Concurrency

and

Directed Algebraic Topology

- MPRI -

thursday the 10^{th} of march 2011

duration: 1h30

Exercise 1: Maximal Subcubes

1) By definition, any subcube of C is included in C, moreover C is a subcube of itself. Thus C is the unique subcube of C.

2) Let C be the unique maximal subcube of X. Since X is a cubical area it can be written as a finite union of n-cubes

$$X = C_1 \cup \cdots \cup C_n$$

Now each C_k is included in some maximal subcube, therefore $C_k \subseteq C$. It follows that $X = C$.

Another proof: let $x \in X \subseteq \mathbb{R}^n$ then $x = (x_1, \ldots, x_n)$ where each $x_k \in \mathbb{R}$. Hence the singleton $\{x\}$ can be written as

$$\{x\} = \{x_1\} \times \cdots \times \{x_n\}$$

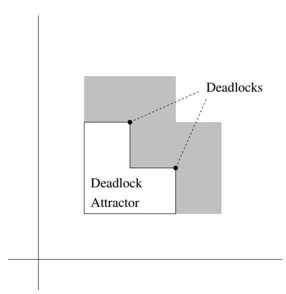
which is thus a subcube of X. Therefore $\{x\}$ is included in the maximal subcube C i.e. $x \in C$. It follows $X \subseteq C$.

3) The cubical area X admits a unique decomposition

$$X = P_1 \times \cdots \times P_n$$

where each P_k is irreducible. Since $M(X)$ is prime we can suppose, without loss of generality, that $M(P_k) = 1$ for $k < n$ and $M(P_n) = M(X)$. According to 2) each P_k for $k < n$ is a cube, however it is also irreducible hence it is an interval. According to 1) $X' := P_n$ is not an interval.

Exercise 2: Staircases, PV programs and Categories of Components

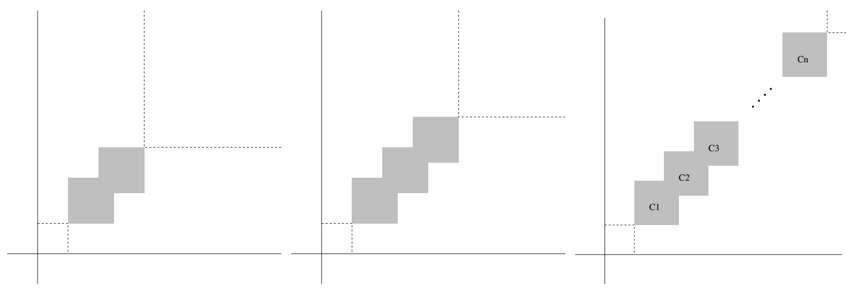


1) The cubical area X has 5 maximal subcubes and Y has 7 maximal subcubes. Since 5 and 7 are prime numbers X and Y are prime cubical areas. The cubical area S_1 has 4 maximal subcubes, 4 is not prime so the preceding argument cannot be applied. Suppose X is not prime, then it is not irreducible and since X is 2-dimensional we have the unique decomposition

$$S_1 = P_1 \times P_2$$

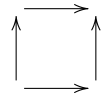
where P_1 and P_2 are prime and 1-dimensional. A 1-dimensional area is the finite union of its connected components which are also its maximal subcubes. Then P_1 and P_2 are connected since so is S_1. But $M(S_1) = 4$ therfore we have $M(P_1) = 2$ or $M(P_1) = 4$ or $M(P_2) = 4$. However in each of the precedening cases P_1 or P_2 is not connected, which is a contradiction.

2) The cubical area X has no deadlock, the deadlocks and the deadlock attractor of Y are sown on the picture above.



S_2 S_3 S_n

3) Let a, b and c be semaphores of arity 2 (can be held be 1 process at most). Then the geometric model of the following PV programs are S_2 and S_3
$P(a).P(b).V(a).V(b)|P(a).P(b).V(a).V(b)$
$P(a).P(b).V(a).P(c).V(b).V(c)|P(a).P(b).V(a).P(c).V(b).V(c)$
5) The symmetry with respect to the main diagonal i.e. the straight line $\{x = y\}$ is a bijection *with no fixpoint* from the set of maximal subcubes to itself whose number of elements is therefore even.
6) The cubical area S_n admits $2n + 2$ maximal subcubes.
4) and 7) The category of components of S_n is the free category generated by the following graph



<u>Exercise 3</u>: Dimension 3
1) The cubical area X is the geometric model of the PV program
$P(a).V(a)|P(a).V(a)|P(a).V(a)$
where a is a semaphore of arity 3 (can be held by 2 processes at most).
2a) Let $\gamma : [0, r] \to X$ be a directed path such that $\gamma(0) = a$ and $\gamma(r) = b$, given

some $t \in [0,r]$ we have $a = \gamma(0) \sqsubseteq \gamma(t) \sqsubseteq \gamma(r) = b$ hence γ is actually a directed path of A. Now suppose h is a directed homotopy from γ to δ, where γ and δ are two directed paths from a to b, by the same argument we prove the image of h is included in A.
2b) Suppose $(1,1,1)$ and $(3,3,3)$ are the bottom corner and the top corner of the cube. The points $a = (0,0,2)$ and $b = (4,4,2)$ answer the question, indeed the hyperplane $A := \{z = 2\}$ meets the grey cube.

Exercise 4: nflcc!
1) If $\mathcal{C}$ is a loopfree category with a single object x, then the only morphism of $\mathcal{C}$ is the identity of x. Then $\mathcal{C}$ is the neutral element of the free commutative monoid of nflcc's, hence it is not prime. Hence a prime nflcc $\mathcal{C}$ has at least two objects x and y. Since $\mathcal{C}$ is connected, we can suppose without loss of generality there is a morphism α from x to y. There are also the identities of x and y, therefore $\mathcal{C}$ has at least 3 elements.

2a and 2b) If $\mathcal{C}$ is not prime we have $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are nflcc's. It follows

$$Ob(\mathcal{C}) = Ob(\mathcal{A}) \times Ob(\mathcal{B}) \text{ and } Mo(\mathcal{C}) = Mo(\mathcal{A}) \times Mo(\mathcal{B})$$

with $Ob(\mathcal{A}) \geqslant 2$, $Ob(\mathcal{A}) \geqslant 2$, $Mo(\mathcal{A}) \geqslant 3$ and $Mo(\mathcal{A}) \geqslant 3$.
3) The posets $\{1 < \cdots < n\}$ for n prime, seen as small categories is such an example.
The n-stars for $n > 1$, i.e. the free categories generated by the graph with vertices $\{0, ..., n\}$ and one arrow from 0 to k for $k > 1$ is prime (as any nflcc which is also a free category).
4) With $\gamma\alpha = \gamma\beta$ and $\delta\alpha = \delta\beta$, there are 4 objects and 10 morphisms.