Exercise 1: PV programs and Geometric Models

Reminder: the geometric model of a PV program is the complement of its forbidden area. We write dipath instead of directed path.

1) [1pt] Write the PV program corresponding to the forbidden area depicted on Figure 1.

A deadlock is a point $d$ of a pospace $X$ such that any directed path starting at $d$ is constant. The attractor of a deadlock $d$ is the subset $A \subset X$ such that any dipath starting in $A$ is contained in $A$ and from any point of $x \in A$ there exists a dipath from $x$ to $d$.

2) [1pt] Find all the deadlocks of the geometric model depicted on Figure 1 and their attractors.

3) [1pt] Find the category of components of the geometric model (a picture suffices).

An $n$-cube is a subset of $\mathbb{R}^n$ of the form $I_1 \times \cdots \times I_n$ where each $I_k$ is an interval. An $n$-cubical area is a finite union of $n$-cubes i.e.

$$X = C_1 \cup \cdots \cup C_p$$  where $p \in \mathbb{N}\{0\}$ and each $C_k$ is an $n$-cube

An $n$-cube $C$ such that $C \subseteq X$ is called a subcube of $X$. Moreover if for all
subcubes $C'$ of $X$ we have $C \subseteq C' \Rightarrow C = C'$ then $C$ is called a maximal subcube of $X$.

Consider the following program and denote by $F$ its forbidden area:

```plaintext
#sem a, b 2
#sem c 3
process definition:
p = P(a).P(c).V(c).V(a)
q = P(b).P(c).V(c).V(b)
program:
p | p | q | q
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4a) [0.5pt] Describe $F_{a,b}$ the forbidden area of the PV program generated by the semaphores $a$ and $b$ (giving the list of its maximal subcubes).

4b) [0.5pt] Describe $F_c$ the forbidden area of the PV program generated by the semaphore $c$ (giving the list of its maximal subcubes).

4c) [1pt] Compare $F_{a,b}$ and $F_c$ then write a simpler PV program whose forbidden area is isomorphic to $F$.

**Exercise 2:** Any nonempty finite loopfree connected category (nflcc) can be written as a product of prime nflcc’s in a unique way (up to permutation of the terms). The number of morphisms of a category $C$ is called the size of $C$ (don’t forget the identities). We denote by $\mathbb{M}$ the free commutative monoid of nflcc’s. We have $\text{size}(A \times B) = \text{size}(A) \times \text{size}(B)$.

1) [1pt] Find the least (i.e. of smallest size) non trivial element of $\mathbb{M}$ and deduce its least non prime element.

2) [4pt] Find all the elements of $\mathbb{M}$ whose size is less or equal than 7. (There are 14 of them up to isomorphism and opposite, 0.25pt each). Deduce the least non free element of $\mathbb{M}$.

3) [1pt] Prove for all prime number $p \neq 2$ there exists some prime element of $\mathbb{M}$ of size $p$.

A semiring is a tuple $(S, \times, 1, +, 0)$ such that $(S, \times, 1)$ and $(S, +, 0)$ are commutative monoid and $\times$ distributes over $+$. A morphism of semiring is a mapping which preserves both monoid structures. For example $\mathbb{N}$ (natural numbers) and $\mathbb{N}[X]$ (polynomials with coefficients in $\mathbb{N}$) are semirings with the usual operations.

The set $\mathcal{S}$ of all finite loopfree categories (empty or disconnected categories are allowed) admits a semiring structure with disjoint union and cartesian product as sum and product. We denote by $\mathbb{N}_0[X]$ and $\mathbb{S}_0$ the commutative monoids (under product) of nonzero elements of $\mathbb{N}[X]$ and $\mathcal{S}$.

4) [1pt] Prove $\mathbb{N}_0[X]$ contains a nonprime irreducible element. Then make a
(relevant) remark about $N_0[X]$.

5) [2pt] Prove for all $C \in \mathcal{M}$ there is a unique morphism $\text{eval}_C : N[X] \to S$ such that $\text{eval}_C(X) = C$.

6) [2pt] Prove the commutative monoid of nonempty finite loopfree categories is not commutative free (in other words the connectedness hypothesis cannot be dropped from the theorem asserting $\mathcal{M}$ is commutative free).

**Exercise 3:** A path on a topological space $X$ is a continuous map $\gamma$ from some compact interval $[a,b]$ to $X$. The path $\gamma$ is called a loop if $\gamma(a) = \gamma(b)$. A pospace is a pair $(X, \subseteq)$ where $X$ is a topological space and $\subseteq$ a partial order on (the underlying set of) $X$. The morphisms of pospace, also called dimaps, are the monotonic continuous maps. The pospaces and their morphisms form the category $\mathcal{P}$. The real line $\mathbb{R}$, with its usual order and topology, is a pospace. A dipath/diloop is a dimap which is also a path/loop.

1) [1pt] Prove any diloop of a pospace is constant.

We generalize the notion of pospace as follows: a d-space is a pair $X, dX$ where $X$ is a topological space and $dX$ a collection of continuous map which is stable under concatenation, contains all the constant maps and such that for all monotonic continuous mapping $\theta : [c, d] \to [a, b]$ and any $\gamma : [a, b] \to X \in dX$, the composite $\gamma \circ \theta$ still belongs to $dX$. A morphism of d-space from $X, dX$ to $Y, dY$ is a continuous map $f : X \to Y$ such that for all $\gamma \in dX$, the composite $f \circ \gamma \in dY$. The d-spaces and their morphisms form the category $\mathcal{D}$.

2) [1pt] Prove for all pospaces $X$, the pair $(X, dX := \{\text{dipaths on } X\})$ is a d-space. Then describe a functor from the category of pospaces $\mathcal{P}$ to the category of d-spaces $\mathcal{D}$.

By definition the dipaths of a d-space $X, dX$ are the elements of $dX$. A subset $F$ of $X, dX$ is said to be future stable when any directed path starting in $F$ is contained in $F$. The subsets $\emptyset$ and $X$ are obviously future stable.

3) [1pt] Prove any union/intersection of future stable subsets of a d-space is future stable. In particular the future stable subsets of a d-space $X, dX$ form a sub-complete lattice of $2^X$ (the complete lattice of subsets of $X$).

4) [1pt] The directed circle is the unit circle $\mathbb{S}$ i.e. $\{z \in \mathbb{C} \mid |z| = 1\}$ the set of complex number of magnitude 1, with $d\mathbb{S}$ the collection of paths $t \in [a, b] \mapsto e^{i\theta(t)} \in \mathbb{S}$ where $\theta$ is a dipath on $\mathbb{R}$. Describe the complete lattice of future stable subsets of the directed circle.

5) [2pt] The directed complex plane is the set of complex number, with $d\mathbb{C}$ the collection of paths $t \in [a, b] \mapsto \rho(t) \cdot e^{i\theta(t)} \in \mathbb{S}$ where $\theta$ is a dipath on $\mathbb{R}$ and $\rho$ a dipath on $\mathbb{R}_+$, Describe the complete lattice of future stable subsets of the directed complex plane.