Loop-free categories and their Components

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special session on Asymmetric Topology

Partially Ordered Space (or Pospace) $\overrightarrow{X}$

S.Eilenberg 41 L.Nachbin 48 65 P.Johnstone 82

1. A topological space X,
2. An order relation $\sqsubseteq$ over $|X|$ whose graph is closed in $X \times X$.

Morphism of pospace from $\overrightarrow{X}$ to $\overrightarrow{Y}$

A map $f : |X| \longrightarrow |Y|$ inducing:

1. a continuous map from X to Y
2. an increasing map from $(|X|, \sqsubseteq_X)$ to $(|Y|, \sqsubseteq_Y)$.

PoTop is the category of pospaces and their morphisms

Examples
of pospaces

1. the real line $\mathbb{R}$ with its classical topology and order (denoted $\overrightarrow{\mathbb{R}}$),
2. the unit segment $[0,1]$ with the induced structure (denoted $\overrightarrow{[0,1]}$),
3. any morphism of PoTop from $\overrightarrow{[0,1]}$ to $\overrightarrow{X}$ is called a directed path over $\overrightarrow{X}$. Formally, the set of directed paths over $\overrightarrow{X}$ is $\mathrm{PoSpc}\left[\overrightarrow{[0,1]}, \overrightarrow{X}\right]$, it is also denoted $d\overrightarrow{X}$.

Categorical Proprerties of PoTop

comparing Top and PoTop

Theorem (E.Haucourt 05)

1. *complete and co-complete,*
2. *symetric monoidal closed,*
3. *the full subcategory of compact pospaces is complete, co-complete and admits* $\overrightarrow{[0,1]}$ *as a cogenerator,*
4. *the full subcategory of compactly generated pospaces is reflective in* PoTop *and cartesian closed.*

Directed homotopy over $\overrightarrow{X}$ from α to β

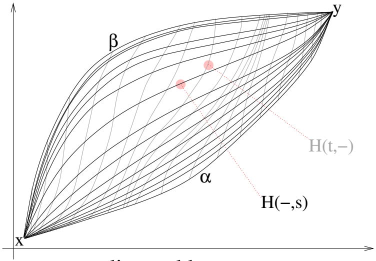
M.Grandis 01 L.Fajstrup/M.Raussen/E.Goubault 98

A morphism h of PoTop from $\overrightarrow{[0,1]} \times \overrightarrow{[0,1]}$ to $\overrightarrow{X}$ such that $U(h)$ is a classical homotopy from $U(\alpha)$ to $U(\beta)$.
Denote $\sim_{\overrightarrow{X}}$ the symetric and transitive closure of

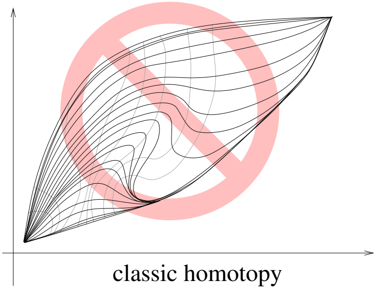
$$\left\{(\alpha, \beta) \in d\overrightarrow{X} \times d\overrightarrow{X} \,\middle|\, \text{there exists a directed homotopy from } \alpha \text{ to } \beta\right\}.$$

Dipaths α and β are said dihomotopic when $\alpha \sim_{\overrightarrow{X}} \beta$.

Directed Homotopy *vs* Classical Homotopy



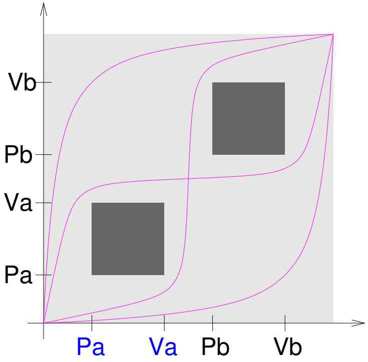
directed homotopy

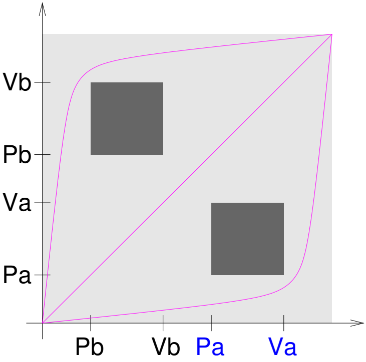


classic homotopy

First subtlety

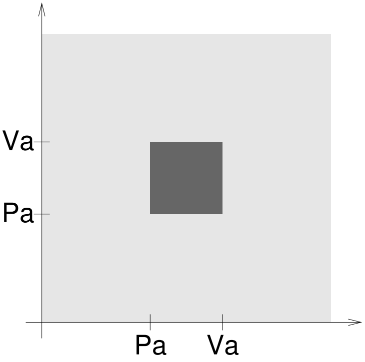
directed homotopy is not classic homotopy

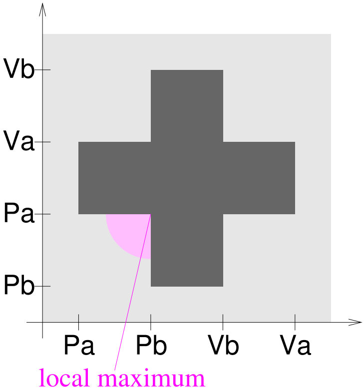




Second subtlety

classic homotopy cannot “see” local extrema

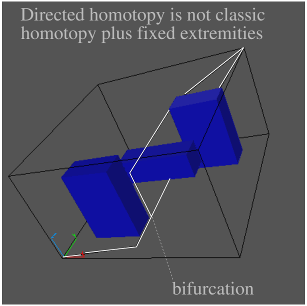




Third subtlety

Floating cube between two pillars

A=Pb.Pc.Vb.Vc
B=Pc.Pa.Vc.Va
C=Pa.Pb.Va.Vb



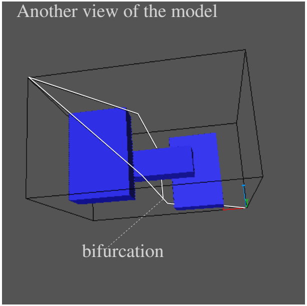


Image of a directed path

A special feature of directed topology

Theorem

1. *The image of a dipath α over a* *pospace* $\overrightarrow{X}$ *is either isomorphic (in* PoTop*) to* $\{\bullet\}$ *or* $\overrightarrow{[0,1]}$
2. *Two dipaths sharing the same image are dihomotopic*
3. *There is no directed* Peano *curve*

Fundamental category $\overrightarrow{\pi_1}(\overrightarrow{X})$ of a pospace $\overrightarrow{X}$

1. its objects are the elements of $|X|$,
2. its set of morphism from x to y, is the collection of $\sim_{\overrightarrow{X}}$-equivalence classes of

$$\left\{\alpha \in d\overrightarrow{X} \middle| \alpha(0) = x \text{ and } \alpha(1) = y\right\}$$

Loop-free categories introduced by *A.Haefliger* as "scwols" 91 instead of groupoids

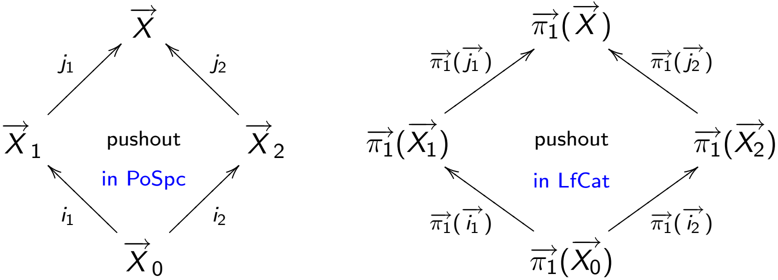
A (small) category $\mathcal{C}$ such that for all objects x and y of $\mathcal{C}$, if $\mathcal{C}[x,y] \neq \emptyset$ and $\mathcal{C}[y,x] \neq \emptyset$ then $x = y$ and $\mathcal{C}[x,x] = \{id_x\}$.
LfCat is the full subcategory (in Cat) of small loop-free categories.

1. LfCat is cartesian closed and reflective in Cat.
2. the fundamental category of a pospace is loop-free, whence the functor

$$\mathrm{PoTop} \xrightarrow{\vec{\pi}_1} \mathrm{LfCat}$$

Van Kampen theorem for fundamental categories

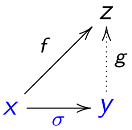
M.Grandis 01 E.Goubault 01 also see P.J.Higgins "Categories and Groupoids"



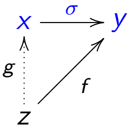
Yoneda morphism
preserving the past and the future I

A morphism $\sigma \in \mathcal{C}[x, y]$ is a *Yoneda* morphism when for any z:

future if $\mathcal{C}[y, z] \neq \emptyset$ then for all $f \in \mathcal{C}[x, z]$, there is a unique $g \in \mathcal{C}[y, z]$ such that



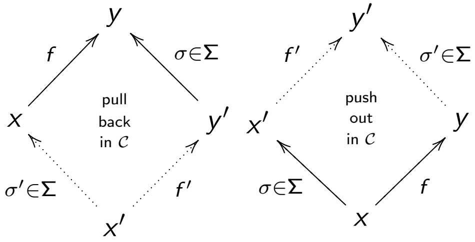
past if $\mathcal{C}[z, x] \neq \emptyset$ then for all $f \in \mathcal{C}[z, y]$, there is a unique $g \in \mathcal{C}[z, x]$ such that



Yoneda system of a small category $\mathcal{C}$ preserving the past and the future II

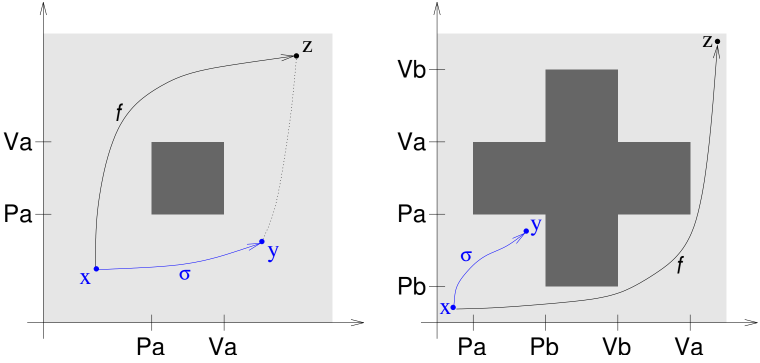
A collection Σ of morphisms of $\mathcal{C}$ such that :

1. Σ is stable under composition,
2. Σ contains all the isomorphisms of $\mathcal{C}$,
3. all the elements of Σ are *Yoneda* morphisms and
4. Σ is stable under change and cochange of base.



Examples

of morphism which do not belong to any *Yoneda* system



Structure of Σ-components

$\mathcal{C}$ loop-free category and Σ *Yoneda* system over $\mathcal{C}$

Theorem (E.Haucourt 05)

1. *the relation $\sim$ over $|\mathcal{C}|$ defined by $x \sim y$ iff $\exists z \in |\mathcal{C}|\ \Sigma[x,z] \neq \emptyset$ and $\Sigma[y,z] \neq \emptyset$ is an equivalence relation*
2. *Given any $\sim$-equivalence class K, the full subcategory of $\mathcal{C}$ whose set of objects is K is a non empty lattice*
3. *If $a \sim b \sim c \sim d$ and $\mathcal{C}[a,b]$, $\mathcal{C}[d,b]$, $\mathcal{C}[c,a]$ and $\mathcal{C}[c,d]$ are not empty, then the following square is both a <u>pullback</u> and a <u>pushout</u> in $\mathcal{C}$.*

$$\begin{array}{ccc} a & \longrightarrow & b \\ \uparrow & & \uparrow \\ c & \longrightarrow & d \end{array}$$

Locale of *Yoneda* systems

topology without point over a loop-free category

Theorem (E.Haucourt 05)

The collection, ordered by inclusion, of the Yoneda *systems of a loop-free category, forms a* locale *whose maximum is denoted $\overline{\Sigma}$. Beside, its minimum is the collection of all identities of $\mathcal{C}$.*

Category of components
generalizing the set of arcwise components

The category of components of a loop-free category $\mathcal{C}$ is the quotient $\mathcal{C}/_{\overline{\Sigma}}$.

Theorem (E.Haucourt 05)

A loop-free category $\mathcal{C}$ is a non empty lattice iff its category of components is $\{\bullet\}$

Fundamental theorem

$\mathcal{C}$ loop-free category and Σ *Yoneda* system over $\mathcal{C}$

Theorem (E.Haucourt 05)

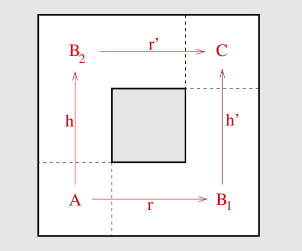
1. *the collection Σ is pure in $\mathcal{C}$ ($\beta \circ \alpha \in \Sigma \Rightarrow \beta, \alpha \in \Sigma$),*
2. *the category $\mathcal{C}/_\Sigma$ is loop-free,*
3. *the categories $\mathcal{C}[\Sigma^{-1}]$ and $\mathcal{C}/_\Sigma$ are equivalent and*
4. *the category $\mathcal{C}[\Sigma^{-1}]$ is fibered over the base $\mathcal{C}/_\Sigma$.*

A detailed example

square with centered hole

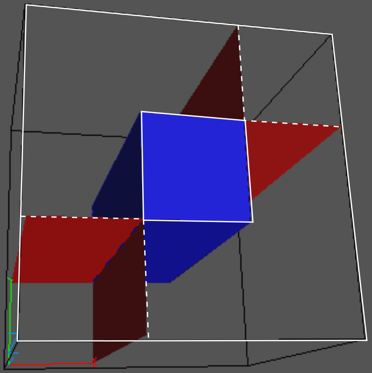
$x \in$	$y \in$	$\overrightarrow{\pi_1}(\overrightarrow{X})[x,y]$
A	A	$\{\sigma_{x,y}\}$
B_1	B_1	$\{\sigma_{x,y}\}$
B_2	B_2	$\{\sigma_{x,y}\}$
C	C	$\{\sigma_{x,y}\}$
A	B_1	$\{r_{x,y}\}$
A	B_2	$\{h_{x,y}\}$
B_1	C	$\{h'_{x,y}\}$
B_2	C	$\{r'_{x,y}\}$
B_1	B_2	$\emptyset$
B_2	B_1	$\emptyset$
A	C	$\{u_{x,y}\ ,\ d_{x,y}\}$

With
$r'_{y,z} \circ h_{x,y} = u_{x,z}$, $h'_{y,z} \circ r_{x,y} = d_{x,z}$
and 3 points x, y, z of the square such that $x \sqsubseteq y \sqsubseteq z$;
if $x \not\sqsubseteq y$ then $\overrightarrow{\pi_1}(\overrightarrow{X})[x,y] = \emptyset$.



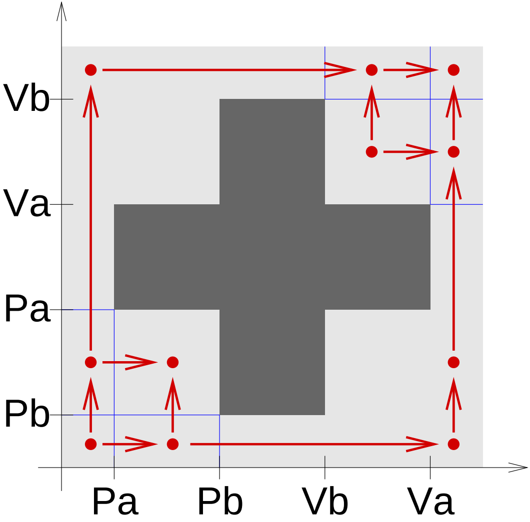
Example of product
parallel "independent" composition

Though their fundamental categories differ...

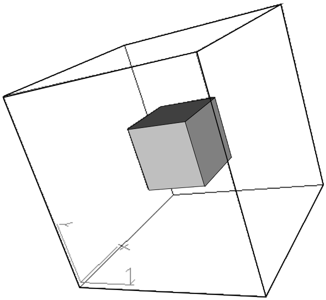


this pospace and the square with centered hole have the same component category

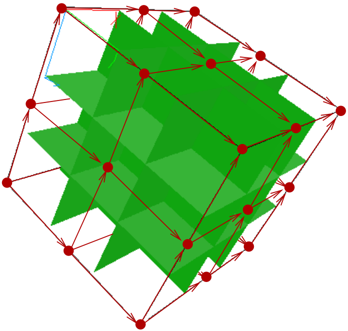
The category of components
of the swiss flag



The components category
of a 2-semaphore

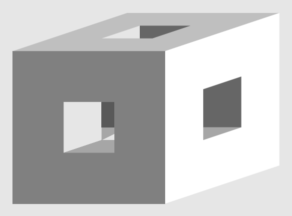


the pospace

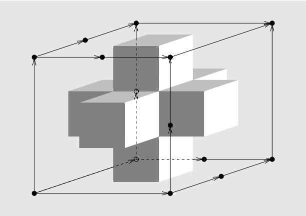


its category of components

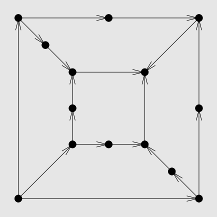
The components category
of the 3D swiss flag



Interior of the pospace



Category of components



Flattened