2.3.1: Concurrency

tuesday, the 26th of february 2019
duration: 3h

Exercise 1: Denote the following (hatched) isothetic region by $X$:

![Hatched Region]

1. Write a program whose forbidden region is $X$.
2. Draw the deadlock attractor of the program.
3. Draw the category of components of the geometric model.

Exercise 2: let $X$ be the geometric model of the following program:

```
sync 2 b
p = \omega(b)
init: 3p
```

1) Make a picture displaying the forbidden region of the program (i.e. the complement of $X$).
2) Give the list of the maximal blocks of the forbidden region of the program.
3) Give the list of all execution traces of the program.
4a) Are the 3 copies of the process $p$ model independent (explain)?
4b) Are the 3 copies of the process $p$ observationally independent (explain)?
4c) Conclude.
5) What are the components of the geometric model (describe them as union of maximal cubes)?

Exercise 3: Denote the following (sequential) program by $P$:

```
sem 2 a
p = P(a)V(a)
init: p
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1) Make a picture of the geometric model of $P|P|P$.
2) Prove that the geometric model of $P|P|P$ is prime.
3) Let $A$, $B$, and $C$ be conservative programs.
Exercise 4: A path is a continuous map defined on \([0, r]\) for some \(r \in \mathbb{R}_+\). A sample of the interval \([0, r]\) is a sequence \(0 = t_0 \leq \cdots \leq t_n = r\) where \(n \in \mathbb{N}\). The length \(\ell(\gamma)\) of a path \(\gamma : [0, r] \to (X, d)\) on a metric space is the least upper bound of the collection of sums

\[
\sum_{i=0}^{n-1} d(\gamma(t_{i+1}), \gamma(t_i))
\]

over all the samples of \([0, r]\). Suppose that \(X\) is the metric space \([0, 1]^2\) where the distance between two points \((x, y)\) and \((x', y')\) is given by

\[
d((x, y), (x', y')) = \max\{|x-x'|, |y-y'|\}
\]

The first and the second projections are denoted by \(\text{proj}_1\) and \(\text{proj}_2\). A path on \(X\) is said to be directed when it is increasing in both coordinates.

1) Denote by \(\delta : [0, r] \to X\) a directed path whose image is the diagonal, i.e.

\[
\{\delta(t) \mid t \in [0, r]\} = \{(x, x) \mid x \in [0, 1]\}.
\]

1a) Assuming that \(X\) is the model of a conservative program, which kind of execution trace is represented by \(\delta\)?

1b) Given a sample \(t_0 \leq \cdots \leq t_n\) of \([0, r]\), calculate the following sum:

\[
\sum_{i=0}^{n-1} d(\delta(t_{i+1}), \delta(t_i))
\]

1c) What is the length of \(\delta\)?

2) A directed path on \(X\) is said to be vertical (resp. horizontal) when \(\text{proj}_1 \circ \gamma\) (resp. \(\text{proj}_2 \circ \gamma\)) is constant. A staircase \(\sigma\) is a (finite) concatenation of vertical paths and horizontal paths on \(X\).

2a) Assuming that \(X\) is the model of a conservative program, which kind of execution trace is represented by a staircase?

2b) Given a staircase \(\sigma : [0, r] \to X\), find a sample \(t_0 \leq \cdots \leq t_n\) of \([0, r]\) such that

\[
\ell(\sigma) = \sum_{i=0}^{n-1} d(\sigma(t_{i+1}), \sigma(t_i))
\]

(Hint: start with the one step staircase and make a picture)

2c) Assuming that \(\sigma(0) = (0, 0)\) and \(\sigma(r) = (1, 1)\), what is the length of \(\sigma\)?

The set of directed paths on \(X\) (which are defined on \([0, r]\)) form a metric space where the distance between \(\gamma\) and \(\gamma'\) is given by

\[
d(\gamma, \gamma') = \max\{d_X(\gamma(t), \gamma'(t)) \mid t \in [0, r]\}
\]

3) Prove that for all \(\varepsilon > 0\) there exists a staircase \(\sigma\) such that \(d(\delta, \sigma) < \varepsilon\). What can be said about the mapping \(\gamma \mapsto \ell(\gamma)\)?