DIRECTED ALGEBRAIC TOPOLOGY
AND
CONCURRENCE

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MPRI : Concurrency (2.3.1)
– Lecture 3 –

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GEOMETRIC MODELS
Cartesian product
Cartesian product in \( Set \)
Cartesian product in $\text{Set}$

$$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$$
Cartesian product in $\mathbf{Set}$

$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$

There exist two mappings $\pi_A$ and $\pi_B$

$\pi_A : A \times B \longrightarrow A$

$(a, b) \longmapsto a$

$\pi_B : A \times B \longrightarrow B$

$(a, b) \longmapsto b$
Cartesian product in $\textbf{Set}$

\[ A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\} \]

There exist two mappings $\pi_A$ and $\pi_B$

\[
\begin{align*}
\pi_A : & \quad A \times B \longrightarrow A \\
(a, b) & \longmapsto a \\
\pi_B : & \quad A \times B \longrightarrow B \\
(a, b) & \longmapsto b
\end{align*}
\]

such that for all sets $X$ the following map is a bijection

\[
\text{Set}[X, A \times B] \longrightarrow \text{Set}[X, A] \times \text{Set}[X, B]
\]

\[
h \longmapsto (\pi_A \circ h, \pi_B \circ h)
\]
Cartesian product in a category $\mathcal{C}$

The object $c$ is the **Cartesian product** (in $\mathcal{C}$) of $a$ and $b$ when there exist two morphisms $\pi_a : c \to a$ and $\pi_b : c \to b$ such that for all objects $x$ of $\mathcal{C}$ the following map is a bijection

$$\begin{align*}
C[x, c] &\to C[x, a] \times C[x, b] \\
h &\mapsto (\pi_a \circ h, \pi_b \circ h)
\end{align*}$$

When such an object $c$ exists we write $c = a \times b$
Cartesian product in the category of graphs ($\text{Grph}$)
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Cartesian product in the category of graphs ($\text{Grph}$)

The Cartesian product in $\text{Grph}$ is deduced from the Cartesian product in $\text{Set}$.
Examples of Cartesian products

- The product of \((X, \Omega_X)\) and \((Y, \Omega_Y)\) in \(\text{Top}\) is given by \(X \times Y\) together with unions of subsets of the form \(U \times V\) with \(U \in \Omega_X\) and \(V \in \Omega_Y\). It is the least topology making the projections continuous.

- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{Pos}\) is given by \(X \times Y\) and the partial order \(\sqsubseteq\) defined by \((x, y) \sqsubseteq (x', y')\) when \(x \sqsubseteq_X x'\) and \(y \sqsubseteq_Y y'\). It is the greatest partial order such that the projections are poset morphisms.

- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{PoSp}\) is given by \(X \times Y\) and the product order \(\sqsubseteq_X \times \sqsubseteq_Y\).

- The product of \((X, \sim_X)\) and \((Y, \sim_Y)\) in \(\text{Lpo}\) is given by \(X \times Y\) together with the collection of ordered charts \(U \times V\) with \(U \in \sim_X\) and \(V \in \sim_Y\).

- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{\text{emb}}\) does not exist.

- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{\text{ctr}}\) is given by \(X \times Y\) together with \(d_{X \times Y}(x, y), (x', y') = \max\{d_X(x, x'), d_Y(y, y')\}\).

- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{\text{top}}\) can also be given by \(X \times Y\) together with the Euclidean product \(d_{X \times Y}(x, y), (x', y') = q d_X^2(x, x') + d_Y^2(y, y')\).
Examples of Cartesian products

- The product of \((X, \Omega_X)\) and \((Y, \Omega_Y)\) in \(\text{Top}\)
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Examples of Cartesian products

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- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{PoSp}\)
Examples of Cartesian products

- The product of \((X, \Omega_X)\) and \((Y, \Omega_Y)\) in \(\text{Top}\) is given by \(X \times Y\) together with unions of subsets of the form \(U \times V\) with \(U \in \Omega_X\) and \(V \in \Omega_Y\). It is the least topology making the projections continuous.

- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{Pos}\) is given by \(X \times Y\) and the partial order \(\leq\) defined by \((x, y) \leq (x', y')\) when \(x \sqsubseteq_X x'\) and \(y \sqsubseteq_Y y'\). It is the greatest partial order such that the projection are poset morphisms.

- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{PoSp}\) is given by \(X \times Y\) and the product order \(\sqsubseteq_X \times \sqsubseteq_Y\).
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- The product of \((X, [U]_\sim)\) and \((Y, [V]_\sim)\) in \(\text{Lpo}\)
Examples of Cartesian products

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- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{PoSp}\) is given by \(X \times Y\) and the product order \(\sqsubseteq_X \times \sqsubseteq_Y\).

- The product of \((X, [U]_\sim)\) and \((Y, [V]_\sim)\) in \(\text{Lpo}\) is given by \(X \times Y\) together with the collection of ordered charts \(U \times V\) with \(U \in \mathcal{U}\) and \(V \in \mathcal{V}\).
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- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{PoSp}\) is given by \(X \times Y\) and the product order \(\sqsubseteq_X \times \sqsubseteq_Y\).
- The product of \((X, [U]_{\sim})\) and \((Y, [V]_{\sim})\) in \(\text{Lpo}\) is given by \(X \times Y\) together with the collection of ordered charts \(U \times V\) with \(U \in \mathcal{U}\) and \(V \in \mathcal{V}\).
- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{\text{emb}}\)
Examples of Cartesian products

- The product of \((X, \Omega_X)\) and \((Y, \Omega_Y)\) in \(\text{Top}\) is given by \(X \times Y\) together with unions of subsets of the form \(U \times V\) with \(U \in \Omega_X\) and \(V \in \Omega_Y\). It is the least topology making the projections continuous.

- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{Pos}\) is given by \(X \times Y\) and the partial order \(\sqsubseteq\) defined by \((x, y) \sqsubseteq (x', y')\) when \(x \sqsubseteq_X x'\) and \(y \sqsubseteq_Y y'\). It is the greatest partial order such that the projection are poset morphisms.

- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{PoSp}\) is given by \(X \times Y\) and the product order \(\sqsubseteq_X \times \sqsubseteq_Y\).

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- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{\text{emb}}\) does not exist.
Examples of Cartesian products

- The product of \((X, \Omega_X)\) and \((Y, \Omega_Y)\) in \(\text{Top}\) is given by \(X \times Y\) together with unions of subsets of the form \(U \times V\) with \(U \in \Omega_X\) and \(V \in \Omega_Y\). It is the least topology making the projections continuous.

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- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{Lpo}\) is given by \(X \times Y\) together with the collection of ordered charts \(U \times V\) with \(U \in \mathcal{U}\) and \(V \in \mathcal{V}\).

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- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{PoSp}\) is given by \(X \times Y\) and the product order \(\sqsubseteq_X \times \sqsubseteq_Y\).

- The product of \((X, [\mathcal{U}]_\sim)\) and \((Y, [\mathcal{V}]_\sim)\) in \(\text{Lpo}\) is given by \(X \times Y\) together with the collection of ordered charts \(U \times V\) with \(U \in \mathcal{U}\) and \(V \in \mathcal{V}\).

- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{emb}\) does not exist.

- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{ctr}\) is given by \(X \times Y\) together with \(d((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}\).
Examples of Cartesian products

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- The product of \((X, \subseteq X)\) and \((Y, \subseteq Y)\) in \(\text{Pos}\) is given by \(X \times Y\) and the partial order \(\subseteq\) defined by \((x, y) \subseteq (x', y')\) when \(x \subseteq_X x'\) and \(y \subseteq_Y y'\). It is the greatest partial order such that the projection are poset morphisms.

- The product of \((X, \subseteq X)\) and \((Y, \subseteq Y)\) in \(\text{PoSp}\) is given by \(X \times Y\) and the product order \(\subseteq_X \times \subseteq_Y\).

- The product of \((X, [U]_\sim)\) and \((Y, [V]_\sim)\) in \(\text{Lpo}\) is given by \(X \times Y\) together with the collection of ordered charts \(U \times V\) with \(U \in \mathcal{U}\) and \(V \in \mathcal{V}\).

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- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{top}\)
Examples of Cartesian products

- The product of \((X, \Omega_X)\) and \((Y, \Omega_Y)\) in \(\text{Top}\) is given by \(X \times Y\) together with unions of subsets of the form \(U \times V\) with \(U \in \Omega_X\) and \(V \in \Omega_Y\). It is the least topology making the projections continuous.

- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{Pos}\) is given by \(X \times Y\) and the partial order \(\sqsubseteq\) defined by \((x, y) \sqsubseteq (x', y')\) when \(x \sqsubseteq_X x'\) and \(y \sqsubseteq_Y y'\). It is the greatest partial order such that the projection are poset morphisms.

- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{PoSp}\) is given by \(X \times Y\) and the product order \(\sqsubseteq_X \times \sqsubseteq_Y\).

- The product of \((X, [\mathcal{U}]_\sim)\) and \((Y, [\mathcal{V}]_\sim)\) in \(\text{Lpo}\) is given by \(X \times Y\) together with the collection of ordered charts \(U \times V\) with \(U \in \mathcal{U}\) and \(V \in \mathcal{V}\).

- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{\text{emb}}\) does not exist.

- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{\text{ctr}}\) is given by \(X \times Y\) together with \(d((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}\).

- The product of \((X, d_X)\) and \((Y, d_Y)\) in \(\text{Met}_{\text{top}}\) can also be given by \(X \times Y\) together with the Euclidean product

\[
d((x, y), (x', y')) = \sqrt{d^2_X(x, x') + d^2_Y(y, y')}
\]
Examples of Cartesian products

- The product of \((X, \Omega_X)\) and \((Y, \Omega_Y)\) in \(\text{Top}\) is given by \(X \times Y\) together with unions of subsets of the form \(U \times V\) with \(U \in \Omega_X\) and \(V \in \Omega_Y\). It is the least topology making the projections continuous.

- The product of \((X, \sqsubseteq_X)\) and \((Y, \sqsubseteq_Y)\) in \(\text{Pos}\) is given by \(X \times Y\) and the partial order \(\sqsubseteq\) defined by \((x, y) \sqsubseteq (x', y')\) when \(x \sqsubseteq_X x'\) and \(y \sqsubseteq_Y y'\). It is the greatest partial order such that the projection are poset morphisms.

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- Categories of models of algebraic theories.
Infinite Cartesian product

The product of a family \((A_i)_{i \in I}\) of objects of a category \(C\), when it exists, is an object \(Y\) together with projections \(\pi_{A_j}: Q_i A_i \rightarrow A_j\) such that the next mapping is a bijection.

\[ C(X, Q_i A_i) \rightarrow C(X, A_i) \]

Infinite products of directed circles do not exist in \(Lpo\).
Infinite Cartesian product

The product of a family \((A_i)_{i \in \mathcal{I}}\) of objects of a category \(C\), when it exists, is an object

\[
\prod_{i} A_i
\]
Infinite Cartesian product

The product of a family \((A_i)_{i \in I}\) of objects of a category \(C\), when it exists, is an object

\[
\prod_i A_i
\]

together with projections

\[
\pi_{A_j} : \prod_i A_i \longrightarrow A_j
\]
Infinite Cartesian product

The product of a family \((A_i)_{i \in I}\) of objects of a category \(C\), when it exists, is an object

\[ \prod_{i} A_i \]

together with projections

\[ \pi_{A_j} : \prod_{i} A_i \longrightarrow A_j \]

such that the next mapping is a bijection.

\[ C(X, \prod_{i} A_i) \longrightarrow \prod_{i} C(X, A_i) \]

\[ h \quad \longmapsto \quad (\pi_{A_i} \circ h) \]
Infinite Cartesian product

The product of a family \((A_i)_{i \in I}\) of objects of a category \(C\), when it exists, is an object

\[
\prod_i A_i
\]

together with projections

\[
\pi_{A_j} : \prod_i A_i \to A_j
\]

such that the next mapping is a bijection.

\[
C(X, \prod_i A_i) \to \prod_i C(X, A_i)
\]

\[
h \mapsto (\pi_{A_i} \circ h)
\]

Infinite products of directed circle does not exist in \(Lpo\).
Turning discrete models into geometric ones
Canonical partition
Canonical partition

$$G : A \xrightarrow{\partial^+} V \xleftarrow{\partial^-}$$
Canonical partition

\[ G : A \xrightarrow{\partial^+} V \subseteq \delta^{-} G \subseteq V \sqcup A \times ]0,1[ \]

\[ |G| = V \sqcup A \times ]0,1[ \]
Canonical partition

\[ G : \partial^+ A \xrightarrow{\partial^-} V \]

\[ |G| = V \sqcup A \times [0,1[ \]

\[ |G_1| \times \cdots \times |G_n| = ( V_1 \sqcup A_1 \times ]0,1[ ) \times \cdots \times ( V_n \sqcup A_n \times ]0,1[ ) \]
Canonical partition

\[ G : A \xrightarrow{\partial^+} V \]

\[ \downarrow G \uparrow = V \sqcup A \times ]0, 1[ \]

\[ \downarrow |G_1| \times \cdots \times |G_n| \]

\[ = ( V_1 \sqcup A_1 \times ]0, 1[ ) \times \cdots \times ( V_n \sqcup A_n \times ]0, 1[ ) \]

\[ \downarrow |G_1| \times \cdots \times |G_n| \]

\[ = \bigsqcup \{ p \} \times ]0, 1[^{\dim(p_1, \ldots, p_n)} \]

where \( p = (p_1, \ldots, p_n) \), \( p_i \in V_i \sqcup A_i \), and \( \dim p = \# \{ i \in \{1, \ldots, n\} \mid p_i \in A_i \} \)
Canonical partition

\[ G : A \xrightarrow{\partial^+} V \quad \downarrow G = V \sqcup A \times ]0,1[ \]

\[ \downarrow G_1 \times \cdots \times \downarrow G_n = ( V_1 \sqcup A_1 \times ]0,1[ ) \times \cdots \times ( V_n \sqcup A_n \times ]0,1[ ) \]

\[ \downarrow G_1 \times \cdots \times \downarrow G_n = \bigsqcup_{\text{points } p \text{ of } G_1, \ldots, G_n} \{ p \} \times ]0,1[^{\dim(p_1, \ldots, p_n)} \]

where \( p = (p_1, \ldots, p_n) \), \( p_i \in V_i \sqcup A_i \), and \( \dim p = \# \{ i \in \{1, \ldots, n\} \mid p_i \in A_i \} \)

\[ B_p = \{ p \} \times ]0,1[^{\dim(p_1, \ldots, p_n)} \] is called a canonical block
**Canonical partition**

\[ G : A \xrightarrow{\partial^+} V \quad \downarrow G \downarrow = V \sqcup A \times ]0, 1[ \]

\[ \downarrow G_1 \downarrow \times \cdots \times \downarrow G_n \downarrow = ( V_1 \sqcup A_1 \times ]0, 1[ ) \times \cdots \times ( V_n \sqcup A_n \times ]0, 1[ ) \]

\[ \downarrow G_1 \downarrow \times \cdots \times \downarrow G_n \downarrow = \bigsqcup_{\text{points } p \text{ of } G_1, \ldots, G_n} \{ p \} \times ]0, 1[^{\dim(p_1, \ldots, p_n)} \]

where \( p = (p_1, \ldots, p_n), p_i \in V_i \sqcup A_i, \) and \( \dim p = \#\{ i \in \{1, \ldots, n\} \mid p_i \in A_i \} \)

\( B_p = \{ p \} \times ]0, 1[^{\dim(p_1, \ldots, p_n)} \) is called a canonical block.

The collection of canonical blocks forms the canonical partition of \( \downarrow G_1 \downarrow \times \cdots \times \downarrow G_n \downarrow. \)
The geometric model of a conservative program

The geometric model of a conservative program $\Pi = (G_1, \ldots, G_n)$ is the disjoint union of canonical blocks $G$ for forbidden points $p$ of $(G_1, \ldots, G_n)$. The geometric model of $\Pi$ is the locally ordered metric space $\uparrow G_1 \downarrow \times \cdots \times \uparrow G_n \downarrow \{\text{forbidden region}\}$, the distance being given by $d(p, p') = \max_{i \in \{1, \ldots, n\}} d_{\uparrow G_i \downarrow}(p_i, p'_i)$ in accordance with the fact that the execution time of the simultaneous execution of many processes is the longest execution time among that of the processes considered individually.
The geometric model of a conservative program

The forbidden region of a conservative program $\Pi = (G_1, \ldots, G_n)$ is the disjoint union of canonical blocks

$$\bigsqcup B_p$$

forbidden points $p$
of $(G_1, \ldots, G_n)$
The geometric model of a conservative program

The forbidden region of a conservative program $\Pi = (G_1, \ldots, G_n)$ is the disjoint union of canonical blocks

$$\bigcup_{\text{forbidden points } p \text{ of } (G_1, \ldots, G_n)} B_p$$

The geometric model of $\Pi$ is the locally ordered metric space

$$\sqcup_{1 \leq i \leq m} |G_i| \times \cdots \times \sqcup_{1 \leq i \leq n} |G_n| \setminus \{\text{forbidden region}\}$$
The geometric model of a conservative program

The forbidden region of a conservative program \( \Pi = (G_1, \ldots, G_n) \) is the disjoint union of canonical blocks

\[
\bigcup \{ B_p \} \quad \text{forbidden points } p \text{ of } (G_1, \ldots, G_n)
\]

The geometric model of \( \Pi \) is the locally ordered metric space

\[
|G_1| \times \cdots \times |G_n| \setminus \{ \text{forbidden region} \}
\]

the distance being given by

\[
d(p, p') = \max \{ d_{|G_i|}(p_i, p'_i) \mid i \in \{1, \ldots, n\} \}
\]

in accordance with the fact that the execution time of the simultaneous execution of many processes is the longest execution time among that of the processes considered individually.
Gallery of examples
From discrete to continuous

sem: 1 a    sync: 1 b
From discrete to continuous

sem: 1 a sync: 1 b
From discrete to continuous

\text{sem: 1 a} \quad \text{sync: 1 b}
From discrete to continuous

sem: 1 a  sync: 1 b
From discrete to continuous

sem: 1 a  sync: 1 b
From discrete to continuous

sem: 1 a  sync: 1 b
From discrete to continuous

sem: 1a  sync: 1b
From discrete to continuous

sem: 1 a  sync: 1 b

V(a)
\[ x := y \]
P(a)
W(b)
\[ z := 1 \]
From discrete to continuous

\[ \text{sem: 1 a sync: 1 b} \]
Square
Square

```
sem 1 a
proc: p = P(a); V(a)
init: 2p
```
Square

sem 1 a
proc: p = P(a);V(a)
init: 2p
Square

sem 1 a
proc: p = P(a); V(a)
init: 2p
Square

sem 1 a
proc:  \( p = P(a); V(a) \)
init:  2p
Square

sem 1 a
proc: \( p = P(a); V(a) \)
init: \( 2p \)
Square

sem 1 a
proc: p = P(a); V(a)
init: 2p
Swiss Cross
Swiss Cross

```plaintext
sem 1 a b
proc:
p = P(a); P(b); V(b); V(a)
q = P(b); P(a); V(a); V(b)
init:  p q
```
Swiss Cross

sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init:  p q
Swiss Cross

sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init:  p q
Swiss Cross

sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init: p q
Swiss Cross

sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init: p q
Swiss Cross

sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init: p q
Swiss Cross

sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init: p q
Swiss Cross

sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init: p q
Swiss Cross

sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init: p q
Binary synchronization
Binary synchronization

sync 1 a
proc:  p = W(a)
init:  2p
Binary synchronization

sync 1 a
proc: p = W(a)
init: 2p
Binary synchronization

\[ \text{sync 1 a} \]
\[ \text{proc: } \ p = W(a) \]
\[ \text{init: } \ 2p \]
Binary synchronization

sync 1 a
proc:  p = W(a)
init:  2p
Binary synchronization

sync 1 a
proc:  p = W(a)
init:  2p
Binary synchronization

sync 1 a
proc: \( p = W(a) \)
init: 2p
Binary synchronization

sync 1 a
proc: p = W(a)
init: 2p
Producer/Consumer

nonlooping
Producer/Consumer
nonlooping

sync 1 a
proc:
  p = x:=x+1 ; W(a)
  c = W(a)  ; x:=x-1
init:  p c
Producer/Consumer
nonlooping

\[
\text{sync 1 a}
\]
\[
\text{proc:}
\]
\[
\begin{align*}
\text{p} &= \mathbf{x} := \mathbf{x} + 1 \ ; \ W(a) \\
\text{c} &= W(a) \ ; \ \mathbf{x} := \mathbf{x} - 1
\end{align*}
\]
\[
\text{init: p c}
\]
Producer/Consumer

nonlooping

sync 1 a
proc:
  p = x:=x+1 ; W(a)
  c = W(a) ; x:=x-1
init: p c
Producer/Consumer

nonlooping

sync 1 a
proc:
  p = x:=x+1 ; W(a)
  c = W(a) ; x:=x-1
init:  p c
Producer/Consumer

nonlooping

sync 1 a
proc:
  p = x:=x+1 ; W(a)
  c = W(a) ; x:=x-1
init:  p c
Producer/Consumer

nonlooping

sync 1 a

proc:
    p = x:=x+1 ; W(a)
    c = W(a) ; x:=x-1

init: p c
Producer/Consumer
nonlooping

sync 1 a
proc:
    p = x:=x+1 ; W(a)
    c = W(a) ; x:=x-1
init:  p c
Producer/Consumer

nonlooping

sync 1 a
proc:
  p = x:=x+1 ; W(a)
  c = W(a) ; x:=x-1
init: p c
Producer/Consumer

looping

\[ \begin{align*}
\text{proc:} & \quad X \ p = x := x + 1 \ ; \ W(a) \ ; \ W(b) \ ; \ J(p) \\
& \quad X \ c = W(a) \ ; \ x := x - 1 \ ; \ W(b) \ ; \ J(c) \\
\text{init:} & \quad p \ c
\end{align*} \]
Producer/Consumer
looping

```
sync 1 a b
proc:
  p = x:=x+1 ; W(a) ; W(b) ; J(p)
  c = W(a) ; x:=x-1 ; W(b) ; J(c)
init:  p c
```
Producer/Consumer
looping

sync 1 a b
proc:
  p = x:=x+1 ; W(a) ; W(b) ; J(p)
  c = W(a) ; x:=x-1 ; W(b) ; J(c)
init:  p c
Producer/Consumer

looping

\[
\text{sync 1 a b}
\]
\[
\text{proc:}
\]
\[
\quad p = \text{x:=x+1 ; W(a) ; W(b) ; J(p)}
\]
\[
\quad c = \text{W(a) ; x:=x-1 ; W(b) ; J(c)}
\]
\[
\text{init: p c}
\]
Producer/Consumer

looping

```
sync 1 a b
proc:
  p = x:=x+1 ; W(a) ; W(b) ; J(p)
  c = W(a)  ; x:=x-1 ; W(b) ; J(c)
init:  p c
```
Producer/Consumer
looping

```
sync 1 a b
proc:
p = x:=x+1 ; W(a) ; W(b) ; J(p)
c = W(a) ; x:=x-1 ; W(b) ; J(c)
init:  p c
```
Producer/Consumer

looping

\[
\begin{align*}
\text{sync } & 1 \ a \ b \\
\text{proc: } & \\
& p = \text{x:=x+1} ; \ W(a) \ ; \ W(b) \ ; \ J(p) \\
& c = W(a) \ ; \ x:=x-1 \ ; \ W(b) \ ; \ J(c) \\
\text{init: } & p \ c
\end{align*}
\]
Producer/Consumer
looping

sync 1 a b
proc:
  p = x:=x+1 ; W(a) ; W(b) ; J(p)
  c = W(a) ; x:=x-1 ; W(b) ; J(c)
init:  p c
Producer/Consumer

looping

```
sync 1 a b
proc:
p = x:=x+1 ; W(a) ; W(b) ; J(p)
c = W(a) ; x:=x-1 ; W(b) ; J(c)
init: p c
```
Producer/Consumer

looping

\[
\text{sync 1 a b}
\]

\[
\text{proc:}
\]

\[
\text{p = x:=x+1 ; W(a) ; W(b) ; J(p)}
\]

\[
\text{c = W(a) ; x:=x-1 ; W(b) ; J(c)}
\]

\[
\text{init: p c}
\]
3D Swiss Cross (tetrahemihexacron) and floating cube
The Lipski algorithm

sem 1: u v w x y z

proc:

\[ p = \text{P}(x);\text{P}(y);\text{P}(z);\text{V}(x);\text{P}(w);\text{V}(z);\text{V}(y);\text{V}(w) \]

\[ q = \text{P}(u);\text{P}(v);\text{P}(x);\text{V}(u);\text{P}(z);\text{V}(v);\text{V}(x);\text{V}(z) \]

\[ r = \text{P}(y);\text{P}(w);\text{V}(y);\text{P}(u);\text{V}(w);\text{P}(v);\text{V}(u);\text{V}(v) \]

init: p q r
Geometric vs Discrete
Justifying the definition of discrete directed paths

Let $B_p$ and $B_p'$ be canonical blocks. If there exists a directed path starting in $B_p$, ending in $B_p'$, and whose image is contained in $B_p \cup B_p'$, then one of the following facts is satisfied:

- for all $i \in \{1, \ldots, n\}$, $p_i = p_i'$ or $p_i$ is the source of the arrow $p_i'$,
- for all $i \in \{1, \ldots, n\}$, $p_i = p_i'$ or $p_i'$ is the target of the arrow $p_i$. 
Justifying the definition of discrete directed paths

Let $B_p$ and $B_{p'}$ be canonical blocks.
Let $B_p$ and $B_{p'}$ be canonical blocks.

If there exists a directed path starting in $B_p$, ending in $B_{p'}$, and whose image is contained in $B_p \cup B_{p'}$, then one of the following facts is satisfied:
Justifying de definition of discrete directed paths

Let $B_p$ and $B_{p'}$ be canonical blocks.

If there exists a directed path starting in $B_p$, ending in $B_{p'}$, and whose image is contained in $B_p \cup B_{p'}$ then one of the following facts is satisfied:

- for all $i \in \{1, \ldots, n\}$, $p_i = p'_i$ or $p_i$ is the source of the arrow $p'_i$,
Let $B_p$ and $B_{p'}$ be canonical blocks.

If there exists a directed path starting in $B_p$, ending in $B_{p'}$, and whose image is contained in $B_p \cup B_{p'}$ then one of the following facts is satisfied:

- for all $i \in \{1, \ldots, n\}$, $p_i = p'_i$ or $p_i$ is the source of the arrow $p'_i$, or
- for all $i \in \{1, \ldots, n\}$, $p_i = p'_i$ or $p'_i$ is the target of the arrow $p_i$. 

Justifying the definition of discrete directed paths
Discretization and lifting

- Given a directed path $\gamma$ on the local pospace $\uparrow G_1 \downarrow \times \cdots \times \uparrow G_n \downarrow$ we have a finite partition $I_0 < \cdots < I_N$ of $\text{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point $p_k$ such that $\gamma(I_k) \subseteq B_{p_k}$.

- The sequence $p_0, \ldots, p_N$ is a directed path on $(G_1, \ldots, G_n)$, it is called the discretization of $\gamma$ and denoted by $D(\gamma)$.

- Given a directed path $\delta$ on $(G_1, \ldots, G_n)$ there exists a directed path $\gamma$ on $\uparrow G_1 \downarrow \times \cdots \times \uparrow G_n \downarrow$ whose discretization is $\delta$, such a directed path $\gamma$ is said to be a lifting of $\delta$. 

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Discretization and lifting

- Given a directed path $\gamma$ on the local pospace $|G_1| \times \cdots \times |G_n|$ we have a finite partition $I_0 < \cdots < I_N$ of $\text{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point $p^k$ such that $\gamma(I_k) \subseteq B_{p^k}$. 

- The sequence $p^0, \ldots, p^N$ is a directed path on $(G_1, \ldots, G_n)$, it is called the discretization of $\gamma$ and denoted by $D(\gamma)$.

- Given a directed path $\delta$ on $(G_1, \ldots, G_n)$ there exists a directed path $\gamma$ on $|G_1| \times \cdots \times |G_n|$ whose discretization is $\delta$, such a directed path $\gamma$ is said to be a lifting of $\delta$. 

Discretization and lifting

- Given a directed path $\gamma$ on the local pospace $\rightarrow G_1 \times \cdots \times \rightarrow G_n$, we have a finite partition $I_0 < \cdots < I_N$ of $\text{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point $p^k$ such that $\gamma(I_k) \subseteq B_{p^k}$.

- The sequence $p^0, \ldots, p^N$ is a directed path on $(G_1, \ldots, G_n)$, it is called the discretization of $\gamma$ and denoted by $D(\gamma)$. 
Discretization and lifting

- Given a directed path $\gamma$ on the local pospace $\downarrow G_1 \times \cdots \times \downarrow G_n$ we have a finite partition $I_0 < \cdots < I_N$ of $\text{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point $p^k$ such that $\gamma(I_k) \subseteq B_{p^k}$.

- The sequence $p^0, \ldots, p^N$ is a directed path on $(G_1, \ldots, G_n)$, it is called the discretization of $\gamma$ and denoted by $D(\gamma)$.

- Given a directed path $\delta$ on $(G_1, \ldots, G_n)$ there exists a directed path $\gamma$ on $\downarrow G_1 \times \cdots \times \downarrow G_n$ whose discretization is $\delta$, such a directed path $\gamma$ is said to be a lifting of $\delta$. 
Example of discretization
Admissible directed paths and execution traces on $\prod_{i=1}^{n} G_i$
Admissible directed paths and execution traces

on $G_1 \times \cdots \times G_n$

The sequence of multi-instructions of a directed path $\gamma$ on $G_1 \times \cdots \times G_n$ is that of its discretization of $D(\gamma)$. 
Admissible directed paths and execution traces

on $|G_1| \times \cdots \times |G_n|$
Admissible directed paths and execution traces

on \( |G_1| \times \cdots \times |G_n| \)

The sequence of multi-instructions of a directed path \( \gamma \) on \( |G_1| \times \cdots \times |G_n| \) is that of its discretization of \( D(\gamma) \).

A directed path on \( |G_1| \times \cdots \times |G_n| \) is admissible (resp. an execution trace) iff so is its discretization.

The action of a directed path \( \gamma \) on \( |G_1| \times \cdots \times |G_n| \) on the right of a state \( \sigma \) is that of its discretization of \( D(\gamma) \).
Example

```plaintext
var x = 0
var y = 0
var z = 0
sync 1 b
sem 1 a

proc p = y:=0 ; W(b) ; P(a) ; x:=z ; V(a)
proc q = z:=1 ; W(b) ; P(a) ; x:=y ; V(a)

init p q
```
Discretization of an execution trace

sem: 1 a  sync: 1 b
Discretization of an execution trace

sem: 1 a  sync: 1 b
Discretization of an execution trace

sem: 1 a  sync: 1 b
If the program under consideration is conservative, then we have the potential function $F: \prod_{i=1}^{n} G_i \rightarrow \{\text{multisets over } \{1, \ldots, n\}\}$.

The function $F$ is constant on each canonical block $B_p$, its value is given by $\tilde{F}(p)$ where $\tilde{F}$ denotes the "discrete" potential function.
Potential function on $|G_1| \times \cdots \times |G_n|$

If the program under consideration is conservative, then we have the potential function

$$F : |G_1| \times \cdots \times |G_n| \times S \rightarrow \{\text{multisets over } \{1, \ldots, n\}\}$$
Potential function on $|G_1| \times \cdots \times |G_n|$

If the program under consideration is conservative, then we have the potential function

$$F : |G_1| \times \cdots \times |G_n| \times S \rightarrow \{\text{multisets over } \{1, \ldots, n\}\}$$

The function $F$ is constant on each canonical block $B_p$. 
Potential function on $|G_1| \times \cdots \times |G_n|$

If the program under consideration is conservative, then we have the potential function

$$F : |G_1| \times \cdots \times |G_n| \times S \to \{\text{multisets over } \{1, \ldots, n\}\}$$

The function $F$ is constant on each canonical block $B_p$, its value is given by $\tilde{F}(p)$ where $\tilde{F}$ denotes the “discrete" potential function.
Geometric models are sound and complete

- Any directed path on a continuous model is admissible.
- Conversely, for each admissible path on a continuous model which meets a forbidden point, there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.
Geometric models are sound and complete

- Any directed path on a continuous model is admissible.
Geometric models are sound and complete

- Any directed path on a continuous model is admissible.
- Conversely, for each admissible path on a continuous model which meets a forbidden point, there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.
Directed paths on the geometric model are admissible

sem: 1 a    sync: 1 b
Directed paths on the geometric model are admissible

\[ \text{sem: 1 a sync: 1 b} \]
Continuous replacement

sem: 1 a  sync: 1 b

\[ x := 0 \]  \[ x := 1 \]
Continuous replacement

sem: 1 a    sync: 1 b
Continuous replacement

sem: 1 a sync: 1 b
Continuous replacement

\[ \text{sem: 1 a sync: 1 b} \]
Continuous replacement

sem: 1 a  
 sync: 1 b 

x := 1
The motivating theorem
Trade off

More mathematics for more properties?
Trade off
More mathematics for more properties?

- Both discrete and geometric models are sound and complete.
Trade off
More mathematics for more properties?

- Both discrete and geometric models are sound and complete.
- The continuous models satisfy extra properties that are “naturally” expressed in terms of metrics.
Uniform distance between directed paths
Uniform distance between directed paths

Given a compact Hausdorff space $K$ and a metric space $(X, d_X)$, the set of continuous maps from $K$ to $X$ can be equipped with the uniform distance

$$d(f, g) = \max\{d_X(f(k), g(k)) \mid k \in K\}.$$
Uniform distance between directed paths

Given a compact Hausdorff space $K$ and a metric space $(X, d_X)$, the set of continuous maps from $K$ to $X$ can be equipped with the uniform distance

$$d(f, g) = \max\{d_X(f(k), g(k)) \mid k \in K\}.$$ 

We consider the case where $K = [0, r]$ is the domain of definition of a directed path and $(X, d_X)$ is the geometric model of a conservative program.
The motivating theorem
The main theorem

Let $B_p$ and $B_{p'}$ be canonical blocks of the geometric model $X$ of a conservative program.
The main theorem

Let $B_p$ and $B_{p'}$ be canonical blocks of the geometric model $X$ of a conservative program.

Let $dX^{[0..r]}(B_p, B_{p'})$ be the set of directed paths on $X$ whose sources and targets lie in $B_p$ and $B_{p'}$ respectively.
The main theorem

Let $B_p$ and $B_{p'}$ be canonical blocks of the geometric model $X$ of a conservative program.

Let $dX^{[0,r]}(B_p, B_{p'})$ be the set of directed paths on $X$ whose sources and targets lie in $B_p$ and $B_{p'}$ respectively.

Let $\gamma$ be an element of $dX^{[0,r]}(B_p, B_{p'})$. 
The main theorem

Let $B_p$ and $B_{p'}$ be canonical blocks of the geometric model $X$ of a conservative program.

Let $dX^{[0,r]}(B_p, B_{p'})$ be the set of directed paths on $X$ whose sources and targets lie in $B_p$ and $B_{p'}$ respectively.

Let $\gamma$ be an element of $dX^{[0,r]}(B_p, B_{p'})$.

There exists an open ball $\Omega$ of $dX^{[0,r]}(B_p, B_{p'})$, centred in $\gamma$, such that all the elements of $\Omega$ induce the same action on valuations. Moreover, if $\gamma$ is an execution trace, then so are all the elements of $\Omega$. 
Illustration
HOMOTOPY OF PATHS
The undirected case
Homotopy of paths

The undirected case

Let $\gamma$ and $\delta$ be two paths on $X$ defined over the segment $[0, r]$. A homotopy from $\gamma$ to $\delta$ is a continuous map $h$ from $[0, r] \times [0, q]$ to $X$ such that:

1. The mappings $h(0, -)$ : $[0, q] \rightarrow X$ and $h(r, -)$ : $[0, q] \rightarrow X$ are constant.
2. The mappings $h(-, 0)$ : $[0, r] \rightarrow X$ and $h(-, q)$ : $[0, r] \rightarrow X$ are $\gamma$ and $\delta$.

As a consequence we have $\gamma(0) = \delta(0)$ and $\gamma(r) = \delta(r)$.
Homotopy of paths

Let $\gamma$ and $\delta$ be two paths on $X$ defined over the segment $[0, r]$. 
Homotopy of paths

Let $\gamma$ and $\delta$ be two paths on $X$ defined over the segment $[0, r]$.

A homotopy from $\gamma$ to $\delta$ is a continuous map $h$ from $[0, r] \times [0, q]$ to $X$ such that

- The mappings $h(0, -) : [0, q] \to X$ and $h(r, -) : [0, q] \to X$ are constant.
- The mappings $h(-, 0) : [0, r] \to X$ and $h(-, q) : [0, r] \to X$ are $\gamma$ and $\delta$ as a consequence we have $\gamma(0) = \delta(0)$ and $\gamma(r) = \delta(r)$. 
Homotopy of paths

Let $\gamma$ and $\delta$ be two paths on $X$ defined over the segment $[0, r]$

A homotopy from $\gamma$ to $\delta$ is a continuous map $h$ from $[0, r] \times [0, q]$ to $X$ such that

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Homotopy of paths

Let $\gamma$ and $\delta$ be two paths on $X$ defined over the segment $[0, r]$

A homotopy from $\gamma$ to $\delta$ is a continuous map $h$ from $[0, r] \times [0, q]$ to $X$ such that

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Homotopy of paths

Let $\gamma$ and $\delta$ be two paths on $X$ defined over the segment $[0, r]$.

A homotopy from $\gamma$ to $\delta$ is a continuous map $h$ from $[0, r] \times [0, q]$ to $X$ such that

- The mappings $h(0, -) : [0, q] \to X$ and $h(r, -) : [0, q] \to X$ are constant

- The mappings $h(-, 0) : [0, r] \to X$ and $h(-, q) : [0, r] \to X$ are $\gamma$ and $\delta$

As a consequence we have $\gamma(0) = \delta(0)$ and $\gamma(r) = \delta(r)$. 
Uniform distance and Curryfication

Suppose that $X$ is a metric space. For all compact Hausdorff space $K$, the homset $\text{Top}(K, X)$ with the (topology induced by the) uniform distance is denoted by $X^K$. The Curryfication $\hat{\cdot}$ induces a homeomorphism from $X^{[0,r]} \times [0,q]$ to $X^{[0,q]}(h: [0,r] \times [0,q] \to X) \to (\hat{h}: [0,q] \to X^{[0,r]})$. 
Uniform distance and Curryfication

Suppose that $X$ is a metric space.
Uniform distance and Curryfication

Suppose that $X$ is a metric space.

For all compact Hausdorff space $K$, the homset $\text{Top}(K, X)$ with the (topology induced by the) uniform distance is denoted by $X^K$. 
Suppose that $X$ is a metric space.

For all compact Hausdorff space $K$, the homset $\text{Top}(K, X)$ with the (topology induced by the) uniform distance is denoted by $X^K$.

The Curryfication ($\hat{\cdot}$) induces a homeomorphism from $X^{[0,r] \times [0,q]}$ to $(X^{[0,r]})^{[0,q]}$

$$(h : [0, r] \times [0, q] \to X) \to (\hat{h} : [0, q] \to X^{[0,r]})$$
The two faces of homotopies

The undirected case

Homotopy of paths
The two faces of homotopies

$h$ is a continuous map from $[0, r] \times [0, q]$ to $X$ i.e. $h \in \mathcal{Top}[[0, r] \times [0, q], X]$
The two faces of homotopies

but is also a path from $\gamma$ to $\delta$ in the space $X^{[0,r]}$ i.e. $h \in \mathcal{Top}[[0,q], X^{[0,r]}]$
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We introduce the following notation
Concatenation of homotopies

vertical composition

Let \( g : [0, r] \times [0, q'] \to X \) and \( h : [0, r] \times [0, q] \to X \) be homotopies from \( \gamma \) to \( \xi \) and from \( \xi \) to \( \delta \). The mapping \( h \ast g : [0, r] \times [0, q + q'] \to X \) defined by

\[
h \ast g(t, s) =
\begin{cases}
g(t, s) & \text{if } 0 \leq s \leq q \\
h(t, s - q) & \text{if } q \leq s \leq q + q'
\end{cases}
\]

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The directed case
Directed homotopy on a locally ordered space

Let $\gamma, \delta \in \mathcal{L}_{po}([0, q], X)$ such that $\partial^- \gamma = \partial^- \delta$ and $\partial^+ \gamma = \partial^+ \delta$.

- A directed homotopy from $\gamma$ to $\delta$ is a local pospace morphism $h : [0, r] \times [0, q] \to X$ whose underlying map $U(h)$ is a homotopy from $U(\gamma)$ to $U(\delta)$.

- An anti-directed homotopy from $\gamma$ to $\delta$ is a homotopy of paths $h : [0, r] \times [0, q] \to X$ such that $(t, s) \mapsto h(t, q - s)$ is a directed homotopy from $\delta$ to $\gamma$.

- An elementary homotopy between $\gamma$ to $\delta$ is a homotopy of paths $h : [0, r] \times [0, q] \to X$ obtained as a finite concatenation of directed homotopies and anti-directed homotopies.

- A weakly directed homotopy from $\gamma$ to $\delta$ is a homotopy of paths $h : [0, r] \times [0, q] \to X$ whose intermediate paths $h(t, s)$, for $s \in [0, q]$, are directed.

- Any elementary homotopy is a weakly directed homotopy. The converse is false.

- Each of the preceding class of homotopies is stable under concatenation.
Directed homotopy on a locally ordered space

Let $\gamma, \delta \in Lpo([0, r], X)$ such that $\partial^- \gamma = \partial^+ \delta$ and $\partial^+ \gamma = \partial^- \delta$. 
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Homotopy of paths

Homotopy and dihomotopy relations

Two paths $\gamma$ and $\gamma'$ are said to be homotopic when there exists a homotopy between them. We have the equivalence relation $\sim_h$ between paths on a topological space. They are said to be dihomotopic when there exists an elementary homotopy between them. We have the equivalence relation $\sim_d$ between directed paths on a locally ordered space. They are said to be weakly dihomotopic when there exists a weakly directed homotopy between them. We have the equivalence relation $\sim_w$ between directed paths on a locally ordered space.
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Homotopy of paths

The directed case

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Reparametrization

An increasing and surjective map \( \theta : [0, r] \to [0, r] \) is called a reparametrization.

The mapping \( h : (t, s) \in [0, r] \times [0, 1] \mapsto \theta(t) + s \cdot (\max(t, \theta(t)) - \theta(t)) \in [0, r] \) is a directed homotopy from \( \theta \) to \( \max(\text{id}_{[0, r]}, \theta) \).

If \( \gamma : [0, r] \to X \) is a directed path on the local pospace \( X \), then \( \gamma \circ h \) is a directed homotopy from \( \gamma \circ \theta \) to \( \gamma \circ \max(\text{id}_{[0, r]}, \theta) \).

Therefore \( \gamma \) and \( \gamma \circ \theta \) are dihomotopic.
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Images of directed paths on a pospace

Theorem
The image of a nonconstant directed path on a pospace is isomorphic to $[0, 1]$.

Corollary
Two directed paths on a pospace having the same image are dihomotopic.

Proof:
Suppose that $\text{im}(\gamma) = \text{im}(\gamma')$. 

$\phi: [0, r] \to \text{im}(\gamma)$ a pospace isomorphism.

$\phi^{-1} \circ \gamma$ and $\phi^{-1} \circ \gamma'$ are reparametrization.

We have $h$ an elementary homotopy from $\phi^{-1} \circ \gamma$ to $\phi^{-1} \circ \gamma'$.

Hence $\phi \circ h$ is an elementary homotopy from $\gamma$ and $\gamma'$. 
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Hence \( \phi \circ h \) is an elementary homotopy from \( \gamma \) and \( \gamma' \).
Relation to geometric models
Main theorem

Two weakly dihomotopic paths on the geometric model of a conservative program induce the same action on valuations. Moreover, if one of them is an execution trace, then so is the other.
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Two weakly dihomotopic paths on the geometric model of a conservative program induce the same action on valuations. Moreover, if one of them is an execution trace, then so is the other.
Weakly directed homotopy

sem: 1 a sync: 1 b
Weakly directed homotopy

sem: 1 a  sync: 1 b

\[ \begin{align*}
\text{z} &:= 1 \\
\text{W(b)} & \quad \text{to} \\
\text{P(a)} & \quad \text{to} \\
\text{V(a)} &
\end{align*} \]
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\begin{align*}
&z := 1 \\
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&x := y
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\[ y := 0 \]
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![Diagram showing weakly directed homotopy]

- \( z := 1 \)
- \( x := y \)
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\[ \times \]
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Proof

By a standard result from general topology, the Curryfication of $\hat{h}^\hat{h}$:

$$s \in [0, q] \rightarrow (t \in [0, r] \rightarrow h(t, s) \in X)$$

is a continuous path on $dX[0, r](p, p')$.

The image of $\hat{h}$ is thus compact, so we cover it with open balls given by the main theorem of geometric models.

By the Lebesgue number theorem there exists a real number $\varepsilon > 0$ such that $|s - s'| \leq \varepsilon$ implies that $\hat{h}(s)$ and $\hat{h}(s')$ belong to the same open ball from the covering.

The conclusion follows considering the sequence $\hat{h}(0), \hat{h}(\varepsilon), \hat{h}(2\varepsilon), \hat{h}(3\varepsilon), \ldots, \hat{h}(n\varepsilon), \hat{h}(q)$ where $n$ is the greatest natural number such that $n\varepsilon \leq q$. 
Proof

By a standard result from general topology, the Curryfication of $h$

$$\hat{h} : s \in [0, q] \mapsto (t \in [0, r] \mapsto h(t, s) \in X)$$

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Homotopy of paths
Relation to geometric models

Programs with mutex only

Directed Homotopy in Non-Positively Curved Spaces, É. Goubault and S. Mimram, LMCS 2020
Let $X$ be the geometric model of a conservative program whose semaphores have arity 1 (mutex), then two directed paths on $X$ are dihomotopic if and only if they are homotopic.
INDEPENDENCE
Compatible programs
Two programs $P$ and $Q$ are said to be compatible when their initial valuations and their arity maps coincide on the intersection of their domains of definition. In that case we define the parallel composition $P|Q$. 
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Two programs $P$ and $Q$ are said to be compatible when their initial valuations and their arity maps coincide on the intersection of their domains of definition. In that case we define the parallel composition $P|Q$.

By extension we define the parallel composition of $P_1, \ldots, P_N$ when the programs are pairwise compatible.
Syntactical independence
Two programs are said to be syntactically independent when the set of resources they use are disjoint:

- they have no variables in common,
- they have no semaphores in common, and
- they have no barriers in common.

Syntactically independent programs are compatible.

Syntactical independence can be decided statically, it is compositional, but it is too restrictive.
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Syntactically independent programs are compatible.
Two programs are said to be syntactically independent when the set of resources they use are disjoint:

- they have no variables in common,
- they have no semaphores in common, and
- they have no barriers in common.

Syntactically independent programs are compatible.

Syntactical independence can be decided statically,
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Syntactically independent programs are compatible.

Syntactical independence can be decided statically, it is compositional, but it is too restrictive.
Model independence
Suppose the programs $P_1, \ldots, P_N$ are conservative. The programs $P_1, \ldots, P_N$ are said to be model independent when $J_{P_1 | \cdots | P_N}^K = J_{P_1}^K \times \cdots \times J_{P_N}^K$. Model independence can be decided statically.
Model Independence

Suppose the programs $P_1, \ldots, P_N$ are conservative.
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Model independence can be decided statically.
Observational independence
Compatible permutations

Assume we have a partition \( \{1, \ldots, n\} = S_1 \sqcup \cdots \sqcup S_N \) of the set \( \{1, \ldots, n\} \). Two multi-instructions \( \mu \) and \( \mu' \) (\( \text{dom}(\mu), \text{dom}(\mu') \subseteq \{1, \ldots, n\} \)) can be swapped when

\[
j \in \{1, \ldots, N\} \mid \text{dom}(\mu) \cap S_j \neq \emptyset \ \text{and} \ \text{dom}(\mu') \cap S_j \neq \emptyset \neq \emptyset,
\]

A permutation \( \pi \) of the set \( \{0, \ldots, q-1\} \) is said to be compatible with the sequence of multi-instructions \( \mu_0, \ldots, \mu_{q-1} \) when it is order preserving on all pairs \( \{k, k'\} \) such that \( \mu_k \) and \( \mu_{k'} \) cannot be swapped.

The permutation \( \pi \) is said to be compatible with the directed path \( \gamma \) when it is compatible with its associated sequence of multi-instructions.
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A permutation $\pi$ of the set $\{0, \ldots, q - 1\}$ is said to be compatible with the sequence of multi-instructions $\mu_0, \ldots, \mu_{q-1}$ when it is order preserving on all pairs $\{k, k'\}$ such that $\mu_k$ and $\mu_{k'}$ cannot be swapped.
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\gamma = \{ \mu_1, \mu_2, \mu_4, \mu_3, \mu_5 \} 
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Independence
Observational independence

Observational independence
related to partial order reduction (?)
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Suppose that the programs $P_1, \ldots, P_N$ are compatible and that $P_j$ has $n_j$ running processes.
Observational independence
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Suppose that the programs $P_1, \ldots, P_N$ are compatible and that $P_j$ has $n_j$ running processes.

The identifiers of the running processes of $P_1|\cdots|P_N$ are the elements of $\{1, \ldots, n\}$ with

$$n = \sum_{j=1}^{N} n_j,$$

and for $j \in \{1, \ldots, N\}$

$$s_j = \sum_{k=1}^{j} n_k.$$
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The programs $P_1, \ldots, P_N$ are said to be observationally independent when:

- for all execution traces $\gamma$
- for all permutations $\pi$ compatible with the sequence of multi-instructions $(\mu_0 \cdots \mu_{q-1})$ associated with $\gamma$, there exists an execution trace $\gamma'$ whose associated sequence of multi-instructions is $\pi \cdot (\mu_0 \cdots \mu_{q-1})$, which has the same action on the system state than $\gamma$, that is to say

$$\sigma \cdot (\mu_0 \cdots \mu_{q-1}) = \sigma \cdot (\mu_{\pi^{-1}(0)} \cdots \mu_{\pi^{-1}(q-1)}) .$$
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Observational independence cannot be decided statically, moreover it is too loose.
Comparison
Main theorem
Main theorem

syntactic independence
↓
model independence
↓
observational independence