Concurrency

and

Directed Algebraic Topology

- MPRI -

monday the 5^{th} of march 2012

duration: 1h30

Answers can be given in french or in english.

The lecture notes from my webpage is the only document allowed.

<u>**Exercise 1**</u>: PV programs and Geometric Models

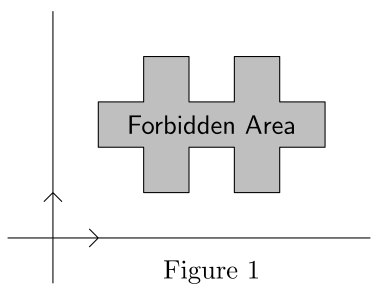

Figure 1

Reminder: the geometric model of a PV program is the *complement* of its forbidden area. We write *dipath* instead of *directed path*.

1) [1pt] Write the PV program corresponding to the forbidden area depicted on Figure 1.

A *deadlock* is a point d of a pospace X such that any directed path starting at d is constant. The *attractor* of a deadlock d is the subset $A \subset X$ such that any dipath starting in A is contained in A and from any point of $x \in A$ there exists a dipath from x to d.

2) [1pt] Find all the deadlocks of the geometric model depicted on Figure 1 and their attractors.

3) [1pt] Find the category of components of the geometric model (a picture suffices).

An *n-cube* is a subset of $\mathbb{R}_+^n$ of the form $I_1 \times \cdots \times I_n$ where each I_k is an interval. An *n-cubical area* is a finite union of n-cubes i.e.

$$X = C_1 \cup \cdots \cup C_p \quad \text{where } p \in \mathbb{N}\backslash\{0\} \text{ and each } C_k \text{ is an } n\text{-cube}$$

An n-cube C such that $C \subseteq X$ is called a *subcube* of X. Moreover if for all subcubes C' of X we have $C \subseteq C' \Rightarrow C = C'$ then C is called a *maximal subcube* of X.

Consider the following program and denote by F its forbidden area:

```
#sem a,b 2
#sem c 3
process definition:
p = P(a).P(c).V(c).V(a)
q = P(b).P(c).V(c).V(b)
program:
p | p | q | q
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4a) [0,5pt] Describe $F_{a,b}$ the forbidden area of the PV program generated by the semaphores a and b (giving the list of its maximal subcubes).

4b) [0.5pt] Describe F_c the forbidden area of the PV program generated by the semaphore c (giving the list of its maximal subcubes).

4c) [1pt] Compare $F_{a,b}$ and F_c then write a simpler PV program whose forbidden area is isomorphic to F.

Exercise 2: Any *nonempty finite loopfree connected category* (nflcc) can be written as a product of prime nflcc's in a unique way (up to permutation of the terms). The number of morphisms of a category $\mathcal{C}$ is called the *size* of $\mathcal{C}$ (don't forget the identities). We denote by $\mathbb{M}$ the free commutative monoid of nflcc's. We have $\mathsf{size}(\mathcal{A} \times \mathcal{B}) = \mathsf{size}(\mathcal{A}) \times \mathsf{size}(\mathcal{B})$.

1) [1pt] Find the least (i.e. of smallest size) non trivial element of $\mathbb{M}$ and deduce its least non prime element.

2) [4pt] Find all the elements of $\mathbb{M}$ whose size is less or equal than 7. (There are 14 of them up to isomorphism and opposite, 0.25pt each). Deduce the least non free element of $\mathbb{M}$.

3) [1pt] Prove for all prime number $p \neq 2$ there exists some prime element of $\mathbb{M}$ of size p.

A semiring is a tuple $(S, \times, 1, +, 0)$ such that $(S, \times, 1)$ and $(S, +, 0)$ are commutative monoid and $\times$ distributes over $+$. A morphism of semiring is a mapping which preserves both monoid structures. For example $\mathbb{N}$ (natural numbers) and $\mathbb{N}[X]$ (polynomials with coefficients in $\mathbb{N}$) are semirings with the usual operations.

The set $\mathbb{S}$ of all *finite loopfree categories* (empty or disconnected categories are allowed) admits a semiring structure with disjoint union and cartesian product as sum and product. We denote by $\mathbb{N}_0[X]$ and $\mathbb{S}_0$ the commutative monoids (under product) of nonzero elements of $\mathbb{N}[X]$ and $\mathbb{S}$.

4) [1pt] Prove $\mathbb{N}_0[X]$ contains a nonprime irreducible element. Then make a

(relevant) remark about $\mathbb{N}_0[X]$.

5) [2pt] Prove for all $\mathcal{C} \in \mathbb{M}$ there is a unique morphism $\mathsf{eval}_{\mathcal{C}} : \mathbb{N}[X] \to \mathbb{S}$ such that $\mathsf{eval}_{\mathcal{C}}(X) = \mathcal{C}$

6) [2pt] Prove the commutative monoid of nonempty finite loopfree categories is not commutative free (in other words the connectedness hypothesis cannot be dropped from the theorem asserting $\mathbb{M}$ is commutative free).

Exercise 3: A *path* on a topological space X is a continuous map γ from some compact interval $[a, b]$ to X. The path γ is called a loop if $\gamma(a) = \gamma(b)$. A *pospace* is a pair $(X, \sqsubseteq)$ where X is a topological space and $\sqsubseteq$ a partial order on (the underlying set of) X. The morphisms of pospace, also called *dimaps*, are the monotonic continuous maps. The pospaces and their morphisms form the category P. The real line $\mathbb{R}$, with its usual order and topology, is a pospace. A *dipath/diloop* is a dimap which is also a path/loop.

1) [1pt] Prove any diloop of a pospace is constant.

We generalize the notion of pospace as follows: a *d-space* is a pair X, dX where X is a topological space and dX a collection of continuous map which is stable under concatenation, contains all the constant maps and such that for all monotonic continuous mapping $\theta : [c, d] \to [a, b]$ and any $\gamma : [a, b] \to X \in dX$, the composite $\gamma \circ \theta$ still belongs to dX. A morphism of d-space from X, dX to Y, dY is a continuous map $f : X \to Y$ such that for all $\gamma \in dX$, the composite $f \circ \gamma \in dY$. The d-spaces and their morphisms form the category D.

2) [1pt] Prove for all pospaces X, the pair $(X, dX := \{\text{dipaths on } X\})$ is a d-space. Then describe a functor from the category of pospaces P to the category of d-spaces D.

By definition the dipaths of a d-space X, dX are the elements of dX. A subset F of X, dX is said to be *future stable* when any directed path starting in F is contained in F. The subsets $\emptyset$ and X are obviously future stable.

3) [1pt] Prove any union/intersection of future stable subsets of a d-space is future stable. In particular the future stable subsets of a d-space X, dX form a sub-complete lattice of 2^X (the complete lattice of subsets of X).

4) [1pt] The *directed circle* is the unit circle $\mathbb{S}$ i.e. $\{z \in \mathbb{C} \mid |z| = 1\}$ the set of complex number of magnitude 1, with $d\mathbb{S}$ the collection of paths $t \in [a, b] \mapsto e^{i\theta(t)} \in \mathbb{S}$ where θ is a dipath on $\mathbb{R}$. Describe the complete lattice of future stable subsets of the directed circle.

5) [2pt] The *directed complex plane* is the set of complex number, with $d\mathbb{C}$ the collection of paths $t \in [a, b] \mapsto \rho(t) \cdot e^{i\theta(t)} \in \mathbb{S}$ where θ is a dipath on $\mathbb{R}$ and ρ a dipath on $\mathbb{R}_+$. Describe the complete lattice of future stable subsets of the directed complex plane.