Directed Algebraic Topology and Concurrency

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MPRI : Concurrency (2.3)

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- We define the homset $\mathcal{C}(x, y) := \left\{ \gamma \in \mathrm{Mo}(\mathcal{C}) \;\middle|\; \partial^{-}\gamma = x \text{ and } \partial^{+}\gamma = y \right\}$

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- The binary composition is a partially defined and often denoted by $\circ$

$$\Big\{(\gamma,\delta) \;\Big|\; \gamma,\delta \text{ morphisms of } \mathcal{C} \text{ s.t. } \partial^{-}\gamma = \partial^{+}\delta\Big\} \xrightarrow{\text{composition}} \mathrm{Mo}(\mathcal{C})$$

$$\begin{tikzcd} & \partial^{+}\delta = \partial^{-}\gamma \arrow[dr, "\gamma"] & \\ \partial^{-}\delta \arrow[ur, "\delta"] \arrow[rr, "\gamma\circ\delta"'] & & \partial^{+}\gamma \end{tikzcd}$$

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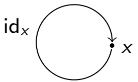

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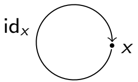

- For all morphisms γ one has $\mathrm{id}_{\partial^{+}\gamma} \circ \gamma = \gamma = \gamma \circ \mathrm{id}_{\partial^{-}\gamma}$

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- The opposite of a category is obtained by reversing all its arrows (i.e. by swapping the roles of the source and the target)

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- any isomorphism is both monomorphism and an epimorphism, the converse is false in general (e.g. $\mathscr{Pos}$).
- if $r \circ s = \mathrm{id}$ then r is called a retract/split epimorphism and s is called a section/split monomorphism.

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with $\mathrm{s}'(\phi_1(\alpha)) = \phi_0(\partial^- \alpha)$ and $\mathrm{t}'(\phi_1(\alpha)) = \phi_0(\partial^+ \alpha)$

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Topological spaces and continuous maps form the category $\mathcal{Top}$

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Every subset of a Hausdorff space is saturated.

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A Hausdorff space is locally compact iff each of its points admits a compact neighbourhood.

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with $\mathrm{s}'(\mathrm{Mo}(f)(\alpha)) = \mathrm{Ob}(f)(\partial^{-}\alpha)$ and $\mathrm{t}'(\mathrm{Mo}(f)(\alpha)) = \mathrm{Ob}(f)(\partial^{+}\alpha)$

Hence it is in particular a morphism of graphs.

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and satisfies $\mathrm{Mo}(f)(\gamma \circ \delta) = \mathrm{Mo}(f)(\gamma) \circ \mathrm{Mo}(f)(\delta)$

$$\begin{tikzcd}
x \arrow[r, "\delta"'] \arrow[rr, bend left, "\gamma\circ\delta"] & y \arrow[r, "\gamma"'] & z
\end{tikzcd} \qquad \begin{tikzcd}
f(x) \arrow[r, "f(\delta)"'] \arrow[rr, bend left, "f(\gamma\circ\delta)"] & f(y) \arrow[r, "f(\gamma)"'] & f(z)
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The small categories and their funtors form a (large) category denoted by $\mathcal{C}at$

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Problem: prove the topological spaces X and Y are *not* the same

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Problem: prove the topological spaces X and Y are *not* the same
Strategy: find a functor F defined over $\mathcal{Top}$ such that $F(X) \ncong F(Y)$

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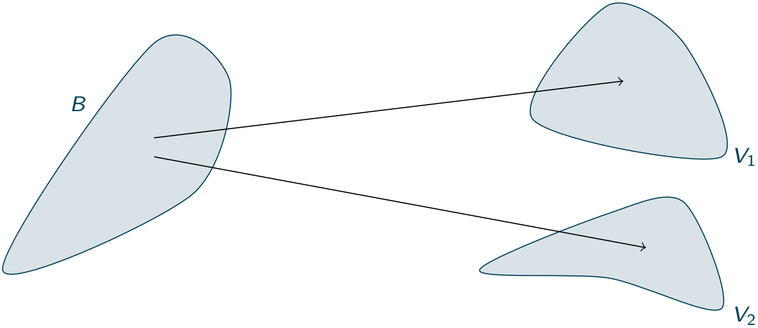
$$\mathcal{T}op \xrightarrow{\pi_0} \mathcal{S}et$$

$$\begin{tikzcd} X \arrow[d, "f"] & & \pi_0(X) \arrow[d] \\ Y \arrow[r, phantom, "f \longmapsto \pi_0(f)"] & & \pi_0(Y) \end{tikzcd}$$

An application involving basic (algebraic) topology

The continuous image of a connected space is connected

The image of the space B is entirely contained in a connected component of the space V.



The set of connected components

is a functorial construction

This situation is abstracted by classifying continuous maps from B to V according to which connected component (V_1 or V_2) the single connected components of B (namely B itself) is sent to. There are exactly two set theoretic maps from the singleton $\{B\}$ to the pair $\{V_1, V_2\}$ hence there is at most (in fact exactly) two kinds of continuous maps from B to V.

$$\{B\} \rightrightarrows \{V_1, V_2\}$$

In particular B and V are not homeomorphic.

Application

The compact interval and the circle are not homeomorphic

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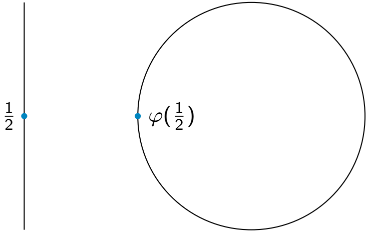
Then φ induces a homeomorphism

$$[0, \tfrac{1}{2}[\cup]\tfrac{1}{2}, 1] \ \to \ \mathbb{S}^1 \backslash \{\varphi(\tfrac{1}{2})\}$$

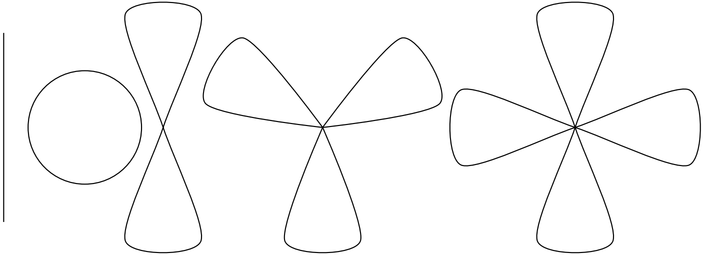
which does not exist!



Generalization

Bouquets of circles

These topological spaces are pairwise not homeomorphic. Why ?



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$$\begin{aligned} f_{x,y} : \ \mathcal{C}[x, y] &\longrightarrow \mathcal{D}[\mathrm{Ob}(f)(x), \mathrm{Ob}(f)(y)] \\ \alpha &\longmapsto \mathrm{Mo}(f)(\alpha) \end{aligned}$$

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- f is fully faithful when it is full and faithful
- f is an embedding when it is faithful and $\mathrm{Ob}(f)$ is one-to-one

Some small functors

(functor between small categories)

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The actions of a monoid M over a set X are the functors from M to $\mathcal{Set}$ which sends the only element of M to X

Some full embeddings in $\mathcal{C}at$

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Metric spaces

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Goal: turn any graph into metric space in a natural way.

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The length $\ell(\gamma)$ of a path $\gamma : [0, r] \to (X, d)$ is the least upper bound of the collection of sums

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The space is said to be geodesic when any two points are related by a geodesic path.

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Metric Spaces of Non-Positive Curvature, *M. R. Bridson, and A. Haefliger*, 1999

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- Any metric space (X, d) is associated with a length space (X, d_ℓ) with

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- Every finite graph with weighted arrows (in $\mathbb{R}_+ \setminus \{0\}$) with can be embedded in $\mathbb{R}^3$.

Neighbours

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- The underlying set of the metric graph is $A \times]0,1[\sqcup V$
- Two points p, p' are said to be neighbours when there is an arrow a such that $p, p' \in \{a\} \times]0,1[\sqcup \{\partial^- a, \partial^+ a\}$

Distance between two neighbours

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- If $\partial^- a \neq \partial^+ a$ there is a canonical bijection

$$\phi : \{a\} \times]0,1[\ \sqcup \{\partial^- a, \partial^+ a\} \to [0,1]$$

In that case $d(p, p') = |t - t'|$ with $t = \phi(p)$ and $t' = \phi(p')$.

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An itinerary on $A\times]0,1[\ \sqcup V$ is a (finite) sequence $p_0, \ldots, p_q$ of points such that p_k and p_{k+1} are neighbours for $k \in \{0, \ldots, q-1\}$.

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The metric graph construction is functorial.

Open balls

Open balls

The open ball of radius $r < 1$ centered at the vertex v is the set

$$\{v\} \quad \cup \quad \{a \mid \partial^- a = v\}\times]0,r[\quad \cup \quad \{a \mid \partial^+ a = v\}\times]1-r,1[$$

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That collection of open balls forms a basis of open sets.

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Topology and Order, *L. Nachbin*, 1965

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is a closed subset of $X \times X$.

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Partially ordered spaces

Topology and Order, *L. Nachbin*, 1965

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The underlying space of a pospace is Hausdorff.

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$$\begin{aligned} d_H(K, K') &= \sup \left\{ d(x, K'), d(x', K) \mid x \in K; x' \in K' \right\} \\ d(x, K) &= \inf \left\{ d(x, k) \mid k \in K \right\} \end{aligned}$$

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- Problem: there is no pospace on the circle whose collection of directed paths is

$$\left\{\rho(t) \cdot e^{i\theta(t)} \mid \rho, \theta : [0, r] \to \mathbb{R}_+ \text{ increasing}\right\}$$

Ordered atlas

Algebraic topology and concurrency, *L. Fajstrup, É. Goubault, and M. Raußen*, 1998

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Any subset of X inherits an ordered atlas from $\mathcal{U}$.

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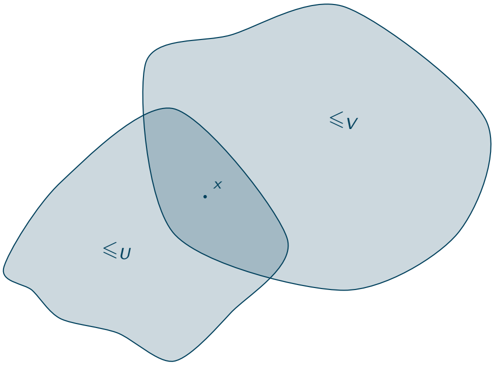

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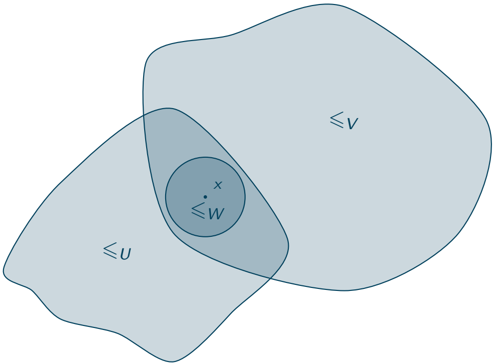

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The locally ordered circle

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- $\{(A, \leqslant) \mid A \text{ open arc}\}$ where $\leqslant$ is the order induced by $\mathbb{R}$ and the restriction of the exponential map to an open subinterval of $\{t \in \mathbb{R} \mid e^{it} \in A\}$ of length at most 2π,

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Local pospace morphisms

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An atlas morphism from $\mathcal{U}$ to $\mathcal{V}$ is a map f (between the underlying sets of $\mathcal{U}$ and $\mathcal{V}$) such that for all $x \in \mathrm{dom}(f)$ there exists an ordered chart $U \in \mathcal{U}$ and an ordered chart $V \in \mathcal{V}$ such that $x \in U$ and f induces a pospace morphism from U to V (implicitly $f(U) \subseteq V$).

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A local pospace morphism defined over a locally ordered compact interval is called a directed path.

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A local pospace has no vortex.

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The locally ordered metric graph construction is functorial.