DIRECTED ALGEBRAIC TOPOLOGY
AND
CONCURRENCY

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MPRI : Concurrency (2.3)

Tuesday, the 4th of December 2018
A QUICK OVERVIEW

OF

CONCURRENCY THEORY
Handbook of Logic in Computer Science

VOLUME 4

S. ABRAMSKY, DOV M. GABBAY,
and T. S. E. MAIBAUM

OXFORD SCIENCE PUBLICATIONS
THE ORIGIN OF CONCURRENT PROGRAMMING
From Semaphores to Remote Procedure Calls

PER BRINCH HANSEN
Editor
PARALLEL AUTOMATA META LANGUAGE
Syntax
Paradigm


---

1Portable Operating Systems Interface, X is a reference to Unix
 Paradigm


- The Dijkstra’s language is a parallel extension of ALGOL60 with P (lock/take), V (unlock/release), and `parbegin` ... `parend`

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- The Carson and Reynolds language is a restriction of Dijkstra’s language:
  - Operator `||` in outermost position: only sequential processes are executed in parallel
  - Neither branchings nor loops

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Features

- shared memory abstract machine (PRAM)
- concurrent read exclusive write (CREW)
- Operator $||$ in outermost position: only sequential processes are executed in parallel
- Branchings, loops, and synchronisation barriers $W$ (wait) are allowed
- no pointer arithmetics
- no function call, only jumps
- no birth nor death of process at runtime
- tokens are owned by processes
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e.g. sem 3 a b c d
- **sync** <int> <set of identifiers>
  
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- **mtx** <set of identifiers>
  
e.g. mtx a b c d
- **var** <identifier> = <constant>
  
e.g. var x = 0
- **proc** <identifier> = <basic block>
- **init** <multiset of identifiers>
  
e.g. init a 2b 3c
Expressions and values
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The set of **expressions** is inductively built on the set of **identifiers** and the following set of operators
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The set of expressions is inductively built on the set of identifiers and the following set of operators.

<table>
<thead>
<tr>
<th>v</th>
<th>content of v ∈ V</th>
<th>x ∈ R</th>
<th>constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>∧</td>
<td>minimum</td>
<td>∨</td>
<td>maximum</td>
</tr>
<tr>
<td>+</td>
<td>addition</td>
<td>−</td>
<td>subtraction</td>
</tr>
<tr>
<td>*</td>
<td>multiplication</td>
<td>/</td>
<td>division</td>
</tr>
<tr>
<td>≤</td>
<td>less or equal</td>
<td>≥</td>
<td>greater of equal</td>
</tr>
<tr>
<td>&lt;</td>
<td>strictly less</td>
<td>&gt;</td>
<td>strictly greater</td>
</tr>
<tr>
<td>=</td>
<td>equal</td>
<td>≠</td>
<td>not equal</td>
</tr>
<tr>
<td>¬</td>
<td>complement</td>
<td>%</td>
<td>modulo</td>
</tr>
<tr>
<td>⊥</td>
<td></td>
<td></td>
<td>bottom</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>nullary</th>
<th>unary</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊥, x ∈ R, v ∈ V</td>
<td>¬</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>binary</th>
</tr>
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<tbody>
<tr>
<td>∧, ∨, +, −, *, /, &lt;, &gt;, ≤, ≥, =, ≠, %</td>
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</tbody>
</table>
Non branching instructions

- identifier := expression
  The expression is evaluated then the result is stored in the identifier

- P(identifier)
  Takes an occurrence of the resource identifier (there are arity available tokens), stops the process otherwise

- V(identifier)
  Release an occurrence of the resource identifier (if such an occurrence is held by the process), ignored otherwise

- W(identifier)
  Stops the execution of the process until arity + 1 of them are stopped by the barrier identifier

- J(identifier)
  The execution of the process is stopped and the one of a copy of identifier starts. There is no return mechanism.

- (L) enclose a list of instructions between parenthesis to make it a single instruction
Non branching instructions

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- `identifier := expression` the expression is evaluated then the result is stored in the identifier
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- `(L)` enclose a list of instructions between parenthesis to make it a single instruction
Branching

The branching is provided by a kind of "match case like" instruction:

\[
(L_1 + e_1 + L_2 + e_2 + \ldots + L_n + e_n + L_{n+1})
\]

- Each \(L_k\) is a basic block
- Each \(e_k\) is an expression
- The triggered branch is \(L_k\) with \(k\) being the first index such that \(e_k\) evaluates to some nonzero value
- If all the expressions evaluate to zero, then \(L_{n+1}\) is triggered.
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Describing a process

- The body of a process is just a (possibly empty) sequence of instructions, i.e., a basic block, separated by semicolons. For example:

  proc p = x:=7; J(q)
  proc q = J(r) + [x<>1] + ()
  proc r = (x:=x/2) + [x%2=0] + (x:=3*x+1) ; J(q)

  init p

- Due to the branchings, basic blocks are actually trees.
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Control flow graphs
Control flow graphs and flowcharts

Control flow analysis, *F. E. Allen*, 1970
Assigning meanings to programs, *R. W. Floyd*, 1967
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- At the core of all modern software dealing with source code e.g. GCC (cf. “basic blocks”), LLVM, Frama-C.
- No such structure exist for parallel programs.
Generators

\[
x := f
\]

\[
\begin{align*}
& x := f \\
& b_1 \\
& a_1 \\
& \text{START} \\
& b_1
\end{align*}
\]
The Hasse-Syracuse algorithm in PAML

\[
\begin{align*}
\text{var } & \ x = 7 \\
\text{proc } & \ p = () + [x=1] + J(q) \\
\text{proc } & \ q = (x:=x/2) + [x\%2=0] + (x:=3*x+1) ; J(p) \\
\text{init } & \ p
\end{align*}
\]
Building the control flow graph

of the Hasse-Syracuse algorithm
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entry point of the basic block of $p$
Building the control flow graph of the Hasse-Syracuse algorithm

entry point of the basic block of $p$

entry point of the basic block of $q$

$x=1$  

$x:=3\times x+1$  

$x:=x/2$  

$x\%2=0$
Building the control flow graph
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entry point of the basic block of \( p \)

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\[ x = 1 \]

\[ J(q) \]

\[ x := 3 \times x + 1 \]

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\[ x := x / 2 \]

\[ x \mod 2 = 0 \]
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Reducing the Control Flow Graph of the Hasse-Syracuse algorithm
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of the Hasse-Syracuse algorithm

entry point

\[ x := x/2 \]
\[ x := 3x + 1 \]
\[ x \mod 2 = 0 \]
\[ x = 1 \]

The current value of \( x \) is 7.
Reducing the Control Flow Graph of the Hasse-Syracuse algorithm

entry point

x := x / 2

x := 3 * x + 1

x % 2 = 0

x = 1

the current value of x is 7
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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- $x := 3x + 1$
- $x \% 2 = 0$
- $x = 1$

The current value of $x$ is 7.
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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the current value of \( x \) is 7
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the current value of $x$ is 22
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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An execution trace on a control flow graph

of the Hasse-Syracuse algorithm

the current value of $x$ is 11
An execution trace on a control flow graph
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the current value of \(x\) is 11
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

x := 3 * x + 1
x := x / 2

the current value of x is 34
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

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the current value of x is 52
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An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 10
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

x := x/2
x := 3*x + 1
x \% 2 = 0

the current value of x is 10
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An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of \( x \) is 10
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of $x$ is 5
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

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An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of $x$ is 5
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 16
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of x is 16
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of $x$ is 16
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 16
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

The current value of \( x \) is 7

\[ x := x / 2 \]

\[ x := 3 \times x + 1 \]

\[ x \text{ mod } 2 = 0 \]

\[ x = 1 \]
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of x is 8
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The control flow graphs

the current value of $x$ is 8
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of $x$ is 8
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 4
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of \( x \) is 4
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of \( x \) is 4
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of \( x \) is 2
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of \( x \) is 2
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of x is 2
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

x := x/2
x := 3*x + 1
x % 2 = 0
x = 1

the current value of x is 2
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 1

$x := x/2$

$x := 3x + 1$

$x \equiv 2 = 0$

$x = 1$
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 1
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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The current value of $x$ is 1
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of \( x \) is 1
Execution traces as paths over a control flow graph
Execution traces as paths over a control flow graph

- Any execution trace induces a path
Execution traces as paths over a control flow graph

- Any execution trace induces a path
- Some paths do not come from an execution trace
Execution traces as paths over a control flow graph

- Any execution trace induces a path
- Some paths do not come from an execution trace
- Therefore the collection of path provides a (strict) overapproximation of the collection of execution traces
Execution traces as paths over a control flow graph

- Any execution trace induces a path
- Some paths do not come from an execution trace
- Therefore the collection of path provides a (strict) overapproximation of the collection of execution traces
- The (infinite) collection of paths is entirely determined by the (finite) control flow graph
The overall idea of static analysis
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Any model of a program should contain a finite representation of an overapproximation of the collection of all its execution traces.
Any model of a program should contain a finite representation of an overapproximation of the collection of all its execution traces.

One of the goal of the course it to provide such a structure for a large class of PAML programs.
Restrictions from the PAML syntax

- There is neither birth nor death of processes at runtime
- The arity of resources cannot be changed at runtime
- There is no pointer arithmetics
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By construction the PAML language enforces the following restrictions
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- There is neither birth nor death of processes at runtime
- The arity of resources cannot be changed at runtime
- There is no pointer arithmetics
An abstract machine
Abstract expressions

- The set of variables of a program is $X$.
- A valuation or memory state is a mapping $\nu: X \to R \perp = R \cup \{\perp\}$.
- An expression is a mapping $\varepsilon: \{\text{valuations}\} \to R$ with $F(\varepsilon) \subseteq X$ such that if the valuations $\nu$ and $\nu'$ match on $F(\varepsilon)$ then $\varepsilon(\nu) = \varepsilon(\nu')$.
- The set of expressions occurring in the program is denoted by $E$. 
Abstract expressions

- The set of variables of a program is $\mathcal{X}$. 
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Abstract expressions

- The set of variables of a program is $\mathcal{X}$.
- A valuation or memory state is a mapping $\nu : \mathcal{X} \rightarrow \mathbb{R}_\bot = \mathbb{R} \cup \{\bot\}$.
- An expression is a mapping $\varepsilon : \{\text{valuations}\} \rightarrow \mathbb{R}$ with $\mathcal{F}(\varepsilon) \subseteq \mathcal{X}$ such that if the valuations $\nu$ and $\nu'$ match on $\mathcal{F}(\varepsilon)$ then $\varepsilon(\nu) = \varepsilon(\nu')$.
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Interpretation of expressions
only depends on the current memory state
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\[ [x]_\nu = \nu(x) \text{ for all } x \in \mathcal{X} \]
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- $[x]_\nu = \nu(x)$ for all $x \in \mathcal{X}$
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  $[\neg](x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$
- $[e] = \bot$ for all expression $e$ in which $\bot$ occurs
Interpretation of expressions
only depends on the current memory state

- \([x]_\nu = \nu(x) \text{ for all } x \in X\)
- Any value in \(\mathbb{R} \setminus \{0\}\) stands for true while 0 stands for false
- \([-] : \mathbb{R}_\perp \to \mathbb{R}_\perp\),
  \([-](0) = 1,\]
  \([-](\perp) = \perp, \text{ and} \]
  \([-](x) = 0 \text{ for all } x \in \mathbb{R} \setminus \{0\}\)
- \([e] = \perp \text{ for all expression } e \text{ in which } \perp \text{ occurs}\)
- the other operators are interpreted as expected
Abstract instructions
Abstract instructions

The sets of semaphores, and barriers of a program are respectively $S$ and $B$. 
Abstract instructions

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- An assignment is an element of $X \times E$ yet we write $x := e$ instead of $(x, e)$. By extension $F(x := e) = F(e)$. 
Abstract instructions

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- Given a graph

$$
G : A \xrightarrow{\partial^+} V
$$

a conditional branching at vertex $v \in V$ is a mapping

$$
\beta : \{\text{valuations}\} \rightarrow \{a \in A \mid \partial^a = v\}
$$

together with a subset $\mathcal{F}(\beta) \subseteq X$ such that if the valuations $\nu$ and $\nu'$ match on $\mathcal{F}(\beta)$ then $\beta(\nu) = \beta(\nu')$. 

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$$G : A \xrightarrow{\partial^-} V$$

a conditional branching at vertex $v \in V$ is a mapping

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together with a subset $\mathcal{F}(\beta) \subseteq X$ such that if the valuations $\nu$ and $\nu'$ match on $\mathcal{F}(\beta)$ then $\beta(\nu) = \beta(\nu')$.
- The synchronisation primitives $P(s)$, $V(s)$, and $W(b)$ for $s \in S$ and $b \in B$
Abstract processes as control flow graphs
Parallel Automata Meta Language

The abstract machine

Abstract processes as control flow graphs

\[ G : A \xrightarrow{\partial^-} V \quad \text{and} \quad \lambda : V \rightarrow \{\text{instructions}\} \]
Abstract processes as control flow graphs

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- An entry point \( v_0 \in V \) such that \( \lambda(v_0) = \text{Skip} \).
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- If \( \lambda(v) \) is not a branching, then \( v \) has at most one outgoing arrow.

The arrows are interpreted as intermediate positions of the instruction pointer so a point on a control flow graph is either a vertex or an arrow.
Abstract program

The abstract machine

- The initial valuation $\nu: X \rightarrow R$ which provides the values of the variables at the beginning of each execution of the program.
- The arity map $\alpha: S \uplus B \rightarrow N \cup \{\infty\}$.
- The tuple $(G_1, \ldots, G_n)$ of processes which are launched simultaneously at the beginning of each execution of the program.
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Points and multi-instructions

*Higher Dimensional Transition Systems*, G. L. Cattani and V. Sassone, 1996
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- A point of \((G_1, \ldots, G_n)\) is an \(n\)-tuple \(p\) whose \(i^{th}\) component, namely \(p_i\), is a point of \(G_i\).
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- A point of \((G_1, \ldots, G_n)\) is an \(n\)-tuple \(p\) whose \(i^{th}\) component, namely \(p_i\), is a point of \(G_i\).
- A multi-instruction is a partial map \(\mu : \{1, \ldots, n\} \rightarrow \{\text{instructions}\}\).
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- A multi-instruction is a partial map \(\mu : \{1, \ldots, n\} \rightarrow \{\text{instructions}\}\).
- \(\lambda(p)\) is the multi-instruction defined by \(\lambda(p)(i) = \lambda_i(p_i)\) over the set below.

\[
\{i \in \{1, \ldots, n\} \mid p_i \text{ is a vertex of } G_i\}
\]
The internal states of the abstract machine
The internal states of the abstract machine

A state is a mapping $\sigma$ defined over the disjoint union $\mathcal{X} \sqcup S$ such that:

- for all $x \in \mathcal{X}$, $\sigma(x) \in \mathcal{R}_\bot$,
- for all $s \in S$, $\sigma(s)$ is a multiset over $\{1, \ldots, n\}$. 
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Admissible multi-instructions

- write-write: \(x := \varepsilon \) vs \(x := \varepsilon'\)
- read-write: \(x := \varepsilon\) vs an instruction in which \(x\) is free

A multi-instruction \(\mu\) is said to be admissible at state \(\sigma\) when:

- for \(i, j \in \text{dom}(\mu)\) with \(i \neq j\), \(\mu(i)\) and \(\mu(j)\) do not conflict,
- for all \(s \in S\), \(\phi(s) \leq \alpha(s)\) where \(\phi(s) = |\sigma(s)| + |\{i \in \text{dom}(\mu) | \mu(i) = P(s)\}| - |\{i \in \text{dom}(\mu) | \mu(i) = V(s); \sigma(s)(i) > 0\}|\)
- for all \(b \in B\), \(\text{card}\{i \in \text{dom}(\mu) | \mu(i) = W(b)\} \notin \{1, \ldots, \alpha(b)\}\)
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Action of a multi-instruction on a state

Assuming that $\mu$ is admissible at $\sigma$

The state $\sigma \cdot \mu$ is defined as follows.

- For all $x \in X$, if $\mu(i) = x := \varepsilon$ for some $i \in \{1, \ldots, n\}$ one has $(\sigma \cdot \mu)(x) = \varepsilon(\sigma|X)$.

In the other case one has $(\sigma \cdot \mu)(x) = \sigma(x)$.

- For all $s \in S$ the multiset $(\sigma \cdot \mu)(s)$, seen as a mapping from $\{1, \ldots, n\}$ to $\mathbb{N}$, is given by

$$i \mapsto \begin{cases} \sigma(s)(i) + 1 & \text{if } i \in \text{dom}(\mu) \text{ and } \mu(i) = P(s) \\ \max(0, \sigma(s)(i) - 1) & \text{if } i \in \text{dom}(\mu) \text{ and } \mu(i) = V(s) \end{cases}$$

A sequence $\mu_0, \ldots, \mu_{q-1}$ of multi-instructions is said to be admissible at state $\sigma$ when for all $k \in \{0, \ldots, q-1\}$ the multi-instruction $\mu_k$ is admissible at state $\sigma \cdot \mu_0 \cdots \mu_{k-1}$.
Action of a multi-instruction on a state

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Action of a multi-instruction on a state
Assuming that $\mu$ is admissible at $\sigma$

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\[
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\]
Action of a multi-instruction on a state

Assuming that $\mu$ is admissible at $\sigma$

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  \sigma(s)(i) & \text{in all other cases}
  \end{cases}$$
Action of a multi-instruction on a state

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Directed paths and sequences of multi-instructions
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A directed path $\gamma$ on $(G_1, \ldots, G_n)$ is a sequence $(\gamma(k))_{k \in \{0, \ldots, q\}}$ of points such that for all $k \in \{0, \ldots, q - 1\}$ we have
A directed path \( \gamma \) on \((G_1, \ldots, G_n)\) is a sequence \((\gamma(k))_{k \in \{0, \ldots, q\}}\) of points such that for all \(k \in \{0, \ldots, q - 1\}\) we have

- \( \gamma_i(k) = \gamma_i(k + 1) \) or \( \partial \gamma_i(k + 1) = \gamma_i(k) \) for all \( i \in \{1, \ldots, n\} \), or

Where \( \partial \) represents the partial transition function.
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\[\]
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Then $\gamma$ is associated with a sequence of multi-instructions $(\mu_k)_{k \in \{0, \ldots, q - 1\}}$ defined for $k \in \{0, \ldots, q - 1\}$ by
Directed paths and sequences of multi-instructions

A directed path $\gamma$ on $(G_1, \ldots, G_n)$ is a sequence $(\gamma(k))_{k \in \{0, \ldots, q\}}$ of points such that for all $k \in \{0, \ldots, q - 1\}$ we have

- $\gamma_i(k) = \gamma_i(k + 1)$ or $\partial^- \gamma_i(k + 1) = \gamma_i(k)$ for all $i \in \{1, \ldots, n\}$, or
- $\gamma_i(k) = \gamma_i(k + 1)$ or $\partial^+ \gamma_i(k) = \gamma_i(k + 1)$ for all $i \in \{1, \ldots, n\}$.

Then $\gamma$ is associated with a sequence of multi-instructions $(\mu_k)_{k \in \{0, \ldots, q - 1\}}$ defined for $k \in \{0, \ldots, q - 1\}$ by

- $\text{dom}(\mu_k) = \{ i \in \{1, \ldots, n\} | \gamma_i(k + 1) = \partial^+ \gamma_i(k) \text{ or } \lambda_i(\gamma_i(k + 1)) = W(...) \}$
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- $\mu_k(i) = \lambda_i(\gamma_i(k+1))$ for all $k \in \{0, \ldots, q-1\}$ and all $i \in \text{dom}(\mu_k)$
Discrete paths are “continuous”
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Admissible paths and execution traces

Given $\sigma$, a state of the program, a directed path is said to be admissible at $\sigma$ when so is its associated sequence of multi-instructions at state $\sigma$. In this case we define the action of $\gamma$ on the right of $\sigma$ as follows.

$\sigma \cdot \gamma = \sigma \cdot \mu_0 \cdots \mu_{q-1}$

An admissible path is an execution trace when all the conditional branchings met along the way are respected: for all $k \in \{0, \ldots, q-2\}$ and all $i \in \{1, \ldots, n\}$ such that $\mu_k(i)$ is a branching.

$(\mu_k(i))_{\sigma \cdot \mu_0 \cdots \mu_{k-1}} = \gamma_i(k+2)$
Admissible paths and execution traces

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$$\left(\mu_k(i)\right)(\sigma \cdot \mu_0 \cdots \mu_{k-1}) = \gamma_i(k + 2)$$
Concurrent access

```plaintext
var x = 0

proc p = x := 1
proc q = x := 2

init p q
```
Admissible execution trace

the value of $x$ is 0
Admissible execution trace

the value of $x$ is 0
Admissible execution trace

the value of $x$ is 0
Admissible execution trace

the value of $x$ is 1
Admissible execution trace

the value of $x$ is 2
Admissible execution trace

the value of $x$ is 2
Admissible execution trace

the value of \( x \) is 2
Admissible execution trace

the value of $x$ is 2
Not admissible execution trace

the value of $x$ is 0
Not admissible execution trace

\( x := 2 \)

the value of \( x \) is 0
Not admissible execution trace

the value of $x$ is 0
Not admissible execution trace

the value of $x$ is ?
Lack of resources

sem 1 a

proc p = P(a); V(a)

init 2p
Admissible concurrent execution trace

\[ \text{sem 1 a} \]
Admissible concurrent execution trace

\[ \text{sem } 1 \ a \]
Admissible concurrent execution trace

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Admissible concurrent execution trace

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Admissible concurrent execution trace

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Not admissible concurrent execution trace

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Not admissible concurrent execution trace

sem 1 a
Not admissible concurrent execution trace

\[
\text{sem } 1 \ a
\]
Synchronisation

\[ \text{sync } 1 \ b \]
\[ \text{proc } p = W(b) \]
\[ \text{init } 2p \]
Concurrent execution trace

\[ \text{sync 1 b} \]
Concurrent execution trace

\texttt{sync 1 b}
Concurrent execution trace

sync 1 b
Concurrent execution trace

\[ \text{sync 1 b} \]
Concurrent execution trace

$\text{sync } 1 \ b$
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\text{sync 1 b}
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Concurrent execution trace

\[ \text{sync } 1 \ b \]
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\texttt{sync 1 b}
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Not admissible concurrent execution trace

sync 1 b
Not admissible concurrent execution trace

\[ \text{sync } 1 \ b \]
Next goal

Encode admissibility into a model.
CONSERVATIVE PROGRAMS
Potential functions
The potential functions of processes and programs

A program $\Pi = (G_1, \ldots, G_n)$ is conservative when for all directed paths starting at the origin, the amount of semaphores held by the program at the end of the path only depends on its arrival point.

For all initial state $\sigma$, for all directed paths $\gamma, \gamma'$ starting at the origin,

$$\partial^+ \gamma \equiv \partial^+ \gamma' \Rightarrow \sigma \cdot \gamma | S = \sigma \cdot \gamma' | S$$

In particular, the program $\Pi$ comes with a potential function $F_\Pi$:

$$\{\text{semaphores}\} \times \{\text{points}\} \rightarrow \mathbb{N}$$

Proposition: The program $\Pi$ is conservative if and only if so are its processes $G_1, \ldots, G_n$ and its potential function is given by

$$F_\Pi(p_1, \ldots, p_d) = \sum_{k=1}^{d} F_{G_k}(p_k)$$
The potential functions of processes and programs

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In particular, the program $\Pi$ comes with a potential function

$$F_{\Pi} : \{\text{semaphores}\} \times \{\text{points}\} \rightarrow \mathbb{N} \cong \{\text{points}\} \rightarrow \{\text{multisets over } S\}$$
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The potential functions of processes and programs

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Conservative process

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Conservativity is decidable

We inductively define a sequence of partial functions $\pi_n$:

- The first term $\pi_0$ is only defined at the origin and $\pi_0(\text{origin})$ is the empty set.
- Assuming that $\pi_n$ is defined, for all pairs of points $(p, p')$ such that:
  - $\pi_n(p)$ is defined but not $\pi_n(p')$,
  - $\partial_- p' = p$ or $p' = \partial_+ p$,
we define a strict extension of $\pi_n$, by setting:

  $p' \mapsto \begin{cases} 
  \pi_n(p) & \text{if } \partial_- p' = p \\
  \lambda(p') & \text{if } p' = \partial_+ p
  \end{cases}$

- If all these extensions are compatible, then $\pi_n + 1$ is their union.
- Otherwise the induction stops and the graph is not conservative.

If all the points have been "visited" we have a finite chain of strict extensions $\pi_0 \subseteq \cdots \subseteq \pi_n \subseteq \pi_n + 1 = \pi$ whose last element is denoted by $\pi$.

If the following holds for all ordered pairs of points $(p, p')$ such that $\partial_- p' = p$ or $p' = \partial_+ p$, then $G$ is conservative, otherwise it is not.
Conservativity is decidable

We inductively define a sequence of partial functions $\pi_n : \{\text{points}\} \rightarrow \mathbb{N}^S$. 
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If all these extensions are compatible, then $\pi_{n+1}$ is their union. Otherwise the induction stops and the graph is not conservative.

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\[
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\end{cases}
\]
Discrete models
The discrete model of a conservative program

A point \( p = (p_1, \ldots, p_n) \) of the conservative program is said to be:

- **conflicting** when \( \lambda_i(p_i) \) and \( \lambda_j(p_j) \) conflict for some \( i \neq j \),
- **exhausting** when there is some semaphore \( s \in S \) such that \( F(p_1, \ldots, p_n, s) > \text{arity}(s) \),
- **desynchronizing** when there is some synchronization barrier \( b \in B \) such that \( 0 < \text{card}\{i \in \{1, \ldots, n\} | \lambda_i(p_i) = W(b)\} \leq \text{arity}(b) \).

The forbidden set gathers all the conflicting, exhausting, and desynchronizing points.

\[ \{\text{forbidden}\} = \{\text{conflicting}\} \cup \{\text{exhausting}\} \cup \{\text{desynchronizing}\} \]

The discrete model is the complement of its forbidden set.

\[ \{\text{points of the program}\} \setminus \{\text{forbidden points}\} \]
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  \]
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\[
\{\text{points of the program}\} \setminus \{\text{forbidden points}\}
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Discrete model

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Discrete model

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## Discrete model

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<tr>
<td>0</td>
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</tr>
</tbody>
</table>

Diagram:

- **P(a)** and **V(a)** are connected with arrows indicating transitions or relationships.
- The states 0 and 1 are distinguished, with 1 being shaded and marked with 'V(a)' and 'P(a)' as labels.
Conservative programs

Discrete models

Discrete model

sem 1 a

Diagram showing the discrete model with states 0 and 1 for variables P(a) and V(a). The diagram includes transitions between states.
Discrete model

$\text{sem 1 a}$
Discrete model

sem 1 a
Discrete Model

\( \text{sync } 1 \ b \)
Discrete Model

sync 1 b
Discrete Model

sync 1 b
Discrete Model

\begin{align*}
\text{sync} & \ 1 \ b
\end{align*}
Discrete Model

sync 1 b
Discrete Model

sync 1 b
Discrete Model

\[
\begin{align*}
W(b) \\
\otimes \\
W(a(b))
\end{align*}
\]
Main theorem of discrete models

– Soundness: any directed path on a discrete model (i.e. which does not meet any forbidden point) is admissible.

– Completeness: for each admissible path which meets a forbidden point there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.
Main theorem of discrete models

- Soundness:
Main theorem of discrete models

- **Soundness**: any directed path on a discrete model (i.e. which does not meet any forbidden point) is admissible.
Main theorem of discrete models

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- **Completeness:**
Main theorem of discrete models

- **Soundness**: any directed path on a discrete model (i.e. which does not meet any forbidden point) is admissible.

- **Completeness**: for each admissible path which meets a forbidden point there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.
Admissible execution trace

the value of $x$ is 0
Admissible execution trace

the value of $x$ is 0
Admissible execution trace

the value of \( x \) is 0
Admissible execution trace

the value of $x$ is 1
Admissible execution trace

the value of $x$ is 2
Admissible execution trace

the value of $x$ is 2
Admissible execution trace

the value of $x$ is 2
Admissible execution trace

the value of $x$ is 2
Admissible execution trace avoiding forbidden points

the value of $x$ is 0
Admissible execution trace avoiding forbidden points

the value of $x$ is 0
Admissible execution trace avoiding forbidden points

the value of $x$ is 0
Admissible execution trace avoiding forbidden points

the value of $x$ is 1
Admissible execution trace avoiding forbidden points

the value of $x$ is 1
Admissible execution trace avoiding forbidden points

the value of $x$ is 2
Admissible execution trace avoiding forbidden points

the value of $x$ is 2
Admissible execution trace avoiding forbidden points

the value of $x$ is 2
Admissible execution trace avoiding forbidden points

The value of $x$ is 2
Replacement

\[ x := 2 \]

\[ x := 1 \]

\[ \frac{50}{50} \]