A BIT OF CATEGORY THEORY
Categories
Category $\mathcal{C}$

Definition (the “underlying graph” part)
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- $\text{Ob}(\mathcal{C})$: collection of objects
Category \( \mathcal{C} \)

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- \( \text{Ob}(\mathcal{C}) \) : collection of objects
- \( \text{Mo}(\mathcal{C}) \) : collection of morphisms
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Definition (the “underlying graph” part)

- $\text{Ob}(\mathcal{C})$: collection of objects
- $\text{Mo}(\mathcal{C})$: collection of morphisms
- $\partial^-, \partial^+$: mappings source, target as follows

$$\text{Mo}(\mathcal{C}) \xrightarrow{\partial^-} \text{Ob}(\mathcal{C}) \xrightarrow{\partial^+}$$
Category $\mathcal{C}$

Definition (the “underlying graph” part)

- $\text{Ob}(\mathcal{C})$: collection of objects
- $\text{Mo}(\mathcal{C})$: collection of morphisms
- $\partial^-, \partial^+ $: mappings source, target as follows

\[
\text{Mo}(\mathcal{C}) \xrightarrow{\partial^-} \text{Ob}(\mathcal{C})
\]

- We define the homset $\mathcal{C}(x,y) := \{\gamma \in \text{Mo}(\mathcal{C}) \mid \partial^- \gamma = x \text{ and } \partial^+ \gamma = y\}$
Category $C$

Definition (the "underlying local monoid" part)
Category \( \mathcal{C} \)

Definition (the “underlying local monoid” part)

- \( \text{id} \) : provides each object with an identity

\[
\text{Mo}(\mathcal{C}) \xleftarrow{id} \xrightarrow{\partial^{-}} \text{Ob}(\mathcal{C}) \xrightarrow{\partial^{+}} \text{Ob}(\mathcal{C})
\]
**Category $\mathcal{C}$**

**Definition (the “underlying local monoid” part)**

- **id**: provides each object with an identity

\[ \text{Mo}(\mathcal{C}) \xleftrightarrow{\text{id}} \text{Ob}(\mathcal{C}) \]

- The binary composition is a partially defined and often denoted by $\circ$

\[
\begin{array}{c}
\left\{ (\gamma, \delta) \mid \gamma, \delta \text{ morphisms of } \mathcal{C} \text{ s.t. } \partial^- \gamma = \partial^+ \delta \right\} \xrightarrow{\text{composition}} \text{Mo}(\mathcal{C})
\end{array}
\]

\[
\begin{array}{c}
\partial^+ \delta = \partial^- \gamma \\
\partial^+ \delta \quad \delta \quad \gamma \quad \gamma \circ \delta \\
\partial \delta \rightarrow \gamma^\circ \delta \rightarrow \partial^- \gamma
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\]
Category $\mathcal{C}$

Definition (the axioms)

- The composition law is associative
- For all objects $x$, one has $\partial - id_x = x = \partial + id_x$
- For all morphisms $\gamma$, one has $id_{\partial} + \gamma \circ \gamma = \gamma = \gamma \circ id_{\partial}$
Category $\mathcal{C}$

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- For all objects $x$ one has $\partial \cdot \text{id}_x = x = \partial \cdot \text{id}_x$

\[
\text{id}_x \quad \circlearrowright \quad x
\]

- For all morphisms $\gamma$ one has $\text{id}_{\partial \cdot \gamma} \circ \gamma = \gamma = \gamma \circ \text{id}_{\partial \cdot \gamma}$
Standard examples

- **Set**: the category of sets.
- **Mon**: the category of monoids.
- **Cmon**: the category of commutative monoids.
- **Gr**: the category of groups.
- **Pre**: the category of preordered sets.
- **Pos**: the category of posets.

- Any preordered set can be seen as a category in which any homset has at most one element.
- Any monoid can be seen as a category with a single object.
- The opposite of a category is obtained by reversing all its arrows (i.e. by swapping the roles of the source and the target).
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Some special kinds of morphisms

- **f** is an isomorphism when there exists **g** such that both **f** ◦ **g** and **g** ◦ **f** are identities.

- Two objects related by an isomorphism are said to be isomorphic.

- **f** is a monomorphism when it is left-cancellative i.e. for all **g**₁, **g**₂, **f** ◦ **g**₁ = **f** ◦ **g**₂ implies **g**₁ = **g**₂.

- **f** is an epimorphism when it is right-cancellative i.e. for all **g**₁, **g**₂, **g**₁ ◦ **f** = **g**₂ ◦ **f** implies **g**₁ = **g**₂.

- Any isomorphism is both monomorphism and an epimorphism, the converse is false in general (e.g. **Pos**).

- If **r** ◦ **s** = id then **r** is called a retract/split epimorphism and **s** is called a section/split monomorphism.
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- any isomorphism is both monomorphism and an epimorphism, the converse is false in general (e.g. $\text{Pos}$).
- if $r \circ s = \text{id}$ then $r$ is called a **retract/split epimorphism** and $s$ is called a **section/split monomorphism**.
The category of graphs ($\mathbf{Grph}$)

The elements of $V$ are the vertices and those of $A$ are the arrows. In particular $A$ and $V$ are sets.
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**Objects**

\[
\begin{array}{c}
\text{\(A\)} \\
\text{\(s\)} \\
\text{\(\downarrow\)} \\
\text{\(V\)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\(t\)} \\
\end{array}
\]

Composition

\[
\begin{array}{c}
\text{\(A\)} \\
\text{\(A'\)} \\
\text{\(V\)} \\
\text{\(V'\)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\(\phi_1\)} \\
\text{\(\phi_0\)} \\
\text{\(\psi_1\)} \\
\text{\(\psi_0\)} \\
\end{array}
\]

with $s'(\phi_1(\alpha)) = \phi_0(\partial^- \alpha)$ and $t'(\phi_1(\alpha)) = \phi_0(\partial^+ \alpha)$.
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The elements of \(V\) are the vertices and those of \(A\) are the arrows.
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**Objects**

\[
\begin{array}{c}
A \\
\downarrow s \\
V \\
\downarrow t
\end{array}
\]

**Morphisms**

\[
\begin{array}{c}
A \\
\downarrow s \\
V \\
\downarrow t
\end{array}
\xrightarrow{\phi_1} \begin{array}{c}
A' \\
\downarrow s' \\
V' \\
\downarrow t'
\end{array}
\]

\[
\begin{array}{c}
V \\
\downarrow \phi_0
\end{array}
\]
The category of graphs ($\text{Grph}$)

The elements of $V$ are the vertices and those of $A$ are the arrows. In particular, $A$ and $V$ are sets.

**Objects**

- $A \xrightarrow{s} V$
- $A' \xleftarrow{s'} V'$

**Morphisms**

- $A \xrightarrow{\phi_1} A'$
- $V \xrightarrow{\phi_0} V'$

**Composition**

- $A \xrightarrow{\phi_1} A' \xrightarrow{\psi_1} A''$
- $V \xrightarrow{\phi_0} V' \xrightarrow{\psi_0} V''$

\[s'(\phi_1(\alpha)) = \phi_0(\partial - \alpha) \quad \text{and} \quad t'(\phi_1(\alpha)) = \phi_0(\partial + \alpha)\]
The category of graphs ($\text{Grph}$)

The elements of $V$ are the vertices and those of $A$ are the arrows. In particular $A$ and $V$ are sets.

<table>
<thead>
<tr>
<th>Objects</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$A \xrightarrow{\phi_1} A'$</td>
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</tr>
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with $s'(\phi_1(\alpha)) = \phi_0(\partial^- \alpha)$ and $t'(\phi_1(\alpha)) = \phi_0(\partial^+ \alpha)$
A base of a topology is a collection of sets $\mathcal{B}$ such that for all $U, V \in \mathcal{B}$, all $p \in U \cap V$, there exists $W \in \mathcal{B}$ such that $p \in W \subseteq U \cap V$. 

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The category of bases of topologies ($\mathcal{B}$)

A map $f : \mathcal{B} \to \mathcal{B}'$ is continuous when
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A map $f : \mathcal{B} \to \mathcal{B}'$ is *continuous* when for every point $p$ of $\mathcal{B}$,
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A map $f : \mathcal{B} \rightarrow \mathcal{B}'$ is continuous when for every point $p$ of $\mathcal{B}$, every $V \in \mathcal{B}'$ with $f(p) \in V$, there exists $U \in \mathcal{B}$ with $p \in U$ such that $f(U) \subseteq V$. 

\[ p \quad f(p) \quad V \]
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The category of topological spaces ($\mathcal{Top}$)

A topological space is a set $X$ and a collection $\Omega_X \subseteq P(X)$ s.t.

1) $\emptyset \in \Omega_X$ and $X \in \Omega_X$
2) $\Omega_X$ is stable under union
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Equivalently, a topological space is a base of a topology stable under union.

A continuous map $f : (X, \Omega_X) \to (Y, \Omega_Y)$ is a map $f : X \to Y$ s.t.

$\forall x \in X$ $\forall V \in \Omega_Y$ s.t. $f(x) \in V$, $\exists U \in \Omega_X$ s.t. $x \in U$ and $f(U) \subseteq V$ or equivalently

$\forall V \in \Omega_Y$ $f^{-1}(V) \in \Omega_X$

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Functors
Functors $f$ from $C$ to $D$

Definition (preserving the “underlying graph”)
Functors \( f \) from \( C \) to \( D \)

Definition (preserving the “underlying graph”)

A functor \( f : C \to D \) is defined by two “mappings” \( \text{Ob}(f) \) and \( \text{Mo}(f) \) such that
Functors $f$ from $\mathcal{C}$ to $\mathcal{D}$

Definition (preserving the “underlying graph”)

A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is defined by two “mappings” $\text{Ob}(f)$ and $\text{Mo}(f)$ such that

$$\begin{align*}
\text{Mo}(\mathcal{C}) & \xrightarrow{\partial^{-}} \text{Ob}(\mathcal{C}) \\
\text{Mo}(\mathcal{D}) & \xrightarrow{\partial'^{-}} \text{Ob}(\mathcal{D}) \\
\text{Mo}(f) & \downarrow \quad \downarrow \\
\text{Ob}(f) & \text{Ob}(f)
\end{align*}$$

with $\partial'^{+}(\text{Mo}(f)(\alpha)) = \text{Ob}(f)(\partial^{+}\alpha)$ and $\partial'^{-}(\text{Mo}(f)(\alpha)) = \text{Ob}(f)(\partial^{-}\alpha)$

Hence it is in particular a morphism of graphs.
Functors $f$ from $C$ to $D$

Definition (preserving the “underlying local monoid”)

The “mappings” $\text{Ob}(f)$ and $\text{Mo}(f)$ also make the following diagram commute:

\[
\begin{array}{ccc}
\text{Ob}(C) & \xrightarrow{\text{Mo}(f)} & \text{Ob}(D) \\
\text{Mo}(C) & \xrightarrow{\text{id}} & \text{Mo}(D) \\
\text{Mo}(f) & \xrightarrow{\text{id}} & \text{id}
\end{array}
\]

and satisfies $\text{Mo}(f)(\gamma \circ \delta) = \text{Mo}(f)(\gamma) \circ \text{Mo}(f)(\delta)$.
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$$
\begin{array}{ccc}
\text{Mo}(\mathcal{C}) & \xhookleftarrow{\text{id}} & \text{Ob}(\mathcal{C}) \\
\downarrow \text{Mo}(f) & & \downarrow \text{Ob}(f) \\
\text{Mo}(\mathcal{D}) & \xhookleftarrow{\text{id}'} & \text{Ob}(\mathcal{D})
\end{array}
$$
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\downarrow & & \downarrow \\
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Hence functors should be thought of as morphisms of categories.

The small categories and their functors form a (large) category denoted by $\mathbf{Cat}$. 
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Some forgetful functors
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\((M, *, e) \in \text{Mon} \mapsto M \in \text{Set}\)

\((X, \Omega) \in \text{Top} \mapsto X \in \text{Set}\)

\((X, \sqsubseteq) \in \text{Pos} \mapsto X \in \text{Set}\)
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\[C \in \text{Cat} \mapsto \text{Ob}(C) \in \text{Set}\]
\[C \in \text{Cat} \mapsto \text{Mo}(C) \in \text{Set}\]
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\[C \in \text{Cat} \mapsto \left( \text{Mo}(C) \xrightarrow{\partial^+} \text{Ob}(C) \right) \in \text{Grph}\]
Some small functors

(functor between small categories)
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The morphisms of monoids are the functors between small categories with a single object
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The morphisms of preordered sets are the functors between small categories whose homsets contain at most one element.
Some small functors
(functor between small categories)

The morphisms of monoids are the functors between small categories with a single object.

The morphisms of preordered sets are the functors between small categories whose homsets contain at most one element.

The actions of a monoid $M$ over a set $X$ are the functors from $M$ to $\text{Set}$ which sends the only element of $M$ to $X$. 
Given a functor $f: C \to D$ and two objects $x$ and $y$ we have the mapping $f x, y: C[x, y] \to D[\text{Ob}(f)(x), \text{Ob}(f)(y)]$.

- $f$ is faithful when for all objects $x$ and $y$ the mapping $f x, y$ is one-to-one (injective).
- $f$ is full when for all objects $x$ and $y$ the mapping $f x, y$ is onto (surjective).
- $f$ is fully faithful when it is full and faithful.
- $f$ is an embedding when it is faithful and $\text{Ob}(f)$ is one-to-one.
- $f$ is an equivalence when it is fully faithful and every object of $D$ is isomorphic to an object of the form $f(C)$ with $C \in C$. 


Given a functor $f : \mathcal{C} \to \mathcal{D}$ and two objects $x$ and $y$ we have the mapping

$$f_{x,y} : \mathcal{C}[x,y] \to \mathcal{D}[\text{Ob}(f)(x), \text{Ob}(f)(y)]$$

$$\alpha \mapsto \text{Mo}(f)(\alpha)$$
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Some full embeddings in \( \text{Cat} \)
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**Remark** : The full embeddings compose
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**Remark**: The full embeddings compose

$\text{Pre} \hookrightarrow \text{Cat}$

$\text{Mon} \hookrightarrow \text{Cat}$

$\text{Pos} \hookrightarrow \text{Pre}$

$\text{Gr} \hookrightarrow \text{Mon}$
Some full embeddings in $\textbf{Cat}$

Remark: The full embeddings compose

\[
\begin{align*}
\text{Pre} &\hookrightarrow \text{Cat} & \text{Cmon} &\hookrightarrow \text{Mon} \\
\text{Mon} &\hookrightarrow \text{Cat} & \text{Ab} &\hookrightarrow \text{Cmon} \\
\text{Pos} &\hookrightarrow \text{Pre} & \text{Ab} &\hookrightarrow \text{Gr} \\
\text{Gr} &\hookrightarrow \text{Mon} & \text{Set} &\hookrightarrow \text{Pos}
\end{align*}
\]
Topological spaces and their bases

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Full embedding $I : \text{Top} \to \text{Bas}$. 
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Space functor $sp : \text{Bas} \to \text{Top}$ sending $B$ to $\{ \bigcup C \mid C \subseteq B \}$. 
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Given $B \in \text{Bas}$, we denote by $UB$ the underlying set of $B$, i.e. the union of all the elements of $B$. E.g.: bases of $\mathbb{R}^2$. 

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Given $B \in \text{Bas}$, the identity map on $UB$ induces an isomorphism from $B$ to $sp(B)$ which we denote by $B \Rightarrow sp(B)$; and an isomorphism from $sp(B)$ to $B$ which we denote by $Sp(B) \Rightarrow B$. We have $(B \Rightarrow sp(B))^{-1} = (Sp(B) \Rightarrow B)$.
Topological spaces and their bases

Full embedding $I : \text{Top} \to \text{Bas}$. 

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Given $B \in \text{Bas}$, the identity map on $UB$ induces an isomorphism from $B$ to $S_p(B)$ which we denote by $B \Rightarrow S_p(B)$; and an isomorphism from $S_p(B)$ to $B$ which we denote by $S_p(B) \Rightarrow B$. We have $(B \Rightarrow S_p(B))^{-1} = (S_p(B) \Rightarrow B)$. 

The functors $I$ and $S_p$ are equivalences of categories.
Natural transformations
Natural Transformations

morphisms of functors from $f : C \to D$ to $g : C \to D$
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morphisms of functors from \( f : C \to D \) to \( g : C \to D \)

A natural transformation \( \eta : f \to g \) is a collection of morphisms \( (\eta_x)_{x \in \text{Ob}(C)} \) where \( \eta_x \in \text{D}[f(x), g(x)] \) and such that for all \( \alpha \in C[x, y] \) we have \( \eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x \) i.e. the following diagram commute
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\[\begin{array}{ccc}
x & \xrightarrow{\alpha} & y \\
\downarrow{\eta_x} & & \downarrow{\eta_y} \\
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\begin{array}{ccc}
x & \xrightarrow{\alpha} & y \\
\downarrow^{f(x)} & & \downarrow^{g(x)} \\
f(y) & \xleftarrow{f(\alpha)} & f(y) \\
\downarrow^{\eta_x} & & \downarrow^{\eta_y} \\
g(y) & \xleftarrow{g(\alpha)} & g(y)
\end{array}
\]

This description is summarized by the following diagram:

\[
\begin{array}{c}
C \xrightarrow{f} D \\
\downarrow \eta \quad \quad \quad \quad \quad \downarrow g
\end{array}
\]
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$$

If every $\eta_x$ is an isomorphism of $D$, then $\eta$ is said to be a natural isomorphism, its inverse $\eta^{-1}$ is $(\eta_x^{-1})_{x \in \text{Ob}(C)}$. 
A functor \( f : \mathcal{C} \to \mathcal{D} \) is an equivalence iff there exists a functor \( g : \mathcal{D} \to \mathcal{C} \) and natural isomorphisms \( \text{id}_\mathcal{C} \cong g \circ f \) and \( \text{id}_\mathcal{D} \cong f \circ g \).
A functor $f : C \to D$ is an equivalence iff there exists a functor $g : D \to C$ and natural isomorphisms $\text{id}_C \cong g \circ f$ and $\text{id}_D \cong f \circ g$.

E.g.: we have $\text{id}_{\text{Top}} = I \circ \text{Sp}$ and the collection $B \Rightarrow \text{Sp}(B)$ for $B \in \text{Bas}$ is a natural isomorphism from $\text{id}_{\text{Bas}}$ to $\text{Sp} \circ I$. 
AN ALGEBRAIC TOPOLOGY TEASER
The overall idea of algebraic topology
The overall idea of algebraic topology

Every functor preserves the isomorphisms
The overall idea of algebraic topology

Every functor preserves the isomorphisms

Problem: prove the topological spaces $X$ and $Y$ are not the same
The overall idea of algebraic topology

Every functor preserves the isomorphisms

Problem: prove the topological spaces $X$ and $Y$ are not the same
Strategy: find a functor $F$ defined over $\text{Top}$ such that $F(X) \not\cong F(Y)$
Compactness
More topological notions

The interior of a subset \( A \) of \( X \) is the greatest open subset of \( X \) contained in \( A \).

Then closure of a subset \( A \) of \( X \) is the least closed subset of \( X \) containing \( A \).

A neighbourhood of a subset \( A \) of \( X \) is a subset of \( X \) whose interior contains \( A \).

A topological space \( X \) is said to be Hausdorff when for all \( x, x' \in X \), if \( x \neq x' \) then \( x \) and \( x' \) have disjoint neighbourhoods.

A subset \( Q \) of \( X \) is said to be saturated when \( Q = \bigcup_{U \text{ open and } Q \subseteq U} U \).

Every subset of a Hausdorff space is saturated.
More topological notions

The interior of a subset $A$ of $X$ is the greatest open subset of $X$ contained in $A$. 
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Every subset of a Hausdorff space is saturated.
Compactness and local compactness

Let $X$ be a topological space.

- An open covering of $X$ is a collection of open subsets of $X$ whose union is $X$.
- $X$ is said to be compact when every open covering of $X$ admits a finite sub-covering.
- $X$ is said to be locally compact when for every $x \in X$, every open neighbourhood $U$ of $x$ contains a saturated compact neighbourhood of $x$.

A Hausdorff space is locally compact iff each of its points admits a compact neighbourhood.
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Connectedness
The connected component functor
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1) A topological space $X$ is said to be connected when its only closed-open subsets are $\emptyset$ and $X$
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4) Any **connected** subset of $X$ is contained in a **connected** component of $X$
The connected component functor

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5) Any continuous direct image of a connected subset of $X$ is connected
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An application
The continuous image of a connected space is connected

The image of the space $B$ is entirely contained in a connected component of the space $V$. 

\[ B \rightarrow V_1 \rightarrow V_2 \]
This situation is abstracted by classifying continuous maps from $B$ to $V$ according to which connected component ($V_1$ or $V_2$) the single connected components of $B$ (namely $B$ itself) is sent to. There are exactly two set theoretic maps from the singleton $\{B\}$ to the pair $\{V_1, V_2\}$ hence there is at most (in fact exactly) two kinds of continuous maps from $B$ to $V$.

$$\{B\} \rightarrow \{V_1, V_2\}$$

In particular $B$ and $V$ are not homeomorphic.
Application

The compact interval and the circle are not homeomorphic
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The compact interval and the circle are not homeomorphic

Let $S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$ be the Euclidean circle and suppose $\varphi : [0, 1] \rightarrow S^1$ is a homeomorphism.
Application

The compact interval and the circle are not homeomorphic

Let $S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$ be the Euclidean circle
and suppose $\varphi : [0, 1] \rightarrow S^1$ is a homeomorphism.

Then $\varphi$ induces a homeomorphism

$[0, \frac{1}{2}] \cup \left[ \frac{1}{2}, 1 \right] \rightarrow S^1 \setminus \{ \varphi(\frac{1}{2}) \}$

which does not exist!
Generalization

Bouquets of circles

These topological spaces are pairwise not homeomorphic. Why?
METRIC SPACES
Categories of Metric Spaces
Metric spaces

A metric space is a set $X$ together with a mapping $d : X \times X \to \mathbb{R}^+ \cup \{\infty\}$ such that:

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

The open balls $B(x, r) = \{x \in X | d(x, c) < r\}$ with $x \in X$ and $r > 0$ form a base of a topology.

Goal: turn any graph into metric space in a functorial way.
A metric space is a set $X$ together with a mapping $d : X \times X \rightarrow \mathbb{R}_+ \cup \{\infty\}$ such that:

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Goal: turn any graph into metric space in a functorial way.
Metric space morphisms

- **Metric space embeddings** $\text{Met} \text{emb}$: $X \to Y$ s.t. $\forall x, x' \in X$, $d_Y(f(x), f(x')) = d_X(x, x')$

- **Metric space contractions** $\text{Met} \text{ctr}$: $X \to Y$ s.t. $\forall x, x' \in X$, $d_Y(f(x), f(x')) \leq d_X(x, x')$

- **Metric space maps** $\text{Met} f$: $X \to Y$ s.t. $\exists r \in ]0, \infty[ \forall x, x' \in X$, $d_Y(f(x), f(x')) \leq r \cdot d_X(x, x')$

- **Metric space topology** $\text{Met} \text{top}$: $X \to Y$ s.t. $\forall x \in X \forall \varepsilon > 0 \exists \eta > 0$, $B(x, \eta) \subseteq B(f(x), \varepsilon)$
Metric space morphisms

- $\text{Met}_{\text{emb}} : X \to Y$ s.t. $\forall x, x' \in X$, $d_Y(f(x), f(x')) = d_X(x, x')$
Metric space morphisms

- \( \text{Met}_{\text{emb}} \) \( f : X \to Y \) s.t. \( \forall x, x' \in X, \ d_Y(f(x), f(x')) = d_X(x, x') \)

- \( \text{Met}_{\text{ctr}} \) \( f : X \to Y \) s.t. \( \forall x, x' \in X, \ d_Y(f(x), f(x')) \leq d_X(x, x') \)
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- $\text{Met} f : X \to Y$ s.t. $\exists r \in ]0, \infty[ \; \forall x, x' \in X, \; d_Y(f(x), f(x')) \leq r \cdot d_X(x, x')$
- $\text{Met}_{\text{top}} f : X \to Y$ s.t. $\forall x \in X \; \forall \varepsilon > 0 \; \exists \eta > 0, \; f(B(x, \eta)) \subseteq B(f(x), \varepsilon)$
Metric space morphisms

- \( \text{Met}_{\text{emb}} \) \( f : X \to Y \) s.t. \( \forall x, x' \in X, \; d_Y(f(x), f(x')) = d_X(x, x') \)
- \( \text{Met}_{\text{ctr}} \) \( f : X \to Y \) s.t. \( \forall x, x' \in X, \; d_Y(f(x), f(x')) \leq d_X(x, x') \)
- \( \text{Met} \) \( f : X \to Y \) s.t. \( \exists r \in ]0, \infty[ \; \forall x, x' \in X, \; d_Y(f(x), f(x')) \leq r \cdot d_X(x, x') \)
- \( \text{Met}_{\text{top}} \) \( f : X \to Y \) s.t. \( \forall x \in X \; \forall \varepsilon > 0 \; \exists \eta > 0, \; f(B(x, \eta)) \subseteq B(f(x), \varepsilon) \)

\( \text{Met}_{\text{emb}} \hookrightarrow \text{Met}_{\text{ctr}} \hookrightarrow \text{Met} \hookrightarrow \text{Met}_{\text{top}} \overset{\text{full}}{\hookrightarrow} \text{Top} \)
Length spaces

The length \( \ell(\gamma) \) of a path \( \gamma : [0,r] \to (X,d) \) is the least upper bound of the collection of sums

\[
\sum_{i=0}^{n} d(\gamma(t_{i}+1), \gamma(t_{i}))
\]

where \( n \in \mathbb{N} \) and \( 0 = t_{0} \leq \cdots \leq t_{n} = r \).

The metric space \((X,d)\) is a length space when the distance between two points \( x, x' \in X \) is the following greatest lower bound

\[
\inf \{ \ell(\gamma) | \gamma \text{ is a path from } x \text{ to } x' \}
\]

A path \( \gamma \) from \( x \) to \( x' \) such that \( \ell(\gamma) = d(x,x') \) is said to be geodesic. The space is said to be geodesic when any two points are related by a geodesic path.
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The length $\ell(\gamma)$ of a path $\gamma : [0, r] \to (X, d)$ is the least upper bound of the collection of sums

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The Hopf-Rinow theorem

Metric Spaces of Non-Positive Curvature, M. R. Bridson, and A. Haefliger, 1999
A metric space is said to be complete when all its Cauchy sequences admit a limit.
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A metric space is said to be complete when all its Cauchy sequences admit a limit.

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If $X$ is complete and locally compact, then
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Isometric embedding in $\mathbb{R}^n$
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Isometric embedding in $\mathbb{R}^n$

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$$d_\ell(x,x') = \inf \{ \ell(\gamma) \mid \gamma \text{ is a path from } x \text{ to } x' \}$$
Metric Graphs
Neighbours

The underlying set of the metric graph is $A \times [0, 1] \sqcup V$. Two points $p, p'$ are said to be neighbours when there is an arrow $a$ such that $p, p' \in \{a\} \times [0, 1] \sqcup \{\partial^- a, \partial^+ a\}$. 

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Neighbours

$G : A \xrightarrow{\partial^-} V \xleftarrow{\partial^+} V$

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Distance between two neighbours

- If $\partial - a \neq \partial + a$ there is a canonical bijection $\phi: \{a\} \times [0,1] \sqcup \{\partial - a, \partial + a\} \to [0,1]$. In that case $d(p,p') = |t - t'|$ with $t = \phi(p)$ and $t' = \phi(p')$.

- If $\partial - a = \partial + a$ there is a canonical bijection $\phi: \{a\} \times [0,1] \sqcup \{\partial - a, \partial + a\} \to [0,1]$. In that case $d(p,p') = \min|t - t'|, 1 - |t - t'|$ with $t = \phi(p)$ and $t' = \phi(p')$. 


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with $t = \phi(p)$ and $t' = \phi(p')$. 
An itinerary on $A \times \mathbb{I}_{0,1}$ is a (finite) sequence $p_0, \ldots, p_q$ of points such that $p_k$ and $p_{k+1}$ are neighbours for $k \in \{0, \ldots, q-1\}$.

The length of that itinerary is $\ell(p_0, \ldots, p_q) = \sum_{k=0}^{q-1} d(p_k, p_{k+1})$.

The distance between two points $p$ and $p'$ of $A \times \mathbb{I}_{0,1} \sqcup V$ is $d(p, p') = \inf \ell(p_0, \ldots, p_q) | p_0, \ldots, p_q$ is a itinerary from $p$ to $p'$.

The metric graph associated with $G$ is the metric space $A \times \mathbb{I}_{0,1} \sqcup V$. 

Itinerary

An itinerary on $A \times [0, 1[ \sqcup V$ is a (finite) sequence $p_0, \ldots, p_q$ of points such that $p_k$ and $p_{k+1}$ are neighbours for $k \in \{0, \ldots, q - 1\}$. 
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$$\left( A \times ]0, 1[ \sqcup V, d \right)$$
Open balls

The open ball of radius $r < 1$ centered at the vertex $v$ is the set 

$$\{v\} \cup \{a | \partial^- a = v\} \times [0, r] \cup \{a | \partial^+ a = v\} \times [1 - r, 1]$$

For $(a, t) \in \{(a) \times [0, 1]\}$ the open ball of radius $r \leq \min\{t, 1 - t\}$ centered at the vertex $(a, t)$ is the set 

$$\{(a) \times [t - r, t + r]\}$$

That collection of open balls forms a base of open sets.

If $r \leq \frac{1}{4}$ then $B(c, r)$ is geodesically stable, i.e. for all $p, q \in B(c, r)$ 

$$\{p, q\} \subseteq \left[\text{im}(\gamma) | \gamma \text{ geodesic from } p \text{ to } q \subseteq B(c, r)\right].$$
Open balls

The open ball of radius $r < 1$ centered at the vertex $v$ is the set

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$$\{p, q\} \subseteq \bigcup \{\text{im}(\gamma) \mid \gamma \text{ geodesic from } p \text{ to } q\} \subseteq B(c, r).$$
The metric graph construction is functorial from $\text{Grph}$ to $\text{Met}$. Every finite graph with weighted arrows (in $\mathbb{R}^+ \setminus \{0\}$) can be embedded in $\mathbb{R}^3$. 
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The metric graph construction is functorial from $\mathcal{Grph}$ to $\mathcal{Met}_{ctr}$. 
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LOCALLY ORDERED METRIC GRAPHS
Ordered Bases
The category of ordered bases \((\mathcal{OB})\)
The category of ordered bases ($\mathcal{OB}$)

We write that $(X, \leq_x)$ is a subposet of $(Y, \leq_y)$, or $(X, \leq_x) \hookrightarrow (Y, \leq_y)$, when $X \subseteq Y$ and $a \leq_x b \Leftrightarrow a \leq_y b$ for all $a, b \in X$. 
The category of ordered bases ($\mathcal{OB}$)

We write that $(X, \leq_X)$ is a subposet of $(Y, \leq_Y)$, or $(X, \leq_X) \hookrightarrow (Y, \leq_Y)$, when $X \subseteq Y$ and $a \leq_X b \iff a \leq_Y b$ for all $a, b \in X$.

An ordered base is a collection of posets $\mathcal{B}$ such that ...
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An ordered base is a collection of posets $\mathcal{B}$ such that for all $(U, \leq_U), (V, \leq_V) \in \mathcal{B}$, ...
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We write that $(X, \leq_X)$ is a subposet of $(Y, \leq_Y)$, or $(X, \leq_X) \hookrightarrow (Y, \leq_Y)$, when $X \subseteq Y$ and $a \leq_X b \iff a \leq_Y b$ for all $a, b \in X$.

An ordered base is a collection of posets $\mathcal{B}$ such that for all $(U, \leq_U), (V, \leq_V) \in \mathcal{B}$, every $p \in U \cap V$, ...
The category of ordered bases \((\mathcal{OB})\)

We write that \((X, \leq_X)\) is a subposet of \((Y, \leq_Y)\), or \((X, \leq_X) \hookrightarrow (Y, \leq_Y)\), when \(X \subseteq Y\) and \(a \leq_X b \Leftrightarrow a \leq_Y b\) for all \(a, b \in X\).

An ordered base is a collection of posets \(\mathcal{B}\) such that for all \((U, \leq_U), (V, \leq_V) \in \mathcal{B}\), every \(p \in U \cap V\), there exists \((W, \leq_W) \in \mathcal{B}\) such that \(p \in (W, \leq_W) \hookrightarrow (U, \leq_U), (V, \leq_V)\).
The category of ordered bases ($OB$)

A map $f : U \to V$ is **locally order-preserving** when
The category of ordered bases ($\mathcal{OB}$)

A map $f : \mathcal{U} \to \mathcal{V}$ is \textit{locally order-preserving} when for every point $p$ of $\mathcal{U}$,
The category of ordered bases \((OB)\)

A map \(f: \mathcal{U} \to \mathcal{V}\) is \textit{locally order-preserving} when for every point \(p\) of \(\mathcal{U}\), every \((V, \leq_V) \in \mathcal{V}\) with \(f(p) \in V\), there exists \((U, \leq_U) \in \mathcal{U}\) with \(p \in U\) such that \(f(U) \subseteq V\) and \(f\) is order-preserving from \((U, \leq_U)\) to \((V, \leq_V)\).
The category of ordered bases ($\mathcal{OB}$)

A map $f : \mathcal{U} \rightarrow \mathcal{V}$ is locally order-preserving when for every point $p$ of $\mathcal{U}$, every $(V, \leq_V) \in \mathcal{V}$ with $f(p) \in V$, there exists $(U, \leq_U) \in \mathcal{U}$ with $p \in U$ such that

$$f(U, \leq_U) \subseteq V$$
The category of ordered bases ($\mathcal{OB}$)

A map $f : \mathcal{U} \rightarrow \mathcal{V}$ is \textit{locally order-preserving} when for every point $p$ of $\mathcal{U}$, every $(\mathcal{V}, \leq_{\mathcal{V}}) \in \mathcal{V}$ with $f(p) \in \mathcal{V}$, there exists $(\mathcal{U}, \leq_{\mathcal{U}}) \in \mathcal{U}$ with $p \in \mathcal{U}$ such that $f(\mathcal{U}) \subseteq \mathcal{V}$ and $f$ is order-preserving from $(\mathcal{U}, \leq_{\mathcal{U}})$ to $(\mathcal{V}, \leq_{\mathcal{V}})$. 
The category of ordered bases ($\mathcal{OB}$)

A map $f : \mathcal{U} \to \mathcal{V}$ is locally order-preserving when for every point $p$ of $\mathcal{U}$, every $(V, \leq_V) \in \mathcal{V}$ with $f(p) \in V$, there exists $\left(U, \leq_U\right) \in \mathcal{U}$ with $p \in U$ such that $f(U) \subseteq V$ and $f$ is order-preserving from $\left(U, \leq_U\right)$ to $(V, \leq_V)$.

Ordered bases and locally order-preserving maps form the category $\mathcal{OB}$. 
The underlying topology of an ordered base

If $B$ is an ordered base, then $\mathcal{U}_B = \{U_B \mid B \in B\}$ is a base of a topology ($\mathcal{U}_B$ denotes the underlying set of the poset $B$).

If $f : B \to B'$ is locally order-preserving, then $\mathcal{U}f : \mathcal{U}_B \to \mathcal{U}_B'$ is continuous; we have a forgetful functor $\mathcal{OB} \to \mathcal{Bas}$.

We have a functor $\mathcal{U} : \mathcal{OB} \to \mathcal{Set}$ obtained as the composite $\mathcal{OB} \to \mathcal{Bas} \to \mathcal{Set}$.

The underlying space functor $\mathcal{Sp} : \mathcal{OB} \to \mathcal{Top}$ is the composite $\mathcal{OB} \to \mathcal{Bas} \to \mathcal{Top}$.

We write $B \sim B'$ when $\mathcal{Sp}(B) = \mathcal{Sp}(B')$ and $B \cup B'$ is still an ordered base; and we say that $B$ and $B'$ are equivalent.

The relation $\sim$ is an equivalence relation on the collection of ordered bases over a given set.

If $A \sim A'$ and $B \sim B'$, then any map $f : \mathcal{U}_A \to \mathcal{U}_B$ is locally order-preserving from $A$ to $B$ iff it is so from $A'$ to $B'$. 
The underlying topology of an ordered base

If $B$ is an ordered base, then $UB = \{ UB \mid B \in B \}$ is a base of a topology ($UB$ denotes the underlying set of the poset $B$).
The underlying topology of an ordered base

If $B$ is an ordered base, then $U B = \{ U B \mid B \in B \}$ is a base of a topology ($U B$ denotes the underlying set of the poset $B$).

If $f : B \to B'$ is locally order-preserving, then $U f : U B \to U B'$ is continuous; we have a forgetful functor $O B \to B a s$. 
The underling topology of an ordered base

If $\mathcal{B}$ is an ordered base, then $UB = \{ UB \mid B \in \mathcal{B} \}$ is a base of a topology ($UB$ denotes the underlying set of the poset $B$).

If $f : \mathcal{B} \to \mathcal{B}'$ is locally order-preserving, then $Uf : UB \to UB'$ is continuous; we have a forgetful functor $OB \to Bas$.

We have a functor $U : OB \to Set$ obtained as the composite $OB \to Bas \to Set$. 
The underlying topology of an ordered base

If $\mathcal{B}$ is an ordered base, then $UB = \{ UB | B \in \mathcal{B} \}$ is a base of a topology ($UB$ denotes the underlying set of the poset $B$).

If $f : \mathcal{B} \rightarrow \mathcal{B}'$ is locally order-preserving, then $Uf : UB \rightarrow UB'$ is continuous; we have a forgetful functor $\mathcal{OB} \rightarrow \mathcal{Bas}$.

We have a functor $U : \mathcal{OB} \rightarrow \mathcal{Set}$ obtained as the composite $\mathcal{OB} \rightarrow \mathcal{Bas} \rightarrow \mathcal{Set}$.

The underlying space functor $Sp : \mathcal{OB} \rightarrow \mathcal{Top}$ is the composite $\mathcal{OB} \rightarrow \mathcal{Bas} \rightarrow \mathcal{Top}$. 
The underlying topology of an ordered base

If \( \mathcal{B} \) is an ordered base, then \( UB = \{ UB \mid B \in \mathcal{B} \} \) is a base of a topology (\( UB \) denotes the underlying set of the poset \( B \)).

If \( f : \mathcal{B} \rightarrow \mathcal{B}' \) is locally order-preserving, then \( Uf : UB \rightarrow UB' \) is continuous; we have a forgetful functor \( OB \rightarrow Bas \).

We have a functor \( U : OB \rightarrow Set \) obtained as the composite \( OB \rightarrow Bas \rightarrow Set \).

The underlying space functor \( Sp : OB \rightarrow Top \) is the composite \( OB \rightarrow Bas \rightarrow Top \).

We write \( \mathcal{B} \sim \mathcal{B}' \) when \( Sp(\mathcal{B}) = Sp(\mathcal{B}') \) and \( \mathcal{B} \cup \mathcal{B}' \) is still an ordered base; and we say that \( \mathcal{B} \) and \( \mathcal{B}' \) are equivalent.
The underlying topology of an ordered base

If $\mathcal{B}$ is an ordered base, then $UB = \{ UB \mid B \in \mathcal{B} \}$ is a base of a topology ($UB$ denotes the underlying set of the poset $B$).

If $f : \mathcal{B} \to \mathcal{B}'$ is locally order-preserving, then $Uf : UB \to UB'$ is continuous; we have a forgetful functor $\mathcal{OB} \to \mathcal{Bas}$.

We have a functor $U : \mathcal{OB} \to \mathcal{Set}$ obtained as the composite $\mathcal{OB} \to \mathcal{Bas} \to \mathcal{Set}$.

The underlying space functor $Sp : \mathcal{OB} \to \mathcal{Top}$ is the composite $\mathcal{OB} \to \mathcal{Bas} \to \mathcal{Top}$.

We write $\mathcal{B} \sim \mathcal{B}'$ when $Sp(B) = Sp(B')$ and $\mathcal{B} \cup \mathcal{B}'$ is still an ordered base; and we say that $\mathcal{B}$ and $\mathcal{B}'$ are equivalent.

The relation $\sim$ is an equivalence relation on the collection of ordered bases over a given set.
The underlying topology of an ordered base

If $\mathcal{B}$ is an ordered base, then $UB = \{ UB \mid B \in \mathcal{B} \}$ is a base of a topology ($UB$ denotes the underlying set of the poset $B$).

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We write $\mathcal{B} \sim \mathcal{B}'$ when $Sp(B) = Sp(B')$ and $\mathcal{B} \cup \mathcal{B}'$ is still an ordered base; and we say that $\mathcal{B}$ and $\mathcal{B}'$ are equivalent.

The relation $\sim$ is an equivalence relation on the collection of ordered bases over a given set.

If $\mathcal{A} \sim \mathcal{A}'$ and $\mathcal{B} \sim \mathcal{B}'$, then any map $f : UA \to UB$ is locally order-preserving from $\mathcal{A}$ to $\mathcal{B}$ iff it is so from $\mathcal{A}'$ to $\mathcal{B}'$. 
An ordered base $B$ is said to be maximal when for every poset $X$, if $UX$ is open in $\text{Sp}(B)$ and $B \cup \{X\}$ is still an ordered base, then $X \in B$.

A locally ordered space is a maximal ordered base.

We denote by $\text{LoSp}$ the full subcategory of $\text{OB}$ whose objects are the locally ordered spaces.

**Lemma:** Every ordered base is contained in a unique maximal ordered base.

**Proposition:** the functor from $\text{OB}$ to $\text{LoSp}$ assigning its locally ordered space to every ordered base is an equivalence of categories whose quasi-inverse is the full embedding $\text{LoSp} \to \text{OB}$. 

Locally ordered spaces
An ordered base $B$ is said to be maximal when for every poset $X$, if $UX$ is open in $Sp(B)$ and $B \cup \{X\}$ is still an ordered base, then $X \in B$. 
An ordered base $\mathcal{B}$ is said to be maximal when for every poset $X$, if $UX$ is open in $Sp(\mathcal{B})$ and $\mathcal{B} \cup \{X\}$ is still an ordered base, then $X \in \mathcal{B}$.

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Locally ordered spaces

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The locally ordered line

Examples of equivalent ordered bases on $\mathbb{R}$
The locally ordered line

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- $\{(I, \leq) \mid I \text{ open interval of } \mathbb{R}\}$,
The locally ordered line

Examples of equivalent ordered bases on $\mathbb{R}$

- $\{ (I, \leq) \mid I \text{ open interval of } \mathbb{R} \}$,
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Suppose that $[0, 1] \cup [2, 3]$ is a locally ordered subspace of $\mathbb{R}$, the map $t \in [0, 1] \cup [2, 3] \mapsto t + 2 \pmod{4} \in [0, 1] \cup [2, 3]$ is locally order-preserving.
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The locally ordered circle

Examples of equivalent ordered bases on $S^1$
The locally ordered circle

Examples of equivalent ordered bases on $S^1$

- $\{(A, \leq) \mid A \text{ open arc} \}$ where $\leq$ is the order induced by $\mathbb{R}$ and the restriction of the exponential map to an open subinterval of $\{t \in \mathbb{R} \mid e^{it} \in A\}$ of length at most $2\pi$, 
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- $\{(U, \sqsubseteq_U) \mid U \text{ proper open subset of } S^1\}$ where $x \sqsubseteq_U y$ means that the anticlockwise compact arc from $x$ to $y$ is included in $U$, 
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- $\{ (U, \sqsubseteq'_U) \mid U \text{ proper open subset of } \mathbb{S}^1 \}$ where $\sqsubseteq'_U$ is any extension of the partial order $\sqsubseteq_U$. 
Ordered spaces
An ordered space is a topological space $X$ together with a partial order $\sqsubseteq$ on (the underlying set of) $X$.

If the relation $\sqsubseteq$ is closed in the sense that $\{ (a, b) \in X \times X \mid a \sqsubseteq b \}$ is a closed subset of $X \times X$, then $X$ is said to be a partially ordered space (or pospace).

An ordered space morphism is an order-preserving continuous map.

Ordered spaces and their morphisms form the category $\mathbf{Ord}$. The underlying space of a pospace is Hausdorff.
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Examples

- The real line with standard topology and order.
- Any subset of a pospace with the induced topology and order.
- The collection of compact subsets of a metric space equipped with the Hausdorff distance is a metric space.

\[ d_{\text{H}}(K, K') = \sup_{x \in K} d(x, K'), \quad d(x', K) | x \in K; x' \in K' \]

The induced topological space ordered by inclusion is a pospace.

- Problem: there is no pospace on the circle whose collection of directed paths is \( e^{i\theta}(t) | \theta : [0, r] \rightarrow \mathbb{R} \) increasing.
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\[ \{ e^{i\theta(t)} \mid \theta : [0, r] \rightarrow \mathbb{R} \text{ increasing} \} \]
Ordered spaces as locally ordered spaces

Each ordered space \((X, \sqsubseteq)\) can be seen as a locally ordered space \((X, (U, \sqsubseteq|U))\) where \(U\) is an open subset of \(X\).

The resulting functor is:
- faithful
- not injective on object (hence not an embedding)
- not full
Ordered spaces as locally ordered spaces

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Directed loops on locally ordered spaces

A locally order-preserving map \( \delta : [a, b] \rightarrow X \) whose image is contained in \( C \in X \) induces an order-preserving map from \([a, b]\) to \(C\).

A directed path \( \delta \) on a local pospace \( X \) is constant iff its extremities are equal and there exists \( C \in X \) that contains the image of \( \delta \).

A vortex is a point every neighbourhood of which contains a non-constant directed loop.

A local pospace has no vortex.
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Ordered bases on metric graphs
A convenient open covering
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Let $\mathcal{B}$ be the collection of open balls $B$ of $|G|$ such that
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Let $\mathcal{B}$ be the collection of open balls $B$ of $|G|$ such that
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Let \( \mathcal{B} \) be the collection of open balls \( B \) of \( |G| \) such that
- \( B \) is centred at a vertex and its radius is \( \leq \frac{1}{3} \), or
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Given $B, B' \in \mathcal{B}$ if $B$ is of the second kind, then so is $B \cap B'$.

If $B, B'$ are centred at $v$ and $v'$ we have
- $v \neq v' \Rightarrow B \cap B' = \emptyset$ and
- $v = v' \Rightarrow B \subseteq B'$ or $B' \subseteq B$
Ordered open stars

An element $B$ of $B$ centred at $v$ of radius $r \leq \frac{1}{3}$ is the disjoint union of $\{v\}$ together with $-\{a\} \times [0, r]$ for each arrow $a$ such that $\partial -a = v - \{a\} \times [1-r, 1]$ for each arrow $a$ such that $\partial +a = v$.

The partial order on $B$ is characterized by the following constraints:

- each branch $\{a\} \times [1-r, 1]$ and $\{a\} \times [0, r]$ inherits its order from $\mathbb{R} - \{v\} \sqsubseteq \{a\} \times [0, r]$ for each arrow $a$ such that $\partial -a = v - \{a\} \times [1-r, 1]$ $\sqsubseteq \{v\}$ for each arrow $a$ such that $\partial +a = v$.

We have $B \cap B' \neq \emptyset \Rightarrow B \cap B' \in B$ and $\sqsubseteq B | B \cap B' = \sqsubseteq B \cap B' = \sqsubseteq B' | B \cap B'$.

The metric graph of $\mathcal{G}$ thus becomes a local pospace.

The locally ordered metric graph construction is functorial.
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We have $B \cap B' \neq \emptyset \Rightarrow B \cap B' \in \mathcal{B}$ and

$$\sqsubseteq_{B \cap B'} = \sqsubseteq_{B \cap B'} = \sqsubseteq_{B' \cap B'}$$
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An element $B$ of $\mathcal{B}$ centred at $v$ of radius $r \leq \frac{1}{3}$ is the disjoint union of $\{v\}$ together with
- $\{a\} \times ]0, r[$ for each arrow $a$ such that $\partial^{-} a = v$
- $\{a\} \times ]1 - r, 1[$ for each arrow $a$ such that $\partial^{+} a = v$

The partial order on $B$ is characterized by the following constraints:
- each branch $\{a\} \times ]1 - r, 1[$ and $\{a\} \times ]0, r[$ inherits its order from $\mathbb{R}$
- $\{v\} \sqsubseteq \{a\} \times ]0, r[$ for each arrow $a$ such that $\partial^{-} a = v$
- $\{a\} \times ]1 - r, 1[ \sqsubseteq \{v\}$ for each arrow $a$ such that $\partial^{+} a = v$

We have $B \cap B' \neq \emptyset \Rightarrow B \cap B' \in \mathcal{B}$ and

$$\sqsubseteq_{B \mid B \cap B'} = \sqsubseteq_{B \cap B'} = \sqsubseteq_{B' \mid B \cap B'}$$

The metric graph of $|G|$ thus becomes a local pospace.
Ordered open stars

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We have $B \cap B' \neq \emptyset \Rightarrow B \cap B' \in B$ and

$$\sqsubseteq_{B|_{B \cap B'}} = \sqsubseteq_{B \cap B'} = \sqsubseteq_{B'|_{B \cap B'}}$$

The metric graph of $|G|$ thus becomes a local pospace.

The locally ordered metric graph construction is functorial.
There exists a (unique) intrinsic metric $d_G$ on $|G|$ such that the open balls of radii $\varepsilon > 0$ about $(a, t)$ and $v$ are
\[
\{a\} \times [t - \varepsilon, t + \varepsilon] \quad \text{if} \quad \varepsilon \leq \min(t, 1 - t),
\]
and $a \in G$ such that
\[
\text{tgt}(a) = v \times [1 - \varepsilon, 1] \cup \{v\} \cup \left\{a \in G \mid \text{src}(a) = v \times [0, \varepsilon]\right\} \quad \text{if} \quad \varepsilon \leq \frac{1}{2}.
\]
The partial order $\sqsubseteq$ and the metric $d_G$ on the ball centered at $v$ of radius $\varepsilon$ are characterized by the following properties:

\[
d_G((a, t), v) = 1 - t \quad \text{if} \quad t \in [1 - \varepsilon, 1],
\]
\[
d_G(v, (a, t)) = t \quad \text{if} \quad t \in [0, \varepsilon],
\]
\[
d_G((a, t), (a, t')) = t' - t \quad \text{if} \quad t \leq t' \text{ and } t, t' \in [0, \varepsilon] \quad \text{or} \quad t, t' \in [1 - \varepsilon, 1],
\]
\[
d_G((a, t), (b, t')) = d_G((a, t), v) + d_G(v, (b, t')) \quad \text{if} \quad a \neq b \quad \text{and} \quad (a, t) \sqsubseteq (b, t') \text{ if } t \in [1 - \varepsilon, 1] \quad \text{and} \quad t' \in [0, \varepsilon].
\]

If $\varepsilon \leq \frac{1}{4}$ then the ball centered at $v$ of radius $\varepsilon$, say $B$, is geodesically stable: for all $p, q \in B$, the union of the images of the geodesics from $p$ to $q$ is nonempty and contained in $B$.

The standard ordered base of $G$ is the collection of ordered open balls of radii $\varepsilon \leq \frac{1}{2}$ with their ‘canonical’ partial order.
Description

There exists a (unique) intrinsic metric $d_G$ on $|G|$ such that the open balls of radii $\varepsilon > 0$ about $(a, t)$ and $v$ are

\[ \{ a \in G^{(1)} \mid \text{tgt}(a) = v \} \times ]1 - \varepsilon, 1[ \cup \{ v \} \cup \{ a \in G^{(1)} \mid \text{src}(a) = v \} \times ]0, \varepsilon[ \text{ if } \varepsilon \leq \frac{1}{2}. \]

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\[
\begin{align*}
d_G((a, t), v) &= 1 - t & (a, t) \sqsubseteq v & \text{if} \quad t \in ]1 - \varepsilon, 1[ \\
d_G(v, (a, t)) &= t & v \sqsubseteq (a, t) & \text{if} \quad t \in ]0, \varepsilon[ \\
d_G((a, t), (a, t')) &= t' - t & (a, t) \sqsubseteq (a, t') & \text{if} \quad t \leq t' \text{ and } (t, t' \in ]0, \varepsilon[ \text{ or } t, t' \in ]1 - \varepsilon, 1[] \\
d_G((a, t), (a, t')) &= \min\{t' - t, 1 - (t' - t)\} & (a, t') \sqsubseteq (a, t) & \text{if} \quad t \in ]0, \varepsilon[ \text{ and } t' \in ]1 - \varepsilon, 1[ \\
d_G((a, t), (b, t')) &= d_G((a, t), v) + d_G(v, (b, t')) & (a, t) \sqsubseteq (b, t') & \text{if} \quad a \neq b \\
d_G((a, t), (b, t')) &= d_G((a, t), v) + d_G(v, (b, t')) & (a, t) \sqsubseteq (b, t') & \text{if} \quad t \in ]1 - \varepsilon, 1[ \text{ and } t' \in ]0, \varepsilon[ \\
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$$d_G((a, t), (a, t')) = t' - t \quad (a, t) \sqsubseteq (a, t') \quad \text{if } t \leq t' \text{ and } (t, t' \in ]0, \varepsilon[ \text{ or } t, t' \in ]1 - \varepsilon, 1[$$
$$d_G((a, t), (a, t')) = \min\{t' - t, 1 - (t' - t)\} \quad (a, t') \sqsubseteq (a, t) \quad \text{if } t \in ]0, \varepsilon[ \text{ and } t' \in ]1 - \varepsilon, 1[$$
$$d_G((a, t), (b, t')) = d_G((a, t), v) + d_G(v, (b, t')) \quad \text{if } a \neq b$$
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$$d_G((a, t), (a, t')) = t' - t \quad \text{if } (a, t) \sqsubseteq (a, t') \text{ and } t, t' \in ]0, \varepsilon[$$

$$d_G((a, t), (a, t')) = \min\{t' - t, 1 - (t' - t)\} \quad \text{if } (a, t') \sqsubseteq (a, t) \text{ and } t, t' \in ]0, \varepsilon[, 1[$$

$$d_G((a, t), (b, t')) = \min\{t' - t, 1 - (t' - t)\} \quad \text{if } a \neq b$$

$$d_G((a, t), v) + d_G(v, (b, t')) \quad \text{if } a \neq b$$

$$d_G((a, t), (b, t')) \quad \text{if } t \in ]1 - \varepsilon, 1[ \text{ and } t' \in ]0, \varepsilon[$$

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