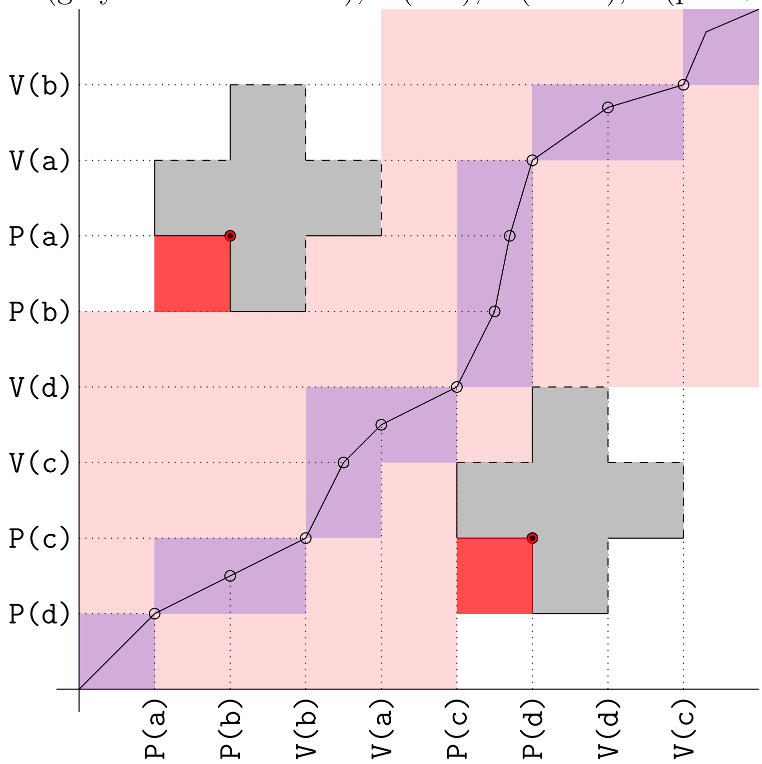
Concurrency
and
Directed Algebraic Topology

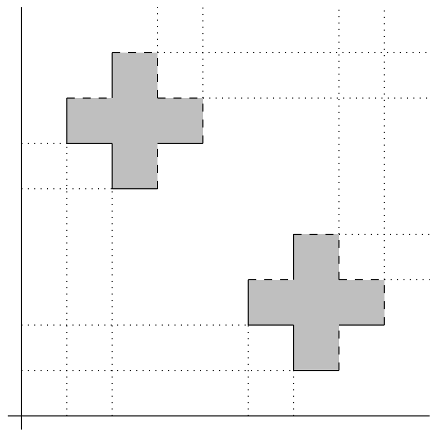
- MPRI -

Examination 2013
Answers

Exercise 1: Paths and Classes
1 - (gray - forbidden area), 2 (line), 3 (violet), 4 (pink+violet), 5 (red)



6 -



Exercise 2: Components of the Cube

1 - The morphism α_1 does not preserve the future cone. Indeed, let S be the square containing α_1 shown on the figure. consider the (unique up to directed homotopy) directed path γ from the target of α_1 to the upper right corner of S. Then consider the (unique up to directed homotopy) path δ (contained in S) running "behind the forbidden cube" from the source of α_1 to the upper right corner of S. Then $\gamma\alpha_1 \not\sim \delta$. Therefore $\alpha_1 \notin \Sigma$.

2 - Let x be the least upper bound (with respect to the product order of $\mathbb{R}^3$) of the target of α_0 and the target of β. Then consider y a point on the vertical line containing x that is above the forbidden cube. One has two commutative square though there neither a directed path from x to y nor from y to x. Hence the pushout does not exist and $\alpha_0 \notin \Sigma$.

3 - Let z be the greatest lower bound of the target of α_1 and the source of α_2. The set Z of common lower bounds of the target of α_1 and the source of α_2 is the closed vertical segment from the origin to z. Given any $z' \in Z$ there are unique (up to directed homotopy) directed paths from z' to the target of α_1, the source of α_2 and z. Thus we have a pushout square. The same arguments apply to both remaining commutative squares.

Exercise 3:
The following facts are pervasively applied in the sequel:

$$\mathsf{card}\{\text{objects of } \mathcal{A} \times \mathcal{B}\} = \mathsf{card}\{\text{objects of } \mathcal{A}\} \times \mathsf{card}\{\text{objects of } \mathcal{B}\}$$

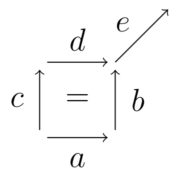
$$\mathsf{card}\{\text{morphisms of } \mathcal{A} \times \mathcal{B}\} = \mathsf{card}\{\text{morphisms of } \mathcal{A}\} \times \mathsf{card}\{\text{morphisms of } \mathcal{B}\}$$

1a - The commutative square.

1b - The totally ordered real line $(\mathbb{R}, \leqslant)$ can be seen as a loop-free category. Its objects are the real numbers and its morphisms are the ordered pairs (x, y) with $x \leqslant y$. The source and the target of (x, y) are x and y. Let G be a generating family and let (x, y) be an element of G. In particular both $(x, \frac{x+y}{2})$ and $(\frac{x+y}{2}, y)$ can be written as composite of elements of G. Hence $G - \{(x, y)\}$ is still generating.

1c - We say that a morphisms f of $\mathcal{C}$ is irreducible when for all morphisms γ and δ, if $f = \gamma \circ \delta$ then either γ or δ is an isomorphism (not both). The least generating family is the collection of irreducible morphisms. Any generating family contains all the irreducible morphisms (no matter $\mathcal{C}$ is finite or loop-free). For $\mathcal{C}$ is loop-free its only isomorphisms are its identities. Let f be a morphism of $\mathcal{C}$ and suppose we have $f = \alpha_n \circ \alpha_0$ with none of the α_i an isomorphism. If any of the morphism α_i is not irreducible we obtain a decomposition $f = \alpha'_{n+1} \circ ... \circ \alpha'_0$ with none of the α_i an isomorphism. Since $\mathcal{C}$ is loop-free the sources of the morphisms $\alpha'_{n+1}, ..., \alpha'_0$ are pairwise distinct, and since the category $\mathcal{C}$ is finite, the induction stops.

2a - The following category is irreducible since its number of objects is a prime number. Hence it is prime. Its least generating family is $\{a, b, c, d, e\}$ and it is not free because $dc = ba$.



2b - Suppose the family G of irreducible elements of $\mathcal{A} \times \mathcal{B}$ is generating (otherwise it is not free and it is done). Given a morphism $(\alpha : a \to b, \beta : x \to y)$ of $\mathcal{A} \times \mathcal{B}$ we have

$$(\mathsf{id}_y, \alpha) \circ (\beta, \mathsf{id}_a) = (\beta, \mathsf{id}_b) \circ (\mathsf{id}_x, \alpha)$$

It follows that any element of G is either (β, id_*) or (id_*, α) with α (β) irreducible in $\mathcal{A}$ ($\mathcal{B}$). The preceding relation then proves that (α, β) admits at least two decomposition. It follows that $\mathcal{A} \times \mathcal{B}$ is not free.

2c - If $\mathcal{C} \in M$ is not prime then it is not irreducible so it can be written as $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ where $\mathcal{A} \neq \mathbb{1}$ and $\mathcal{B} \neq \mathbb{1}$. Then $\mathcal{C}$ is not free by 2b.

3a - Any element of M' is a finite category hence admits finitely many objects. Moreover, if $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ with $\mathcal{A} \neq \mathbb{1}$ and $\mathcal{B} \neq \mathbb{1}$ then $\mathsf{card}\{\text{objects of } \mathcal{A}\}$ and $\mathsf{card}\{\text{objects of } \mathcal{B}\}$ are both strictly less than $\mathsf{card}\{\text{objects of } \mathcal{C}\}$.

3b - We have in $\mathbb{N}[X]$

$$\sum_{i=0}^{5} X^i = (1+X)(1+X^2+X^3) = (1+X^3)(1+X+X^2)$$

Substitute the disjoint union $\sqcup$ of categories to the polynomial addition $+$, the cartesian product of categories to the polynomial product, and the category $(\cdot \to \cdot)$ to the variable X. Then $\mathbb{1} \sqcup (\cdot \to \cdot)$ is irreducible since its number of objects is prime. Yet it is not prime because neither it divides $\mathbb{1} \sqcup (\cdot \to \cdot) \sqcup (\cdot \to \cdot)^2$ (since it has 7 objects and 3 does not divide 7) nor $\mathbb{1} \sqcup (\cdot \to \cdot)^3$ (since it has 10 morphisms and 4 does not divide 10).