Concurrency

and

Directed Algebraic Topology

MPRI

thursday, the 12^{th} of february 2009

examination duration : 1h30

<u>Exercice 1</u>

In this exercise, `a` and `b` denote two mutex (a ressource that cannot be shared by two processes or more). Consider the program

```
P(a).V(a).P(a).V(a)|P(a).V(a).P(a).V(a)
```

1 - Determine the forbidden area of this program and check that it can be factorized as a product of two cubical areas of dimension 1.

2 - Do the two copies of the process `P(a).V(a).P(a).V(a)` run independently from each other ?

Let P and Q be the following PV programs, recall that the geometric model of a program is the complement of its forbidden area.

$P =$ `P(a).P(b).V(b).V(a)|P(b).P(a).V(a).V(b)` and

$Q =$ `P(a).P(b).V(a).V(b)|P(b).P(a).V(b).V(a)`

3 - Draw the geometric models $[\![P]\!]$ and $[\![Q]\!]$ of these programs and their components.

4 - What can be said about the fundamental categories $\overrightarrow{\pi_1}([\![P]\!])$ and $\overrightarrow{\pi_1}([\![Q]\!])$?

<u>Exercice 2</u>

1 - Suppose `a` and `b` are semaphores of arity 2 (also called mutex). Determine (by means of a picture on which the forbidden area is colored in grey) the geometric model of the following program. What can be said about the fundamental category of the model ?

```
P(a).P(b).V(b).P(b).V(b).V(a)|
P(b).P(a).V(a).P(a).V(a).V(b)
```

Denote by $\overrightarrow{X}$ the corresponding pospace and suppose that all the points on the frontier of the forbidden area belong to the model.

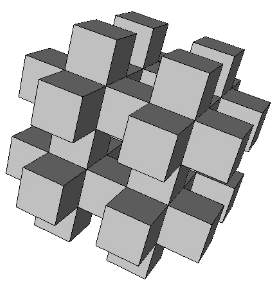
2 - A **directed path** on a pospace $\overrightarrow{X}$ is a continuous increasing map from $\overrightarrow{[0,1]}$ to $\overrightarrow{X}$. A point $x \in \overrightarrow{X}$ is called a **deadlock** when any directed path δ s.t. $\delta(0) = x$ is constant. Draw the deadlocks on the picture.

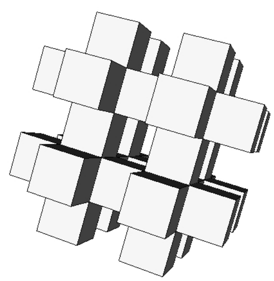
3 - The **deadlock attractor** is the biggest subset B of $\overrightarrow{X}$ s.t. for all directed paths δ, if $\delta(0) \in B$ then there exists some directed path γ s.t. $\gamma(0) = \delta(1)$ and $\gamma(1)$ is a deadlock. Represent the deadlock attractor on the picture.

4 - Write a PV program P whose <u>forbidden area</u> is

$$\begin{array}{lclcl}
]1,2[\times]1,2[\times\mathbb{R} & \cup &]1,2[\times\mathbb{R}\times]1,2[& \cup & \mathbb{R}\times]1,2[\times]1,2[\;\cup \\
]1,2[\times]3,4[\times\mathbb{R} & \cup &]1,2[\times\mathbb{R}\times]3,4[& \cup & \mathbb{R}\times]1,2[\times]3,4[\;\cup \\
]3,4[\times]1,2[\times\mathbb{R} & \cup &]3,4[\times\mathbb{R}\times]1,2[& \cup & \mathbb{R}\times]3,4[\times]1,2[\;\cup \\
]3,4[\times]3,4[\times\mathbb{R} & \cup &]3,4[\times\mathbb{R}\times]3,4[& \cup & \mathbb{R}\times]3,4[\times]3,4[
\end{array}$$

The forbidden area is represented on the pictures below (note that it has been truncated).





5 - By construction, the model of the program $[\![P]\!]$ is the complement of the forbidden region. Denote by $\mathcal{C}$ the fundamental category of $[\![P]\!]$ (i.e. $\mathcal{C} = \overrightarrow{\pi_1}([\![P]\!])$). Then determine the following integer (with some lines of explanation)

$$\max\left\{\mathsf{Card}\big(\mathcal{C}[x,y]\big) \;\middle|\; x,y \in \overrightarrow{X}\right\}$$

Recall that $\mathcal{C}[x,y]$ is the set of directed paths from x to y up to directed homotopy.

<u>Exercice 3</u>

Along this exercise, by “category” we mean “isomorphism class” of category. Denote by **1** the unique category with exactly 1 morphism and by $\mathcal{A} \times \mathcal{B}$ the

Cartesian product of the categories $\mathcal{A}$ and $\mathcal{B}$. A category $\mathcal{C}$ is said to be **reducible** when there exist two categories $\mathcal{A} \neq \mathbf{1}$ and $\mathcal{B} \neq \mathbf{1}$ such that $\mathcal{C} \cong \mathcal{A} \times \mathcal{B}$, otherwise we say that $\mathcal{C}$ is **irreducible**. When a list of categories is asked, it is understood up to opposite (if both $\mathcal{C}$ and $\mathcal{C}^{op}$ belong to the expected list, you just need to give one of them).

1 - Give the list of all connected loop-free categories of size 8

2 - Denote by $\mathbb{M}_f$ the monoid of finite categories and $\mathbb{M}_f^*$ the submonoid of non-empty finite categories. Give two morphisms of monoids from $\mathbb{M}_f^*$ to $(\mathbb{N}\backslash\{0\}, \times, 1)$

3 - Given $m, x \in \mathbb{N}$ we define $\Phi(m, x)$ as the number of loop-free connected categories with m morphisms and x objects.

3a) For all $m \in \mathbb{N}$ determine $\Phi(m, 1)$ and $\Phi(m, 2)$.

3b) For all $m \in \mathbb{N}$ prove $\lim_{x\to\infty} \Phi(m, x) = 0$

4 - For all objects x and y of a loop-free category $\mathcal{C}$, write $x \preccurlyeq y$ when $\mathcal{C}[x, y] \neq \emptyset$. Denote by $\mathcal{P}os$ the category of partially ordered sets and $\mathcal{L}fc$ the category of loop-free categories.

4a) Prove that $\preccurlyeq$ is a partial order.

4b) Deduce a functor R from $\mathcal{L}fc$ to $\mathcal{P}os$.

4c) Conversely, describe a full embeding I of $\mathcal{P}os$ in $\mathcal{L}fc$ (i.e. check that any partially ordered set X can be seen as a loop-free category $I(X)$, that any morphism of posets induces a functor between the corresponding loop-free categories and conversely that any functor from $I(X)$ to $I(Y)$ comes from a morphism of posets from X to Y.)

4d) Give a natural transformation from $\mathsf{id}_{\mathcal{L}fc}$ to $I \circ R$.

5 - By definition, a set X of morphisms of $\mathcal{C}$ is said to be generating when any morphism of $\mathcal{C}$ can be written as a composite of elements of X.

5a) Prove that any finite loop-free category has a minimum generating set (i.e. a generating set which is contained in all the other ones).

5b) Suppose A and B are the least generating sets of the finite loop-free categories $\mathcal{A}$ and $\mathcal{B}$. Determine the least generating set of $\mathcal{A} \times \mathcal{B}$.

6 - Prove that the commutative monoid of non-empty finite loop-free categories is *not* free. Hint : a non-connected finite loop-free category is a "sum" of connected ones, therefore finite loop-free categories behave like polynomials.