Introduction to Directed Algebraic Topology
with a view towards modelling Concurrency
II
Mathematical Structures of Computations - Lyon 2014

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Summary

Geometric realization

Directed Topology
   Local pospaces
   Realization of graphs
   Continuous interpretation
   Geometric model

Fundamental category
   Precubical sets
   Local pospaces
   Some calculations
Diagram in \( \textbf{Top} \)
from a precubical set \( K \)

- \( \partial_i \equiv (x \cdots x_{i}^{0} x \cdots x) \) and \( \partial^+_i \equiv (x \cdots x_{i}^{1} x \cdots x) \)
Diagram in \textbf{Top}
from a precubical set $K$

$- \partial_i \cong (x \cdots x_0 \cdots x)$ and $\partial_i^+ \cong (x \cdots x_1 \cdots x)$

$- \text{for all } n \in \mathbb{N} \text{ for all } x \in K_n \text{ for all } i \in \{0, \ldots, n-1\}$

and for $\varepsilon \in \{0, 1\}$ we have the inclusion map

$\phi_{i,n,x}^\varepsilon : \{\partial_i^\varepsilon(x)\} \times [0,1]^{n-1} \rightarrow \{x\} \times [0,1]^n$

$(t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_{i-1}, \varepsilon, t_i, \ldots, t_{n-1})$
Diagram in \textbf{Top}

from a precubical set $K$

- $\partial_i \cong (x \cdots x_0 x \cdots x)$ and $\partial_i^+ \cong (x \cdots x_1 x \cdots x)$

- for all $n \in \mathbb{N}$ for all $x \in K_n$ for all $i \in \{0, \ldots, n-1\}$
  and for $\varepsilon \in \{0, 1\}$ we have the inclusion map

$$\phi_{i,n,x}^\varepsilon : \{\partial_i^\varepsilon (x)\} \times [0, 1]^{n-1} \rightarrow \{x\} \times [0, 1]^n$$

$$(t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_{i-1}, \varepsilon, t_i, \ldots, t_{n-1})$$

- $|K|$: the geometric realization of $K$ is
  the colimit of this diagram in \textbf{Top}
Diagram in \textbf{Top} from a precubical set \( K \)

- \( \partial_i \cong (x \cdots x \underbrace{0}_{i^{th}} x \cdots x) \) and \( \partial^+_i \cong (x \cdots x \underbrace{1}_{i^{th}} x \cdots x) \)

- for all \( n \in \mathbb{N} \) for all \( x \in K_n \) for all \( i \in \{0, \ldots, n-1\} \)
  and for \( \varepsilon \in \{0, 1\} \) we have the inclusion map

\[
\phi_{i,n,x}^\varepsilon : \{ \partial_i^\varepsilon(x) \} \times [0, 1]^{n-1} \rightarrow \{ x \} \times [0, 1]^n
\]

\[
(t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_{i-1}, \varepsilon, t_i, \ldots, t_{n-1})
\]

- \( |K| \): the geometric realization of \( K \) is
  the colimit of this diagram in \textbf{Top}

- for all \( K, K' \) precubical sets, \( |K \otimes K'| \cong |K| \times |K'| \)
Geometric realization in $\textbf{Top}$

a calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$
- $\partial^– \alpha = \partial^– \beta = a$ and $\partial^+ \alpha = \partial^+ \beta = b$
Geometric realization in \textbf{Top}

a calculation

\[- K_0 = \{a, b\} \text{ and } K_1 = \{\alpha, \beta\}\]
\[- \partial^{-}\alpha = \partial^{-}\beta = a \text{ and } \partial^{+}\alpha = \partial^{+}\beta = b\]
Geometric realization in $\textbf{Top}$

A calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$
  \[ \partial^- \alpha = \partial^- \beta = a \text{ and } \partial^+ \alpha = \partial^+ \beta = b \]

\[ \partial^- \alpha = a \quad \Rightarrow \quad b = \partial^+ \alpha \]
Geometric realization in \textbf{Top} \\

a calculation \\

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$ \\

$\partial \alpha = \partial \beta = a$ and $\partial^+ \alpha = \partial^+ \beta = b$
Geometric realization in Top

another calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$
  \[ \partial^- \alpha = \partial^+ \beta = a \text{ and } \partial^+ \alpha = \partial^- \beta = b \]
Geometric realization in $\textbf{Top}$

another calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$
  \[ \partial^{-}\alpha = \partial^{+}\beta = a \text{ and } \partial^{+}\alpha = \partial^{-}\beta = b \]
Geometric realization in \textbf{Top}

another calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$
  
  $\partial^- \alpha = \partial^+ \beta = a$ and $\partial^+ \alpha = \partial^- \beta = b$

\[
\begin{align*}
\partial^- \alpha &= a \\
\cdot &
\end{align*}
\begin{align*}
\cdot &= \partial^+ \alpha \\
b &= b
\end{align*}
\]
Geometric realization in Top

another calculation

- \( K_0 = \{a, b\} \) and \( K_1 = \{\alpha, \beta\} \)

\( \partial^- \alpha = \partial^+ \beta = a \) and \( \partial^+ \alpha = \partial^- \beta = b \)

\( \partial^+ \beta = \partial^- \alpha = a \) \hspace{2cm} b = \partial^+ \alpha = \partial^- \beta \)
Partially Ordered Spaces - pospaces

Eilenberg 41 / Nachbin 48

- A topological space $X$ together with a closed partial order
Partially Ordered Spaces - pospaces

Eilenberg 41 / Nachbin 48

- A topological space $X$ together with a closed partial order
- morphisms: increasing continuous maps
Partially Ordered Spaces - pospaces

- A topological space $X$ together with a closed partial order
- morphisms: increasing continuous maps
- e.g. $\mathbb{R}$ with its standard topology and order
- A topological space $X$ together with a closed partial order
- morphisms: increasing continuous maps
- e.g. $\mathbb{R}$ with its standard topology and order
- Potop is complete and cocomplete
  but its colimits do not preserve the topology
Directed geometric realization in PoTop

A calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$

$\partial^- \alpha = \partial^- \beta = a$ and $\partial^+ \alpha = \partial^+ \beta = b$
Directed geometric realization in \textbf{PoTop}

a calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$
  $\partial^- \alpha = \partial^- \beta = a$ and $\partial^+ \alpha = \partial^+ \beta = b$

\[
\begin{array}{c}
a \cdot \quad \cdot \quad b
\end{array}
\]
Directed geometric realization in PoTop

a calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$
  \[ \partial^- \alpha = \partial^- \beta = a \text{ and } \partial^+ \alpha = \partial^+ \beta = b \]

$\partial^- \alpha = a \wedge b = \partial^+ \alpha$
Directed geometric realization in \textbf{PoTop}

a calculation

- \( K_0 = \{ a, b \} \) and \( K_1 = \{ \alpha, \beta \} \)

\[ \partial \alpha = \partial \beta = a \text{ and } \partial^+ \alpha = \partial^+ \beta = b \]

\[ \partial \beta = \partial \alpha = a \quad \times \quad b = \partial^+ \alpha = \partial^+ \beta \]
Directed geometric realization in PoTop

another calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$

$\partial^{-}\alpha = \partial^{+}\beta = a$ and $\partial^{+}\alpha = \partial^{-}\beta = b$
Directed geometric realization in PoTop

another calculation

- \( K_0 = \{a, b\} \) and \( K_1 = \{\alpha, \beta\} \)

\( \partial \alpha = \partial^+ \beta = a \) and \( \partial^+ \alpha = \partial \beta = b \)

\[
\begin{align*}
  a & \cdot \\
  \cdot &\cdot \\
  \cdot & b
\end{align*}
\]
Directed geometric realization in $\text{PoTop}$

another calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$
  \[\partial^- \alpha = \partial^+ \beta = a \text{ and } \partial^+ \alpha = \partial^- \beta = b\]
Directed geometric realization in \textbf{PoTop}

another calculation

\[ K_0 = \{a, b\} \text{ and } K_1 = \{\alpha, \beta\} \]
\[ \partial^-\alpha = \partial^+\beta = a \text{ and } \partial^+\alpha = \partial^-\beta = b \]
Directed geometric realization in PoTop

another calculation

- \( K_0 = \{a, b\} \) and \( K_1 = \{\alpha, \beta\} \)
  \[ \partial^- \alpha = \partial^+ \beta = a \text{ and } \partial^+ \alpha = \partial^- \beta = b \]

  just one point remains
Locally Partially Ordered Spaces - local pospaces

Fajstrup, Goubault, and Raussen 98 (original version)

- $X$ underlying topological space
Locally Partially Ordered Spaces - local pospaces

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- $X$ underlying topological space
- ordered chart on $X$: pospace over some open subset of $X$
Locally Partially Ordered Spaces - local pospaces
Fajstrup, Goubault, and Raussen 98 (original version)

- $X$ underlying topological space
- ordered chart on $X$: pospace over some open subset of $X$
- ordered atlas on $X$: collection $\mathcal{U}$ of ordered charts s.t.
  i) for all $U, U' \in \mathcal{U}$ and $x \in U \cap U'$ there exists $U'' \in \mathcal{U}$ s.t.
     $x \in U'' \subseteq U \cap U'$ and the order on $U''$
     matches both orders on $U$ and $U'$
  ii) $\mathcal{U}$ induces a basis of topology of $X$
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      $x \in U'' \subseteq U \cap U'$ and the order on $U''$
      matches both orders on $U$ and $U'$
  ii) $\mathcal{U}$ induces a basis of topology of $X$
- morphism of atlases $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$:
  a continuous map $f : X \rightarrow Y$ such that for all $x \in X$
  there exists $U \in \mathcal{U}$, $V \in \mathcal{V}$ neighborhoods of $x$ and $f(x)$
  such that $f$ induces a morphism of pospaces from $U$ to $V$
Atlastes $\mathcal{U}$ and $\mathcal{U}'$ on $X$ are equivalent when their union is still an atlas.
Locally Partially Ordered Spaces - local pospaces

Fajstrup, Goubault, and Raussen 98 (original version)

- Atlases \( \mathcal{U} \) and \( \mathcal{U}' \) on \( X \) are equivalent when their union is still an atlas
- The union of all atlases equivalent to \( \mathcal{U} \) is an atlas

- \( U \) and \( U' \) on \( X \) are equivalent when their union is still an atlas
- The union of all atlases equivalent to \( U \) is an atlas
Locally Partially Ordered Spaces - local pospaces

Fajstrup, Goubault, and Raussen 98 (original version)

- Atlases $\mathcal{U}$ and $\mathcal{U}'$ on $X$ are equivalent when their union is still an atlas
- The union of all atlases equivalent to $\mathcal{U}$ is an atlas
- Local pospace: equivalence class of atlases
Locally Partially Ordered Spaces - local pospaces
Fajstrup, Goubault, and Raussen 98 (original version)

- Atlases $\mathcal{U}$ and $\mathcal{U}'$ on $X$ are equivalent when their union is still an atlas
- The union of all atlases equivalent to $\mathcal{U}$ is an atlas
- Local pospace: equivalence class of atlases
- If $\mathcal{U} \sim \mathcal{U}'$, $\mathcal{V} \sim \mathcal{V}'$, and $f : \mathcal{U} \to \mathcal{V}$ morphism of atlases then $f : \mathcal{U}' \to \mathcal{V}'$ morphism of atlases
- Atlases $\mathcal{U}$ and $\mathcal{U}'$ on $X$ are equivalent when their union is still an atlas.
- The union of all atlases equivalent to $\mathcal{U}$ is an atlas.
- Local pospace: equivalence class of atlases.
- If $\mathcal{U} \sim \mathcal{U}'$, $\mathcal{V} \sim \mathcal{V}'$, and $f : \mathcal{U} \to \mathcal{V}$ morphism of atlases, then $f : \mathcal{U}' \to \mathcal{V}'$ morphism of atlases.
- e.g. the exponential map $t \in \mathbb{R} \mapsto e^{it} \in S^1$.
Locally Partially Ordered Spaces - local pospaces

Fajstrup, Goubault, and Raussen 98 (original version)

- Atlases $\mathcal{U}$ and $\mathcal{U}'$ on $X$ are equivalent when their union is still an atlas.
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- If $\mathcal{U} \sim \mathcal{U}'$, $\mathcal{V} \sim \mathcal{V}'$, and $f : \mathcal{U} \to \mathcal{V}$ morphism of atlases, then $f : \mathcal{U}' \to \mathcal{V}'$ morphism of atlases.
- e.g. the exponential map $t \in \mathbb{R} \mapsto e^{it} \in S^1$
- Lpotop is finitely complete but misses some infinite products.
  Its cocompleteness is an open question.
  Its colimits do not preserve the topology.
Directed geometric realization in $LpoTop$

- For all finite precubical sets $K$, the directed geometric realization $|K|_{LpoTop}$ exists.
Directed geometric realization in $\mathbf{LpoTop}$

- For all finite precubical sets $K$, the directed geometric realization $\mid K \mid_{\mathbf{LpoTop}}$ exists
- and preserves the topology

$$U(\mid K \mid_{\mathbf{LpoTop}}) = \mid K \mid$$
Directed geometric realization in $\mathbf{LpoTop}$

a claim

- For all finite precubical sets $K$, the directed geometric realization $\lvert K \rvert_{\mathbf{LpoTop}}$ exists
- and preserves the topology

$$U(\lvert K \rvert_{\mathbf{LpoTop}}) = \lvert K \rvert$$

- therefore

$$\lvert K \otimes K' \rvert_{\mathbf{LpoTop}} \cong \lvert K \rvert_{\mathbf{LpoTop}} \times \lvert K' \rvert_{\mathbf{LpoTop}}$$
Realization of graphs as local pospaces

\[ G : A \xrightarrow{\partial^-} V \xrightarrow{\partial^+} \]

- underlying set \( V \sqcup A \times ]0, 1[ \)
Realization of graphs
as local pospaces

\[
G : A \xrightarrow{\partial^-} V \xleftarrow{\partial^+}
\]

- underlying set \( V \sqcup A \times ]0, 1[ \)
- \( v^+_\varepsilon \) union of \( \{\alpha\} \times ]0, \varepsilon[ \)
  for all \( \alpha \in A \) such that \( \partial^- \alpha = v \) and \( 0 < \varepsilon < 1 \)
- \( v^-\varepsilon \) union of \( \{\alpha\} \times ]1 - \varepsilon, 1[ \)
  for all \( \alpha \in A \) such that \( \partial^+ \alpha = v \) and \( 0 < \varepsilon < 1 \)
Realization of graphs
as local pospaces

\[ G : A \xrightarrow{\partial^-} V \xrightarrow{\partial^+} \]

- underlying set \( V \sqcup A \times ]0, 1[ \)
- \( v_\epsilon^+ \) union of \( \{ \alpha \} \times ]0, \epsilon[ \)
  for all \( \alpha \in A \) such that \( \partial^- \alpha = v \) and \( 0 < \epsilon < 1 \)
- \( v_\epsilon^- \) union of \( \{ \alpha \} \times ]1 - \epsilon, 1[ \)
  for all \( \alpha \in A \) such that \( \partial^+ \alpha = v \) and \( 0 < \epsilon < 1 \)
- directed atlas
  \( \{ \alpha \} \times ]a, b[ \) with \( \alpha \in A \) and \( 0 \leq a < b \leq 1 \), and
  \( \{ v \} \cup v_\epsilon^+ \cup v_\epsilon^- \) with \( v \in V \) and \( 0 < \epsilon < 1 \)
  with obvious partial order
Realization of graphs
as local pospaces

\[ G : A \xrightarrow{\partial} V \]

- underlying set \( V \sqcup A \times ]0, 1[ \)
- \( v_\varepsilon^+ \) union of \( \{ \alpha \} \times ]0, \varepsilon[ \)
  for all \( \alpha \in A \) such that \( \partial^- \alpha = v \) and \( 0 < \varepsilon < 1 \)
- \( v_\varepsilon^- \) union of \( \{ \alpha \} \times ]1 - \varepsilon, 1[ \)
  for all \( \alpha \in A \) such that \( \partial^+ \alpha = v \) and \( 0 < \varepsilon < 1 \)
- directed atlas
  \( \{ \alpha \} \times ]a, b[ \) with \( \alpha \in A \) and \( 0 \leq a < b \leq 1 \), and
  \( \{ v \} \cup v_\varepsilon^+ \cup v_\varepsilon^- \) with \( v \in V \) and \( 0 < \varepsilon < 1 \)
  with obvious partial order
- denoted by \( |G| \)
Continuous sequential virtual machine

- The labelling $\lambda : D \rightarrow A$, with $D = \{(\alpha, \frac{1}{2}) \mid \alpha \in A\}$
Continuous sequential virtual machine

- The labelling $\lambda : D \to A$, with $D = \{ (\alpha, \frac{1}{2}) \mid \alpha \in A \}$
- for $\gamma : [0, r] \to \mathcal{G}$ the set $\gamma^{-1}(D)$ is a finite union of disjoint compact intervals $[a_1, b_1] \cup \cdots \cup [a_n, b_n]$
Continuous sequential virtual machine

- The labelling \( \lambda : D \to A \), with \( D = \{ (\alpha, \frac{1}{2}) \mid \alpha \in A \} \)
- for \( \gamma : [0, r] \to \downarrow G \uparrow \) the set \( \gamma^{-1}(D) \) is a finite union of disjoint compact intervals \([a_1, b_1] \cup \cdots \cup [a_n, b_n]\)
- Instructions are performed when they are touched so \( \llbracket \gamma \rrbracket = \gamma(a_n), \ldots, \gamma(a_1) \) is associated with \( \gamma \) therefore the action of \( \gamma \) upon a distribution \( \delta \) is \( \llbracket \gamma \rrbracket \cdot \delta \)
Continuous sequential virtual machine

- The labelling $\lambda : D \to A$, with $D = \{ (\alpha, \frac{1}{2}) \mid \alpha \in A \}$
- for $\gamma : [0, r] \to \mathbb{G}$ the set $\gamma^{-1}(D)$ is a finite union of disjoint compact intervals $[a_1, b_1] \cup \cdots \cup [a_n, b_n]$
- Instructions are performed when they are touched so $\llbracket \gamma \rrbracket = \gamma(a_n), \ldots, \gamma(a_1)$ is associated with $\gamma$
  therefore the action of $\gamma$ upon a distribution $\delta$ is $\llbracket \gamma \rrbracket \cdot \delta$
- for any execution trace $s$, there exists a dipath $\gamma$ such that $\llbracket \gamma \rrbracket = s$
Continuous parallel dynamics

- if the process is conservative then for any \( \delta \),

\[ [\gamma] \cdot \delta \text{ only depends on } \partial^-\gamma \text{ and } \partial^+\gamma \]
Continuous parallel dynamics

- if the process is conservative then for any \( \delta \), \( \lbrack \gamma \rbrack \cdot \delta \) only depends on \( \partial^- \gamma \) and \( \partial^+ \gamma \)
- therefore we have a potential function

\[ F : \lvert G \rvert \times \mathcal{R} \rightarrow \mathbb{N} \]
Areas

definition

- $G_1, \ldots, G_d$ finite graphs
Areas

definition

- $G_1, \ldots, G_d$ finite graphs
- $(G_1, \ldots, G_d)$-block: $B_1 \times \cdots \times B_n$ with $B_k$ connected subset of $|G_k|$
Areas

definition

- $G_1, \ldots, G_d$ finite graphs
- $(G_1, \ldots, G_d)$-block: $B_1 \times \cdots \times B_n$ with $B_k$ connected subset of $|G_k|$
- $(G_1, \ldots, G_d)$-areas: finite union of blocks
Areas

definition

- $G_1, \ldots, G_d$ finite graphs
- $(G_1, \ldots, G_d)$-block: $B_1 \times \cdots \times B_n$ with $B_k$ connected subset of $|G_k|$
- $(G_1, \ldots, G_d)$-areas: finite union of blocks
- The collection of $(G_1, \ldots, G_d)$-areas forms a boolean subalgebra of $2^{[G_1] \times \cdots \times [G_d]}$
Race conditions
conflicts in variable access

- $G_1, \ldots, G_d$ the control flow graphs of each process
Race conditions
conflicts in variable access

- $G_1, \ldots, G_d$ the control flow graphs of each process
- Race conditions is the subset of $|G_1| \times \cdots \times |G_d|$ s.t. there is $1 \leq i < j \leq d$ such that $\lambda(v_i)$ and $\lambda(v_j)$ are actions sharing some variable
Forbidden area
via potential function

- $F_1, \ldots, F_d$ the associated potential functions
Forbidden area via potential function

- $F_1, \ldots, F_d$ the associated potential functions
- $F : |G_1| \times \cdots \times |G_d| \times \mathcal{R} \to \mathbb{N}$ the potential function

$$F(v_1, \ldots, v_d, x) = \sum_{k=1}^{d} F_k(v_k, x)$$
Forbidden area
via potential function

- $F_1, \ldots, F_d$ the associated potential functions
- $F : |G_1| \times \cdots \times |G_d| \times \mathcal{R} \to \mathbb{N}$ the potential function

$$F(v_1, \ldots, v_d, x) = \sum_{k=1}^{d} F_k(v_k, x)$$

- Forbidden area is the subset of $|G_1| \times \cdots \times |G_d|$

$$\{(v_1, \ldots, v_d) \mid \exists x \in \mathcal{R}, \ F(v_1, \ldots, v_d, x) \geq \text{arity}(x)\}$$
Walls
and geometric model

- **Walls** is the subset of $(v_1, \ldots, v_d) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_d$ s.t.
  there exists a synchronization $x$ s.t.
  the cardinal of $\{k \in \{1, \ldots, d\} \mid \lambda(v_k) = W(x)\}$
  is neither 0 nor the arity of $x$
Walls 

and geometric model

- **Walls** is the subset of \((v_1, \ldots, v_d) \in |G_1| \times \cdots \times |G_d|\) s.t. there exists a synchronization \(x\) s.t.
  
  the cardinal of \(\{k \in \{1, \ldots, d\} \mid \lambda(v_k) = W(x)\}\)
  
  is neither 0 nor the arity of \(x\)

- The geometric model is then defined as
  
  \(|G_1| \times \cdots \times |G_d| \setminus (\text{Race} \cup \text{Forbidden} \cup \text{Walls})|
Geometric model: an example

\[ y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) \mid z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a) \]
Geometric model: an example

\[ y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) \mid z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a) \]
Geometric model: an example

\[
y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) \mid z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a)
\]
Geometric model: an example

\[ y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) | z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a) \]
Geometric model: an example

\[
y := 0. W(b) . P(a) . x := z . V(a) | z := 0 . W(b) . P(a) . x := y . V(a)
\]
Geometric model: an example

\( y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) \mid z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a) \)
Geometric model: an example

\[ y := 0 \cdot W(b) \cdot P(a) \cdot x := z \cdot V(a) \mid z := 0 \cdot W(b) \cdot P(a) \cdot x := y \cdot V(a) \]
Comparing
Discrete vs Continuous

MSC - Lyon 2014
Geometric realization
Directed Topology
Local pospaces
Realization
Continuous interpretation
Geometric model
Fundamental category
Precubical sets
Local pospaces
Some calculations
Fundamental category of a precubical set $K$

- $F(trunc_1(K))$ the category of paths on the underlying graph of $K$
Fundamental category
of a precubical set $K$

- $F(trunc_1(K))$ the category of paths on the underlying graph of $K$
- the congruence $\sim$ over $F(trunc_1(K))$ generated by $\gamma \sim \delta$ when $\gamma$ and $\delta$ start and finish at the lower and upper corners of the same $n$-cube
Fundamental category
of a precubical set $K$

- $F(\text{trunc}_1(K))$ the category of paths on the underlying graph of $K$
- the congruence $\sim$ over $F(\text{trunc}_1(K))$ generated by $\gamma \sim \delta$ when $\gamma$ and $\delta$ start and finish at the lower and upper corners of the same $n$-cube
- define $\overrightarrow{\pi_1} K = F(\text{trunc}_1(K))/\sim$
Fundamental category of a precubical set $K$

- $F(\text{trunc}_1(K))$ the category of paths on the underlying graph of $K$

- the congruence $\sim$ over $F(\text{trunc}_1(K))$ generated by $\gamma \sim \delta$ when $\gamma$ and $\delta$ start and finish at the lower and upper corners of the same $n$-cube

- define $\overrightarrow{\pi_1} K = F(\text{trunc}_1(K)) / \sim$

- $\overrightarrow{\pi_1} K = \overrightarrow{\pi_1}(\text{trunc}_2(K))$
**Dipath**
on $X \in \text{LpoTop}$

- Dipath: morphism $\gamma : [0, r] \to X$ with $r \geq 0$
  $\partial^- \gamma = \gamma(0)$ and $\partial^+ \gamma = \gamma(r)$
Dipath on $X \in \mathbf{LpoTop}$

- Dipath: morphism $\gamma : [0, r] \to X$ with $r \geq 0$
  \[ \partial^- \gamma = \gamma(0) \text{ and } \partial^+ \gamma = \gamma(r) \]

- Concatenation $\gamma \cdot \delta : [0, r + r'] \to X$ when $\partial^- \gamma = \partial^+ \delta$;
  \[
  \gamma \cdot \delta(t) = \begin{cases} 
  \delta(t) & \text{if } t \leq r \\
  \gamma(t) & \text{if } r \leq t
  \end{cases}
  \]
Dipath

on $X \in \textbf{LpoTop}$

- Dipath: morphism $\gamma : [0, r] \to X$ with $r \geq 0$
  $\partial^- \gamma = \gamma(0)$ and $\partial^+ \gamma = \gamma(r)$
- Concatenation $\gamma \cdot \delta : [0, r + r'] \to X$ when $\partial^- \gamma = \partial^+ \delta$;
  $\gamma \cdot \delta(t) = \begin{cases} 
\delta(t) & \text{if } t \leq r \\
\gamma(t) & \text{if } r \leq t
\end{cases}$
- Dipath functor $P : \textbf{LpoTop} \to \textbf{Cat}$
Dipath on $X \in \text{LpoTop}$

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- Dipath functor $P : \text{LpoTop} \to \text{Cat}$
- If $X$ is the model of a program
  then the dipaths on $X$ is an overapproximation of the execution traces
Dipath
on $X \in \mathbf{LpoTop}$

- Dipath: morphism $\gamma : [0, r] \to X$ with $r \geq 0$
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  $\gamma \cdot \delta(t) = \begin{cases} 
\delta(t) & \text{if } t \leq r \\
\gamma(t) & \text{if } r \leq t 
\end{cases}$
- Dipath functor $P : \mathbf{LpoTop} \to \mathbf{Cat}$
- If $X$ is the model of a program
  then the dipaths on $X$ is an overapproximation of the execution traces
- Infinitely many paths between two points
Dihomotopy between dipaths on $X$

- morphism $h : [0, r] \times [0, \rho] \to X$ s.t. $h(0, \_)$ and $h(r, \_)$ are both constant
Dihomotopy between dipaths on $X$

- morphism $h : [0, r] \times [0, \rho] \to X$ s.t.
  $h(0, \_)$ and $h(r, \_)$ are both constant
- 2-dimensional precubical set...

- \[ \partial^+_1 h = \delta \]
- \[ \partial^+_0 h = \text{cst} \]
- \[ \partial^-_1 h = \gamma \]
- \[ \partial^-_0 h = \text{cst} \]
Dihomotopy between dipaths on $X$

- morphism $h : [0, r] \times [0, \rho] \to X$ s.t. $h(0, \_)$ and $h(r, \_)$ are both constant
- 2-dimensional precubical set...

\[ \partial^+_1 h = \delta \]
\[ \partial^+_0 h = \text{cst} \]
\[ \partial^-_1 h = \gamma \]
\[ \partial^-_0 h = \text{cst} \]

...and even more.
Dihomotopy

2-category

- $h : [0, r] \times [0, \rho] \to X$ and $g : [0, r] \times [0, \rho'] \to X$

with $h(\_ \_ , \rho) = g(\_ \_ , 0)$
Dihomotopy

2-category

- \( h : [0, r] \times [0, \rho] \to X \) and \( g : [0, r] \times [0, \rho'] \to X \)
  with \( h(\_ , \rho) = g(\_ , 0) \)

\( g \ast h : [0, r] \times [0, \rho + \rho'] \to X \) defined by

\[
g \ast h(t, x) = \begin{cases} 
  h(t, x) & \text{if } x \leq \rho \\
  g(t, x - \rho) & \text{if } \rho \leq x
\end{cases}
\]
Dihomotopy

2-category

- \( h : [0, r] \times [0, \rho] \to X \) and \( g : [0, r] \times [0, \rho'] \to X \) with \( h(\_ , \rho) = g(\_ , 0) \)

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Dihomotopy

2-category

- \( h : [0, r] \times [0, \rho] \to X \) and \( g : [0, r] \times [0, \rho'] \to X \)

with \( h(-, \rho) = g(-, 0) \)

\[ g \ast h : [0, r] \times [0, \rho + \rho'] \to X \text{ defined by} \]

\[ g \ast h(t, x) = \begin{cases} h(t, x) & \text{if } x \leq \rho \\ g(t, x - \rho) & \text{if } \rho \leq x \end{cases} \]
Dihomotopy

2-category

- $h : [0, r] \times [0, \rho] \to X$ and $g : [0, r] \times [0, \rho'] \to X$
  with $h(_-, \rho) = g(_-, 0)$

$g * h : [0, r] \times [0, \rho + \rho'] \to X$ defined by

$g * h(t, x) = \begin{cases} 
  h(t, x) & \text{if } x \leq \rho \\
  g(t, x - \rho) & \text{if } \rho \leq x
\end{cases}$
Dihomotopy

2-category

- \( h : [0, r] \times [0, \rho] \to X \) and \( h' : [0, r'] \times [0, \rho] \to X \)

with \( \partial^+ h = \partial^- h' \) i.e. \( h(r, \_ ) = h'(0, \_ ) \)
Dihomotopy

2-category

- $h : [0, r] \times [0, \rho] \to X$ and $h' : [0, r'] \times [0, \rho] \to X$

with $\partial^+ \circ h = \partial^- \circ h'$ i.e. $h(r, _) = h'(0, _)$

$h' \cdot h : [0, r] \times [0, \rho + \rho'] \to X$ defined by

$h' \cdot h(t, x) = \begin{cases} 
 h(t, x) & \text{if } t \leq r \\
 h'(t - r, x) & \text{if } r \leq t 
\end{cases}$
Dihomotopy

2-category

- \( h : [0, r] \times [0, \rho] \rightarrow X \) and \( h' : [0, r'] \times [0, \rho] \rightarrow X \)
  with \( \partial^+_0 h = \partial^-_0 h' \) i.e. \( h(r, -) = h'(0, -) \)

\( h' \cdot h : [0, r] \times [0, \rho + \rho'] \rightarrow X \) defined by

\[
\begin{align*}
  h' \cdot h(t, x) &= \begin{cases} 
  h(t, x) & \text{if } t \leq r \\
  h'(t - r, x) & \text{if } r \leq t
  \end{cases}
\end{align*}
\]
Dihomotopy

2-category

\(-\ h : [0, r] \times [0, \rho] \to X\ \text{and}\ h' : [0, r'] \times [0, \rho] \to X\)

with \(\partial^+_0 h = \partial^-_0 h'\) i.e. \(h(r, \_ ) = h'(0, \_ )\)

\(h' \cdot h : [0, r] \times [0, \rho + \rho'] \to X\) defined by

\[h' \cdot h(t, x) = \begin{cases} h(t, x) & \text{if } t \leq r \\ h'(t - r, x) & \text{if } r \leq t \end{cases}\]
Godement

Exchange property
Godement
Exchange property

\[ (g' \ast h') \ast (g \ast h) = (g' \cdot g) \ast (h' \cdot h) \]
## Directed topology vs Category

### 2-category

<table>
<thead>
<tr>
<th>Directed topology</th>
<th>Category</th>
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<tbody>
<tr>
<td>point</td>
<td>category</td>
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<tr>
<td>dipath</td>
<td>functor</td>
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<td>dihomotopy</td>
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<tr>
<td>path concatenation</td>
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<tr>
<td>‘piled up’ homotopies</td>
<td>composition of natural transformations</td>
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<td>‘side-by-side’ homotopies</td>
<td>juxtaposition of natural transformations</td>
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</tbody>
</table>
Elementary homotopy

- anti-dihomotopy \( h : [0, r] \times [0, \rho] \to X \) such that \((t, x) \mapsto h(t, -x)\) is a dihomotopy
Elementary homotopy

- anti-dihomotopy $h : [0, r] \times [0, \rho] \to X$ such that $(t, x) \mapsto h(t, -x)$ is a dihomotopy
- elementary homotopy $h_n \ast \cdots \ast h_1$ where each $h_k$ is either a dihomotopy or an antidihomotopy
Elementary homotopy

- anti-dihomotopy $h : [0, r] \times [0, \rho] \to X$ such that $(t, x) \mapsto h(t, -x)$ is a dihomotopy
- elementary homotopy $h_n \ast \cdots \ast h_1$ where each $h_k$ is either a dihomotopy or an antidihomotopy
- a finite juxtaposition of dihomotopies and anti-dihomotopies can be ‘replaced’ by an elementary homotopy
The dihomotopy relation
γ and δ dipaths defined over [0, r] and [0, r']

- Write γ ∼ δ when ∂−γ = ∂−δ, ∂+γ = ∂+δ
  and there is an elementary homotopy
  between c · γ and d · δ where c (resp. d)
  is constant over [0, (r ∨ r') − r] (resp. [0, (r ∨ r') − r'])
The dihomotopy relation

γ and δ dipaths defined over [0, r] and [0, r']

- Write γ ∼ δ when \( \partial^- \gamma = \partial^- \delta \), \( \partial^+ \gamma = \partial^+ \delta \)
  and there is an elementary homotopy
  between \( c \cdot \gamma \) and \( d \cdot \delta \) where \( c \) (resp. \( d \))
  is constant over \([0, (r \lor r') - r]\) (resp. \([0, (r \lor r') - r']\))
- The relation ∼ is a congruence over \( PX \)
The fundamental category

- By definition $\overrightarrow{\pi_1}X = PX/\sim$
The fundamental category

- By definition $\pi_1 X = PX/\sim$
- $f \circ (h_n \ast \cdots \ast h_1) = (f \circ h_n) \ast \cdots \ast (f \circ h_1)$
  therefore $\gamma \sim \delta$ implies $f \circ \gamma \sim f \circ \delta$
The fundamental category

- By definition $\overrightarrow{\pi_1}X = PX/\sim$
- $f \circ (h_n * \cdots * h_1) = (f \circ h_n) * \cdots * (f \circ h_1)$
  therefore $\gamma \sim \delta$ implies $f \circ \gamma \sim f \circ \delta$
- Hence a functor $\overrightarrow{\pi_1} : \text{LpoTop} \rightarrow \text{Cat}$
The fundamental category

- By definition $\overrightarrow{\pi_1}X = PX/\sim$
- $f \circ (h_n \star \cdots \star h_1) = (f \circ h_n) \star \cdots \star (f \circ h_1)$
  therefore $\gamma \sim \delta$ implies $f \circ \gamma \sim f \circ \delta$
- Hence a functor $\overrightarrow{\pi_1} : \text{LpoTop} \to \text{Cat}$
- $\overrightarrow{\pi_1}(A \times B) \cong \overrightarrow{\pi_1}A \times \overrightarrow{\pi_1}B$
The fundamental category

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- $f \circ (h_n \ast \cdots \ast h_1) = (f \circ h_n) \ast \cdots \ast (f \circ h_1)$
  therefore $\gamma \sim \delta$ implies $f \circ \gamma \sim f \circ \delta$

- Hence a functor $\overrightarrow{\pi_1} : \text{LpoTop} \rightarrow \text{Cat}$

- $\overrightarrow{\pi_1}(A \times B) \cong \overrightarrow{\pi_1}A \times \overrightarrow{\pi_1}B$

- for all dipaths $\gamma : [0, r] \rightarrow X$ for all $\theta$ morphisms
  from $[0, r']$ onto $[0, r]$, $\gamma \sim \gamma \circ \theta$
The fundamental category
of the $n$-cube

- $\text{Obj}(\overset{\rightarrow}{\pi_1}[0, 1]) = [0, 1]$
The fundamental category of the $n$-cube

- $\text{Obj}(\vec{\pi}_1 [0, 1]) = [0, 1]$
- $(\vec{\pi}_1 [0, 1])[a, b] = \begin{cases} \{(a, b)\} & \text{if } a \leq b \\ \emptyset & \text{otherwise} \end{cases}$
The fundamental category
of the $n$-cube

- $\text{Obj}(\pi_1^\rightarrow [0, 1]) = [0, 1]$
- $(\pi_1^\rightarrow [0, 1])[a, b] = \begin{cases} \{(a, b)\} & \text{if } a \leq b \\ \emptyset & \text{otherwise} \end{cases}$
- $\pi_1^\rightarrow [0, 1]^n = ([0, 1], \leq)^n = ([0, 1]^n, \leq^n)$
The fundamental category of the realization of a graph $G$ as a local pospace

- A presentation is given by the graph
The fundamental category 
of the realization of a graph $G$ as a local pospace

- A presentation is given by the graph 
  vertex: $V \sqcup A \times [0, 1]$
The fundamental category
of the realization of a graph $G$ as a local pospace

- A presentation is given by the graph
  vertex: $V \sqcup A \times [0, 1[$
  arrows: $(t, \alpha, t')$ with $\alpha$ arrow of $G$ and $t < t' \in [0, 1]$
The fundamental category
of the realization of a graph $G$ as a local pospace

- A presentation is given by the graph
  vertex: $V \sqcup A \times [0, 1[$
  arrows: $(t, \alpha, t')$ with $\alpha$ arrow of $G$ and $t < t' \in [0, 1]$  
  $\partial^- (t, \alpha, t') = (\alpha, t)$ if $t > 0$; $\partial^- \alpha$ otherwise
The fundamental category
of the realization of a graph $G$ as a local pospace

- A presentation is given by the graph
  vertex: $V \sqcup A \times ]0,1[$
  arrows: $(t, \alpha, t')$ with $\alpha$ arrow of $G$ and $t < t' \in [0,1]$
  $\partial^-(t, \alpha, t') = (\alpha, t)$ if $t > 0$; $\partial^- \alpha$ otherwise
  $\partial^+(t, \alpha, t') = (\alpha, t')$ if $t < 1$; $\partial^+ \alpha$ otherwise
The fundamental category
of the realization of a graph $G$ as a local pospace

- A presentation is given by the graph
  \[ \text{vertex: } V \sqcup A \times ]0, 1[ \]
  \[ \text{arrows: } (t, \alpha, t') \text{ with } \alpha \text{ arrow of } G \text{ and } t < t' \in [0, 1] \]
  \[ \partial^- (t, \alpha, t') = (\alpha, t) \text{ if } t > 0; \partial^- \alpha \text{ otherwise} \]
  \[ \partial^+ (t, \alpha, t') = (\alpha, t') \text{ if } t < 1; \partial^+ \alpha \text{ otherwise} \]
- with the relations \((t', \alpha, t'') \circ (t, \alpha, t') = (t'', \alpha, t)\)
  for $\alpha$ arrow of $G$ and $t < t' < t'' \in [0, 1]$
The fundamental category
of the directed circle

\[ S^1 = \{ z \in \mathbb{C} \mid z \text{ of magnitude } 1 \} \]
The fundamental category of the directed circle

- $S^1 = \{z \in \mathbb{C} \mid z \text{ of magnitude } 1\}$
- $\text{Obj}(\pi_1 S^1) = S^1$
The fundamental category
of the directed circle

- $S^1 = \{z \in \mathbb{C} \mid z \text{ of magnitude } 1\}$
- $\text{Obj}(\pi_1 S^1) = S^1$
- $\pi_1 S^1 [a, b] \cong \{a\} \times \mathbb{N} \times \{b\}$

MSC - Lyon 2014
Geometric realization
Directed Topology
Local pospaces
Realization
Continuous interpretation
Geometric model
Fundamental category
Precubical sets
Local pospaces
Some calculations
The fundamental category
of the directed circle

- $S^1 = \{ z \in \mathbb{C} \mid z \text{ of magnitude } 1 \}$
- $\text{Obj}(\overrightarrow{\pi_1} S^1) = S^1$
- $\overrightarrow{\pi_1} S^1 [a, b] \cong \{a\} \times \mathbb{N} \times \{b\}$
- $(b, m, c) \circ (a, n, b) = \begin{cases} (a, n + m, c) & \text{if } ab \cup bc \neq S^1 \\ (a, n + m + 1, c) & \text{otherwise} \end{cases}$
The fundamental category of the directed complex plane

- The directed complex plane is not a local pospace yet it contains the directed circle
The fundamental category
of the directed complex plane

- The directed complex plane is not a local pospace yet it contains the directed circle
- Obj($\pi_1^\rightarrow C$) = $\mathbb{C}$
The fundamental category
of the directed complex plane

- The directed complex plane is **not** a local pospace
  yet it contains the directed circle
- $\text{Obj}(\vec{\pi}_1(C)) = C$
- $\vec{\pi}_1 C[a, b] \cong \begin{cases} 
\{a\} \times \mathbb{N} \times \{b\} & \text{if } a \neq 0 \text{ and } |a| \leq |b| \\
\{(0, b)\} & \text{if } a = 0 \\
\emptyset & \text{otherwise}
\end{cases}$
The fundamental category
of the directed complex plane

- The directed complex plane is not a local pospace yet it contains the directed circle
- \( \text{Obj}(\pi_1 \mathbb{C}) = \mathbb{C} \)

- \( \pi_1 \mathbb{C}[a, b] \cong \begin{cases} 
\{ a \} \times \mathbb{N} \times \{ b \} & \text{if } a \neq 0 \text{ and } |a| \leq |b| \\
\{ (0, b) \} & \text{if } a = 0 \\
\emptyset & \text{otherwise}
\end{cases} \)

- \( (b, m, c) \circ (a, n, b) = \begin{cases} 
(a, n + m, c) & \text{if } ab \cup bc \neq S^1 \text{ and } a \neq 0 \\
(a, n + m + 1, c) & \text{if } ab \cup bc = S^1 \text{ and } a \neq 0 \\
(0, c) & \text{if } a = 0
\end{cases} \)
The fundamental category
of the directed complex plane

- The directed complex plane is not a local pospace yet it contains the directed circle
- \( \text{Obj}(\pi_1 \mathbb{C}) = \mathbb{C} \)
- \( \pi_1 \mathbb{C} [a, b] \cong \left\{ \begin{array}{ll} \{a\} \times \mathbb{N} \times \{b\} & \text{if } a \neq 0 \text{ and } |a| \leq |b| \\ \{(0, b)\} & \text{if } a = 0 \\ \emptyset & \text{otherwise} \end{array} \right. \)
- \( (b, m, c) \circ (a, n, b) = \left\{ \begin{array}{ll} (a, n + m, c) & \text{if } ab \cup bc \neq S^1 \text{ and } a \neq 0 \\ (a, n + m + 1, c) & \text{if } ab \cup bc = S^1 \text{ and } a \neq 0 \\ (0, c) & \text{if } a = 0 \end{array} \right. \)
- The fundamental category of the directed Riemann sphere is analogous
Fundamental categories of cubical areas - a conjecture

- Cubical area $X$: finite union of $n$-cubes
Fundamental categories
of cubical areas - a conjecture

- Cubical area $X$: finite union of $n$-cubes
- There exists a finite family $\mathcal{K}$ of sub-cubical areas of $X$ such that $\forall \gamma, \delta$ dipaths on $X$ sharing their extremities, $\gamma \sim \delta$ iff $\forall K \in \mathcal{K}$ s.t. $\text{img}(\gamma) \subseteq K \iff \text{img}(\delta) \subseteq K$
Fundamental categories
of cubical areas - a conjecture

- Cubical area $X$: finite union of $n$-cubes
- There exists a finite family $\mathcal{K}$ of sub-cubical areas of $X$ such that $\forall \gamma, \delta$ dipaths on $X$ sharing their extremities, $\gamma \sim \delta$ iff $\forall K \in \mathcal{K}$ s.t. $\text{img}(\gamma) \subseteq K \iff \text{img}(\delta) \subseteq K$
- it fails if $\pi_1 X$ contains loops