Compactifications of d-spaces and vector fields

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D-spaces

Directed Homotopy Theory I, Cah. Top. Géom. Diff. Cat., Marco Grandis (2003)

- A Hausdorff space X together with a collection dX of paths on it such that
  - any constant path belongs to dX,
  - the collection dX is stable under concatenation, and
  - if $\gamma \in dX$, $\operatorname{dom}\gamma = [0, r]$ and $\theta : [0, r'] \to [0, r]$ is continuous and increasing, then $\gamma \circ \theta \in dX$
- The elements of dX are called the d-paths while the collection dX is called a direction on X. The collection of all directions over X is a complete lattice.
- A d-map from (X, dX) to (Y, dY) is a continuous map $f : X \to Y$ s.t. $f \circ dX \subseteq dY$
- The category of d-spaces is denoted by **dTop**

D-spaces

Examples

- Any subspace of $\mathbb{R}^n$ with increasing paths.
- The d-complex plane $\mathbb{C}$ (*i.e.* the d-paths are $t \mapsto \rho(t)e^{i\theta(t)}$ with $\rho \geqslant 0$ and θ, ρ nondecreasing)
- The d-Riemann sphere Σ (*i.e.* the d-paths are $t \mapsto \rho(t)e^{i\theta(t)}$ with $\rho \in \mathbb{R}_+ \cup \{+\infty\}$ and θ, ρ nondecreasing)
- The d-circle $\mathbb{S}^1$ as a d-subspace of $\mathbb{C}$ (or Σ).
- The direction of a product of d-spaces is given by paths whose projections are d-paths.

The fundamental category

of a d-space (X, dX)

A d-homotopy (resp. anti-d-homotopy) from a dipath γ to a dipath δ is a d-map h of some rectangle $[a,b] \times [c,d]$ (resp. $[a,b] \times [c,d]^{op}$) such that Uh is a homotopy from $U\gamma$ to $U\delta$.

An elementary homotopy is a finite concatenation of d-homotopies and anti-d-homotopies.

Then γ and δ are d-homotopic when there exists an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \theta'$ for some reparametrizations $\theta : [a,b] \to \mathrm{dom}(\gamma)$ and $\theta' : [a,b] \to \mathrm{dom}(\delta)$. We write $\gamma \sim \delta$.

The relation $\sim$ defines a congruence over PX, the path category of X, and the fundamental category of X, denoted by $\overrightarrow{\pi_1}X$, is the quotient $PX/\sim$. This construction extends to a functor

$$\overrightarrow{\pi_1} : \mathbf{dTop} \to \mathbf{Cat}$$

Compactification

- A compactification of a topological space X is an embedding $k : X \hookrightarrow K$ such that K is Hausdorff compact and $k(X)$ is dense in K.
- Some examples:
  - $]0,1[^n \hookrightarrow [0,1]^n$ and $]0,1[^n \hookrightarrow \mathbb{S}^{n+1}$
  - The Stone-Čech compactification for Tychonoff spaces given by β, the left adjoint to **CHaus** $\hookrightarrow$ **Top** (*e.g.* $\beta\mathbb{R}$ has $2^{2^{\aleph_0}}$ elements).
  - The Alexandroff compactification for locally compact Hausdorff spaces adds one point ∞ and its neighborhoods are the complement of the compact subspaces (*e.g.* $\mathbb{R}^n \cup \{\infty\} \cong \mathbb{S}^{n+1}$).
  - The Freudenthal compactification for σ-locally compact, locally connected, Hausdorff spaces with finitely many connected components, which adds a new point for each end of the space (*e.g.* $\mathbb{R} \cup \{\text{ends}\} \cong \mathbb{R} \cup \{-\infty, +\infty\} \cong [0,1]$ and $\mathbb{R}^n \cup \{\text{ends}\} \cong \mathbb{S}^{n+1}$).

Compactifying d-spaces

A problem

Suppose X and K are d-spaces such that

- $k : UX \hookrightarrow UK$ is a compactification
- The direction dK of K is the least one that makes the preceding inclusion a d-map (*i.e.* that contains $k \circ dX$)

Problem: No path starting or ending at a point of $K \setminus X$ is a d-path (*e.g.* $]0,1[\hookrightarrow [0,1]$).
Consequence: $\overrightarrow{\pi_1}K \cong \overrightarrow{\pi_1}X \sqcup \overrightarrow{\pi_1}(K \setminus X)$ the second one being discrete.
A solution: A d-space is said to be complete when

- for all d-maps $\delta : \mathbb{R} \to X$, if both following limits exist then δ extends to a d-map $\overline{\delta} : \mathbb{R} \cup \{-\infty, +\infty\} \to X$.

$$\lim_{t \to -\infty} \delta(t) \quad \text{and} \quad \lim_{t \to +\infty} \delta(t)$$

dTop$_c$ $\subseteq$ **dTop** the full subcategory whose objects are complete.
A compactification of a complete d-space X is a d-space K s.t. UK is compactification of UX and dK is the least complete direction on UK that contains dX.

Examples

of d-compactifications

- $(\mathbb{R}\times\mathbb{S}^1)\cup\{\text{ends}\}\cong$ the d-Riemann sphere $\cong\mathbb{C}\cup\{\infty\}$
- $(\mathbb{R}\times\mathbb{S}^1)\cup\{\infty\}$ is the d-Riemann sphere with north and south poles identified ... make a picture !

Direction

from a single vector field

Given a vector field f over a manifold $\mathcal{M}$ and a point $x \in \mathcal{M}$, there is a unique maximal integral curve γ that goes through x at time 0 i.e.

$$\gamma(0) = x \quad \text{and} \quad \forall t \in \operatorname{dom}(\gamma), \frac{d\gamma}{dt}(t) = f(\gamma(t))$$

In particular the traces of the maximal integral curves form a partition of $\mathcal{M}$.

Then consider the direction $d\mathcal{M}$ on $\mathcal{M}$ generated by the proper integral curves

$$\left\{\delta \mid \delta = \gamma|_{[a,b]} \text{ for some maximal intergal curve } \gamma \text{ and some compact interval } [a,b] \subseteq \operatorname{dom}(\gamma)\right\}$$

Then $\overrightarrow{\pi_1}(\mathcal{M}, d\mathcal{M})$ is isomorphic with a disjoint union of copies of $\{0\}$, $(\mathbb{R}, \leqslant)$ and $\overrightarrow{\pi_1}\mathbb{S}^1$.

Direction

from several vector fields

Given an n-uple of vector fields $f_1, \ldots, f_k$ over a manifold $\mathcal{M}$, consider for all points $x \in \mathcal{M}$, the set

$$F_x := \Big\{ \sum_{i=1}^{k} \lambda_i \cdot f_i(x) \mid \lambda_i \geqslant 0 \text{ for } i = 1, \ldots, k \Big\}$$

as the forward cone of $\mathcal{M}$ at x.

A curve γ is said to be forward (with respect to $f_1, \ldots, f_k$) when its derivative at time t belongs to $F_{\gamma(t)}$ for all $t \in \operatorname{dom}\gamma$:

$$\frac{\partial\gamma}{\partial t}(t) \in F_{\gamma(t)}$$

The d-space generated by the vector fields $f_1, \ldots, f_k$ on the manifold $\mathcal{M}$ is the least direction on $\mathcal{M}$ that contains all the forward curves, it is denoted by $d\mathcal{M}_f$ with f being understood as the set $\{f_1, \ldots, f_k\}$.

Example: $\mathbb{R}^n$ with the constant vector fields $f_k(x) = (\ldots, 0, 1, 0, \ldots)$

Singular points are disconnected

Problem: If $f_1(x) = \cdots = f_n(x) = 0$ at some point x, then x is isolated in $\overrightarrow{\pi_1}(\mathcal{M}, d\mathcal{M})$.

Examples:

- the vector fields $f(t) = 1$ and $g(t) = t$ induce the d-spaces $d\mathbb{R}_f$ and $d\mathbb{R}_g$ and $\overrightarrow{\pi_1}(d\mathbb{R}_f) \cong (\mathbb{R}, \leqslant)$ and $\overrightarrow{\pi_1}(d\mathbb{R}_g) \cong (\mathbb{R}_- \setminus \{0\}, \leqslant) \sqcup \{0\} \sqcup (\mathbb{R}_+ \setminus \{0\}, \leqslant)$
- if Σ is equipped with the vector fields $f_1(z) = z$ and $f_2(z) = z \cdot e^{\frac{i\pi}{2}}$ then

$$\overrightarrow{\pi_1}\mathbb{C} \cong \left(\overrightarrow{\pi_1}\mathbb{S}^1 \times (\mathbb{R}, \leqslant)\right) \sqcup \{0\} \sqcup \{\infty\}$$

As before we consider the complete direction generated by the forward curves.

Direction from an *n*-uple of vector fields

vs *n*-join of the directions for each vector field

The collection of (complete) directions form a complete lattice and one easily sees that

$$d\mathcal{M}_{f_1} \vee \cdots \vee d\mathcal{M}_{f_n} \subseteq d\mathcal{M}_f$$

problem: The example of $\mathbb{R}^n$ with the constant vector fields $f_k(x) = (\ldots, 0, 1, 0, \ldots)$ proves that the converse inclusion does not hold.

One can fix it by considering the d-spaces X such that for all paths γ,
if for all open subsets U, all $[a, b] \subseteq \gamma^{-1}(U)$ there exists a d-path δ from $\gamma(a)$ to $\gamma(b)$ such that $\mathrm{img}(\delta) \subseteq U$,
then γ is a d-path.
Such a d-space is said to be filled.

Conjecture: If $d\mathcal{M}_f$ is defined as the least complete filled d-space containing the forward curves, then

$$d\mathcal{M}_{f_1} \vee \cdots \vee d\mathcal{M}_{f_n} = d\mathcal{M}_f$$

Pospace atlases

Fajstrup, Goubault, and Raußen (1998)

A pospace is a topological space X together with a closed parital order on it (Nachbin (1948)).
The underlying space UX of a pospace X is Hausdorff.

A pospace atlas on a Hausdorff space X is a family $\mathcal{U}$ of pospace such that:

- the collection $\{UW \mid W \in \mathcal{U}\}$ is an open covering of UX, and
- for all $W_0, W_1 \in \mathcal{U}$ and all $x \in W_0 \cap W_1$, there exists $W_2 \in \mathcal{U}$ such that $x \in W_2 \subseteq W_0 \cap W_1$ and

$$\sqsubseteq_{W_0} |_{UW_2} \quad = \quad \sqsubseteq_{W_2} \quad = \quad \sqsubseteq_{W_1} |_{UW_2}$$

The pospace atlases $\mathcal{U}$ and $\mathcal{U}'$ are equivalent when their union is still a pospace atlas.

A local pospace is an equivalence class of pospace atlases.

Local pospaces

Fajstrup, Goubault, and Raußen (1998)

Every equivalence class has a greatest element (namely the greatest pospace atlas).

A pospace atlas morphism from $\mathcal{U}$ to $\mathcal{U}'$ is a mapping f s.t. for all x and all $W' \in \mathcal{U}'$ containing $f(x)$ there exists $W \in \mathcal{U}$ containing x s.t. $f(W) \subseteq W'$.

If $\mathcal{U}_0 \sim \mathcal{U}_1$ and $\mathcal{U}'_0 \sim \mathcal{U}'_1$ and f is a pospace atlas morphism from $\mathcal{U}_0$ to $\mathcal{U}'_0$, then it is also a pospace atlas morphism from $\mathcal{U}_1$ to $\mathcal{U}'_1$.

The category of local pospaces is denoted by **LpoTop**.

There is an inclusion **LpoTop** $\hookrightarrow$ $\mathbf{dTop}_{\mathrm{cf}}$ in the category of complete filled d-spaces.

Fundamental category of local pospaces

Let X be a local pospace

- A local pospace has no vortex (*i.e.* each point has a neighborhood without d-loop)
- Given a d-loop α at x, α is d-homotopic with the constant path x iff α is the constant path x.
- Conjecture: Given a nonconstant d-loop $\alpha \in \overrightarrow{\pi_1}X(x,x)$, one has $\{\alpha^n \mid n \in \mathbb{N}\} \cong (\mathbb{N}, +, 0)$

Parallelizable manifolds

A parallelization of a manifold $\mathcal{M}$ of dimension n is an n-uple of vector fields $(f_1, \ldots, f_n)$ s.t. for all $x \in \mathcal{M}$, $(f_1(x), \ldots, f_n(x))$ is a vector basis of the tangent space of $\mathcal{M}$ at x namely $T_x\mathcal{M}$.

Conjecture: There exits an open covering $\mathcal{U}$ of $\mathcal{M}$ such that

- for all $W \in \mathcal{U}$, the relation $x \sqsubseteq_W y$ defined by the existence of a forward curve δ from x to y with $\mathrm{img}(\delta) \subseteq W$ defines a pospace on W
- These pospaces induce a local pospace

This local pospace induces $d\mathcal{M}_f$.

A manifold $\mathcal{M}$ is said to be parallelizable when it admits a parallelization.

Parallelizable manifolds

All the linear groups of the tangent spaces $T_x\mathcal{M}$, for $x \in \mathcal{M}$, are gathered in a single manifold called the frame manifold $GL\mathcal{M}$.

Then $GL\mathcal{M}$ "transitively acts" on the parallelizations of $\mathcal{M}$ in the following sense: if g is a section of $GL\mathcal{M}$ then $g \cdot (f_1, \ldots, f_n)$ is another parallelization of $\mathcal{M}$ and all of them can be obtained that way.

Conjecture: Up to isomorphism, the local pospace structure induced by a parallelization of a manifold $\mathcal{M}$ (and therefore $\overrightarrow{\pi_1}\mathcal{M}_f$), does not depend on the specific parallelization.
In that case we can define "the" fundamental category of a parallelizable manifold.

Example: Every Lie group is parallelizable.