Precubical and Continuous Control Flow

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Control Flow Graphs of Sequential Processes

*Control Flow Analysis*, Frances E. Allen, SIGPLAN Notices 1970

Source Code

\[\text{Lexer + Parser} \]

Abstract Syntaxic Tree

\[\text{Control Flow Analysis} \]

middle-end

Control Flow Graph

\[\text{Compiler} \quad \text{Static Analyzer} \]

executable  abstract model
The overall idea
of Static Analysis

The model of a program should be a finite representation of an overapproximation of the collection of all its execution traces.
Precubical sets
as presheaves over $\square^+$

$K_0$
Precubical sets
as presheaves over $\square^+$

\[ \partial^-_0 \ \partial^+_0 \]

\[
K_0 \quad \partial^+_0 \quad \partial^-_0 \quad K_1
\]
Precubical sets
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Tensor product of precubical sets

Given precubical sets $K$ and $K'$ of dimension $p$ and $q$, the set of $n$-cubes for $0 \leq n \leq p + q$

$$(K \otimes K')_n = \bigsqcup_{i+j=n} K_i \times K_j$$

For $x \otimes y \in K_i \times K'_j$ with $i + j = n$ the $k^{th}$ face map, with $0 \leq k < n$, is given by

$$\partial_k^\pm (x \otimes y) = \begin{cases} 
\partial_k^\pm (x) \otimes y & \text{if } 0 \leq k < i \\
x \otimes \partial_k^\pm (y) & \text{if } i \leq k < n 
\end{cases}$$
Example of tensor product of precubical sets
Example of tensor product
of precubical sets
Example of tensor product
of precubical sets
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of precubical sets
The PV language
Dijkstra 68 - Input language for ALCOOL in an extended form

- **Sem**: set of semaphores with arity in $\mathbb{N} \setminus \{0, 1\}$
- **Mtx**: set of mutex, an alias for a semaphore of arity 2
- A semaphore $x$ of arity $n$ is a resource offering $n - 1$ tokens, each process can hold one token or more
- A process acquire a token executing the instruction $P(x)$ and release it executing the instruction $V(x)$
- A mutex can be held by only one process at the time
- Trying to perform $P(x)$ though $x$ is not available blocks the execution unless $x$ is a mutex already held by the process
- The instruction $V(x)$ is not blocking
- **Wait**: set of synchronization barriers with arity in $\mathbb{N} \setminus \{0, 1\}$
- Instruction $W(x)$ blocks the execution of the process until $n$ (arity of $x$) processes are blocked by $x$ then all the execution are resumed at the same time
Extending the middle-end representation

Conservative process

A process is said to be conservative when for all paths $\gamma$, the amount of resources available at the arrival of $\gamma$ only depends on the amounts of resources that were available at the origin of $\gamma$.

$$\partial^- \gamma = \partial^- \gamma' \quad \text{and} \quad \partial^+ \gamma = \partial^+ \gamma' \implies [\gamma] \cdot \delta(x) = [\gamma'] \cdot \delta(x)$$

Being conservative is decidable and induces a potential function.
The potential function
of a PV program $P_1 | \cdots | P_d$

- assume each $P_k$ is conservative
  and $F_k$ the associated potential function
- let $K_0 = V_1 \times \cdots \times V_d$ the 0-dimensional cubes of the
tensor product of the cfgs
- The potential function $F : K_0 \times \mathcal{R} \rightarrow \mathbb{N}$ is

$$F(v_1, \ldots, v_d, x) = \sum_{k=1}^{d} F_k(v_k, x)$$
Control Flow Precubical Set: an example

\[ y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) | z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a) \]
Control Flow Precubical Set: an example

\[\begin{align*}
y &:= 0 . W(b) . P(a) . x := z . V(a) \mid z := 0 . W(b) . P(a) . x := y . V(a)
\end{align*}\]
Control Flow Precubical Set: an example

\[ y := 0, W(b).P(a).x := z, V(a) | z := 0, W(b).P(a).x := y, V(a) \]
Control Flow Precubical Set: an example

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Geometric model: an example

$y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) | z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a)$
Geometric model: an example

\[ y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) \mid z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a) \]
Geometric model: an example

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Geometric model: an example

\[
y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) | z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a)
\]

\[
y := 0 \quad W(b) \quad P(a) \quad x := z \quad V(a) \quad z := 1 \quad W(b) \quad P(a) \quad x := y \quad V(a)
\]

conflict in z
Geometric model: an example

\[ y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) | z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a) \]
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Comparing
Discrete vs Continuous
Cubical areas

- A pospace is a topological space with a closed partial order
- The Morphisms of pospace are the continuous increasing maps
- A $n$-cube is the product of a $n$-uple of intervals of $\mathbb{R}$
- A $n$-cubical area is a finite union of $n$-cubes
- A $n$-cubical area inherits a pospace structure from $\mathbb{R}^n$
Prime decomposition theorem for areas

- areas form a commutative monoid with cartesian product
- this commutative monoid is free
- prime decomposition of $[P]$ provide information about parallel decomposition of $P$. 
- Dipath are continuous increasing maps $\gamma : [0, r] \to X$ with $r \geq 0$, $\partial^- \gamma = \gamma(0)$ and $\partial^+ \gamma = \gamma(r)$
- Concatenation $\gamma \cdot \delta : [0, r + r'] \to X$ when $\partial^- \gamma = \partial^+ \delta$;
  $$\gamma \cdot \delta(t) = \begin{cases} 
\delta(t) & \text{if } t \leq r \\
\gamma(t) & \text{if } r \leq t
\end{cases}$$

- If $X$ is the model of a program
  then the dipaths on $X$ is an overapproximation of the execution traces

- Infinitely many paths between two points
Elementary homotopy

- dihomotopy $h : [0, r] \times [0, \rho] \rightarrow X$ a morphism s.t. $h(0, -)$ and $h(r, -)$ are both constant

- anti-dihomotopy $h : [0, r] \times [0, \rho] \rightarrow X$ such that $(t, x) \mapsto h(t, -x)$ is a dihomotopy

- elementary homotopy $h_n \ast \cdots \ast h_1$ where each $h_k$ is either a dihomotopy or an antidihomotopy

- $\gamma \sim \delta$ when there exist an elementary homotopy between $\gamma\theta$ and $\delta\psi$ for some $\theta$ and $\psi$ both increasing and surjective, and sharing their domain of definition.
Characterizing dihomotopy classes through areas

- $X$ cubical area, for all dipath $\gamma$ there exist a cubical area s.t. $\delta \sim \gamma$ iff $\text{img}(\delta) \subseteq A_{\gamma}$
- In fact $\gamma \sim \delta$ iff $A_{\gamma} = A_{\delta}$
- Further there is a finite collection $\mathcal{K}$ of subareas of $X$ such that for all $\gamma$ and $\delta$ sharing their extremities, $\gamma \sim \delta$ iff for all $K \in \mathcal{K}$, $\text{img}(\gamma) \subseteq K \iff \text{img}(\delta) \subseteq K$
Dihomotopy classes as cubical areas
Dihomotopy classes as cubical areas
Dihomotopy classes as cubical areas
Thank you