AN ALGEBRAIC TOPOLOGY TEASER
Categories
Category $C$

Definition (the “underlying graph” part)
Category $\mathcal{C}$

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- $\text{Ob}(\mathcal{C})$ : collection of objects
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- $\text{Mo}(\mathcal{C})$ : collection of morphisms
- $s, t$ : mappings source, target as follows

\[
\text{Mo}(\mathcal{C}) \xrightarrow{s} \text{Ob}(\mathcal{C}) \xleftarrow{t}
\]
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- $s, t$ : mappings source, target as follows

$$\text{Mo}(\mathcal{C}) \xrightarrow{s} \text{Ob}(\mathcal{C}) \quad t$$

- We define the homset $\mathcal{C}(x, y) := \{ \gamma \in \text{Mo}(\mathcal{C}) \mid \partial^- \gamma = x \text{ and } \partial^+ \gamma = y \}$
Category $\mathcal{C}$

Definition (the “underlying local monoid” part)
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- id : provides each object with an identity

$$\text{Mo}(\mathcal{C}) \xrightarrow{\text{id}} \text{Ob}(\mathcal{C})$$
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$$\text{Mo}(\mathcal{C}) \xrightarrow{s} \text{Ob}(\mathcal{C})$$

- The binary composition is a partially defined and often denoted by $\circ$

$$\left\{ (\gamma, \delta) \mid \gamma, \delta \text{ morphisms of } \mathcal{C} \text{ s.t. } \partial \gamma = \partial^+ \delta \right\} \xrightarrow{\text{composition}} \text{Mo}(\mathcal{C})$$

Diagram:

$$\partial^+ \delta = \partial \gamma$$

$$\partial \delta \xrightarrow{\gamma \circ \delta} \partial^+ \gamma$$
Category $\mathcal{C}$

Definition (the axioms)

- The composition law is associative
- For all objects $x$, one has $\partial - \text{id}_x = x = \partial + \text{id}_x$
- For all morphisms $\gamma$, one has $\text{id}_\partial \circ \gamma + \gamma \circ \gamma = \gamma = \gamma \circ \text{id}_\partial$
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- For all morphisms $\gamma$ one has $\text{id}_{\partial^+ \gamma} \circ \gamma = \gamma = \gamma \circ \text{id}_{\partial^- \gamma}$
Standard examples

- Set: the category of sets.
- Mon: the category of monoids.
- Cmon: the category of commutative monoids.
- Gr: the category of groups.
- Pre: the category of preordered sets.
- Pos: the category of posets.

- Any preordered set can be seen as a category in which any homset has at most one element.
- Any monoid can be seen as a category with a single object.
- The opposite of a category is obtained by reversing all its arrows (i.e. by swapping the roles of the source and the target).
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Some special kinds of morphisms

- *f* is an isomorphism when there exists *g* such that both *f* ◦ *g* and *g* ◦ *f* are identities.

- Two objects related by an isomorphism are said to be isomorphic.

- *f* is a monomorphism when it is left-cancellative i.e. for all *g*₁, *g*₂, *f* ◦ *g*₁ = *f* ◦ *g*₂ implies *g*₁ = *g*₂.

- *f* is a epimorphism when it is right-cancellative i.e. for all *g*₁, *g*₂, *g*₁ ◦ *f* = *g*₂ ◦ *f* implies *g*₁ = *g*₂.

- any isomorphism is both monomorphism and an epimorphism, the converse is false in general (e.g. *Pos*).

- if *r* ◦ *s* = id then *r* is called a retract/split epimorphism and *s* is called a section/split monomorphism.
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The category of graphs

\[ \text{Grph} \]

The elements of \( V \) are the \textit{vertices} and those of \( A \) are the \textit{arrows}

In particular \( A \) and \( V \) are \textit{sets}
The category of graphs

*Graph*

The elements of $V$ are the *vertices* and those of $A$ are the *arrows*
In particular $A$ and $V$ are *sets*

**Objects**

![Diagram of category of graphs](image-url)
The category of graphs

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### Objects

\[
\begin{array}{ccc}
A & \xrightarrow{s} & V \\
\downarrow{s} & & \downarrow{t} \\
V & & V
\end{array}
\]

### Morphisms

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_1} & A' \\
\downarrow{s} & & \downarrow{t} \\
V & & V \\
\downarrow{s'} & & \downarrow{t'} \\
V & \xrightarrow{\phi_0} & V'
\end{array}
\]
The category of graphs

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The elements of $V$ are the *vertices* and those of $A$ are the *arrows*. In particular $A$ and $V$ are *sets*.
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A & \xrightarrow{\phi_1} & A' \\
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V & \xrightarrow{\phi_0} & V'
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\quad \text{Composition}
\[
\begin{array}{ccc}
A & \xrightarrow{\phi_1} & A' & \xrightarrow{\psi_1} & A'' \\
\downarrow{s} & & \downarrow{s'} & & \downarrow{s''} \\
V & \xrightarrow{\phi_0} & V' & \xrightarrow{\psi_0} & V''
\end{array}
\]

with \( s'(\phi_1(\alpha)) = \phi_0(\partial^- \alpha) \) and \( t'(\phi_1(\alpha)) = \phi_0(\partial^+ \alpha) \)
Topology
Topological spaces

A topological space is a set $X$ and a collection $\Omega_{X} \subseteq P(X)$ s.t.

1) $\emptyset \in \Omega_{X}$ and $X \in \Omega_{X}$
2) $\Omega_{X}$ is stable under union
3) $\Omega_{X}$ is stable under finite intersection

A continuous map $f : (X, \Omega_{X}) \to (Y, \Omega_{Y})$ is a map $f : X \to Y$ s.t.

$\forall x \in X \forall V \in \Omega_{Y}$ s.t. $f(x) \in V$, $\exists U \in \Omega_{X}$ s.t. $x \in U$ and $f(U) \subseteq V$ or equivalently $\forall V \in \Omega_{Y}$ $f^{-1}(V) \in \Omega_{X}$

The elements of $\Omega_{X}$ are called the open subsets of $X$.

The complement of an open subset is said to be closed.

$B \subseteq \Omega_{X}$ is a base of topology when each open subset is a union of elements of $B$. 

Topological spaces and continuous maps form the category Top.
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Related definitions

- **The interior of a subset** $A$ of $X$ is the greatest open subset of $X$ contained in $A$.

- **The closure of a subset** $A$ of $X$ is the least closed subset of $X$ containing $A$.

- **A neighbourhood of a subset** $A$ of $X$ is a subset of $X$ whose interior contains $A$.

- A topological space $X$ is said to be **Hausdorff** when for all $x, x' \in X$, if $x \neq x'$ then $x$ and $x'$ have disjoint neighbourhoods.

- A subset $Q$ of $X$ is said to be **saturated** when $Q = \bigcap U$, $U$ is open and $Q \subseteq U$.

Every subset of a Hausdorff space is saturated.
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Let $X$ be a topological space.

- An open covering of $X$ is a collection of open subsets of $X$ whose union is $X$.
- $X$ is said to be compact when every open covering of $X$ admits a finite sub-covering.
- $X$ is said to be locally compact when for every $x \in X$, every open neighbourhood $U$ of $x$ contains a saturated compact neighbourhood of $x$.

A Hausdorff space is locally compact iff each of its points admits a compact neighbourhood.
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Compactness and local compactness

Let $X$ be a topological space.
- An open covering of $X$ is a collection of open subsets of $X$ whose union is $X$.
- $X$ is said to be compact when every open covering of $X$ admit a finite sub-covering.
- $X$ is said to be locally compact when for every $x \in X$, every open neighbourhood $U$ of $x$ contains a saturated compact neighbourhood of $x$.

A Hausdorff space is locally compact iff each of its points admits a compact neighbourhood.
Functors
Functors $f$ from $C$ to $D$

Definition (preserving the “underlying graph”)
Functors $f$ from $\mathcal{C}$ to $\mathcal{D}$

Definition (preserving the “underlying graph”)

A functor $f : \mathcal{C} \to \mathcal{D}$ is defined by two “mappings” $\text{Ob}(f)$ and $\text{Mo}(f)$ such that

$$s'(\text{Mo}(f)(\alpha)) = \text{Ob}(f)(\partial^- \alpha)$$

$$t'(\text{Mo}(f)(\alpha)) = \text{Ob}(f)(\partial^+ \alpha)$$

Hence it is in particular a morphism of graphs.
Functors $f$ from $\mathcal{C}$ to $\mathcal{D}$

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\[
\begin{array}{ccc}
\text{Mo}(\mathcal{C}) & \xrightarrow{s} & \text{Ob}(\mathcal{C}) \\
\text{Mo}(\mathcal{D}) & \xrightarrow{s'} & \text{Ob}(\mathcal{D})
\end{array}
\]

\[
\begin{array}{ccc}
\text{Mo}(f) & \xrightarrow{\text{t}} & \text{Ob}(f) \\
\text{t'} & \xrightarrow{\text{t'}} & \text{Ob}(f)
\end{array}
\]

with $s'(\text{Mo}(f)(\alpha)) = \text{Ob}(f)(\partial^- \alpha)$ and $t'(\text{Mo}(f)(\alpha)) = \text{Ob}(f)(\partial^+ \alpha)$

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Definition (preserving the “underlying local monoid”)
Functors $f$ from $\mathcal{C}$ to $\mathcal{D}$

Definition (preserving the “underlying local monoid”)

The “mappings” $\text{Ob}(f)$ and $\text{Mo}(f)$ also make the following diagram commute:

\[
\begin{array}{ccc}
\text{Mo}(\mathcal{C}) & \xleftarrow{id} & \text{Ob}(\mathcal{C}) \\
\downarrow \text{Mo}(f) & & \downarrow \text{Ob}(f) \\
\text{Mo}(\mathcal{D}) & \xleftarrow{id'} & \text{Ob}(\mathcal{D})
\end{array}
\]
Functors $f$ from $\mathcal{C}$ to $\mathcal{D}$

Definition (preserving the “underlying local monoid”)

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\text{Mo}(\mathcal{C}) & \xrightarrow{\text{id}} & \text{Ob}(\mathcal{C}) \\
\downarrow \text{Mo}(f) & & \downarrow \text{Ob}(f) \\
\text{Mo}(\mathcal{D}) & \xleftarrow{\text{id}'} & \text{Ob}(\mathcal{D})
\end{array}
\]

and satisfies $\text{Mo}(f)(\gamma \circ \delta) = \text{Mo}(f)(\gamma) \circ \text{Mo}(f)(\delta)$
Functors compose as morphisms of graphs do
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The small categories and their functors form a (large) category denoted by $\mathbf{Cat}$.
Functors compose as morphisms of graphs do

Hence the functors should be thought of as the morphisms of categories
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Hence the functors should be thought of as the morphisms of categories

The small categories and their functors form a (large) category denoted by \( \text{Cat} \)
Connectedness
The overall idea of algebraic topology
The overall idea of algebraic topology

Every functor preserves the isomorphisms
The overall idea of algebraic topology

Every functor preserves the isomorphisms

Problem: prove the topological spaces $X$ and $Y$ are not the same
The overall idea of algebraic topology

Every functor preserves the isomorphisms

Problem: prove the topological spaces $X$ and $Y$ are *not* the same
Strategy: find a functor $F$ defined over $\text{Top}$ such that $F(X) \not\cong F(Y)$
The connected component functor
The connected component functor

1) A topological space $X$ is said to be connected when its only closed-open subsets are $\emptyset$ and $X$
The connected component functor

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\[
\begin{align*}
\text{Top} & \xrightarrow{\pi_0} \text{Set} \\
X & \xrightarrow{\pi_0(X)} \\
Y & \xrightarrow{\pi_0(Y)}
\end{align*}
\]
An application

The continuous image of a connected space is connected

The image of the space $B$ is entirely contained in a connected component of the space $V$. 
The set of connected components

is a functorial construction

This situation is abstracted by classifying continuous maps from $B$ to $V$ according to which connected component ($V_1$ or $V_2$) the single connected components of $B$ (namely $B$ itself) is sent to. There are exactly two set theoretic maps from the singleton $\{B\}$ to the pair $\{V_1, V_2\}$ hence there is at most (in fact exactly) two kinds of continuous maps from $B$ to $V$.

$$\{B\} \rightarrow \{V_1, V_2\}$$

In particular $B$ and $V$ are not homeomorphic.
Application

The compact interval and the circle are not homeomorphic
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Let $\mathbb{S}^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$ be the Euclidean circle and suppose $\varphi : [0, 1] \rightarrow \mathbb{S}^1$ is a homeomorphism.
Application

The compact interval and the circle are not homeomorphic

Let $S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$ be the Euclidean circle and suppose $\varphi : [0, 1] \to S^1$ is a homeomorphism.

Then $\varphi$ induces a homeomorphism

$$[0, \frac{1}{2} \cup \frac{1}{2}, 1] \to S^1 \setminus \{ \varphi(\frac{1}{2}) \}$$

which does not exist!
Generalization

Bouquets of circles

These topological spaces are pairwise not homeomorphic. Why?
METRIC SPACES
Functor Terminology
Given a functor \( f : C \to D \) and two objects \( x \) and \( y \) we have the mapping \( f_x, y : C \times X \to D \times X \). \( \alpha \) is faithful when for all objects \( x \) and \( y \) the mapping \( f_x, y \) is one-to-one (injective). \( f \) is full when for all objects \( x \) and \( y \) the mapping \( f_x, y \) is onto (surjective). \( f \) is fully faithful when it is full and faithful. \( f \) is an embedding when it is faithful and \( \text{Ob}(f) \) is one-to-one.
Given a functor $f : C \to D$ and two objects $x$ and $y$ we have the mapping

$$f_{x,y} : C[x, y] \to D[\text{Ob}(f)(x), \text{Ob}(f)(y)]$$

$$\alpha \mapsto \text{Mo}(f)(\alpha)$$
Given a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ and two objects $x$ and $y$ we have the mapping

$$f_{x,y} : \mathcal{C}[x, y] \rightarrow \mathcal{D}[\text{Ob}(f)(x), \text{Ob}(f)(y)]$$

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- $f$ is faithful when for all objects $x$ and $y$ the mapping $f_{x,y}$ is one-to-one (injective)
Given a functor $f : \mathcal{C} \to \mathcal{D}$ and two objects $x$ and $y$ we have the mapping

$$f_{x,y} : \mathcal{C}[x, y] \longrightarrow \mathcal{D}[\text{Ob}(f)(x), \text{Ob}(f)(y)]$$

$$\alpha \longmapsto \text{Mo}(f)(\alpha)$$

- $f$ is faithful when for all objects $x$ and $y$ the mapping $f_{x,y}$ is one-to-one (injective)
- $f$ is full when for all objects $x$ and $y$ the mapping $f_{x,y}$ is onto (surjective)
Given a functor \( f : \mathcal{C} \to \mathcal{D} \) and two objects \( x \) and \( y \) we have the mapping

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 f_{x,y} : \mathcal{C}[x,y] \to \mathcal{D}[\text{Ob}(f)(x), \text{Ob}(f)(y)] \\
 \alpha \mapsto \text{Mo}(f)(\alpha)
\]

- \( f \) is **faithful** when for all objects \( x \) and \( y \) the mapping \( f_{x,y} \) is one-to-one (injective)
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- $f$ is an embedding when it is faithful and $\text{Ob}(f)$ is one-to-one
Some small functors
(functor between small categories)
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The morphisms of monoids are the functors between small categories with a single object.
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The morphisms of preordered sets are the functors between small categories whose homsets contain at most one element.
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(functor between small categories)

The morphisms of monoids are the functors between small categories with a single object.

The morphisms of preordered sets are the functors between small categories whose homsets contain at most one element.

The actions of a monoid $M$ over a set $X$ are the functors from $M$ to $\text{Set}$ which sends the only element of $M$ to $X$. 
Some full embeddings in $\mathbf{Cat}$
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**Remark**: The full embeddings compose
Some full embeddings in $\textbf{Cat}$

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$\text{Pre} \hookrightarrow \text{Cat}$
Some full embeddings in $\textbf{Cat}$

Remark: The full embeddings compose

$\text{Pre} \hookrightarrow \textbf{Cat}$
$\text{Mon} \hookrightarrow \textbf{Cat}$
Some full embeddings in $\textit{Cat}$

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$\text{Pos} \hookrightarrow \text{Pre}$
Some full embeddings in $\text{Cat}$

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- $\text{Ab} \hookrightarrow \text{Cmon}$
- $\text{Ab} \hookrightarrow \text{Gr}$
- $\text{Set} \hookrightarrow \text{Pos}$
Some forgetful functors
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\((M, \ast, e) \in \mathcal{Mon} \mapsto M \in \mathcal{Set}\)
Some forgetful functors

$$(M, *, e) \in \text{Mon} \mapsto M \in \text{Set}$$
$$(X, \Omega) \in \text{Top} \mapsto X \in \text{Set}$$
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\(C \in \text{Cat} \mapsto \text{Ob}(C) \in \text{Set}\)
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\(C \in \text{Cat} \mapsto \left( \begin{array}{c}
\text{Mo}(C) \\
\text{Ob}(C)
\end{array} \right) \in \text{Grph}\)
Categories of Metric Spaces
Metric spaces

A metric space is a set $X$ together with a mapping $d: X \times X \rightarrow \mathbb{R}^+ \cup \{\infty\}$ such that:

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

Goal: turn any graph into metric space in a natural way.
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- $\text{Met}_{emb} f : X \to Y$ s.t. $\forall x, x' \in X$, $d_Y(f(x), f(x')) = d_X(x, x')$
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Metric space morphisms

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- $\text{Met}_{\text{ctr}} f : X \to Y$ s.t. $\forall x, x' \in X, \ d_Y(f(x), f(x')) \leq d_X(x, x')$
- $\text{Met} f : X \to Y$ s.t. $\exists r \in ]0, \infty[, \forall x, x' \in X, \ d_Y(f(x), f(x')) \leq r \cdot d_X(x, x')$
Metric space morphisms

- $\text{Met}_\text{emb} \ f : X \to Y \text{ s.t. } \forall x, x' \in X, d_Y(f(x), f(x')) = d_X(x, x')$
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- $\text{Met}_\text{top} \ f : X \to Y \text{ s.t. } \forall x \in X \forall \varepsilon > 0 \exists \eta > 0, f(B(x, \eta)) \subseteq B(f(x), \varepsilon)$
Metric space morphisms

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- $\text{Met}_{\text{top}} : X \to Y$ s.t. $\forall x \in X \ \forall \varepsilon > 0 \ \exists \eta > 0, \ f(B(x, \eta)) \subseteq B(f(x), \varepsilon)$

$$\text{Met}_{\text{emb}} \leftrightarrow \text{Met}_{\text{ctr}} \leftrightarrow \text{Met} \leftrightarrow \text{Met}_{\text{top}} \rightarrow \text{Top}$$
Length spaces

The length $\ell(\gamma)$ of a path $\gamma : [0, r] \to (X, d)$ is the least upper bound of the collection of sums $\sum_{i=0}^{n} d(\gamma(t_{i}+1), \gamma(t_{i}))$, where $n \in \mathbb{N}$ and $0 = t_{0} \leq \ldots \leq t_{n} = r$. The metric space $(X, d)$ is a length space when the distance between two points $x, x' \in X$ is the greatest lower bound $\inf \{\ell(\gamma) | \gamma \text{ is a path from } x \text{ to } x'\}$.

A path $\gamma$ from $x$ to $x'$ such that $\ell(\gamma) = d(x, x')$ is said to be geodesic. The space is said to be geodesic when any two points are related by a geodesic path.
**Length spaces**

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A path $\gamma$ from $x$ to $x'$ such that $\ell(\gamma) = d(x, x')$ is said to be geodesic.
Length spaces

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The Hopf-Rinow theorem

Metric Spaces of Non-Positive Curvature, M. R. Bridson, and A. Haefliger, 1999
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A metric space is said to be complete when all its Cauchy sequences admit a limit.
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Let $X$ be a length space.
The Hopf-Rinow theorem

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Let \( X \) be a length space. If \( X \) is complete and locally compact, then
The Hopf-Rinow theorem

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Let \( X \) be a length space. If \( X \) is complete and locally compact, then
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A metric space is said to be complete when all its Cauchy sequences admit a limit.

Let $X$ be a length space.

If $X$ is complete and locally compact, then

- every closed bounded subset of $X$ is compact, and
- $X$ is a geodesic space.
Isometric embedding in $\mathbb{R}^n$
Isometric embedding in $\mathbb{R}^n$

- $\mathbb{R}^n$ is a geodesic space with the distance inherited from $\mathbb{R}^n$. $\mathbb{R}^n \setminus \{0\}$ is a length space, not a geodesic one.

- $\mathbb{R}^n \setminus [0, 1]$ with the distance inherited from $\mathbb{R}^n$ is not a length space.

- Any metric space $(X, d)$ is associated with a length space $(X, d^\ell)$ with $d^\ell(x, x') = \inf \ell(\gamma)$ where $\gamma$ is a path from $x$ to $x'$. 

\[ 25 / 42 \]
Isometric embedding in $\mathbb{R}^n$

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- $\mathbb{R}^n \setminus [0, 1]^n$ with the distance inherited from $\mathbb{R}^n$ is not a length space.
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Isometric embedding in $\mathbb{R}^n$

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$$d_\ell(x, x') = \inf \{ \ell(\gamma) \mid \gamma \text{ is a path from } x \text{ to } x' \}$$
Metric Graphs
**Neighbours**

- The underlying set of the metric graph is $A \times [0,1] \sqcup V$.
- Two points $p, p'$ are said to be neighbours when there is an arrow $a$ such that $p, p' \in \{a\} \times [0,1] \sqcup \{\partial- a, \partial+ a\}$. 

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Metric spaces | Metric graphs
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$$G : A \xrightarrow{\partial^-} V \xleftarrow{\partial^+}$$
Distance between two neighbours
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- If $\partial^- a \neq \partial^+ a$ there is a canonical bijection

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Itinerary

An itinerary on $A \times [0,1]$ is a (finite) sequence $p_0, \ldots, p_q$ of points such that $p_k$ and $p_{k+1}$ are neighbours for $k \in \{0, \ldots, q-1\}$.

The length of that itinerary is $\ell(p_0, \ldots, p_q) = q - 1 \sum_{k=0}^{q-1} d(p_k, p_{k+1})$.

The distance between two points $p$ and $p'$ of $A \times [0,1] \sqcup V$ is $d(p, p') = \inf \ell(p_0, \ldots, p_q) | p_0, \ldots, p_q$ is a itinerary from $p$ to $p'$.

The metric graph associated with $G$ is the metric space $A \times [0,1] \sqcup V$, $d$.
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$$\left( A \times ]0,1[ \sqcup V , d \right)$$
Open balls

The open ball of radius $r < 1$ centered at the vertex $v$ is the set 
\[
\{ v \} \cup \{ a : \partial - a = v \} \times [0, r] \cup \{ a : \partial + a = v \} \times [1 - r, 1] \]

For $(a, t) \in \{ a \} \times [0, 1]$ the open ball of radius $r \leq \min\{t, 1 - t\}$ centered at the vertex $(a, t)$ is the set 
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\{ a \} \times [t - r, t + r] \]

That collection of open balls forms a basis of open sets.
The open ball of radius $r < 1$ centered at the vertex $v$ is the set

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The metric graph construction is functorial from $\text{Grph}$ to $\text{Met rac}$.

Every finite graph with weighted arrows (in $\mathbb{R}^+$) can be embedded in $\mathbb{R}^3$. 
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Every finite graph with weighted arrows (in $\mathbb{R}_+ \setminus \{0\}$) with can be embedded in $\mathbb{R}^3$. 
LOCALLY ORDERED METRIC GRAPHS
Partially Ordered Spaces
Partially ordered spaces

*Topology and Order*, L. Nachbin, 1965

A partially ordered space (or pospace) is a topological space $X$ together with a partial order $\sqsubseteq$ on (the underlying set of) $X$ such that $\{(a, b) \in X \times X \mid a \sqsubseteq b\}$ is a closed subset of $X \times X$. A pospace morphism is an order-preserving continuous map. Pospaces and their morphisms form the category $\text{PoSp}$. The underlying space of a pospace is Hausdorff.
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Pospaces and their morphisms form the category $\text{PoSp}$.

The underlying space of a pospace is Hausdorff.
Examples

- The real line with standard topology and order.
- Any subset of a pospace with the induced topology and order.
- The collection of compact subsets of a metric space equipped with the Hausdorff distance is a metric space.

$$d_{H}(K, K') = \sup_{x \in K, x' \in K'} \left( d(x, K') + d(x', K) \right)$$

The induced topological space ordered by inclusion is a pospace.

- Problem: there is no pospace on the circle whose collection of directed paths is $e^{i\theta}(t)$ for $\theta : [0, \pi] \rightarrow \mathbb{R}$ increasing.
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\[ d_H(K, K') = \sup \{ d(x, K'), d(x', K) \mid x \in K; x' \in K' \} \]

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- **Problem**: there is no pospace on the circle whose collection of directed paths is

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\{e^{i\theta(t)} | \theta : [0, r] \to \mathbb{R} \text{ increasing}\}
\]
Ordered Atlases
Let $X$ be a Hausdorff space. An (ordered) chart on $X$ is a pospace $U$ whose underlying space is an open subset of $X$. An (ordered) atlas is a collection $U$ of ordered charts on $X$ such that:

- the underlying spaces of the charts form a basis of the topology of $X$, and
- for all $U, V \in U$ for all $x \in U \cap V$ there exists $W \in U$ such that $x \in W \subseteq U \cap V$ and denoting by $\preceq_U|W$ the relation induced by $\preceq_U$ on the underlying set of $W$, the restrictions of $\preceq_U$ and $\preceq_V$ to $W$ match $\preceq_W$.

$\preceq_U|W = \preceq_W = \preceq_V|W$

Any subset of $X$ inherits an ordered atlas from $U$.
Ordered atlas

Let $X$ be a Hausdorff space.
Ordered atlas

*Ordered manifolds, invariant cone fields, and semigroups*, J. D. Lawson, 1989

*Algebraic topology and concurrency*, L. Fajstrup, É. Goubault, and M. Raußen, 1998

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An *(ordered) atlas* is a collection $\mathcal{U}$ of ordered charts on $X$ such that:

- the underlying spaces of the charts form a basis of the topology of $X$, and
- for all $U, V \in \mathcal{U}$, for all $x \in U \cap V$ there exists $W \in \mathcal{U}$ such that $x \in W \subseteq U \cap V$ and denoting by $\preceq_{U|W}$ the relation induced by $\preceq_U$ on the underlying set of $W$, the restrictions of $\preceq_U$ and $\preceq_V$ to $W$ match $\preceq_W$. 

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$$\sqsubseteq_{U \upharpoonright W} = \sqsubseteq_W = \sqsubseteq_{V \upharpoonright W}$$

Any subset of $X$ inherits an ordered atlas from $\mathcal{U}$. 
Ordered atlas
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Locally ordered space

Two atlases on the same space are compatible when their union is still an atlas. The relation of compatibility is an equivalence relation. The union of all the atlases of a given equivalence class is still an atlas i.e. every equivalence class contains a greatest element for inclusion.

A local pospace is a Hausdorff space together with an equivalence class of ordered atlas.
Two atlases on the same space are compatible when their union is still an atlas.
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The locally ordered line

Examples of equivalent atlases on $\mathbb{R}$
The locally ordered line

Examples of equivalent atlases on \( \mathbb{R} \)

- \( \{(I, \le) \mid I \text{ open interval of } \mathbb{R}\} \),
The locally ordered line

Examples of equivalent atlases on $\mathbb{R}$

- $\{(I, \leq) \mid I \text{ open interval of } \mathbb{R}\}$,
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Examples of equivalent atlases on \( \mathbb{R} \)

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- $\{(I, \leq) \mid I \text{ open interval of } \mathbb{R}\}$,
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- $\{(U, \sqsubseteq'_U) \mid U \text{ open subset of } \mathbb{R}\}$ where $x \sqsubseteq'_U y$ is any extension of $\sqsubseteq_U$. 
The locally ordered circle

Examples of equivalent atlases on $S^1$
The locally ordered circle

Examples of equivalent atlases on $\mathbb{S}^1$

- $\{(A, \preceq) \mid A \text{ open arc} \}$ where $\preceq$ is the order induced by $\mathbb{R}$ and the restriction of the exponential map to an open subinterval of $\{t \in \mathbb{R} \mid e^{it} \in A\}$ of length at most $2\pi$, 
The locally ordered circle

Examples of equivalent atlases on $S^1$

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- $\{(U, \sqsubseteq_U) \mid U \text{ proper open subset of } S^1\}$ where $x \sqsubseteq_U y$ means that the anticlockwise compact arc from $x$ to $y$ is included in $U$, 
The locally ordered circle

Examples of equivalent atlases on $\mathbb{S}^1$

- $\{(A, \leq) \mid A \text{ open arc}\}$ where $\leq$ is the order induced by $\mathbb{R}$ and the restriction of the exponential map to an open subinterval of $\{t \in \mathbb{R} \mid e^{it} \in A\}$ of length at most $2\pi$,

- $\{(U, \sqsubseteq_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $x \sqsubseteq_U y$ means that the anticlockwise compact arc from $x$ to $y$ is included in $U$,

- $\{(U, \sqsubseteq'_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $\sqsubseteq'_U$ is any extension of the partial order $\sqsubseteq_U$. 
Basic Properties
Morphisms

An atlas morphism from $U$ to $V$ is a map $f$ (between the underlying sets of $U$ and $V$) such that for all $x \in \text{dom}(f)$ there exists an ordered chart $U \in U$ and an ordered chart $V \in V$ such that $x \in U$ and $f$ induces a pospace morphism from $U$ to $V$ (implicitly $f(U) \subseteq V$).

e.g. $t \in [0, 1] \cup [2, 3] \mapsto t + 2 \pmod{4} \in [0, 1] \cup [2, 3]$

Let $f$ be an atlas morphism from $U$ to $V$. If $U \sim U'$ and $V \sim V'$ then $f$ is also an atlas morphism from $U'$ to $V'$.

A local pospace morphism from $(X, [U]) \sim (Y, [V])$ is a map from $X$ to $Y$ inducing an atlas morphism from $U$ to $V$.

A local pospace morphism defined over a locally ordered compact interval is called a directed path.
Morphisms

An atlas morphism from $\mathcal{U}$ to $\mathcal{V}$ is a map $f$ (between the underlying sets of $\mathcal{U}$ and $\mathcal{V}$) such that for all $x \in \text{dom}(f)$ there exists an ordered chart $U \in \mathcal{U}$ and an ordered chart $V \in \mathcal{V}$ such that $x \in U$ and $f$ induces a pospace morphism from $U$ to $V$ (implicitly $f(U) \subseteq V$).
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If $\mathcal{U} \sim \mathcal{U}'$ and $\mathcal{V} \sim \mathcal{V}'$ then $f$ is also an atlas morphism from $\mathcal{U}'$ to $\mathcal{V}'$.

A local pospace morphism from $(X, [\mathcal{U}]_{\sim})$ to $(Y, [\mathcal{V}]_{\sim})$ is a map from $X$ to $Y$ inducing an atlas morphism from $\mathcal{U}$ to $\mathcal{V}$. 
Morphisms

An atlas morphism from $\mathcal{U}$ to $\mathcal{V}$ is a map $f$ (between the underlying sets of $\mathcal{U}$ and $\mathcal{V}$) such that for all $x \in \text{dom}(f)$ there exists an ordered chart $U \in \mathcal{U}$ and an ordered chart $V \in \mathcal{V}$ such that $x \in U$ and $f$ induces a pospace morphism from $U$ to $V$ (implicitly $f(U) \subseteq V$).

e.g. $t \in [0, 1] \cup [2, 3] \Rightarrow t + 2 \pmod{4} \in [0, 1] \cup [2, 3]$

Let $f$ be an atlas morphism from $\mathcal{U}$ to $\mathcal{V}$.
If $\mathcal{U} \sim \mathcal{U}'$ and $\mathcal{V} \sim \mathcal{V}'$ then $f$ is also an atlas morphism from $\mathcal{U}'$ to $\mathcal{V}'$.

A local pospace morphism from $(X, [\mathcal{U}]_\sim)$ to $(Y, [\mathcal{V}]_\sim)$ is a map from $X$ to $Y$ inducing an atlas morphism from $\mathcal{U}$ to $\mathcal{V}$.

A local pospace morphism defined over a locally ordered compact interval is called a directed path.
Pospaces as local pospaces

Each pospace \((X, \subseteq)\) can be seen as a local pospace \((X, \subseteq | U)\) where \(U\) is an open subset of \(X\). The resulting functor is:

- faithful
- not injective on object (hence not an embedding)
- not full
Locally ordered metric graphs

Basic properties

Pospaces as local pospaces

Each pospace \((X, \sqsubseteq)\) can be seen as a local pospace

\[
\left( X, \left\{ (U, \sqsubseteq|_U) \mid U \text{ open subset of } X \right\} \right)
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The resulting functor is:
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Directed loops on local pospaces.
A local pospace morphism $\delta : [a, b] \to X$ whose image is contained in a chart $C$ induces a pospace morphism from $[a, b]$ to $C$. 
Directed loops on local pospaces

A local pospace morphism $\delta : [a, b] \to X$ whose image is contained in a chart $C$ induces a pospace morphism from $[a, b]$ to $C$.

A directed path $\delta$ on a local pospace $X$ is constant iff its extremities are equal and there exists an ordered chart of some atlas of $X$ that contains the image of $\delta$. 
A local pospace morphism $\delta : [a, b] \to X$ whose image is contained in a chart $C$ induces a pospace morphism from $[a, b]$ to $C$.

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A vortex is a point every neighbourhood of which contains a non-constant directed loop.
Directed loops on local pospaces

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A vortex is a point every neighbourhood of which contains a non-constant directed loop.

A local pospace has no vortex.
Ordered Atlas on Metric Graphs
A convenient open covering

Let $B$ be the collection of open balls $B$ of $|G|$ such that $-B$ is centred at a vertex and its radius is $\leq 1/3$, or $-B = \{a\} \times U$ for some arrow $a$ and some open interval $U \subseteq [0,1]$ of length $\leq 1/3$.

Given $B, B' \in B$ if $B$ is of the second kind, then so is $B \cap B'$.

If $B, B'$ are centred at $v$ and $v'$ we have $-v \neq v' \Rightarrow B \cap B' = \emptyset$ and $-v = v' \Rightarrow B \subseteq B'$ or $B' \subseteq B$.
A convenient open covering

Let $\mathcal{B}$ be the collection of open balls $B$ of $|G|$ such that
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If $B, B'$ are centred at $v$ and $v'$ we have
- $v \neq v' \Rightarrow B \cap B' = \emptyset$ and
- $v = v' \Rightarrow B \subseteq B'$ or $B' \subseteq B$
An element $B$ of $B$ centred at $v$ of radius $r \leq 1/3$ is the disjoint union of $\{v\}$ together with $-\{a\} \times [0, r]$ for each arrow $a$ such that $\partial^- a = v - \{a\} \times [1-r, 1]$ for each arrow $a$ such that $\partial^+ a = v$.

The partial order on $B$ is characterized by the following constraints:

- each branch $\{a\} \times [1-r, 1]$ and $\{a\} \times [0, r]$ inherits its order from $\{v\} \sqsubseteq \{a\} \times [0, r]$ for each arrow $a$ such that $\partial^- a = v - \{a\} \times [1-r, 1]$.

We have $B \cap B' \neq \emptyset$ $\Rightarrow$ $B \cap B' \in B$ and $\sqsubseteq B | B \cap B' = \sqsubseteq B \cap B' = \sqsubseteq B' | B \cap B'$. The metric graph of $|G|$ thus becomes a local pospace.

The locally ordered metric graph construction is functorial.
Ordered open stars

An element $B$ of $B$ centred at $v$ of radius $r \leq \frac{1}{3}$ is the disjoint union of $\{v\}$ together with
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We have $B \cap B' \neq \emptyset \Rightarrow B \cap B' \in \mathcal{B}$ and

$$\sqsubseteq_{B|_{B \cap B'}} = \sqsubseteq_{B \cap B'} = \sqsubseteq_{B'|_{B \cap B'}}$$
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The metric graph of $|G|$ thus becomes a local pospace.
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The partial order on $B$ is characterized by the following constraints:
- each branch $\{a\} \times ]1 - r, 1[$ and $\{a\} \times ]0, r[ \text{ inherits its order from } \mathbb{R}$
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