DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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MPRI : Concurrency (2.3.1)
– Lecture 1 –

2020 – 2021
A QUICK OVERVIEW

OF

CONCURRENCY THEORY
Handbook of Logic in Computer Science

VOLUME 4

S. ABRAMSKY, DOV M. GABBAY,
and T. S. E. MAIBAUM

OXFORD SCIENCE PUBLICATIONS
PARALLEL AUTOMATA META LANGUAGE
Syntax
Paradigm


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1 Portable Operating Systems Interface, X is a reference to Unix
Paradigm


- The Dijkstra’s language is a parallel extension of ALGOL60 with P (lock/take), V (unlock/release), and parbegin ... parend

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- e.g. POSIX\(^1\) Threads

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Parallel Automata Meta Language

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x := 0 ; y := 0 ; (x := 1 || y := 1)
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  \[ x:=0 \ ; \ y:=0 \ ; \ (x:=1 \ || \ y:=1) \]

- The Carson and Reynolds language is a restriction of Dijkstra's language:
  \- Operator || in outermost position: only sequential processes are executed in parallel
  \- Neither branchings nor loops

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Features

- shared memory abstract machine (PRAM)
- concurrent read exclusive write (CREW)
- Operator || in outermost position: only sequential processes are executed in parallel
- Branchings, loops, and synchronisation barriers W (wait) are allowed
- no pointer arithmetics
- no function call, only jumps
- no birth nor death of process at runtime
- tokens are owned by processes
- conservative processes
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Declarations
Declarations

A basic block is defined as a (finite) sequence of instructions. A program is a list of declarations, the available declarations are:

- `sem <int> <set of identifiers>`
  ```
  sem 3 a b c d
  ```

- `sync <int> <set of identifiers>`
  ```
  sync 3 a b c d
  ```

- `mtx <set of identifiers>`
  ```
  mtx a b c d
  ```

- `var <identifier> = <constant>`
  ```
  var x = 0
  ```

- `proc <identifier> = <basic block>`

- `init <multiset of identifiers>`
  ```
  init a 2b 3c
  ```
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e.g. $\text{sem} 3 \ a \ b \ c \ d$

- $\text{sync} <\text{int}> <\text{set of identifiers}>$
  
e.g. $\text{sync} 3 \ a \ b \ c \ d$

- $\text{mtx} <\text{set of identifiers}>$
  
e.g. $\text{mtx} \ a \ b \ c \ d$

- $\text{var} <\text{identifier}> = <\text{constant}>$
  
e.g. $\text{var} \ x = 0$

- $\text{proc} <\text{identifier}> = <\text{basic block}>$

- $\text{init} <\text{multiset of identifiers}>$
  
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e.g. `init a 2b 3c`
Expressions and values
Expressions and values

The set of expressions is inductively built on the set of identifiers and the following set of operators:

- expressions
- identifiers
- operators: 
  - \(\land\) and 
  - \(\lor\)
  - \(+\)
  - \(-\)
  - \(*\)
  - \(/\)
  - \(\leq\)
  - \(\geq\)
  - \(<\)
  - \(>\)
  - \(=\)
  - \(\neq\)
  - \(\neg\)
  - \(\%\)
  - \(\bot\)

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Expressions and values

The set of expressions is inductively built on the set of identifiers and the following set of operators

<table>
<thead>
<tr>
<th>Operator</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>v</code></td>
<td>content of <code>v ∈ V</code></td>
</tr>
<tr>
<td><code>x</code></td>
<td><code>x ∈ R</code> constant</td>
</tr>
<tr>
<td><code>∧</code></td>
<td>minimum</td>
</tr>
<tr>
<td><code>∨</code></td>
<td>maximum</td>
</tr>
<tr>
<td><code>+</code></td>
<td>addition</td>
</tr>
<tr>
<td><code>−</code></td>
<td>substraction</td>
</tr>
<tr>
<td><code>∗</code></td>
<td>multiplication</td>
</tr>
<tr>
<td><code>/</code></td>
<td>division</td>
</tr>
<tr>
<td><code>⩽</code></td>
<td>less or equal</td>
</tr>
<tr>
<td><code>⩾</code></td>
<td>greater of equal</td>
</tr>
<tr>
<td><code>&lt;</code></td>
<td>strictly less</td>
</tr>
<tr>
<td><code>&gt;</code></td>
<td>strictly greater</td>
</tr>
<tr>
<td><code>=</code></td>
<td>equal</td>
</tr>
<tr>
<td><code>≠</code></td>
<td>not equal</td>
</tr>
<tr>
<td><code>¬</code></td>
<td>complement</td>
</tr>
<tr>
<td><code>%</code></td>
<td>modulo</td>
</tr>
<tr>
<td>⊥</td>
<td>bottom</td>
</tr>
</tbody>
</table>

### Syntax

<table>
<thead>
<tr>
<th>Syntax</th>
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</tr>
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<tbody>
<tr>
<td><code>⊥, x ∈ R, v ∈ V</code></td>
<td>nullary</td>
</tr>
<tr>
<td><code>¬</code></td>
<td>unary</td>
</tr>
<tr>
<td><code>∧, ∨, +, −, ∗, /, &lt;, &gt;, ⩽, ⩾, =, ≠, %</code></td>
<td>binary</td>
</tr>
</tbody>
</table>
Non branching instructions

- \textbf{identifier} := \textbf{expression} 
  the expression is evaluated then the result is stored in the identifier

- \textbf{P(identifier)} 
  takes an occurrence of the resource \textbf{identifier} (there are \textbf{arity} available tokens), stops the process
  otherwise

- \textbf{V(identifier)} 
  release an occurrence of the resource \textbf{identifier} (if such an occurrence is held by the process), ignored
  otherwise

- \textbf{W(identifier)} 
  stops the execution of the process until \textbf{arity} + 1 of them are stopped by the barrier

- \textbf{J(identifier)} 
  the execution of the process is stopped and the one of a copy of \textbf{identifier} starts. There is no return mechanism.

- \((L)\) enclose a list of instructions between parenthesis to make it a single instruction
Non branching instructions

- \textit{identifier:}=\textit{expression} the expression is evaluated then the result is stored in the identifier
Non branching instructions

- **identifier**:=`expression` the expression is evaluated then the result is stored in the identifier

- \( P(identifier) \) takes an occurrence of the resource \( identifier \) (there are \( arity \) available tokens), stops the process otherwise
Non branching instructions

- \texttt{identifier:=expression} the expression is evaluated then the result is stored in the identifier
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Non branching instructions

- \( \text{identifier} := \text{expression} \) the expression is evaluated then the result is stored in the identifier
- \( P(\text{identifier}) \) takes an occurrence of the resource \( \text{identifier} \) (there are \( \text{arity} \) available tokens), stops the process otherwise
- \( V(\text{identifier}) \) release an occurrence of the resource \( \text{identifier} \) (if such an occurrence is held by the process), ignored otherwise
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- \( (L) \) enclose a list of instructions between parenthesis to make it a single instruction
Branching

The branching is provided by a kind of "match case like" instruction:

\[(L_1)^+ [e_1] + (L_2)^+ [e_2] + \cdots + (L_n)^+ [e_n] + (L_{n+1})\]

- Each \(L_k\) is a basic block
- Each \(e_k\) is an expression
- The triggered branch is \(L_k\) with \(k\) being the first index such that \(e_k\) evaluates to some nonzero value
- If all the expressions evaluate to zero, then \(L_{n+1}\) is triggered.
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Describing a process

The body of a process is just a (possibly empty) sequence of instructions, i.e. a basic block, separated by semicolons. For example, the Hasse/Syracuse algorithm with input value 7:

```
proc p = x:=7;J(q)
proc q = J(r)+[x<>1]+()
proc r = (x:=x/2)+[x%2=0]+(x:=3*x+1) ; J(q)
```

Due to the branchings, basic blocks are actually trees.
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init p
```
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Control Flow Graphs
Control flow graphs and flowcharts
Control flow analysis, *F. E. Allen*, 1970
Assigning meanings to programs, *R. W. Floyd*, 1967
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- At the core of all modern software dealing with source code
e.g. GCC (cf. “basic blocks”), LLVM, Frama-C.
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- No theoretical definition yet control flow graphs must be finite for practical reasons.
- At the core of all modern software dealing with source code
  e.g. GCC (cf. “basic blocks”), LLVM, Frama-C.
- No such structure exist for parallel programs.
Generators

\[
x := f
\]

\[
\begin{align*}
\text{START} & \quad \downarrow \quad a_1 \quad \downarrow \quad b_1 \\
\quad \quad \downarrow \quad x := f \\
\quad \downarrow \quad b_1 \\
\end{align*}
\]

\[
\begin{align*}
\text{Yes} & \quad \downarrow \quad a_1 \quad \downarrow \quad b_1 \\
\text{No} & \quad \downarrow \quad b_2
\end{align*}
\]

\[
\text{HALT}
\]

\[
\begin{align*}
\phi? & \quad \downarrow \quad a_1 \quad \downarrow \quad b_1 \\
\quad \downarrow \quad a_1 \\
\quad \downarrow \quad b_1 \\
\end{align*}
\]

\[
\begin{align*}
\quad \downarrow \quad a_1 \\
\quad \downarrow \quad a_2 \\
\quad \downarrow \quad b_1
\end{align*}
\]
The Hasse-Syracuse algorithm in PAML

```plaintext
var x = 7

proc p = ()+[x=1]+J(q)

proc q = (x:=x/2) + [x%2=0] + (x:=3*x+1) ; J(p)

init p
```
Building the control flow graph
of the Hasse-Syracuse algorithm
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Building the control flow graph of the Hasse-Syracuse algorithm

entry point of the basic block of \( p \)

\[ x = 1 \]

entry point of the basic block of \( q \)

\[ x := 3 \times x + 1 \]

\[ x \mod 2 = 0 \]

\[ x := x / 2 \]
Building the control flow graph of the Hasse-Syracuse algorithm
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Building the control flow graph
of the Hasse-Syracuse algorithm

\[ J(q) \]

\[ x = 1 \]

\[ J(p) \]

\[ x = \frac{x}{2} \]

\[ x \mod 2 = 0 \]

\[ J(p) \]

\[ x = 3x + 1 \]

entry point of the basic block of \( p \)

entry point of the basic block of \( q \)
Building the control flow graph
of the Hasse-Syracuse algorithm
Reducing the Control Flow Graph
of the Hasse-Syracuse algorithm
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of the Hasse-Syracuse algorithm
Reducing the Control Flow Graph
of the Hasse-Syracuse algorithm

entry point

entry point

x:=x/2

x:=3*x+1

x=1

x%2=0

x%2=0

the current value of x is 7
An execution trace on a control flow graph of the Hasse-Syracuse algorithm
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of $x$ is 7
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 7
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of $x$ is 22
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of x is 22
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of x is 22
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 11
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of $x$ is 11
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

entry point

\[
x := \frac{x}{2}
\]

\[
x := 3 \times x + 1
\]

the current value of \( x \) is 11
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of $x$ is 34
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

\[ x := x/2 \]
\[ x := 3 \times x + 1 \]
\[ x \mod 2 = 0 \]
\[ x = 1 \]

the current value of \( x \) is 34
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of $x$ is 34
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of $x$ is 34
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of $x$ is 17
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

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The current value of $x$ is 52
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The current value of $x$ is 26
An execution trace on a control flow graph
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the current value of $x$ is 26
An execution trace on a control flow graph
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entry point

\[ x := x/2 \]
\[ x := 3 \times x + 1 \]

the current value of \( x \) is 26
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 26
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of \( x \) is 13
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The current value of \( x \) is 40
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 40
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

x := x/2

x := 3*x + 1

x = 1

entry point

x \% 2 = 0

the current value of x is 40
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of $x$ is 40
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 20
An execution trace on a control flow graph
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An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 10
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of x is 10
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the current value of $x$ is 10
An execution trace on a control flow graph
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An execution trace on a control flow graph of the Hasse-Syracuse algorithm

- $x = \frac{x}{2}$
- $x = 3x + 1$
- $x \mod 2 = 0$
- $x = 1$

The current value of $x$ is 5
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

The current value of \( x \) is 5
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of $x$ is 5
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 5
An execution trace on a control flow graph of the Hasse-Syracuse algorithm.

The current value of $x$ is 16.
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of \( x \) is 16
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

entry point

x := x / 2

x := 3 * x + 1

x % 2 = 0

the current value of x is 16
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

entry point

x := x/2

x := 3*x + 1

the current value of x is 16
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

The current value of $x$ is 8
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of $x$ is 8
An execution trace on a control flow graph
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\[ \begin{align*}
    x &= x/2 \\
    x &= 3x + 1 \\
    x \% 2 &= 0
\end{align*} \]

the current value of \( x \) is 8
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 8
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

The current value of $x$ is 4
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

entry point

\[ x := x/2 \]

\[ x := 3 \times x + 1 \]

\[ x \equiv 0 \]

the current value of \( x \) is 4
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of $x$ is 4
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of $x$ is 4
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entry point

the current value of $x$ is 2
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of x is 2
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of $x$ is 2
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

the current value of \( x \) is 1
An execution trace on a control flow graph
of the Hasse-Syracuse algorithm

the current value of $x$ is 1
An execution trace on a control flow graph of the Hasse-Syracuse algorithm

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Execution traces as paths over a control flow graph
Execution traces as paths over a control flow graph

- Any execution trace induces a path
Execution traces as paths over a control flow graph

- Any execution trace induces a path
- Some paths do not come from an execution trace
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- Therefore the collection of path provides a (strict) overapproximation of the collection of execution traces
Execution traces as paths over a control flow graph

- Any execution trace induces a path
- Some paths do not come from an execution trace
- Therefore the collection of path provides a (strict) overapproximation of the collection of execution traces
- The (infinite) collection of paths is entirely determined by the (finite) control flow graph
The overall idea of static analysis
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Any model of a program should contain a finite representation of an overapproximation of the collection of all its execution traces.
The overall idea of static analysis

Any model of a program should contain a finite representation of an overapproximation of the collection of all its execution traces.

One of the goal of the course it to provide such a structure for a large class of PAML programs.
Parallel Automata Meta Language

Restrictions from the PAML syntax

By construction the PAML language enforces the following restrictions:

- There is neither birth nor death of processes at runtime.
- The arity of resources cannot be changed at runtime.
- There is no pointer arithmetics.
Restrictions from the PAML syntax

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Restrictions from the PAML syntax

By construction the PAML language enforces the following restrictions

- There is neither birth nor death of processes at runtime
- The arity of resources cannot be changed at runtime
- There is no pointer arithmetics
Abstract Machine
Abstract expressions

- The set of variables of a program is $X$.
- A valuation or memory state is a mapping $\nu: X \rightarrow R \cup \{\perp\}$.
- An expression is a mapping $\varepsilon: \{\text{valuations}\} \rightarrow R$ with $F(\varepsilon) \subseteq X$ such that if the valuations $\nu$ and $\nu'$ match on $F(\varepsilon)$ then $\varepsilon(\nu) = \varepsilon(\nu')$.
- The set of expressions occurring in the program is denoted by $E$. 

Parallel Automata Meta Language

The abstract machine
Abstract expressions

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Interpretation of expressions

only depends on the current memory state
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Interpretation of expressions
only depends on the current memory state

- $[x]_\nu = \nu(x)$ for all $x \in \mathcal{X}$
- Any value in $\mathbb{R} \setminus \{0\}$ stands for true while 0 stands for false
Interpretation of expressions
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- Any value in $\mathbb{R} \setminus \{0\}$ stands for true while 0 stands for false
- $\llbracket \neg \rrbracket : \mathbb{R}_\bot \to \mathbb{R}_\bot$, where $\mathbb{R}_\bot$ is the set of non-real numbers.
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- \([\neg] : \mathbb{R}_\bot \rightarrow \mathbb{R}_\bot\),
  \([\neg](0) = 1\),
  \([\neg](\bot) = \bot\), and
  \([\neg](x) = 0\) for all \(x \in \mathbb{R} \setminus \{0\}\)
- \([e] = \bot\) for all expression \(e\) in which \(\bot\) occurs
Interpretation of expressions
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  \([\neg](0) = 1,\]
  \([\neg](\bot) = \bot,\)
  and \([\neg](x) = 0\) for all \(x \in \mathbb{R} \setminus \{0\}\)
- \([e] = \bot\) for all expression \(e\) in which \(\bot\) occurs
- the other operators are interpreted as expected
Abstract instructions
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The sets of semaphores, and barriers of a program are respectively $S$ and $B$. 
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- An assignment is an element of $\mathcal{X} \times \mathcal{E}$ yet we write $x := \varepsilon$ instead of $(x, \varepsilon)$. By extension $F(x := \varepsilon) = F(\varepsilon)$. 
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- Given a graph

$$G : A \xrightarrow{\partial^-} V \xrightarrow{\partial^+}$$

a conditional branching at vertex $v \in V$ is a mapping

$$\beta : \{\text{valuations}\} \rightarrow \{a \in A \mid \partial^- a = v\}$$

together with a subset $F(\beta) \subseteq X$ such that if the valuations $\nu$ and $\nu'$ match on $F(\beta)$ then $\beta(\nu) = \beta(\nu')$. 
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- The synchronisation primitives $P(s)$, $V(s)$, and $W(b)$ for $s \in S$ and $b \in B$
Abstract processes as control flow graphs
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\[ G : A \xrightarrow{\partial} V \quad \text{and} \quad \lambda : V \rightarrow \{\text{instructions}\} \]
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\[ G : A \xrightarrow{\partial^{-}} V \quad \text{and} \quad \lambda : V \rightarrow \{\text{instructions}\} \]

- An entry point \( v_0 \in V \) such that \( \lambda(v_0) = \text{Skip} \).
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The arrows are interpreted as intermediate positions of the instruction pointer so a point on a control flow graph is either a vertex or an arrow.
Abstract program
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- The arity map $\alpha : S \sqcup B \rightarrow \mathbb{N} \cup \{\infty\}$. 
Abstract program

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- The arity map $\alpha : S \sqcup B \rightarrow \mathbb{N} \cup \{\infty\}$.
- The tuple $(G_1, \ldots, G_n)$ of processes which are launched simultaneously at the beginning of each execution of the program.
Points and multi-instructions

*Higher Dimensional Transition Systems*, G. L. Cattani and V. Sassone, 1996
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- A point of \((G_1, \ldots, G_n)\) is an \(n\)-tuple \(p\) whose \(i^{\text{th}}\) component, namely \(p_i\), is a point of \(G_i\).
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The internal states of the abstract machine

A state is a mapping $\sigma$ defined over the disjoint union $X \sqcup S$ such that:
- for all $x \in X$, $\sigma(x) \in R_\perp$, and
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Parallel Automata Meta Language
The abstract machine
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A state is a mapping \( \sigma \) defined over the disjoint union \( X \sqcup S \) such that:

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Admissible multi-instructions

The possible conflicts are:

- **write-write**: \[ x := \epsilon \text{ vs } x := \epsilon' \]
- **read-write**: \[ x := \epsilon \text{ vs an instruction in which } x \text{ is free} \]

A multi-instruction \( \mu \) is said to be admissible at state \( \sigma \) when:

1. For \( i, j \in \text{dom}(\mu) \) with \( i \neq j \), \( \mu(i) \) and \( \mu(j) \) do not conflict,
2. For all \( s \in S \), \( \phi(s) \leq \alpha(s) \) where \( \phi(s) = |\sigma(s)| + \text{card}\{i \in \text{dom}(\mu) | \mu(i) = P(s)\} - \text{card}\{i \in \text{dom}(\mu) | \mu(i) = V(s); \sigma(s)(i) > 0\} \)
3. For all \( b \in B \), \( \text{card}\{i \in \text{dom}(\mu) | \mu(i) = W(b)\} \notin \{1, \ldots, \alpha(b)\} \)
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Action of a multi-instruction on a state

Assuming that $\mu$ is admissible at $\sigma$
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- For all $x \in \mathcal{X}$, if $\mu(i)$ is $x := \varepsilon$ for some $i \in \{1, \ldots, n\}$ one has

$$
(\sigma \cdot \mu)(x) = \varepsilon(\sigma|\mathcal{X})
$$
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In the other case one has \( (\sigma \cdot \mu)(x) = \sigma(x) \).
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- For all $s \in S$ the multiset $(\sigma \cdot \mu)(s)$, seen as a mapping from $\{1, \ldots, n\}$ to $\mathbb{N}$, is given by

$$ i \mapsto \begin{cases} 
\sigma(s)(i) + 1 & \text{if } i \in \text{dom}(\mu) \text{ and } \mu(i) = P(s) \\
\max(0, \sigma(s)(i) - 1) & \text{if } i \in \text{dom}(\mu) \text{ and } \mu(i) = V(s) \\
\sigma(s)(i) & \text{in all other cases}
\end{cases} $$
Action of a multi-instruction on a state

Assuming that $\mu$ is admissible at $\sigma$

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$$
i \mapsto \begin{cases} 
\sigma(s)(i) + 1 & \text{if } i \in \text{dom}(\mu) \text{ and } \mu(i) = P(s) \\
\max(0, \sigma(s)(i) - 1) & \text{if } i \in \text{dom}(\mu) \text{ and } \mu(i) = V(s) \\
\sigma(s)(i) & \text{in all other cases}
\end{cases}
$$

A sequence $\mu_0, \ldots, \mu_{q-1}$ of multi-instructions is said to be admissible at state $\sigma$ when for all $k \in \{0, \ldots, q - 1\}$ the multi-instruction $\mu_k$ is admissible at state $\sigma \cdot \mu_0 \cdots \mu_{k-1}$.
Directed paths and sequences of multi-instructions
Directed paths and sequences of multi-instructions

A directed path $\gamma$ on $(G_1, \ldots, G_n)$ is a sequence $(\gamma(k))_{k \in \{0, \ldots, q\}}$ of points such that for all $k \in \{0, \ldots, q - 1\}$ we have
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- $\gamma_i(k) = \gamma_i(k+1)$ or $\gamma_i(k) = \partial \gamma_i(k+1)$ for all $i \in \{1, \ldots, n\}$, or
Directed paths and sequences of multi-instructions

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- \( \gamma_i(k) = \gamma_i(k + 1) \) or \( \partial^+ \gamma_i(k) = \gamma_i(k + 1) \) for all \( i \in \{1, \ldots, n\} \).
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Then $\gamma$ is associated with a sequence of multi-instructions $(\mu_k)_{k \in \{0, \ldots, q-1\}}$ defined for $k \in \{0, \ldots, q - 1\}$ by
Directed paths and sequences of multi-instructions

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Directed paths and sequences of multi-instructions

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- $\mu_k(i) = \lambda_i(\gamma_i(k + 1))$ for all $k \in \{0, \ldots, q - 1\}$ and all $i \in \text{dom}(\mu_k)$
Discrete paths are “continuous”
Discrete paths are “continuous”
Discrete paths are “continuous”
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Admissible paths and execution traces
Admissible paths and execution traces

Given $\sigma$ a state of the program, a directed path is said to be admissible at $\sigma$ when so is its associated sequence of multi-instructions at state $\sigma$. In this case we define the action of $\gamma$ on the right of $\sigma$ as follows.

$$\sigma \cdot \gamma = \sigma \cdot \mu_0 \cdots \mu_{q-1}$$
Admissible paths and execution traces

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An admissible path is an execution trace when all the conditional branchings met along the way are respected: for all $k \in \{0, \ldots, q - 2\}$ and all $i \in \{1, \ldots, n\}$ such that $\mu_k(i)$, which is by definition $\lambda_i(\gamma_i(k + 1))$, is a branching, and

$$(\mu_k(i))(\sigma \cdot \mu_0 \cdots \mu_{k-1}) = \gamma_i(k + 2)$$
Concurrent access

```plaintext
var x = 0

proc p = x:=1

proc q = x:=2

init p q
```
Admissible execution trace

the value of \( x \) is 0
Admissible execution trace

the value of $x$ is 0
Admissible execution trace

the value of $x$ is 0
Admissible execution trace

the value of \(x\) is 1
Admissible execution trace

the value of $x$ is 2
Admissible execution trace

x := 2

the value of x is 2
Admissible execution trace

The value of $x$ is 2
Admissible execution trace

the value of $x$ is 2
Not admissible execution trace

the value of $x$ is 0
Not admissible execution trace

The value of $x$ is 0
Not admissible execution trace

the value of $x$ is 0
Not admissible execution trace

the value of $x$ is $\ ?$
Lack of resources

\text{sem 1 a}

\text{proc p = P(a);V(a)}

\text{init 2p}
Admissible concurrent execution trace

\[ \text{sem 1 a} \]
Admissible concurrent execution trace

\[
\text{sem } 1 a
\]
Admissible concurrent execution trace

sem 1 a
Admissible concurrent execution trace

\( \text{sem } 1 \ a \)
Admissible concurrent execution trace

sem 1 a
Admissible concurrent execution trace

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Admissible concurrent execution trace

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Admissible concurrent execution trace

sem 1 a
Admissible concurrent execution trace

\[ \text{sem} \ 1 \ a \]
Admissible concurrent execution trace

\[ \text{sem 1 a} \]
Admissible concurrent execution trace

sem 1 a
Admissible concurrent execution trace

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Not admissible concurrent execution trace

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Not admissible concurrent execution trace

\[ \text{sem 1 a} \]
Not admissible concurrent execution trace

\[ \text{sem } 1 \ a \]
Not admissible concurrent execution trace

\text{sem 1 a}
Synchronisation

sync 1 b

proc p = W(b)

init 2p
Concurrent execution trace

\[ \text{sync } 1 \ b \]
Concurrent execution trace

sync 1 b
Concurrent execution trace

sync 1 b
Concurrent execution trace

sync 1 b
Concurrent execution trace

sync 1 b
Concurrent execution trace

sync 1 b
Concurrent execution trace

sync 1 b
Concurrent execution trace

sync 1 b
Not admissible concurrent execution trace

sync 1 b
Not admissible concurrent execution trace

sync 1 b
Not admissible concurrent execution trace

\( \text{sync } 1 \ b \)
Not admissible concurrent execution trace

sync 1 b
Next goal

Encode admissibility into a model.
CONSERVATIVE PROGRAMS
Potential Functions
The potential functions of processes and programs

A program $\Pi = (G_1, \ldots, G_n)$ is conservative when for all directed paths starting at the origin, the amount of semaphores held by the program at the end of the path only depends on its arrival point.

For all initial state $\sigma$, for all directed paths $\gamma, \gamma'$ starting at the origin,

$$\partial^+ \gamma = \partial^+ \gamma' \Rightarrow \sigma \cdot \gamma | S = \sigma \cdot \gamma' | S$$

In particular, the program $\Pi$ comes with a potential function $F_\Pi$:

$$\{\text{semaphores}\} \times \{\text{points}\} \rightarrow \mathbb{N} \sim \{\text{points}\} \rightarrow \{\text{multisets over S}\}$$

Proposition: The program $\Pi$ is conservative if and only if so are its processes $G_1, \ldots, G_n$ and its potential function is given by

$$F_\Pi(p_1, \ldots, p_d) = \sum_{k=1}^d F_{G_k}(p_k)$$
A program \( \Pi = (G_1, \ldots, G_n) \) is \textit{conservative} when for all directed paths starting at the origin, the amount of semaphores held by the program at the end of the path \textit{only} depends on its arrival point.
The potential functions of processes and programs

A program $\Pi = (G_1, \ldots, G_n)$ is conservative when for all directed paths starting at the origin, the amount of semaphores held by the program at the end of the path only depends on its arrival point.

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In particular, the program $\Pi$ comes with a potential function

$$F_\Pi : \{\text{semaphores}\} \times \{\text{points}\} \to \mathbb{N} \cong \{\text{points}\} \to \{\text{multisets over } S\}$$
The potential functions of processes and programs

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F_\Pi(p_1, \ldots, p_d) = \ldots
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Conservative process

example
Conservative process

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Conservative process

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Conservative programs

Potential functions

Conservative process

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Conservative process

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Conservative process

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Conservative programs

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Conservative programs
Potential functions
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\[ P(s) \]
Not conservative process

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Conservative programs
Potential functions

\( P(s) \)
Conservativity is decidable

We inductively define a sequence of partial functions $\pi_n: \{\text{points}\} \to \mathbb{N}$.

- The first term $\pi_0$ is only defined at the origin and $\pi_0(\text{origin})$ is the empty.

- Assuming that $\pi_n$ is defined, for all pairs of points $(p, p')$ such that:
  - $\pi_n(p)$ is defined but not $\pi_n(p')$,
  - $\partial_+ p' = p$ or $p' = \partial_- p$,

we define a strict extension of $\pi_n$, by setting:

$$p' \mapsto \begin{cases} \pi_n(p) & \text{if } \partial_+ p' = p \\ \lambda(p') & \text{if } p' = \partial_- p \end{cases}$$

- If all these extensions are compatible, then $\pi_n + 1$ is their union.

- If all the points have been "visited" we have a finite chain of strict extensions $\pi_0 \subseteq \cdots \subseteq \pi_n \subseteq \pi_n + 1 = \pi$ whose last element is denoted by $\pi$.

- If the following holds for all ordered pairs of points $(p, p')$ such that $\partial_+ p' = p$ or $p' = \partial_- p$, then $G$ is conservative, otherwise it is not.

$$\pi(p') = \begin{cases} \pi(p) & \text{if } \partial_+ p' = p \\ \lambda(p') & \text{if } p' = \partial_- p \end{cases}$$
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  Otherwise the induction stops and the graph is not conservative.
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Conservativity is decidable

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Discrete Models
The discrete model of a conservative program

A point \( p = (p_1, \ldots, p_n) \) of the conservative program is said to be:

- conflicting when \( \lambda_i(p_i) \) and \( \lambda_j(p_j) \) conflict for some \( i \neq j \),
- exhausting when there is some semaphore \( s \in S \) such that \( F(p_1, \ldots, p_n, s) > \text{arity}(s) \),
- desynchronizing when there is some synchronization barrier \( b \in B \) such that \( 0 < \text{card} \{ i \in \{1, \ldots, n\} | \lambda_i(p_i) = W(b) \} \leq \text{arity}(b) \).

The forbidden set gathers all the conflicting, exhausting, and desynchronizing points:

\[
\{\text{forbidden}\} = \{\text{conflicting}\} \cup \{\text{exhausting}\} \cup \{\text{desynchronizing}\}
\]

The discrete model is the complement of its forbidden set.
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\{\text{points of the program}\} \setminus \{\text{forbidden points}\}
\]
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- exhausting when there is some semaphore \( s \in S \) such that
  \[
  F(p_1, \ldots, p_n, s) > \text{arity}(s),
  \]
where
  \[
  \text{arity}(s) = |\text{arity}(\lambda_i(p_i))|.
  \]
The discrete model of a conservative program

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The **forbidden** set gathers all the conflicting, exhausting, and desynchronizing points.

\[
\{\text{forbidden}\} = \{\text{conflicting}\} \cup \{\text{exhausting}\} \cup \{\text{desynchronizing}\}
\]
The discrete model of a conservative program

A point \( p = (p_1, \ldots, p_n) \) of the conservative program is said to be:
- conflicting when \( \lambda_i(p_i) \) and \( \lambda_j(p_j) \) conflict for some \( i \neq j \),
- exhausting when there is some semaphore \( s \in S \) such that
  \[
  F(p_1, \ldots, p_n, s) > \text{arity}(s),
  \]
- desynchronizing when there is some synchronization barrier \( b \in B \) such that
  \[
  0 < \text{card}\{i \in \{1, \ldots, n\} | \lambda_i(p_i) = W(b)\} \leq \text{arity}(b),
  \]

The forbidden set gathers all the conflicting, exhausting, and desynchronizing points.

\[
\{\text{forbidden}\} = \{\text{conflicting}\} \cup \{\text{exhausting}\} \cup \{\text{desynchronizing}\}
\]

The discrete model is the complement of its forbidden set.

\[
\{\text{points of the program}\} \setminus \{\text{forbidden points}\}
\]
Discrete model

sem 1 a
Discrete model

\[ P(a) \otimes V(a) \]
Discrete model

sem 1 a
Discrete model

$P(a) \otimes V(a)$
Discrete model

sem 1 a
Discrete model

sem 1 a
Discrete model

sem 1 a
Discrete Model

sync 1 b
Discrete Model

sync 1 b
Discrete Model

sync \( 1 \ b \)
Discrete Model

sync 1 b
Discrete Model

sync 1 b

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & W(b) & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]
Discrete Model

csync 1 b
Discrete Model

sync 1 b
Main theorem of discrete models

– Soundness: any directed path on a discrete model (i.e. which does not meet any forbidden point) is admissible.

– Completeness: for each admissible path which meets a forbidden point there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.
Main theorem of discrete models

– Soundness:
Main theorem of discrete models

- **Soundness**: any directed path on a discrete model (i.e. which does not meet any forbidden point) is ...
Main theorem of discrete models

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Main theorem of discrete models

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Main theorem of discrete models

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Admissible execution trace

the value of $x$ is 0
Admissible execution trace

The value of $x$ is 0
Admissible execution trace

the value of $x$ is 0
Admissible execution trace

the value of $x$ is 1
Admissible execution trace

the value of $x$ is 2
Admissible execution trace

the value of $x$ is 2
Admissible execution trace

the value of x is 2
Conservative programs

Discrete models

Admissible execution trace

the value of \( x \) is 2
Admissible execution trace avoiding forbidden points

the value of $x$ is 0
Admissible execution trace avoiding forbidden points

the value of \( x \) is 0

\[
\begin{align*}
    x &= 1 \\
    x &= 2 \\
\end{align*}
\]
Admissible execution trace avoiding forbidden points

the value of x is 0
Conservative programs
Discrete models

Admissible execution trace avoiding forbidden points

\[ x := 2 \]

the value of \( x \) is 1
Admissible execution trace avoiding forbidden points

the value of $x$ is 1
Admissible execution trace avoiding forbidden points

the value of $x$ is 2
Admissible execution trace avoiding forbidden points

the value of $x$ is 2
Admissible execution trace avoiding forbidden points

the value of $x$ is 2
Conservative programs

Discrete models

Admissible execution trace avoiding forbidden points

the value of $x$ is 2
Replacement

$x := 2$

$x := 1$

Conservative programs

Discrete models