DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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GEOMETRIC MODELS

Cartesian product

Cartesian product in $\mathcal{S}et$

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There exist two mappings π_A and π_B

$$\begin{aligned} \pi_A :\ A \times B &\longrightarrow A \\ (a, b) &\longmapsto a \end{aligned} \qquad \begin{aligned} \pi_B :\ A \times B &\longrightarrow B \\ (a, b) &\longmapsto b \end{aligned}$$

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such that for all sets X the following map is a bijection

$$\begin{aligned} \mathcal{S}et[X, A \times B] & \longrightarrow \mathcal{S}et[X, A] \times \mathcal{S}et[X, B] \\ h & \longmapsto (\ \pi_A \circ h\ ,\ \pi_B \circ h\ ) \end{aligned}$$

Cartesian product in a category $\mathcal{C}$

The object c is the Cartesian product (in $\mathcal{C}$) of a and b when there exist two morphisms $\pi_a : c \to a$ and $\pi_b : c \to b$ such that for all objects x of $\mathcal{C}$ the following map is a bijection

$$\begin{array}{ccc} \mathcal{C}[x,c] & \longrightarrow & \mathcal{C}[x,a] \times \mathcal{C}[x,b] \\ h & \longmapsto & (\ \pi_a \circ h\ ,\ \pi_b \circ h\ ) \end{array}$$

When such an object c exists we write $c = a \times b$

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The Cartesian product in $\mathcal{Grph}$ is deduced form the Cartesian product in $\mathcal{Set}$

Examples of Cartesian products

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- Categories of models of algebraic theories.

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Infinite products of directed circle does not exist in $\mathcal{L}po$.

Turning discrete models into geometric ones

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where $p = (p_1, \ldots, p_n)$, $p_i \in V_i \sqcup A_i$, and $\dim p = \#\big\{ i \in \{1, \ldots, n\} \mid p_i \in A_i \big\}$

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The collection of canonical blocks forms the canonical partition of $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$.

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the distance being given by

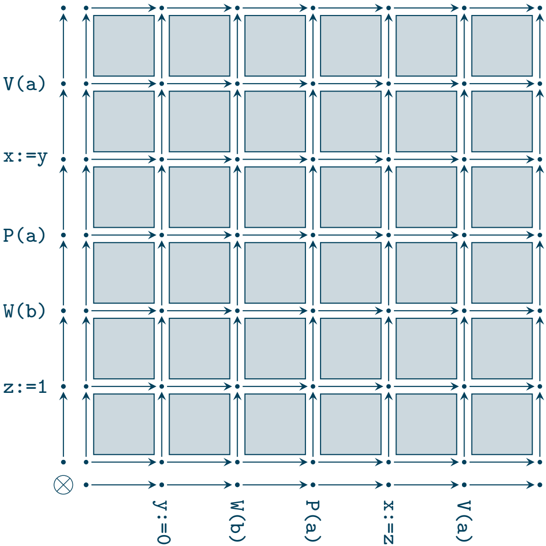
$$d(p, p') \quad = \quad \max\left\{ d_{\upharpoonleft G_i \downharpoonright}(p_i, p'_i) \mid i \in \{1, \dots, n\} \right\}$$

in accordance with the fact that the execution time of the simultaneous execution of many processes is the longest execution time among that of the processes considered individually.

Gallery of examples

From discrete to continuous

`sem: 1 a` `sync: 1 b`



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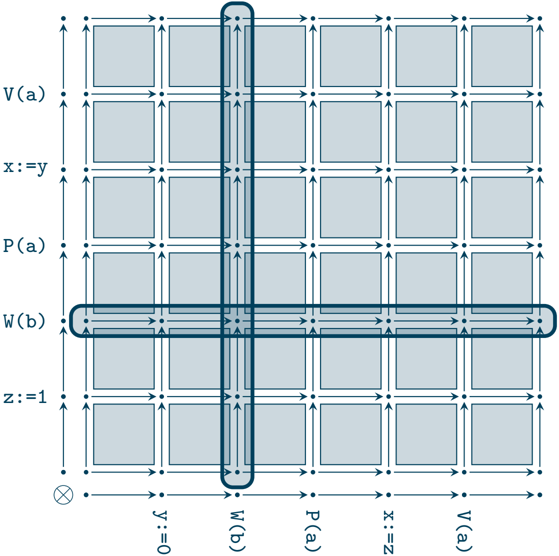

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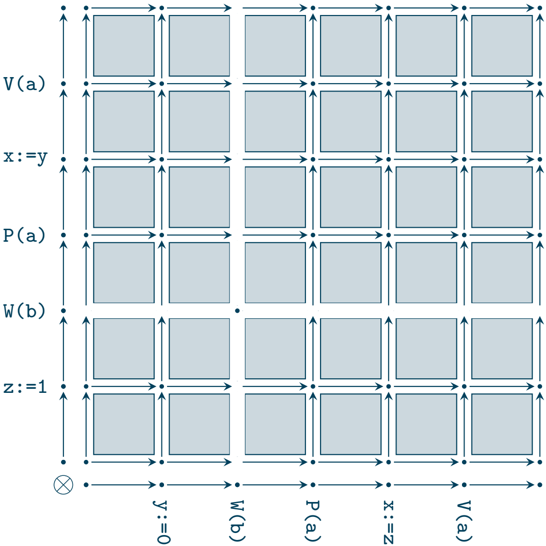
sem: 1 a sync: 1 b



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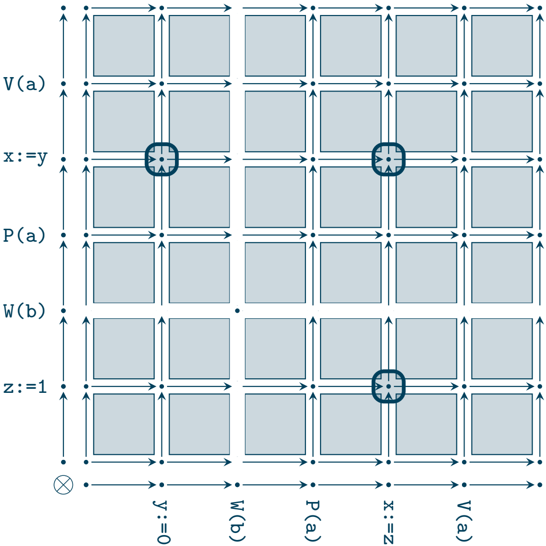

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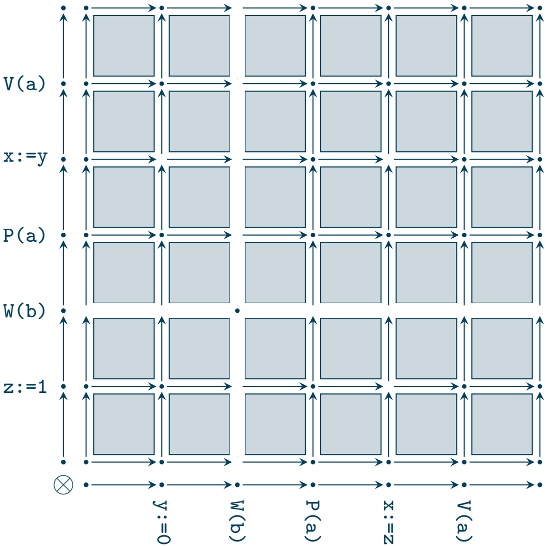
sem: 1 a sync: 1 b



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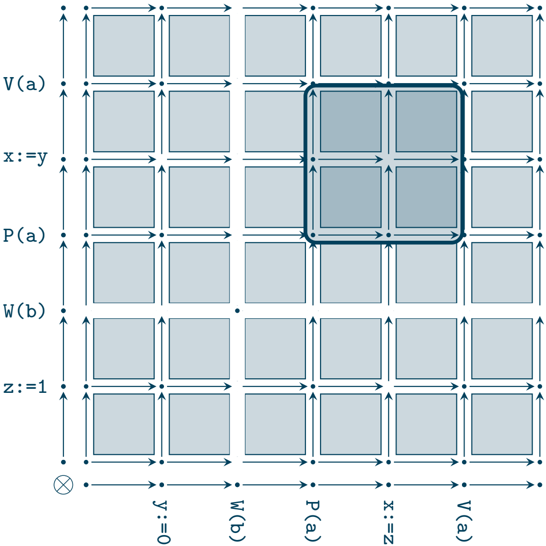
```
sem:  1 a       sync:  1 b
```



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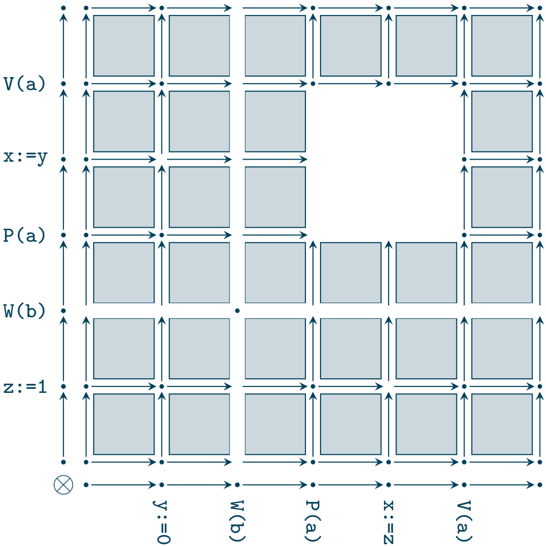
sem: 1 a sync: 1 b



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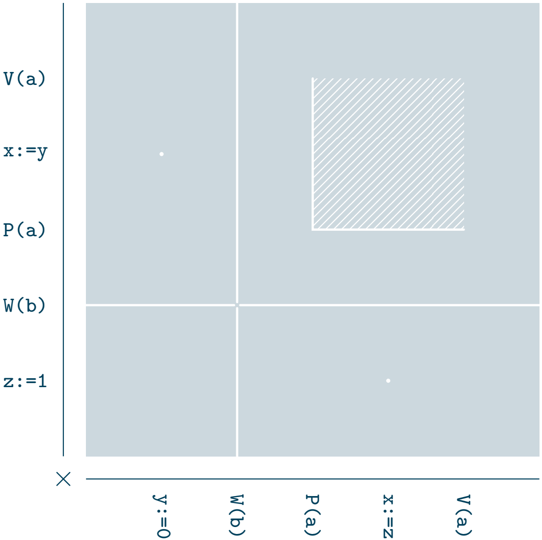
sem: 1 a   sync: 1 b



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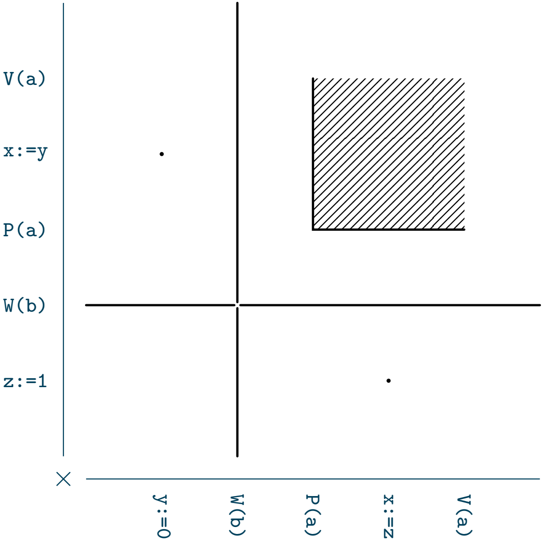

Square

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```
sem 1 a
proc:  p = P(a);V(a)
init:  2p
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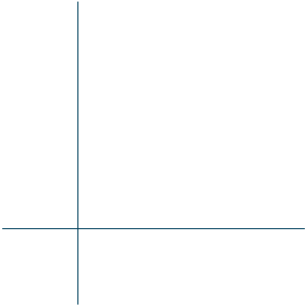

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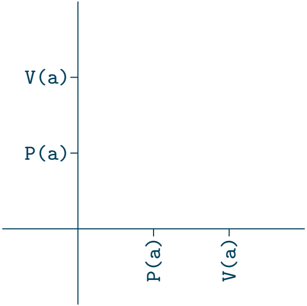

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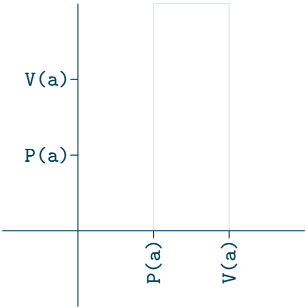

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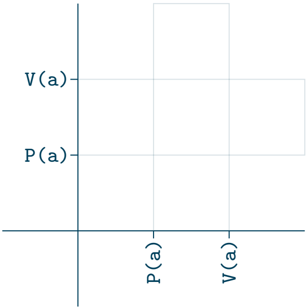

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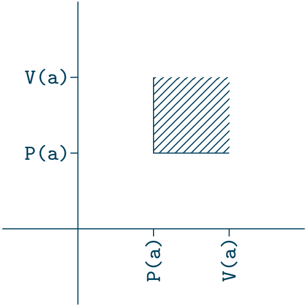

Swiss Cross

Swiss Cross

```
sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init:  p q
```

Swiss Cross

```
sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
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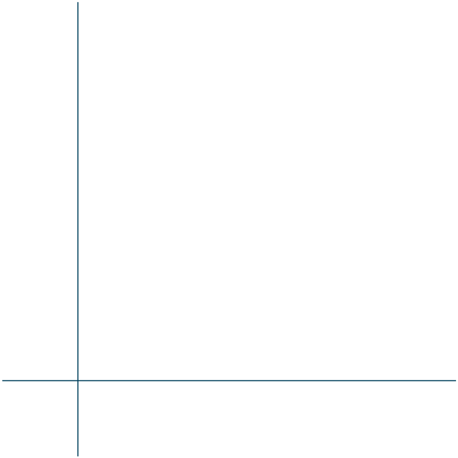
```



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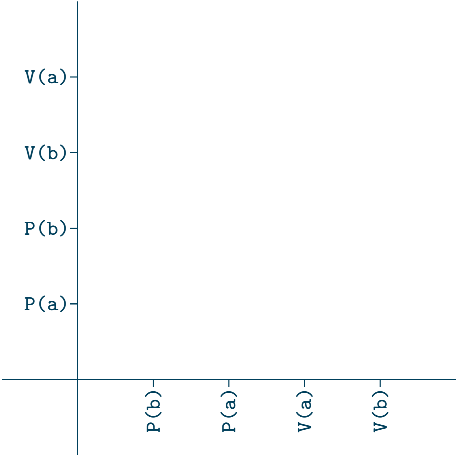

Swiss Cross

```
sem 1 a b
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p = P(a);P(b);V(b);V(a)
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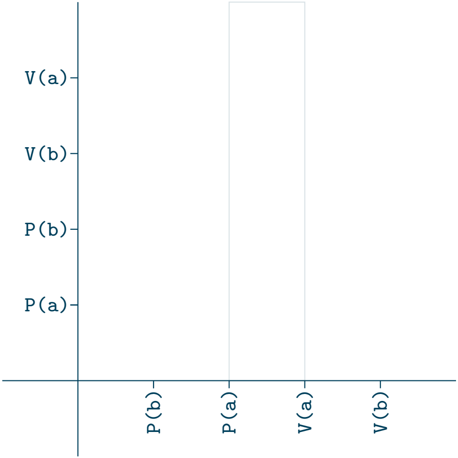

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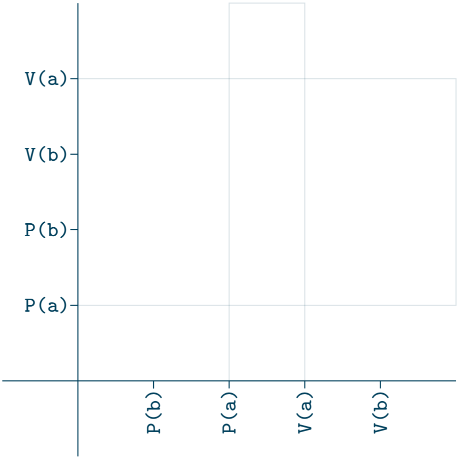

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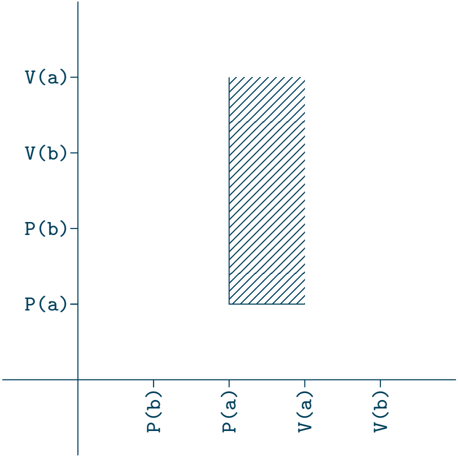

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p = P(a);P(b);V(b);V(a)
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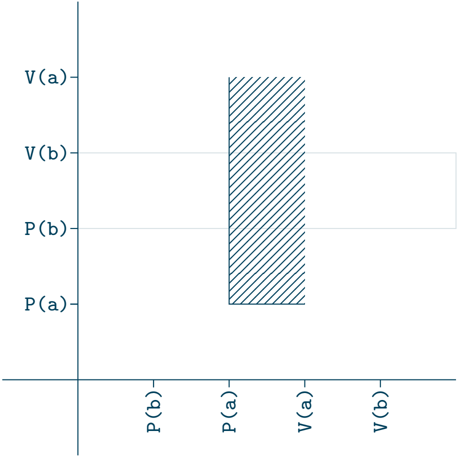

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p = P(a);P(b);V(b);V(a)
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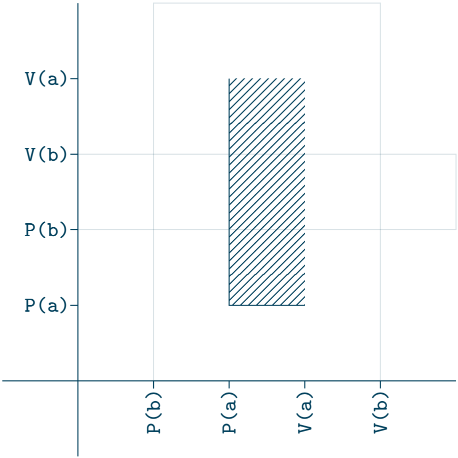

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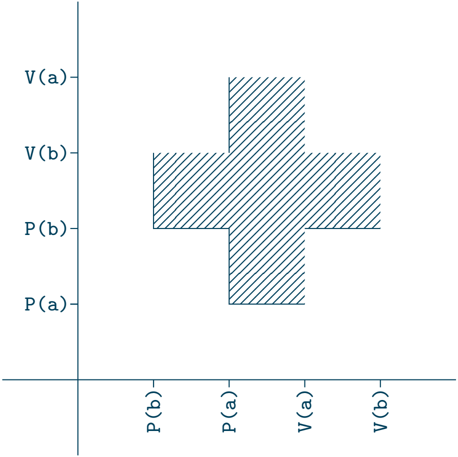

Binary synchronization

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sync 1 a
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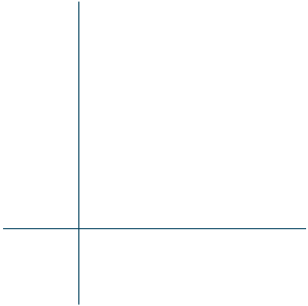

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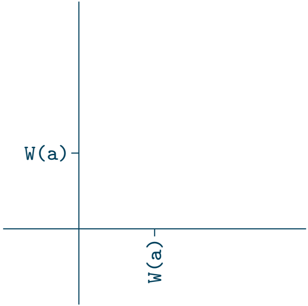

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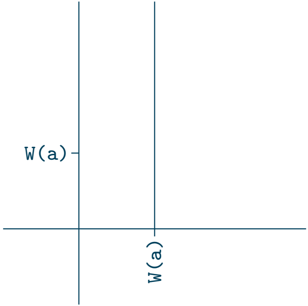

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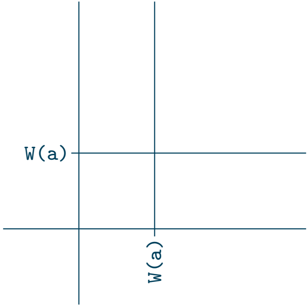

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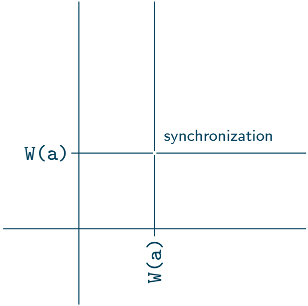

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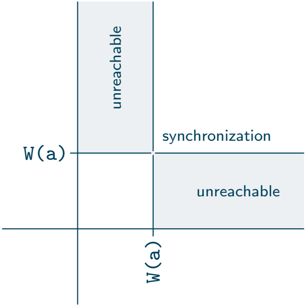

Producer/Consumer

nonlooping

Producer/Consumer

nonlooping

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proc:
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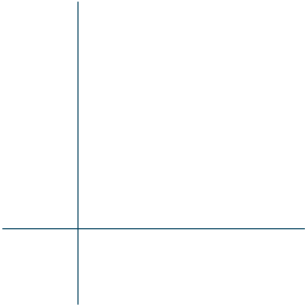

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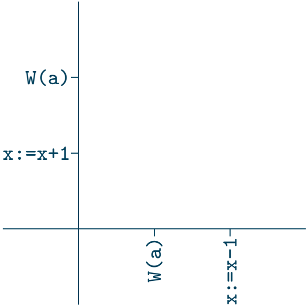

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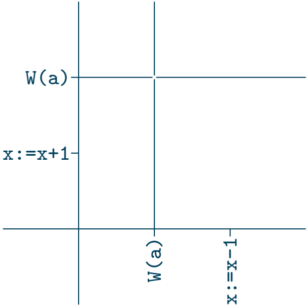

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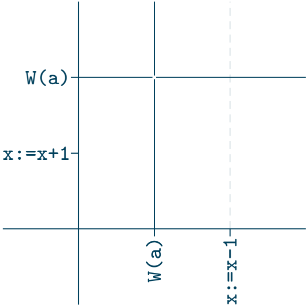

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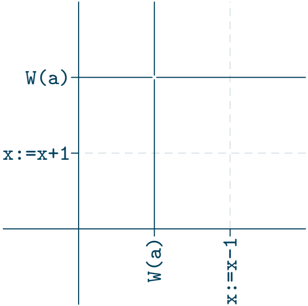

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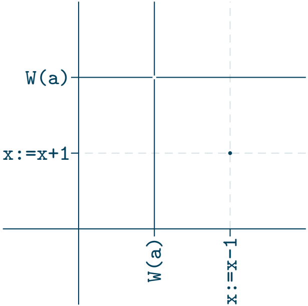

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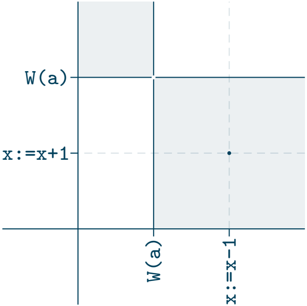

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looping

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```
sync 1 a b
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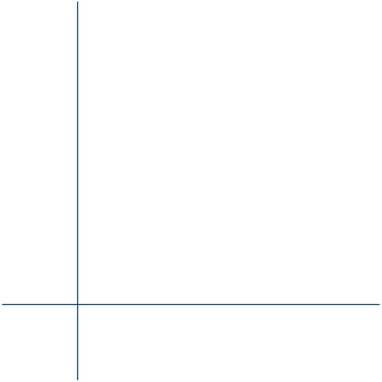

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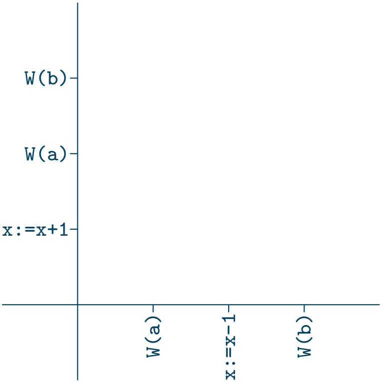

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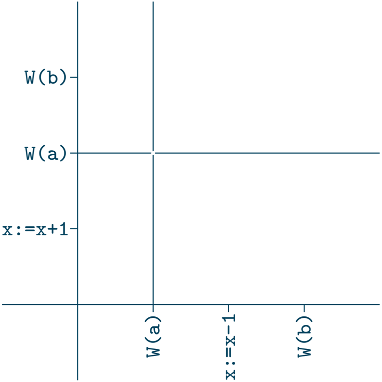

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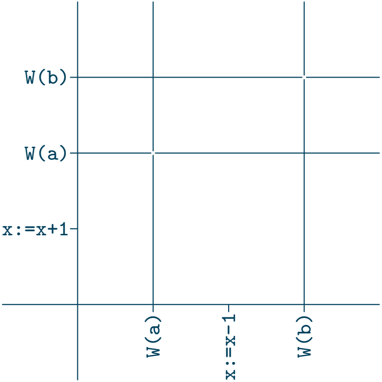

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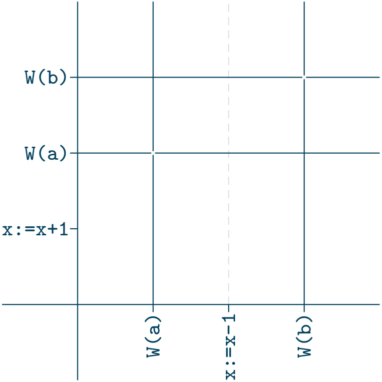

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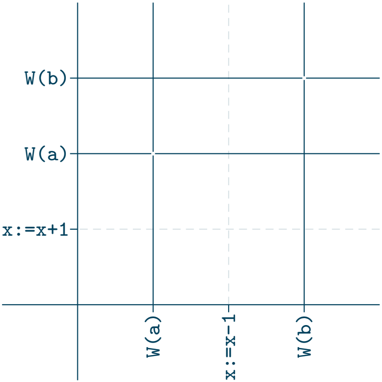

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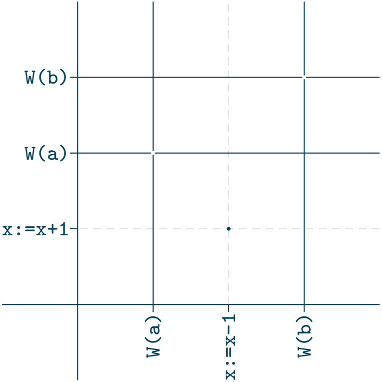

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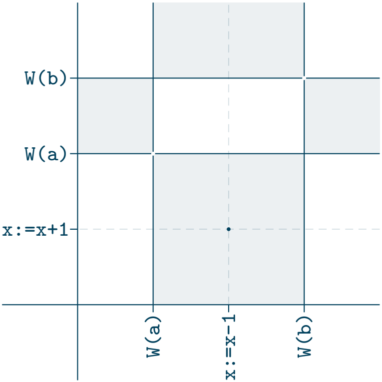
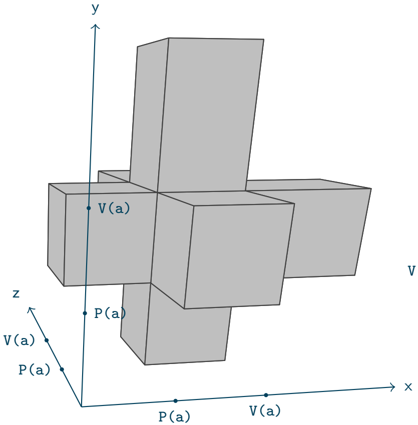
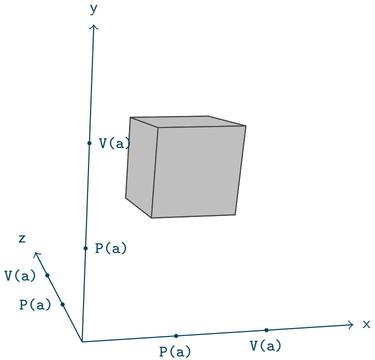
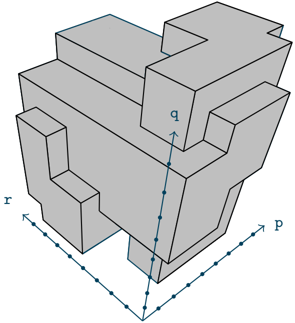


3D Swiss Cross (tetrahemihexacron) and floating cube





The Lipski algorithm




```
sem 1:  u v w x y z
proc:
  p = P(x);P(y);P(z);V(x);P(w);V(z);V(y);V(w)
  q = P(u);P(v);P(x);V(u);P(z);V(v);V(x);V(z)
  r = P(y);P(w);V(y);P(u);V(w);P(v);V(u);V(v)
init:  p q r
```

Geometric vs Discrete

Justifying de definition of discrete directed paths

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Let B_p and $B_{p'}$ be canonical blocks.

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If there exists a directed path starting in B_p, ending in $B_{p'}$, and whose image is contained in $B_p \cup B_{p'}$ then one of the following facts is satisfied:

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- for all $i \in \{1, \ldots, n\}$, $p_i = p'_i$ or p'_i is the target of the arrow p_i.

Discretization and lifting

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- Given a directed path γ on the local pospace $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$ we have a finite partition $I_0 < \cdots < I_N$ of $\mathrm{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point p^k such that $\gamma(I_k) \subseteq B_{p^k}$.

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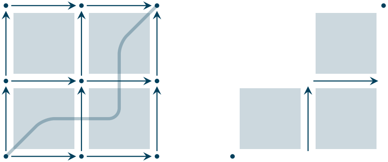
- The sequence $p^0, \ldots, p^N$ is a directed path on $(G_1, \ldots, G_n)$, it is called the discretization of γ and denoted by $D(\gamma)$.

Discretization and lifting

- Given a directed path γ on the local pospace $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$ we have a finite partition $I_0 < \cdots < I_N$ of $\mathrm{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point p^k such that $\gamma(I_k) \subseteq B_{p^k}$.
- The sequence $p^0, \ldots, p^N$ is a directed path on $(G_1, \ldots, G_n)$, it is called the discretization of γ and denoted by $D(\gamma)$.
- Given a directed path δ on $(G_1, \ldots, G_n)$ there exists a directed path γ on $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$ whose discretization is δ, such a directed path γ is said to be a lifting of δ.

Example of discretization



Admissible directed paths and execution traces

on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$

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The action of a directed path γ on $\lceil G_1 \rfloor \times \cdots \times \lceil G_n \rfloor$ on the right of a state σ is that of its discretization of $D(\gamma)$.

Example

```
var x = 0
var y = 0
var z = 0
sync 1 b
sem 1 a
```

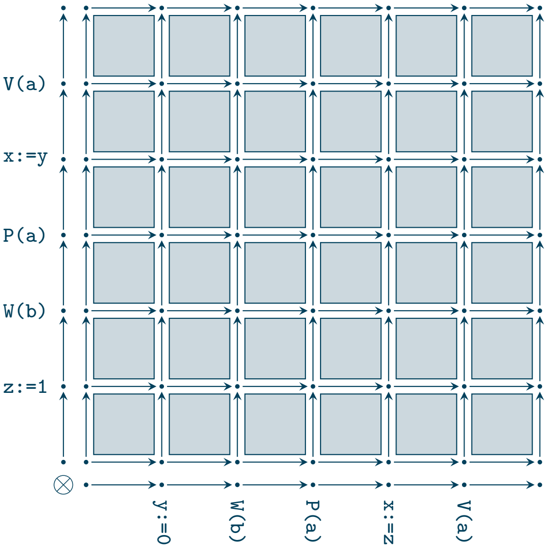
```
proc p = y:=0 ; W(b) ; P(a) ; x:=z ; V(a)

proc q = z:=1 ; W(b) ; P(a) ; x:=y ; V(a)
```

```
init p q
```

Discretization of an execution trace

`sem: 1 a` `sync: 1 b`



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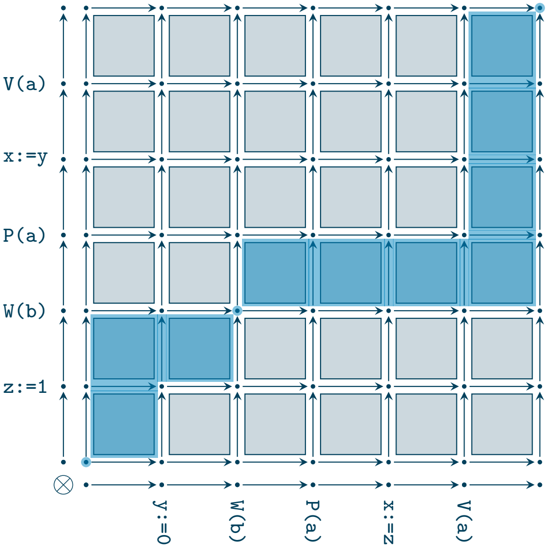

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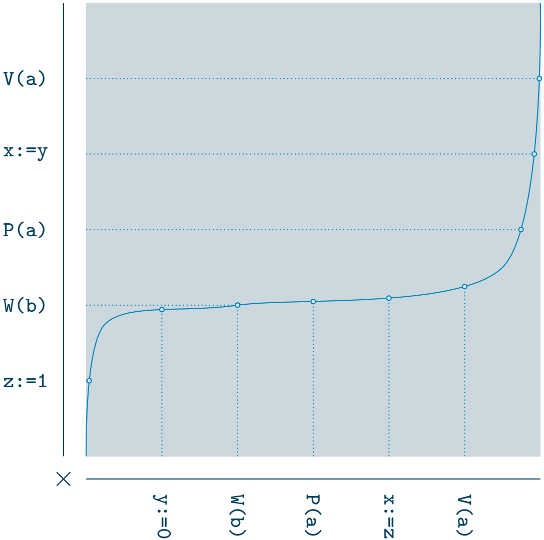

Potential function on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$

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If the program under consideration is conservative, then we have the potential function

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The function F is constant on each canonical block B_p, its value is given by $\tilde{F}(p)$ where $\tilde{F}$ denotes the "discrete" potential function.

Geometric models are sound and complete

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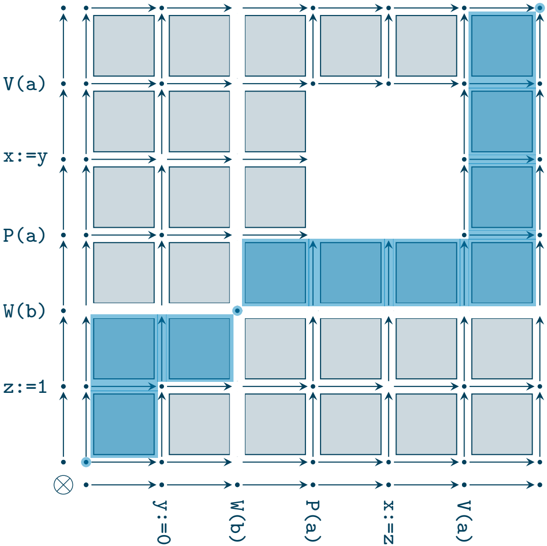
- Any directed path on a continuous model is admissible.
- Conversely, for each admissible path on a continuous model which meets a forbidden point, there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.

Directed paths on the geometric model are admissible

sem: 1 a sync: 1 b



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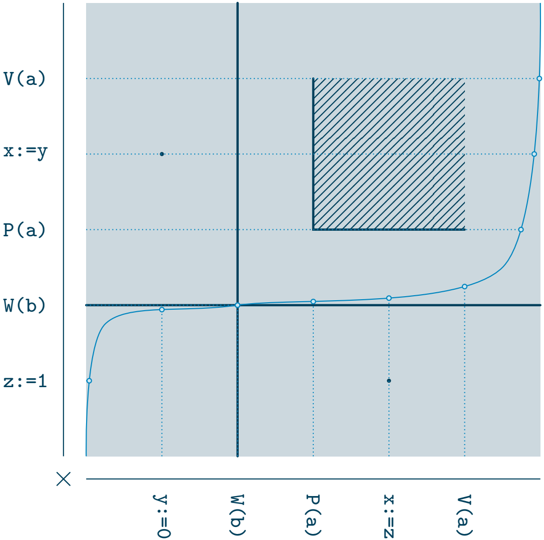
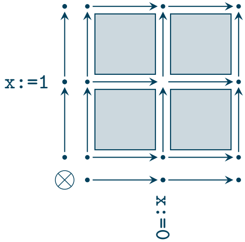


Continuous replacement

sem: 1 a sync: 1 b



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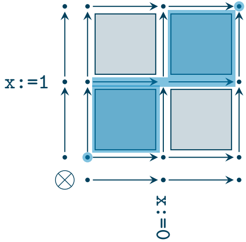

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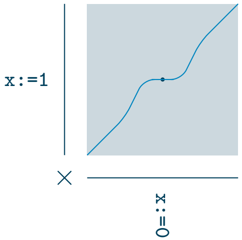

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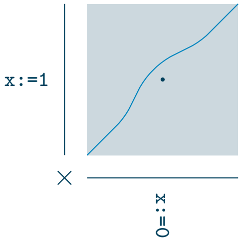

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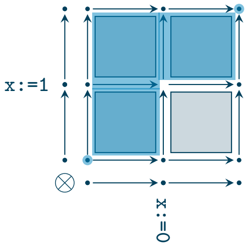

The motivating theorem

Trade off

More mathematics for more properties?

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- Both discrete and geometric models are sound and complete.

Trade off

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- Both discrete and geometric models are sound and complete.
- The continuous models satisfy extra properties that are “naturally” expressed in terms of metrics.

Uniform distance between directed paths

Uniform distance between directed paths

Given a compact Hausdorff space K and a metric space (X, d_X), the set of continuous maps from K to X can be equipped with the uniform distance

$$d(f,g) \quad = \quad \max\{d_X(f(k),g(k)) \mid k \in K\} \; .$$

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We consider the case where $K = [0, r]$ is the domain of definition of a directed path and (X, d_X) is the geometric model of a conservative program.

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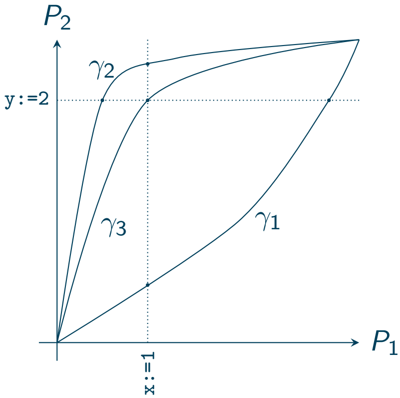
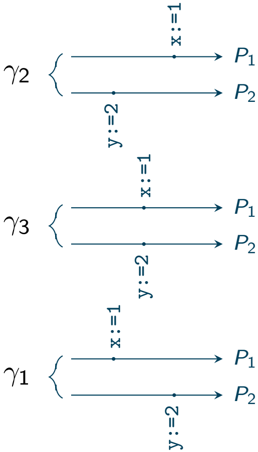
There exists an open ball Ω of $dX^{[0,r]}(B_p, B_{p'})$, centred in γ, such that all the elements of Ω induce the same action on valuations. Moreover, if γ is an execution trace, then so are all the elements of Ω.

Illustration





HOMOTOPY OF PATHS

The undirected case

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As a consequence we have $\gamma(0) = \delta(0)$ and $\gamma(r) = \delta(r)$.

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The Curryfication $(\hat{\_})$ induces a homeomorphism from $X^{[0,r]\times[0,q]}$ to $\left(X^{[0,r]}\right)^{[0,q]}$

$$(h : [0,r] \times [0,q] \to X) \to (\hat{h} : [0,q] \to X^{[0,r]})$$

The two faces of homotopies

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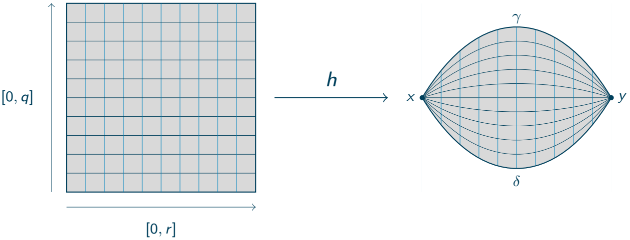
h is a continuous map from $[0,r] \times [0,q]$ to X i.e. $h \in \mathcal{Top}\big[[0,r] \times [0,q], X\big]$



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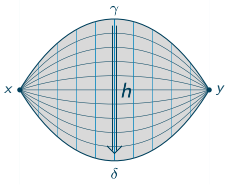
but is also a path from γ to δ in the space $X^{[0,r]}$ i.e. $h \in \mathcal{Top}\big[[0,q], X^{[0,r]}\big]$



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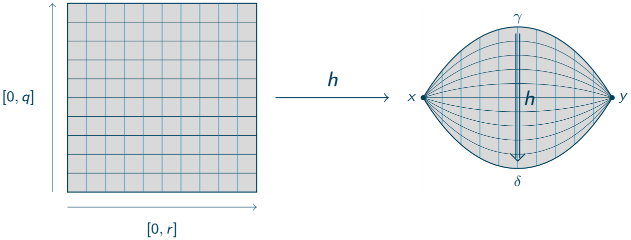
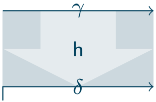


We introduce the following notation



$$\begin{tikzcd} x \arrow[r, bend left, "\gamma"{name=U}] \arrow[r, bend right, "\delta"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "h"] \end{tikzcd}$$

Concatenation of homotopies

vertical composition

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Let $g : [0, r] \times [0, q'] \to X$ and $h : [0, r] \times [0, q] \to X$ be homotopies from γ to ξ and from ξ to δ.

The mapping $h * g : [0, r] \times [0, q + q'] \to X$ defined by

$$h * g(t, s) \quad = \quad \begin{cases} g(t, s) & \text{if } 0 \leqslant s \leqslant q \\ h(t, s - q) & \text{if } q \leqslant s \leqslant q + q' \end{cases}$$

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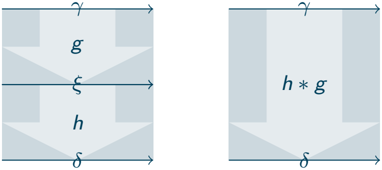

The directed case

Directed homotopy on a locally ordered space

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- A weakly directed homotopy from γ to δ is a homotopy of paths $h : [0,r] \times [0,q] \to X$ whose intermediate paths $h(\_, s)$, for $s \in [0,q]$, are directed.

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- Each of the preceding class of homotopies is stable under concatenation.

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They are said to be weakly dihomotopic when there exists a weakly directed homotopy between them. We have the equivalence relation $\sim_w$ between directed paths on a locally ordered space.

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is a directed homotopy from θ to $\max(\mathrm{id}_{[0,r]}, \theta)$.

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If $\gamma : [0, r] \to X$ is a directed path on the local pospace X, then $\gamma \circ h$ is a directed homotopy from $\gamma \circ \theta$ to $\gamma \circ \max(\mathrm{id}_{[0,r]}, \theta)$

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Therefore γ and $\gamma \circ \theta$ are dihomotopic.

Images of directed paths on a pospace

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Theorem

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Corollary

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Hence $\phi \circ h$ is an elementary homotopy from γ and γ'.

Relation to geometric models

Main theorem

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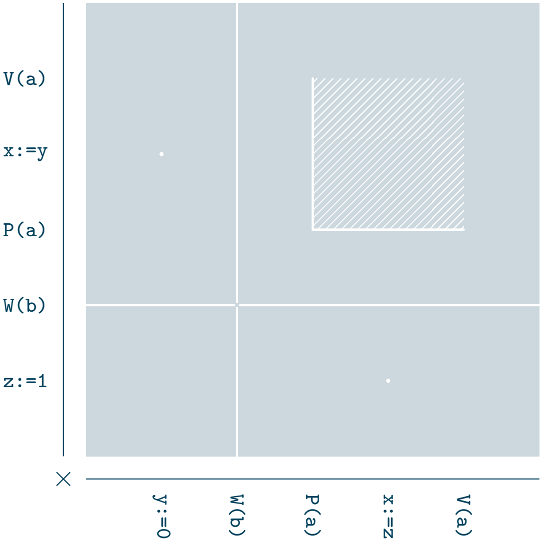
Two weakly dihomotopic paths on the geometric model of a conservative program induce the same action on valuations. Moreover, if one of them is an execution trace, then so is the other.

Weakly directed homotopy

sem: 1 a sync: 1 b



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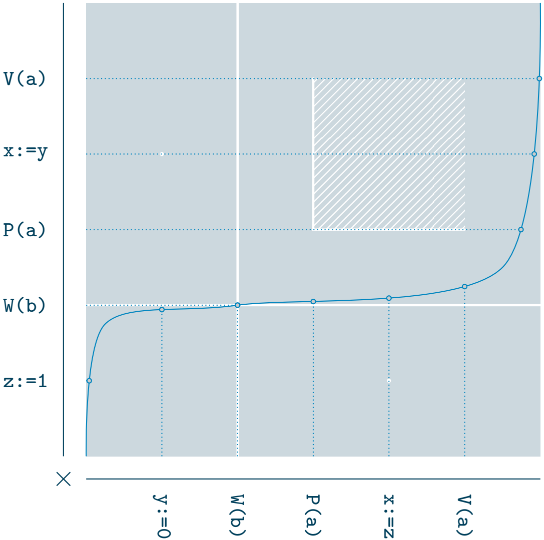

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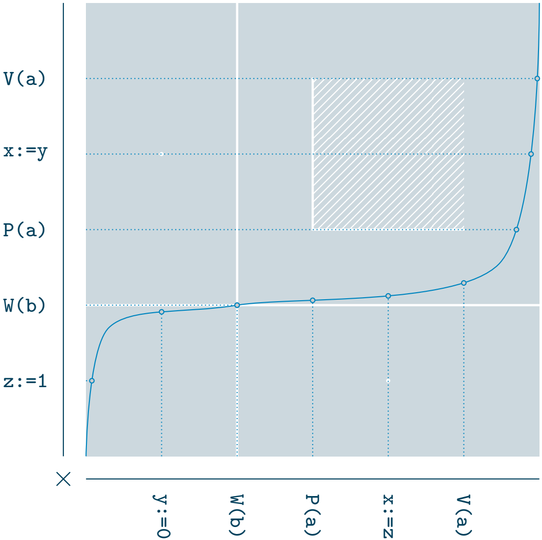

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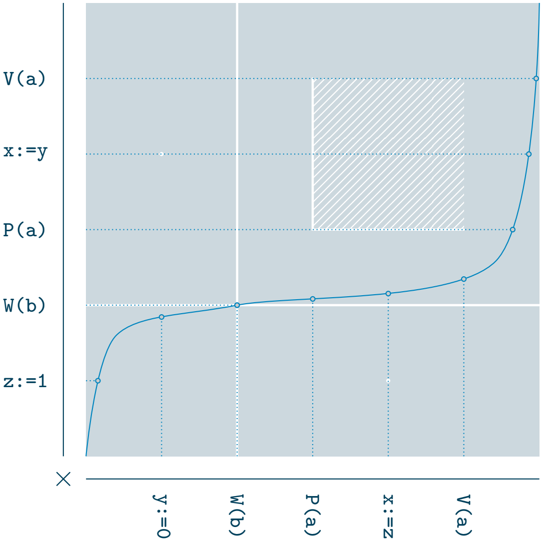

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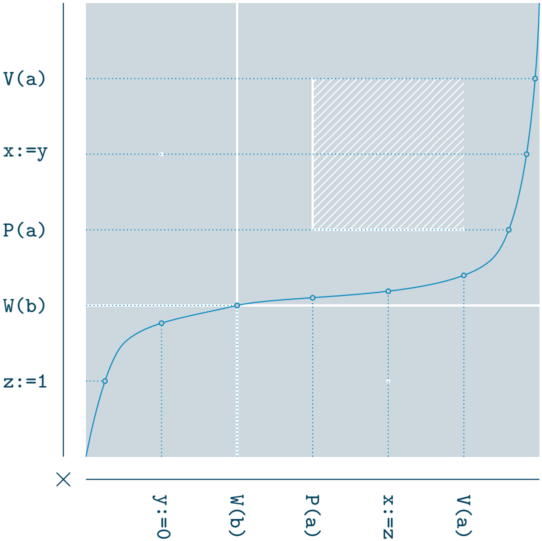

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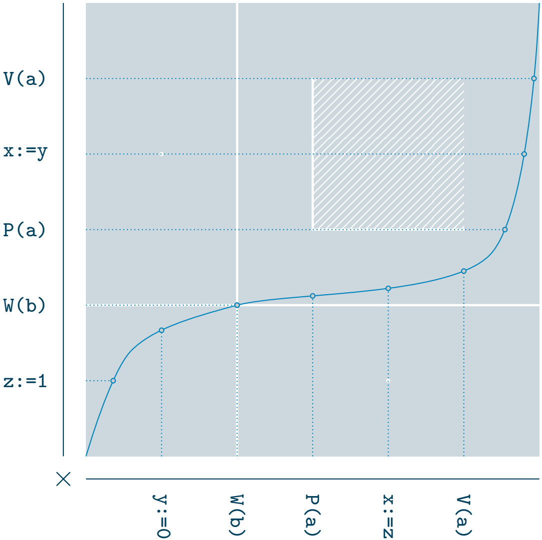

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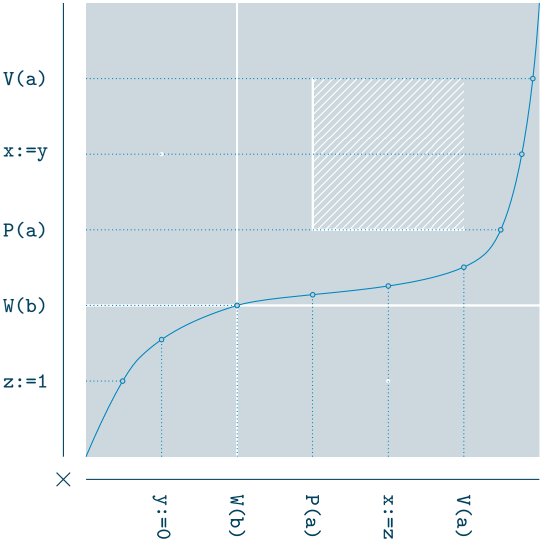

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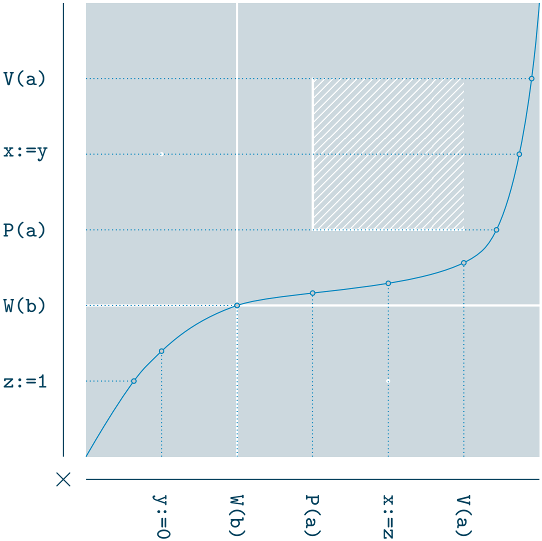

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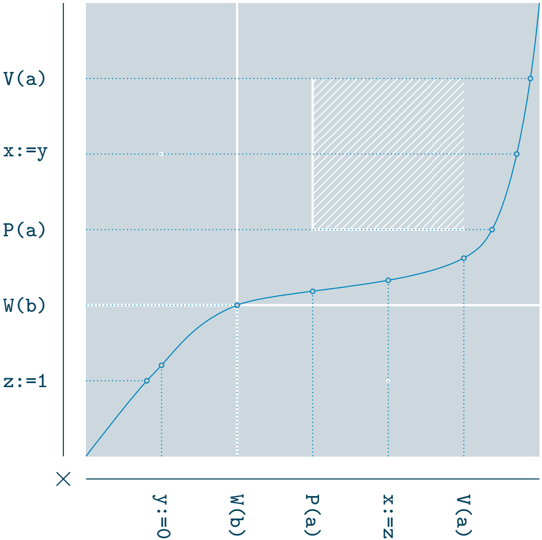

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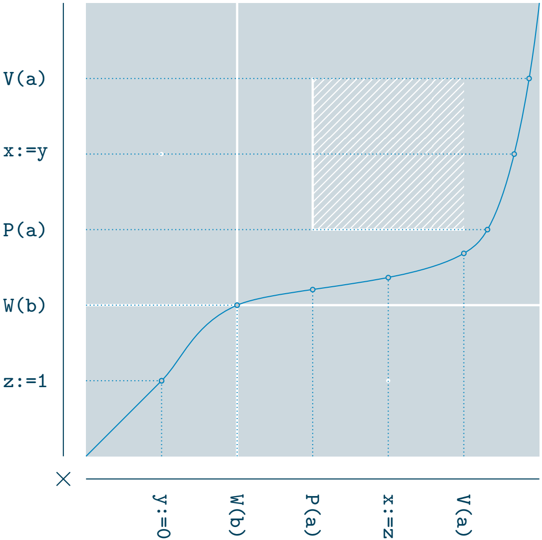

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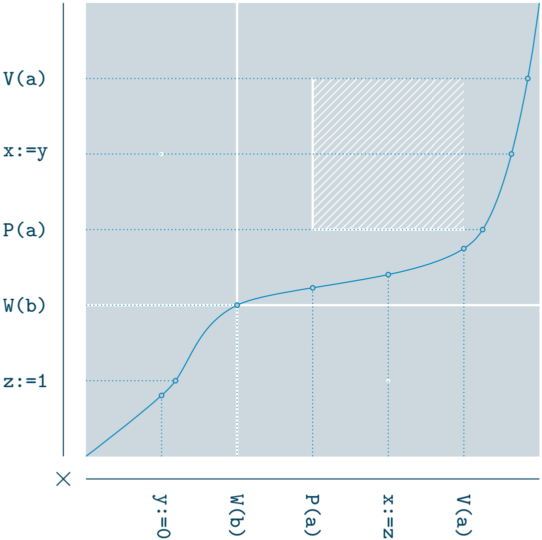

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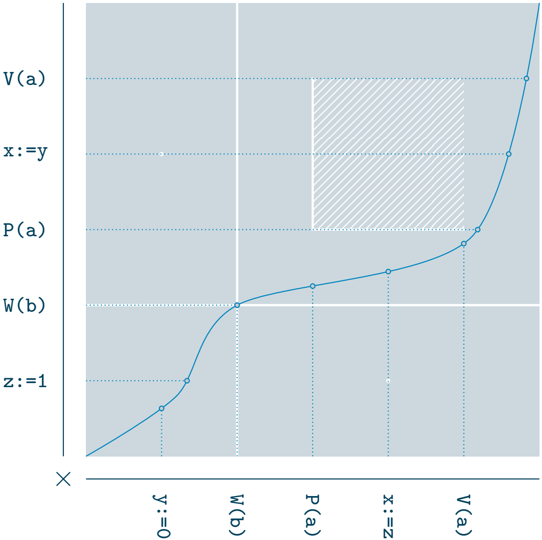

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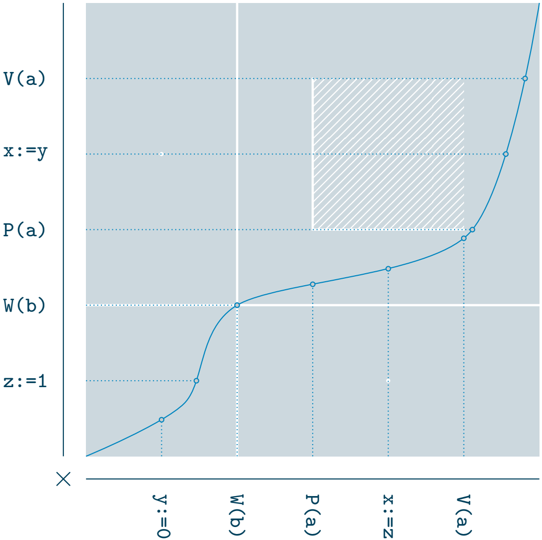

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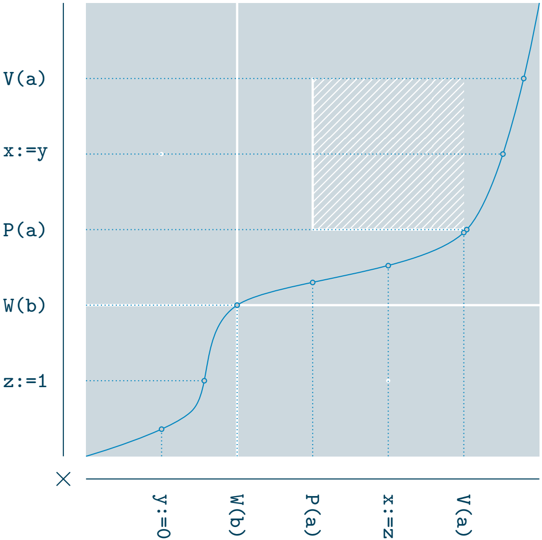

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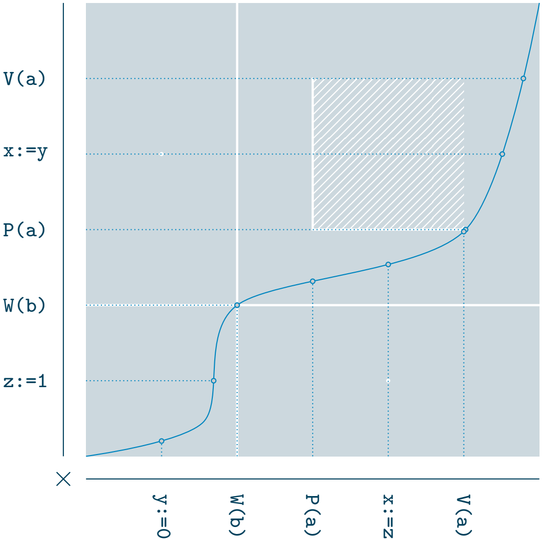

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The conclusion follows considering the sequence

$$\hat{h}(0), \hat{h}(\varepsilon), \hat{h}(2\varepsilon), \hat{h}(3\varepsilon), \cdots, \hat{h}(n\varepsilon), \hat{h}(q)$$

where n is the greatest natural number such that $n\varepsilon \leqslant q$.

Programs with mutex only

a result by É. Goubault and S. Mimram

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Let X be the geometric model of a conservative program whose semaphores have arity 1 (mutex), then two directed paths on X are dihomotopic if and only if they are homotopic.

INDEPENDENCE

Compatible programs

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By extension we define the parallel composition of $P_1, \ldots, P_N$ when the programs are pairwise compatible.

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Syntactical independence can be decided statically, it is compositional, but it is too restrictive.

Model independence

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The permutation π is said to be compatible with the directed path γ when it is compatible with its associated sequence of multi-instructions.

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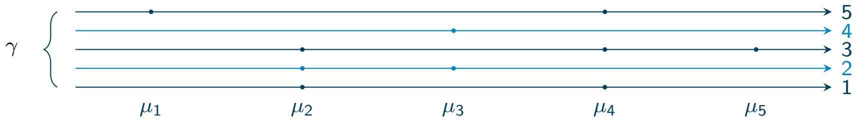

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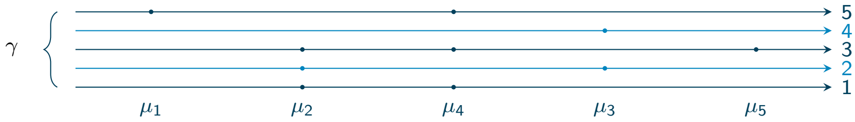

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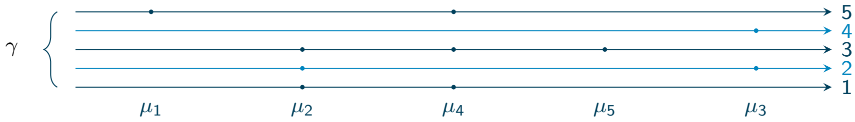

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Observational independence cannot be decided statically, moreover it is too loose.

Comparison

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syntactic independence

$\Downarrow$

model independence

$\Downarrow$

observational independence