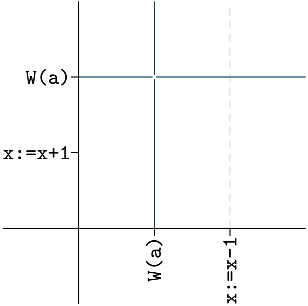

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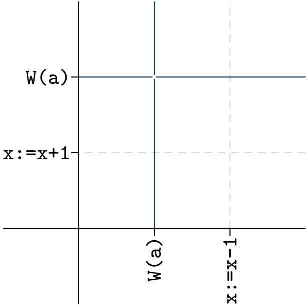

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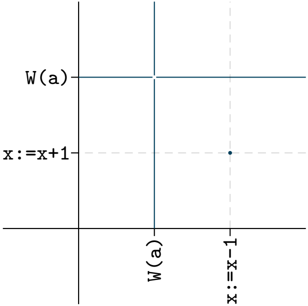

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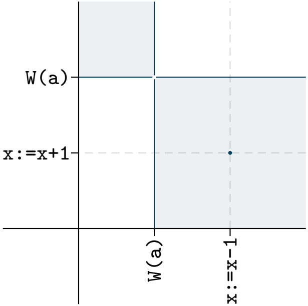

Producer/Consumer

looping

Producer/Consumer

looping

```
sync 1 a b
proc:
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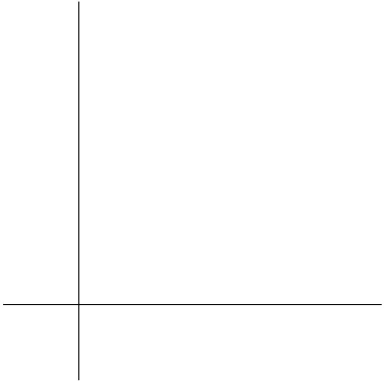

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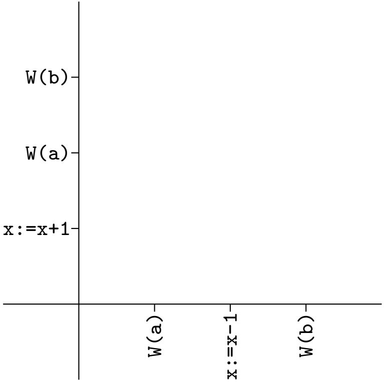

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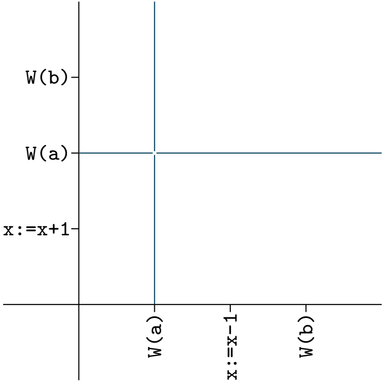

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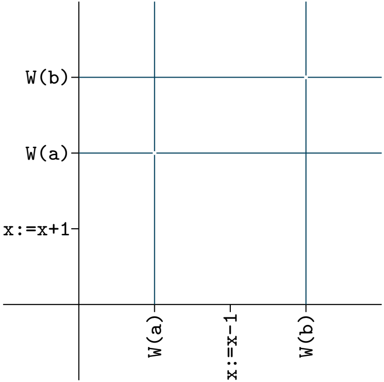

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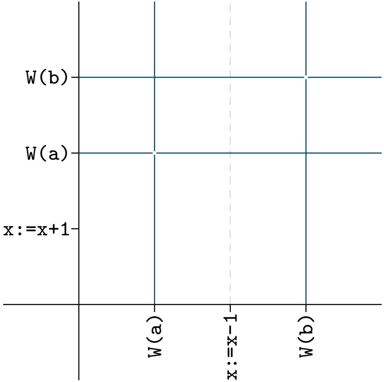

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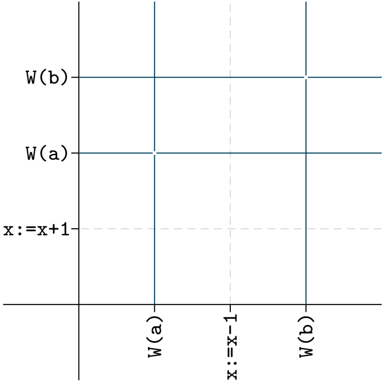

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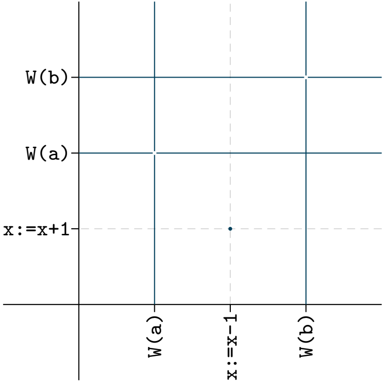

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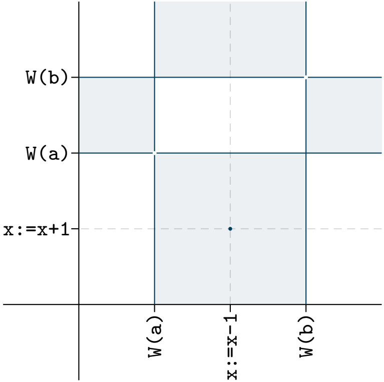
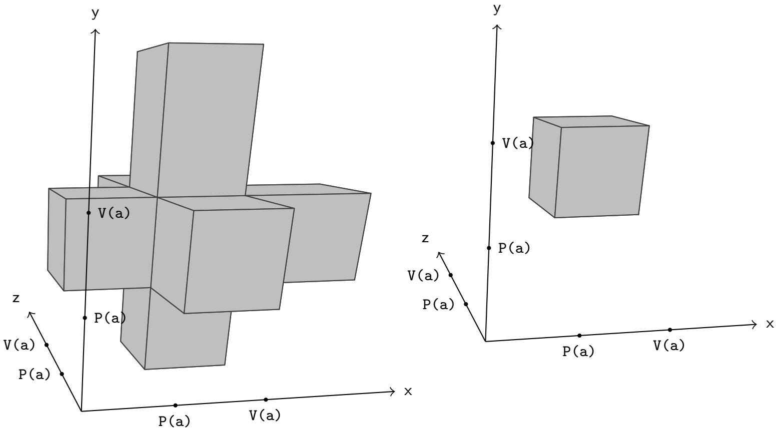


3D Swiss Cross (tetrahemihexacron) and floating cube



Geometric models are sound and complete

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- Any directed path on a continuous model is admissible.

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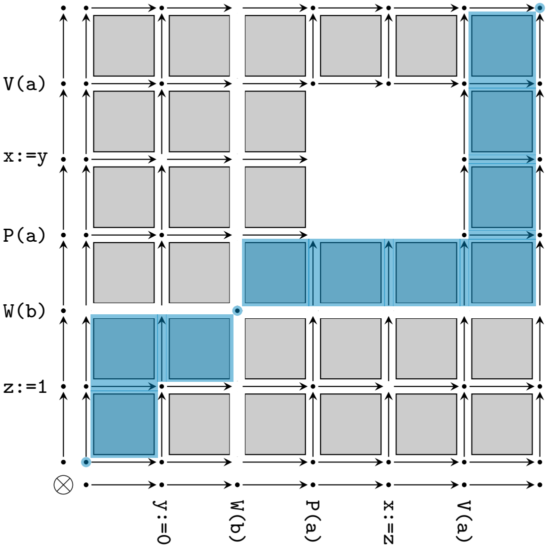
- Any directed path on a continuous model is admissible.
- Conversely, for each admissible path on a continuous model which meets a forbidden point, there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.

Directed paths on the geometric model are admissible

`sem: 1 a` `sync: 1 b`



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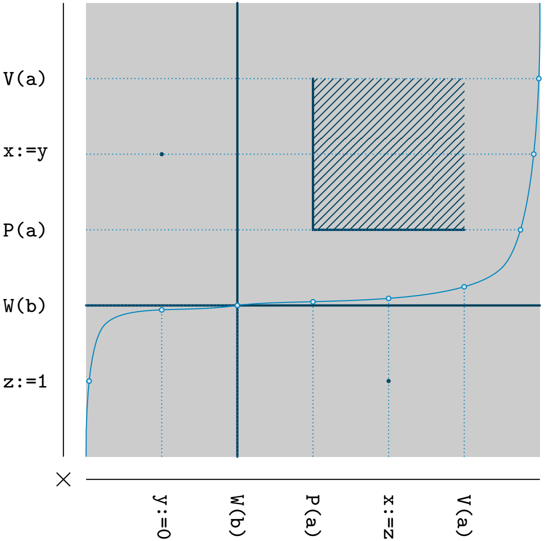

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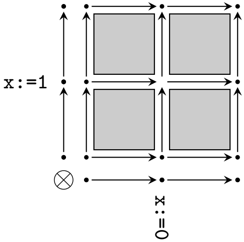

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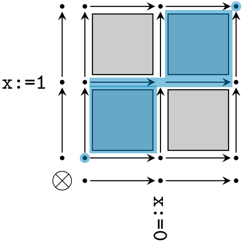

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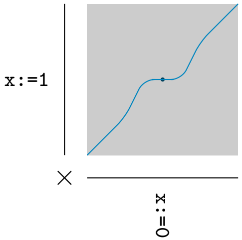

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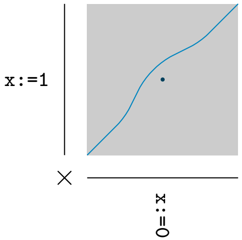

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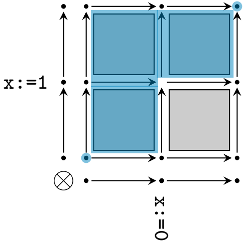

Trade off

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- Both discrete and geometric models are sound and complete.

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- Both discrete and geometric models are sound and complete.
- The continuous models satisfy extra properties that are “naturally” expressed in terms of metrics.

The geometric model of a conservative program

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The continuous model X of a conservative program whose running processes are $G_1, \ldots, G_n$ is a sub-local pospace of $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$.

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The continuous model X inherits a distance d_X from the distances $d_{\upharpoonleft G_i \downharpoonright}$ of the metric graphs $\upharpoonleft G_i \downharpoonright$

$$d_X(p, p') \quad = \quad \max \left\{ d_{\upharpoonleft G_i \downharpoonright}(p_i, p'_i) \mid i \in \{1, \ldots, n\} \right\}$$

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The distance d_X is in accordance with the fact that the execution time of the simultaneous execution of many processes is the longest execution time among that of the processes considered individually.

Uniform distance between directed paths

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Given a compact Hausdorff space K and a metric space (X, d_X), the set of continuous maps from K to X can be equipped with the uniform distance

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We consider the case where $K = [0, r]$ is the domain of definition of a directed path and (X, d_X) is the geometric model of a conservative program.

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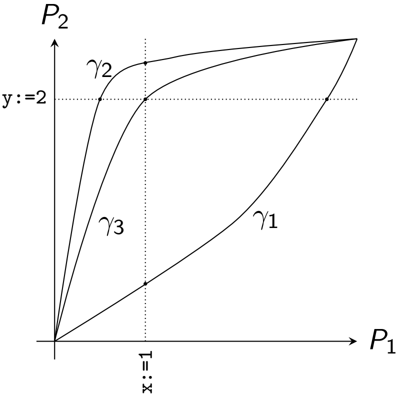
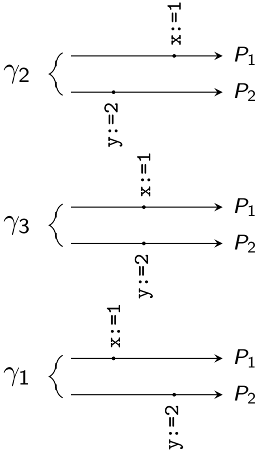
Let γ be an element of $dX^{[0,r]}(B_p, B_{p'})$.

There exists an open ball Ω of $dX^{[0,r]}(B_p, B_{p'})$, centred in γ, such that all the elements of Ω induce the same action on valuations. Moreover, if γ is an execution trace, then so are all the elements of Ω.

Desynchronization

one of the artifact used in the proof





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As a consequence we have $\gamma(0) = \delta(0)$ and $\gamma(r) = \delta(r)$.

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The Curryfication $(\hat{\_})$ induces a homeomorphism from $X^{[0,r]\times[0,q]}$ to $\left(X^{[0,r]}\right)^{[0,q]}$

$$(h : [0,r] \times [0,q] \to X) \to (\hat{h} : [0,q] \to X^{[0,r]})$$

The two faces of homotopies

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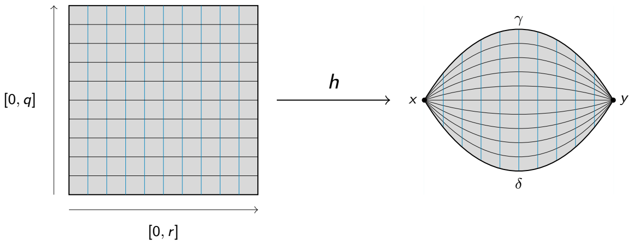
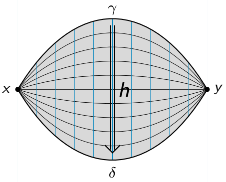
h is a continuous map from $[0, r] \times [0, q]$ to X i.e. $h \in \mathcal{T}op\big[[0, r] \times [0, q], X\big]$



The two faces of homotopies

but is also a path from γ to δ in the space $X^{[0,r]}$ i.e. $h \in \mathcal{Top}\big[[0,q], X^{[0,r]}\big]$



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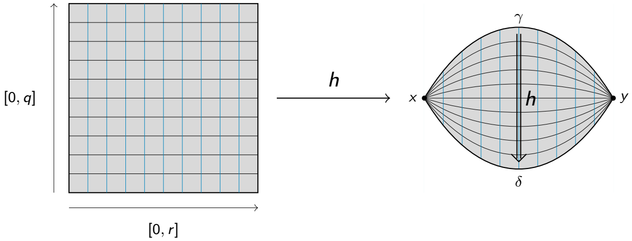

The second point of view leads us to introduce the following notation

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\delta"'] \arrow[r, phantom, "\Downarrow h"] & y \end{tikzcd}$$

Directed homotopy on a locally ordered space

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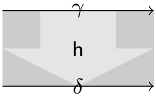

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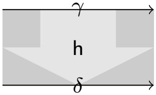

- A weakly directed homotopy from γ to δ is a homotopy of paths $h : [0, r] \times [0, q] \to X$ whose intermediate paths $h(\_, s)$, for $s \in [0, q]$, are directed.

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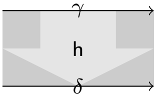

- A weakly directed homotopy from γ to δ is a homotopy of paths $h : [0,r] \times [0,q] \to X$ whose intermediate paths $h(\_, s)$, for $s \in [0,q]$, are directed.
- Any directed homotopy is a weakly directed homotopy. The converse is false.

Theorem

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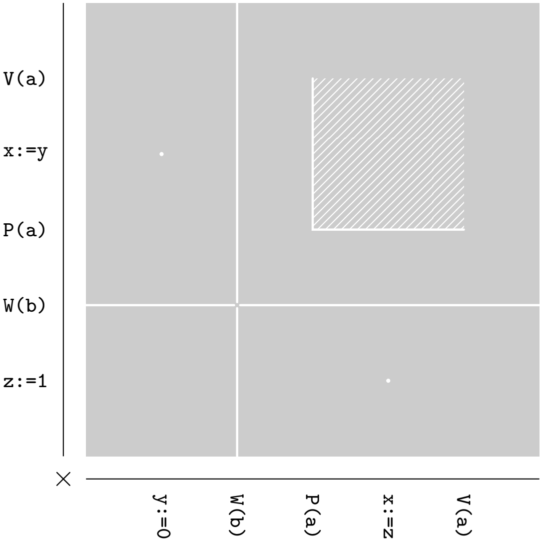
Two weakly dihomotopic paths on the geometric model of a conservative program induce the same action on valuations. Moreover, if one of them is an execution trace, then so is the other.

Weakly directed homotopy

sem: 1 a sync: 1 b



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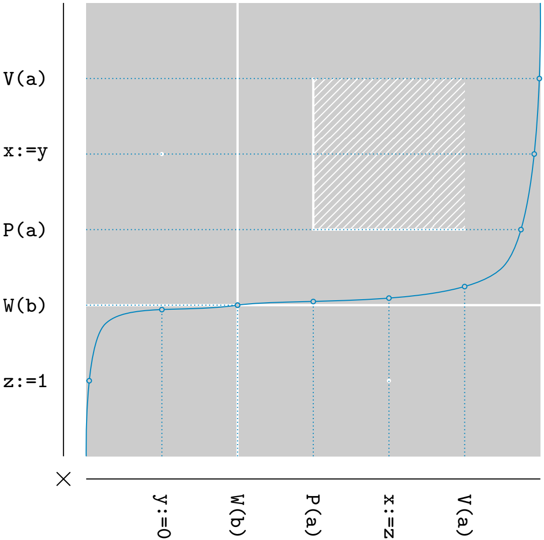

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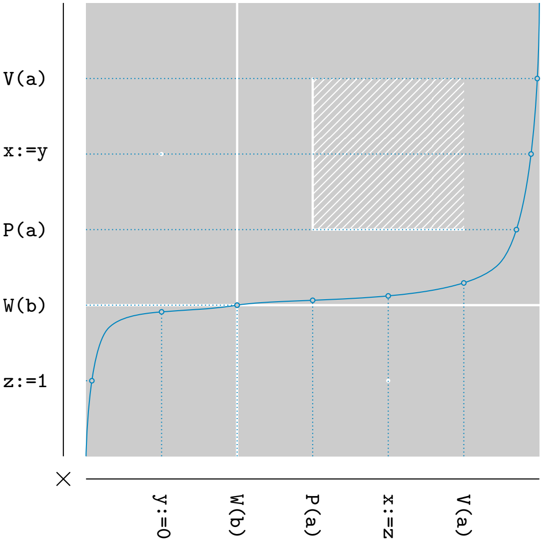

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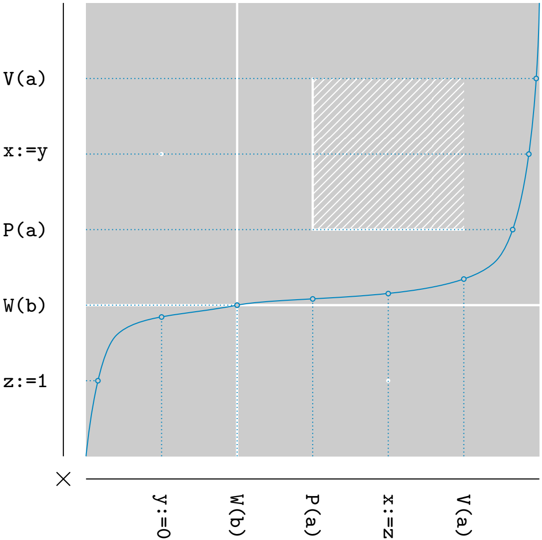

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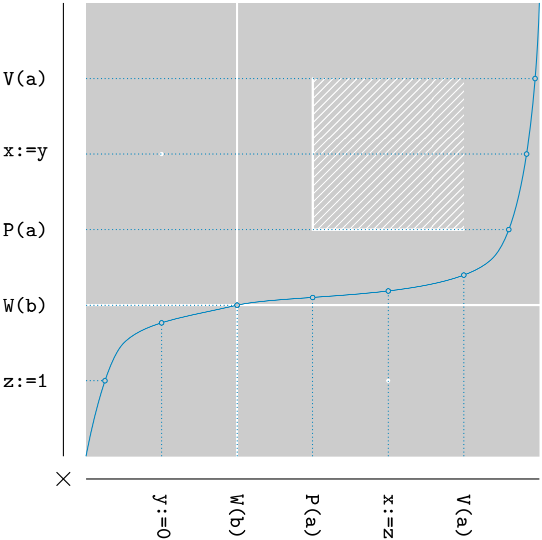

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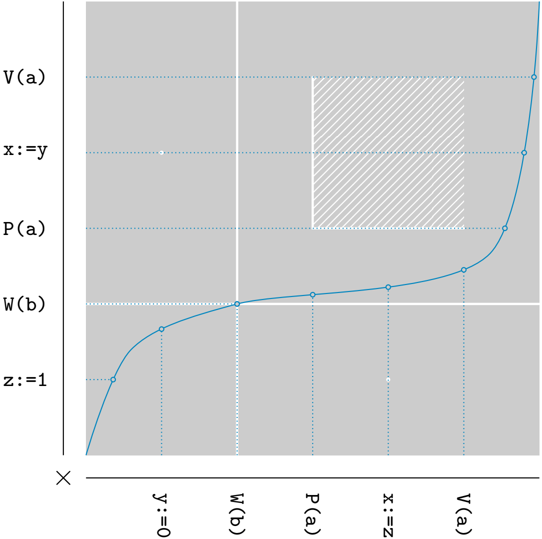

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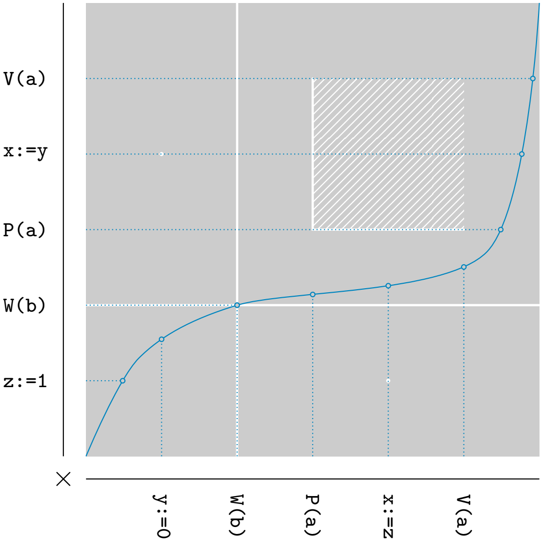

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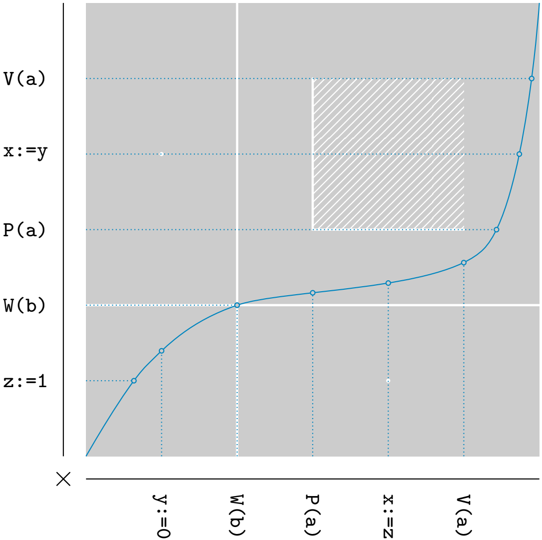

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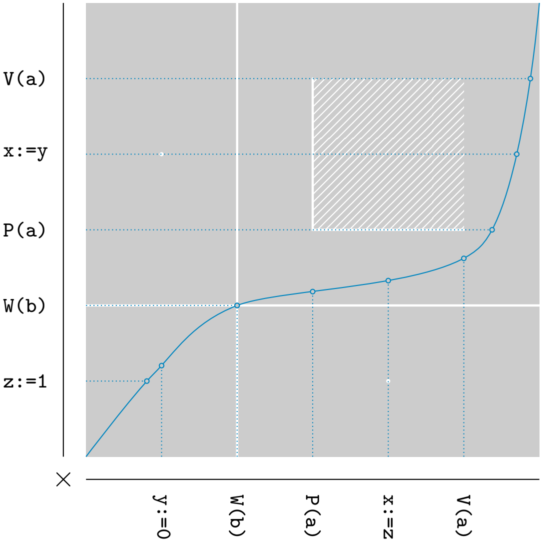

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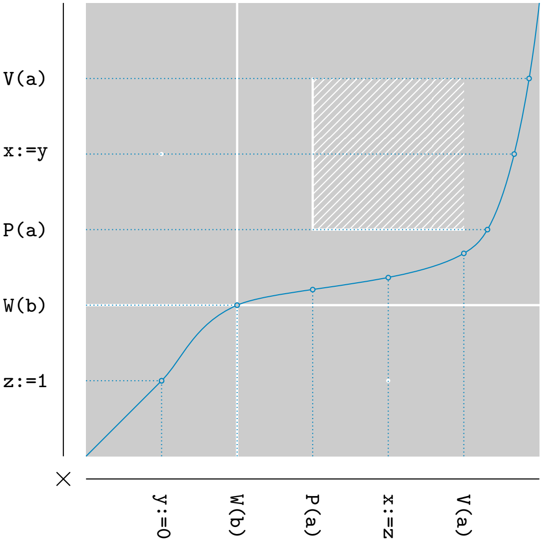

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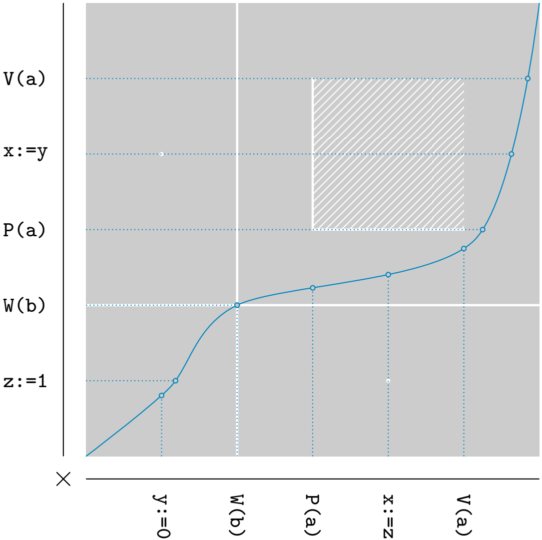

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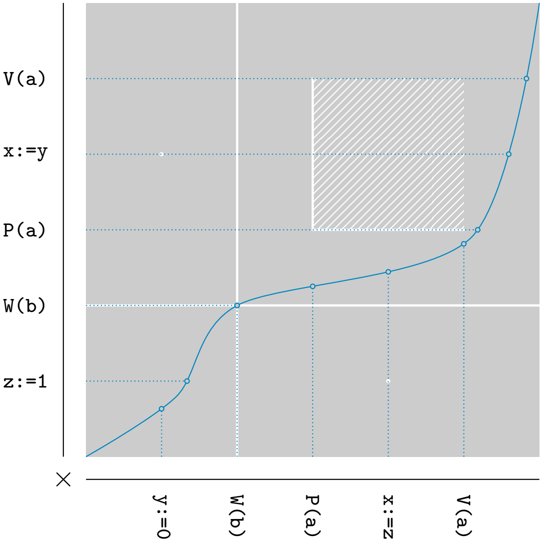

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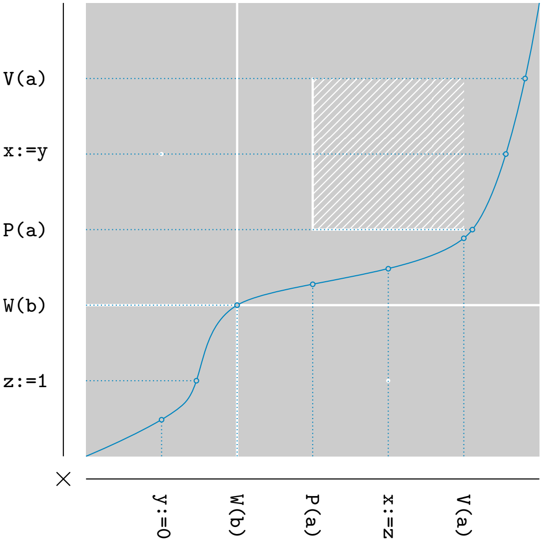

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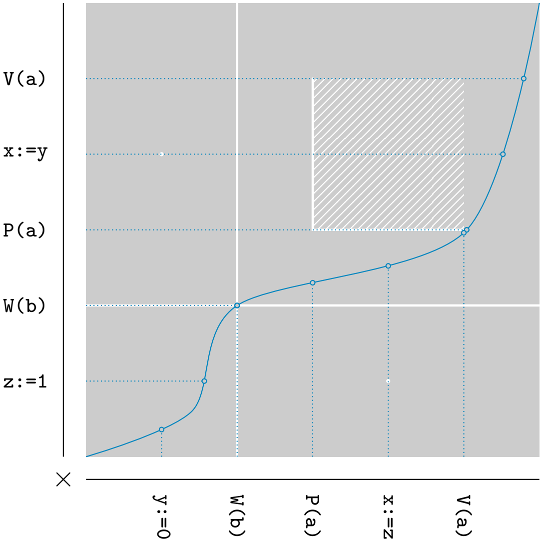

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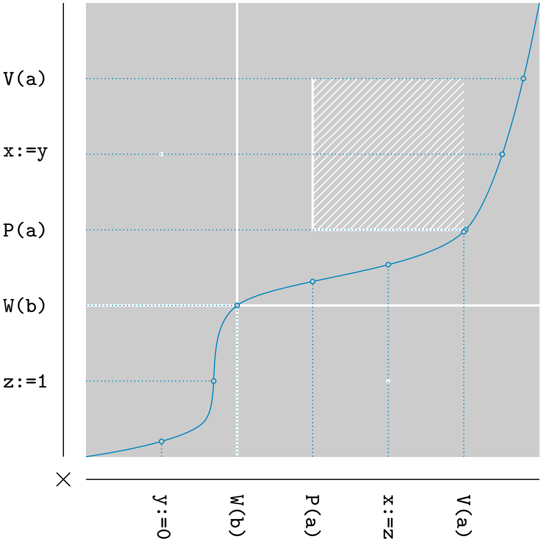

Proof

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By a standard result from general topology and the Curryfication of h, namely

$$\hat{h} : s \in [0,q] \mapsto (t \in [0,r] \mapsto h(t,s) \in X)$$

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The conclusion follows considering the sequence

$$\hat{h}(0), \hat{h}(\varepsilon), \hat{h}(2\varepsilon), \hat{h}(3\varepsilon), \cdots, \hat{h}(n\varepsilon), \hat{h}(q)$$

where n is the greatest natural number such that $n\varepsilon \leqslant q$.

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vertical composition

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The mapping $h * g : [0, r] \times [0, q + q'] \to X$ defined by

$$h * g(t, s) \quad = \quad \begin{cases} g(t, s) & \text{if } 0 \leqslant s \leqslant q \\ h(t, s - q) & \text{if } q \leqslant s \leqslant q + q' \end{cases}$$

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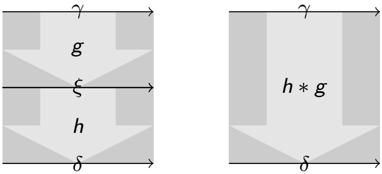

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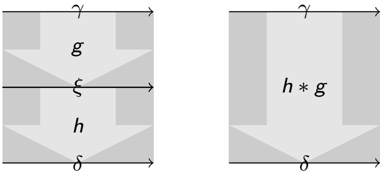

If g and h are (weakly) directed homotopies, then so is their concatenation $h * g$.

Homotopy and dihomotopy relations

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An anti-directed homotopy from γ to δ is a homotopy of paths $h : [0, r] \times [0, q] \to X$ such that $(t, s) \mapsto h(t, q - s)$ is a directed homotopy from δ to γ.

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An elementary homotopy between γ to δ is a homotopy of paths $h : [0, r] \times [0, q] \to X$ obtained as a finite concatenation of directed homotopies and anti-directed homotopies.

Homotopy and dihomotopy relations

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An elementary homotopy between γ to δ is a homotopy of paths $h : [0, r] \times [0, q] \to X$ obtained as a finite concatenation of directed homotopies and anti-directed homotopies.

Two paths γ and γ' are said to be homotopic when there exists a homotopy between them. We have the equivalence relation $\sim_h$ between paths on a topological space.

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They are said to be weakly dihomotopic when there exists a weakly directed homotopy between them. We have the equivalence relation $\sim_w$ between directed paths on a locally ordered space.

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If h is a (weakly) directed homotopy from γ to γ' on the local pospace space X and $f : X \to Y$ is a local pospace morphism, then $f \circ h$ is a (weakly) directed homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the local pospace space Y.

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If $\gamma, \gamma' : [0, r] \to X$ are ((weakly) di)homotopic, then so are $f \circ \gamma, f \circ \gamma' : [0, r] \to Y$.

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The mapping

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Therefore γ and $\gamma \circ \theta$ are dihomotopic.

Images of directed paths on a pospace

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Theorem

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Hence $\phi \circ h$ is an elementary homotopy from γ and γ'.

Programs with mutex only

a result by É. Goubault and S. Mimram

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Let X be the geometric model of a conservative program whose semaphores have arity 1 (mutex), then two directed paths on X are dihomotopic if and only if they are homotopic.