Exercise 1:
According to the virtual machine of the PAML language give all the possible outputs \((x, y)\) of the following program.

```
#variable x = 3
#process p1 = x:=1
#process p2 = x:=2
#process q = (y:=x)+[x=1]+(y:=x)+[x<>1]+(y:=x)
#init p1 p2 q
```

Exercise 2:
a) Provides the following process with a conservative control flow graph

```
#process p = P(a)+[e = 0]+P(b) with e begin some expression.
```

b) Prove the following process has no finite conservative control flow graph:

```
#process p = P(a).C(p)
```

Exercise 3:
Give the potential function of the following program

```
#mutex a
#process p = P(a).V(a)
#init 2p
```

Exercise 4:
Extend the PAML language and its virtual machine so the instruction \(F(name)\), \(name\) being an identifier, allows dynamic creation of processes. By convention, if \(name\) was not associated to a body of instructions by some line

```
#process name = body of instructions
```

then its is considered as the empty process.

Exercise 5:
Give two precubical sets \(C\) and \(M\) whose realization in the category of topological spaces are respectively the cylinder and the Möbius band (pictures with some explanations suffice).
Exercise 6:
Let \( P \) be the following program

\[
\begin{align*}
\text{#mutex a b c} \\
\text{#process x = Pb.Pa.Va.Pc.Vb} \\
\text{#init x y}
\end{align*}
\]

a) Draw the geometric model (i.e. actually draw the forbidden region in grey) of \( P \) (make a rather large picture)

b) Draw the deadlock attractor of \( P \)

c) Draw the category of components of \( P \)

Observe that the forbidden region of \( P \) is actually contained in some square \([0, r]^2\) with \( r > 0 \). A dipath is then said to be maximal when its image is contained in \([0, r]^2\) and it cannot be extended (by nonconstant dipaths). On a new picture:

d) Draw one representative of each dihomotopy class of the maximal dipaths on the model of \( P \)

e) Give a finite collection \( H_1, \ldots, H_n \) of subregions of the model such that:
- the image of a maximal dipath is contained in a unique \( H_k \)
- two maximal dipaths are dihomotopic iff their image are contained in the same \( H_k \). The subregions may overlap so it is recommended to draw several pictures. Also note that a maximal dipath may not start at the origin.

Exercise 7:
Consider the following PAML program \( P \)

\[
\begin{align*}
\text{#synchronization 2 a} \\
\text{#process p = Wa} \\
\text{#init 2p}
\end{align*}
\]

a) Draw \([P]\) the geometric model of \( P \)

b) Compute the fundamental category \( \pi_1([P]) \)

c) Compute the category of components \( \pi_0(\pi_1([P])) \)

Exercise 8:
Consider the following PAML program \( P \)

\[
\begin{align*}
\text{#mutex a b} \\
\text{#semaphore c} \\
\text{#process p = Pa.Pc.Vc.Va} \\
\text{#process q = Pb.Pc.Vc.Vb} \\
\text{#init 2p 2q}
\end{align*}
\]

a) Compute the forbidden region generated by each resource.

b) Give the maximal cubes of the forbidden region.
c) Compute a decomposition of the state space.

d) Denote by $X$ the geometric model of $P$ and write a PAML program whose geometric model is isomorphic with $X$.

Exercise 9: An $n$-grid, for $n \in \mathbb{N}$ is a subset $S \subseteq \mathbb{Z}^n$. A path of length $n \in \mathbb{N}$ on the grid $S$ is a finite sequence of points $p_0, \ldots, p_n$ of $S$ such that for all $k \in \{1, \ldots, n\}$, $p_{k+1} - p_k$ is a vector whose unique nonzero coordinate is 1.

a) We want to define a category $P(S)$ whose morphisms are the paths on the grid $S$. Describe the sources, the targets, the compositions, and the identities.

b) Prove that $P(S)$ is freely generated by some graph $G(S)$ (describe this graph).

Two paths of length 2 with the same source and the same target are declared to be equivalent so we have a congruence $\sim$ over $P(S)$. Define $F(S)$ as the quotient $P(S)/\sim$.

c) Prove that $F(S)$ is loop-free.

Let $S$ be $\{0, 1, 2\}^3 \setminus \{(1, 1, 1)\}$.

d) Make a picture of the graph $G(S)$.

e) What is the collection of arrows of $G(S)$ that preserve both future cones and past cones in the category $F(S)$?

f) Which square of the graph $G(S)$ are not pushouts of $F(S)$? Same question with pullbacks.

g) What is the greatest system of weak isomorphisms of $F(S)$? What is the category of components of $F(S)$?

Exercise 10:

Recall that $\mathcal{C} \text{Set} = \mathcal{Set}^{\square_{+}^{\text{op}}}$ is the category of precubical sets. For any $n \in \mathbb{N}$ and any precubical set $K$, define $\text{trunc}_n(K)$ as the precubical sets obtained by discarding all the cubes of dimension greater or equal than $n$. For $n \in \mathbb{N}$, define $\mathcal{C} \text{Set}_n$ as the full subcategory of $\mathcal{C} \text{Set}$ whose objects are the precubical sets of dimension $n$ i.e. $K_d = \emptyset$ for $d > n$. By convention $\mathcal{C} \text{Set}_{-1}$ is the category with only one morphism (and therefore only one object). Let $I_n$ be the inclusion functor of $\mathcal{C} \text{Set}_n$ into $\mathcal{C} \text{Set}$.

a) Explain why $\text{trunc}_n$ actually extends to a functor $\mathcal{C} \text{Set} \to \mathcal{C} \text{Set}_{n-1}$.

b) Prove that $\text{trunc}_n$ is right adjoint to $I_{n-1}$.

The standard $n$-cube is denoted by $\square^n_+$. In particular $\square^1_+$ is the graph with one arrow between two vertex, $\square^2_+$ is the square, $\square^3_+$ is the cube. In general $\square^n_+$ is the $n$-fold tensor product of $\square^1_+$.

b) What is the geometric realization (i.e. in $\mathcal{Top}$) of $\text{trunc}_n(\square^n_+)$?

Let $\mathcal{T}_1 : \mathcal{C} \text{Set} \to \mathcal{Sh} \mathcal{Top}$ denotes the realization in $\mathcal{Top}$ and $\overline{\pi}_1 : \mathcal{Sh} \mathcal{Top} \to \mathcal{Cat}$ be the fundamental category functor. Then for all precubical sets $K$, define $F(K)$ as the full subcategory of $\overline{\pi}_1(\mathcal{K}_1)$ whose set of objects is $K_0$. 

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c) What is $F(K)$ when $K$ is a graph (i.e. a 1-dimensional precubical set)

d) Compute $F(□^1)$ and $F(\text{trunc}_3(□^3))$

e) Given a precubcial set $K$, what is $F(\text{trunc}_3(K))$ (all the cubes of dimension greater of equal than 3 are dropped)

f) Explain why $F$ actually extends to a functor from $\mathcal{C}\text{Set}$ to $\mathcal{C}\text{at}$

g) Prove that $F(K \otimes K') \cong F(K) \times F(K')$

h) Give a direct description of $F$ (i.e. without topological arguments)