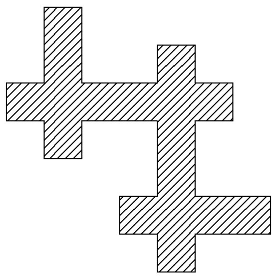
2.3.1: Concurrency

tuesday, the 26^{th} of february 2019
duration: 3h

Exercise 1: Denote the following (hatched) isothetic region by X:



1. Write a program whose forbidden region is X.
2. Draw the deadlock attractor of the program.
3. Draw the category of components of the geometric model.

Exercise 2: let X be the geometric model of the following program:

```
sync 2 b
p = W(b)
init:  3p
```

1) Make a picture displaying the forbidden region of the program (i.e. the complement of X).

2) Give the list of the maximal blocks of the forbidden region of the program.

3) Give the list of all execution traces of the program.

4a) Are the 3 copies of the process p model independent (explain)?
4b) Are the 3 copies of the process p observationally independent (explain)?
4c) Conclude.

5) What are the components of the geometric model (describe them as union of maximal cubes)?

Exercise 3: Denote the following (sequential) program by P :

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sem 2 a
p = P(a)V(a)
init:  p
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1) Make a picture of the geometric model of $P|P|P$.

2) Prove that the geometric model of $P|P|P$ is prime.

3) Let A, B, and C be conservative programs.

3a) Assuming that A, B, and C are pairwise model independent (i.e. $[\![A|B]\!] = [\![A]\!] \times [\![B]\!]$, $[\![B|C]\!] = [\![B]\!] \times [\![C]\!]$, and $[\![C|A]\!] = [\![C]\!] \times [\![A]\!]$), can we deduce that $[\![A|B|C]\!] = [\![A]\!] \times [\![B]\!] \times [\![C]\!]$ (explain)?

3b) Assuming that A, B, and C are pairwise observationally independent, can we deduce that $[\![A|B|C]\!] = [\![A]\!] \times [\![B]\!] \times [\![C]\!]$ (explain)?

<u>Exercise 4</u>: A path is a continuous map defined on $[0, r]$ for some $r \in \mathbb{R}_+$. A sample of the interval $[0, r]$ is a sequence $0 = t_0 \leqslant \cdots \leqslant t_n = r$ where $n \in \mathbb{N}$. The length $\ell(\gamma)$ of a path $\gamma : [0, r] \to (X, d)$ on a metric space is the least upper bound of the collection of sums

$$\sum_{i=0}^{n-1} d\big(\gamma(t_{i+1}), \gamma(t_i)\big)$$

over all the samples of $[0, r]$. Suppose that X is the metric space $[0, 1]^2$ where the distance between two points (x, y) and (x', y') is given by

$$d((x, y), (x', y')) \quad = \quad \max\{|x - x'|, |y - y'|\}$$

The first and the second projections are denoted by proj_1 and proj_2. A path on X is said to be directed when it is increasing in both coordinates.

1) Denote by $\delta : [0, r] \to X$ a directed path whose image is the diagonal, i.e.

$$\{\delta(t) \mid t \in [0, r]\} \quad = \quad \{(x, x) \mid x \in [0, 1]\} \ .$$

1a) Assuming that X is the model of a conservative program, which kind of execution trace is represented by δ?

1b) Given a sample $t_0 \leqslant \cdots \leqslant t_n$ of $[0, r]$, calculate the following sum:

$$\sum_{i=0}^{n-1} d\big(\delta(t_{i+1}), \delta(t_i)\big)$$

1c) What is the length of δ?

2) A directed path on X is said to be vertical (resp. horizontal) when $\mathrm{proj}_1 \circ \gamma$ (resp. $\mathrm{proj}_2 \circ \gamma$) is constant. A staircase σ is a (finite) concatenation of vertical paths and horizontal paths on X.

2a) Assuming that X is the model of a conservative program, which kind of execution trace is represented by a staircase?

2b) Given a staircase $\sigma : [0, r] \to X$, find a sample $t_0 \leqslant \cdots \leqslant t_n$ of $[0, r]$ such that

$$\ell(\sigma) \quad = \quad \sum_{i=0}^{n-1} d\big(\sigma(t_{i+1}), \sigma(t_i)\big)$$

(Hint: start with the one step staircase and make a picture)
2c) Assuming that $\sigma(0) = (0, 0)$ and $\sigma(r) = (1, 1)$, what is the length of σ?

The set of directed paths on X (which are defined on $[0, r]$) form a metric space where the distance between γ and γ' is given by

$$d(\gamma, \gamma') \quad = \quad \max\{d_X(\gamma(t), \gamma'(t)) \mid t \in [0, r]\}$$

3) Prove that for all $\varepsilon > 0$ there exists a staircase σ such that $d(\delta, \sigma) < \varepsilon$. What can be said about the mapping $\gamma \mapsto \ell(\gamma)$?