Two equivalent ways of directing the spaces

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The *Pakken-Vrijlaten* language

Edsger Wybe Dijkstra (1968)

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#mutex a b

P(a).P(b).V(b).V(a) | P(b).P(a).V(a).V(b)
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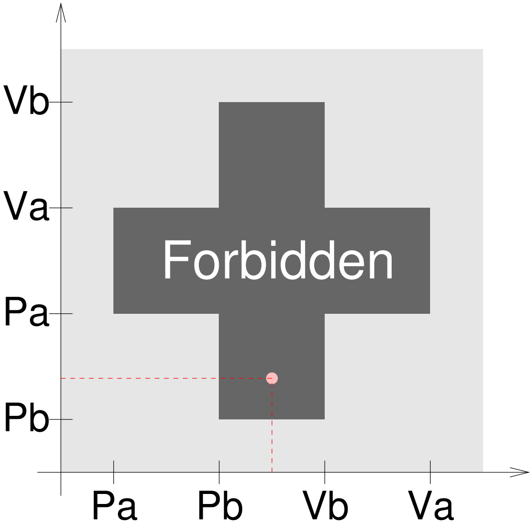
The geometric interpretation of the *PV* language

Scott D. Carson and Paul F. Reynolds (1987)



Partially Ordered Spaces $\mathcal{P}o$

Leopoldo Nachbin (1948,1965)

$$\text{pospace } \overrightarrow{X} : \begin{cases} X & \text{topological space} \\ \sqsubseteq & \text{partial order closed in } X \times X \end{cases}$$

morphism f from $\overrightarrow{X}$ to $\overrightarrow{X'}$: continuous and order preserving maps.

Directed real line $\overrightarrow{\mathbb{R}}$ and the sub-objects of its products.

The directed loops are not allowed in $\mathcal{P}o$.

Locally Ordered Spaces $\mathcal{L}po$

Lisbeth Fajstrup, Eric Goubault and Martin Raußen (1998)

$$\overrightarrow{X} : \begin{cases} X & \text{topological space} \\ \mathcal{U}_X & \text{open covering}^1 \text{ of } X \\ (U, \sqsubseteq_U) & \text{pospace for all } U \in \mathcal{U}_X \end{cases}$$

$$(\sqsubseteq_U)|_{U\cap V} = (\sqsubseteq_V)|_{U\cap V} \quad \text{for all } U, V \in \mathcal{U}_X$$

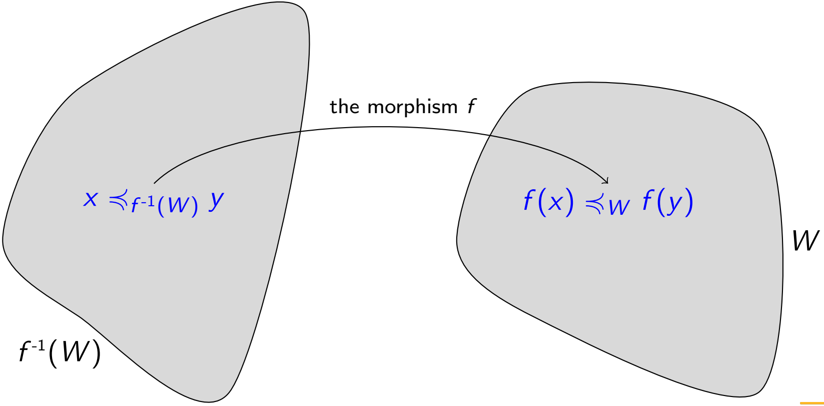
$f : \overrightarrow{X} \to \overrightarrow{X}'$ continuous and locally order preserving maps

i.e. $x \sqsubseteq_U y \;\Rightarrow\; f(x) \sqsubseteq_{U'} f(y)$ for all $U \in \mathcal{U}_X$ and $U' \in \mathcal{U}_{X'}$

such that $U \subseteq f^{-1}(U')$

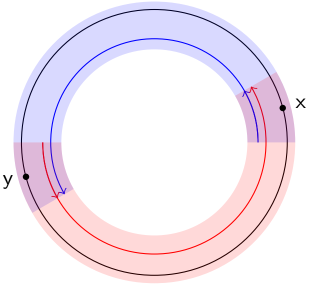
[1] Actually one can even suppose that $\mathcal{U}_X$ is a $\subseteq$-ideal.

Morphisms of $\mathcal{L}po$



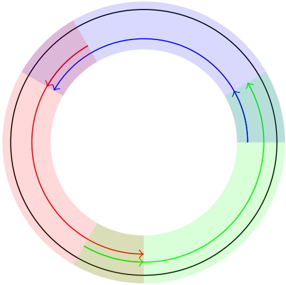
Locally Ordered Spaces

Directed circle $\overrightarrow{\mathbb{S}^1}$ and the sub-objects of its products



$x \sqsubseteq y$ and $y \sqsubseteq x$

Problem



Colimits in $\mathcal{L}po$ are ill-behaved

since $\mathcal{L}po$ does not allow vortex

- $\overrightarrow{\mathbb{C}\backslash\{|z|<1\}}$ has a local pospace structure such that $(r,\theta) \in \overrightarrow{[1,+\infty[} \times \overrightarrow{\mathbb{R}} \longmapsto re^{i\theta} \in \mathbb{C}\backslash\{|z|<1\}$ is a morphism of $\mathcal{L}po$.
- $\mathbb{C}$ has no local pospace structure such that $(r,\theta) \in \overrightarrow{\mathbb{R}}_+ \times \overrightarrow{\mathbb{R}} \longmapsto re^{i\theta} \in \mathbb{C}$ is a morphism of $\mathcal{L}po$.
- The following is a pushout in $\mathcal{L}po$

$$
\begin{array}{ccc}
\overrightarrow{\mathbb{C}\backslash\{|z|<1\}} & \xrightarrow{z \mapsto |z|} & \overrightarrow{\mathbb{R}}_+ \\
\uparrow & & \uparrow \\
\overrightarrow{\{|z|=1\}} & \xrightarrow[!]{} & \{0\}
\end{array}
$$

Streams $\mathcal{S}tr$

Sanjeevi Krishnan (2006)

A stream is a topological space X equiped with a circulation i.e. a mapping defined over the collection Ω_X of open subsets of X

$$W \in \Omega_X \;\mapsto\; \preccurlyeq_W \text{ preorder on } W$$

such that for all $W \in \Omega_X$ and all open coverings $(O_i)_{i\in I}$ of W

$$(W, \preccurlyeq_W) = \bigvee_{i\in I} (O_i, \preccurlyeq_{O_i})$$

$f : \overrightarrow{X} \to \overrightarrow{X}'$ continuous and locally order preserving maps

i.e. $x \preccurlyeq_{f^{-1}(W')} y \;\Rightarrow\; f(x) \preccurlyeq_{W'} f(y)$ for all $W' \in \Omega_{X'}$

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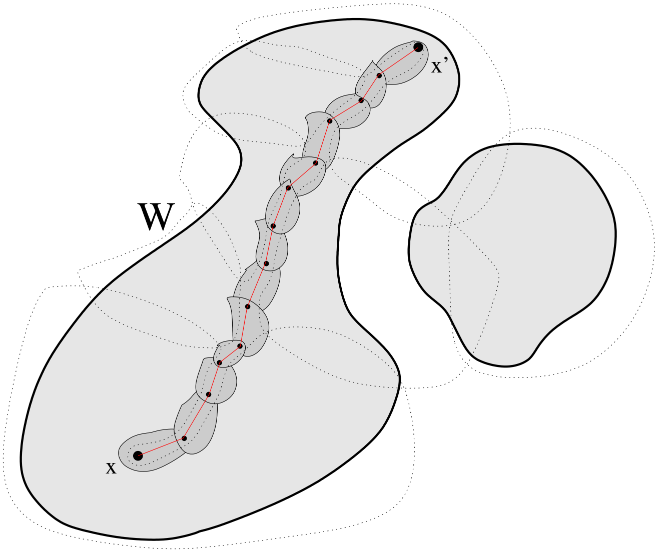
The stream condition



Moore paths and Concatenation

on a topological space X

A Moore path is a continuous mapping $\delta : [0, r] \to X$ ($r \in \mathbb{R}_+$)
Its source $s(\delta)$ and its target $t(\delta)$ are $\delta(0)$ and $\delta(r)$
A subpath of δ is a path $\delta \circ \theta$ where $\theta : [0, r] \to [0, r']$ is increasing
Given a path $\gamma : [0, s] \to X$ such that $s(\gamma) = t(\delta)$ we have the concatenation of δ followed by γ

$$\gamma * \delta : [0, r+s] \longrightarrow X$$

$$t \longmapsto \begin{cases} \delta(t) & \text{if } t \in [0, r] \\ \gamma(t - r) & \text{if } t \in [r, r+s] \end{cases}$$

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The path category functor
from $\mathcal{T}op$ to $\mathcal{C}at$

- The points of X together with the Moore paths of X and their concatenation form a category $P(X)$ whose identities are the paths defined on $\{0\}$
- This construction is functorial $P : \mathcal{T}op \to \mathcal{C}at$

d-Spaces $d\mathcal{T}op$

Marco Grandis (2001)

A topological space X and a collection dX of paths on X s.t.

- dX contains all constant paths
- dX is stable under concatenation
- dX is stable under subpath

$f : \overrightarrow{X} \to \overrightarrow{X}'$ continuous and $f \circ \delta \in dX'$ for all $\delta \in dX$

Examples of d-spaces

- the compact interval $[0, r]$ with all the continuous increasing maps on it : denoted by $\uparrow\mathbb{I}_r$
- the Euclidean circle with paths $t \in [0, r] \mapsto e^{i\theta(t)}$ where θ is any increasing continuous map to $\mathbb{R}$: denoted by $\uparrow\mathbb{S}^1$
- the directed complex plane $\uparrow\mathbb{C}$ with paths $t \in [0, r] \mapsto \rho(t)e^{i\theta(t)}$ where ρ and θ are any increasing continuous map to $\mathbb{R}_+$ and $\mathbb{R}$

Examples of streams

- the compact interval $[0, r]$ with $x \preccurlyeq_U x'$ when $x \leqslant x'$ and $[x, x'] \subseteq U$: denoted by $\overrightarrow{\mathbb{I}}_r$
- the Euclidean circle with $x \preccurlyeq_U x'$ when $x \curvearrowleft x' \subseteq U$ denoted by $\overrightarrow{\mathbb{S}^1}$ [2]

[2] $x \curvearrowleft x'$ denotes the anticlockwise arc from x to x'.

Alternative approaches

- Enriching small categories in $\mathcal{Top}$ (Philippe Gaucher)
- Completing $\mathcal{Lpo}$ by means of Sheaves and Localization (Krzysztof Worytkiewicz)
- Using locally presentable category methods to obtain a subcategory of $d\mathcal{Top}$ in which the notion of "directed universal covering" makes sense (Lisbeth Fajstrup/jiri Rosicky)

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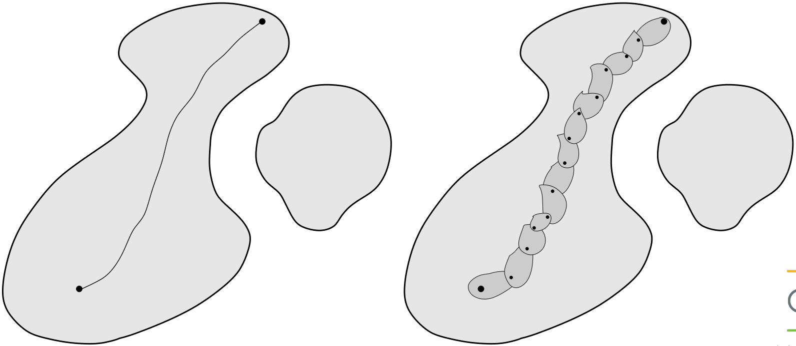
From $d\mathcal{T}op$ to $\mathcal{S}tr$

The functor S

Let (X, dX) be a d-space and put $x \preccurlyeq_U x'$ when there exists $\delta \in dX$ such that

- $\exists t, t' \in$dom(δ) s.t. $t \leqslant t'$, $\delta(t) = x$ and $\delta(t') = x'$
- img$(\delta) \subseteq U$



From $\mathcal{S}tr$ to $d\mathcal{T}op$

The functor D

Let $(X, (\preccurlyeq_U)_{U \in \Omega_X})$ be a stream and consider the following collection of paths on the underlying space of X

$$\bigcup_{r \in \mathbb{R}_+} \mathcal{S}tr[\overrightarrow{\mathbb{I}}_r, X]$$

Theorem (Sanjeevi Krishnan)

$$\big(S : d\mathcal{T}op \to \mathcal{S}tr\big) \dashv \big(D : \mathcal{S}tr \to d\mathcal{T}op\big)$$

Denote the unit and the co-unit by η and ε

The cores of $\mathcal{S}tr$ and $d\mathcal{T}op$

- Let $\overline{\mathcal{S}tr}$ be the full subcategory of $\mathcal{S}tr$ whose collection of objects is
$$\{S(X) \mid X \text{ d-space}\}$$
- Let $\overline{d\mathcal{T}op}$ be the full subcategory of $d\mathcal{T}op$ whose collection of objects is
$$\{D(X) \mid X \text{ stream}\}$$

By restricting the codomains of S and D we have the functors $S' : d\mathcal{T}op \to \overline{\mathcal{S}tr}$ and $D' : \mathcal{S}tr \to \overline{d\mathcal{T}op}$

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Some objects of $d\mathcal{T}op$ and $\mathcal{S}\bar{t}r$

Directed versions of some usual spaces

- Compact Interval : $S(\uparrow\mathbb{I}_1) = \overrightarrow{\mathbb{I}}_1$ and $\uparrow\mathbb{I}_1 = D(\overrightarrow{\mathbb{I}}_1)$
- Hypercubes : $S\big((\uparrow\mathbb{I}_1)^n\big) = (\overrightarrow{\mathbb{I}}_1)^n$ and $D\big((\overrightarrow{\mathbb{I}}_1)^n\big) = (\uparrow\mathbb{I}_1)^n$ for all $n \in \mathbb{N}$
- Euclidean Circle : $S(\uparrow\mathbb{S}^1) = \overrightarrow{\mathbb{S}^1}$ and $\uparrow\mathbb{S}^1 = D(\overrightarrow{\mathbb{S}^1})$
- Complex plane : $S(\uparrow\mathbb{C}) = \overrightarrow{\mathbb{S}^1}$ and $\uparrow\mathbb{S}^1 = D(\overrightarrow{\mathbb{C}})$
- Riemann Sphere : $S(\uparrow\Sigma) = \overrightarrow{\Sigma}$ and $\uparrow\Sigma = D(\overrightarrow{\Sigma})$

Properties

- The natural transformations $\eta * D$, $S * \eta$, $D * \varepsilon$ and $\varepsilon * S$ are identities ($S \dashv D$ is an idempotent adjunction), in particular $S \circ D \circ S = S$ and $D \circ S \circ D = D$
- the adjoint pair $S \dashv D$ induces a pair of isomorphisms $(\overline{S}, \overline{D})$

$$\overline{S} \circ \overline{D} = id_{\overline{\mathcal{S}tr}} \qquad \overline{D} \circ \overline{S} = id_{\overline{d\mathcal{T}op}}$$

More properties

- $d\overline{\mathcal{T}op}$ is a mono and epi reflective subcategory of $d\mathcal{T}op$: the reflector being $\overline{D} \circ S'$
- $\mathcal{S}\overline{tr}$ is a mono and epi coreflective subcategory of $\mathcal{S}tr$: the coreflector being $\overline{S} \circ D'$
- $d\overline{\mathcal{T}op}$ and $\mathcal{S}\overline{tr}$ are complete and cocomplete
- the following diagrams commute

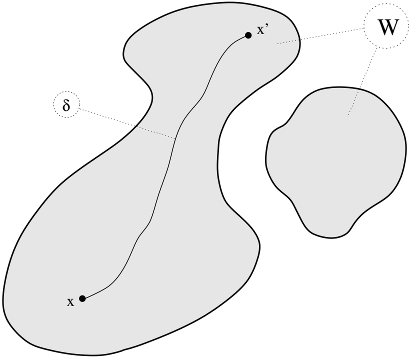
$$\begin{array}{ccc} \mathcal{S}tr & \xrightarrow{D} & d\mathcal{T}op \\ \uparrow & & \downarrow{\scriptstyle \overline{D}\circ S'} \\ \mathcal{S}\overline{tr} & \xrightarrow[\overline{D}]{} & d\overline{\mathcal{T}op} \end{array} \qquad \begin{array}{ccc} \mathcal{S}tr & \xleftarrow{S} & d\mathcal{T}op \\ {\scriptstyle \overline{S}\circ D'}\downarrow & & \uparrow \\ \mathcal{S}\overline{tr} & \xleftarrow[\overline{S}]{} & d\overline{\mathcal{T}op} \end{array}$$

Describing the coreflector $\overline{S} \circ D'$

Let X be a stream and UX its underlying space

$$\text{For all } W \in \Omega_{UX} \text{ we have } x \preccurlyeq_W^{(\overline{S}\circ D'(X))} x' \text{ iff}$$

$$\exists \delta \in \mathcal{S}tr[\overrightarrow{\mathbb{I}}_1, X] \text{ s.t. } s(\delta) = x,\ t(\delta) = x' \text{ and } \mathrm{img}(\delta) \subseteq W$$



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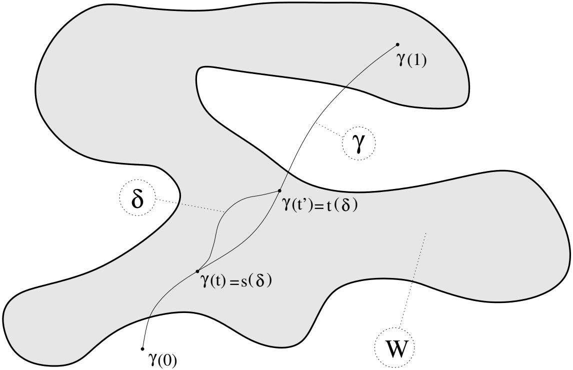
Describing the reflector $\overline{D} \circ S'$

Let X be a d-space and UX its underlying space

Given a path $\gamma \in \mathcal{T}op[[0, r], UX]$, $\gamma \in d(\overline{D} \circ S'(X))$ iff
$\forall W \in \Omega_{UX}$, $\forall t \leqslant t'$ s.t. $[t, t'] \subseteq \gamma^{-1}(W)$, $\exists \delta \in dX$ s.t.
$s(\delta) = \gamma(t),\ t(\delta) = \gamma(t')$ and $\mathrm{img}(\delta) \subseteq W$



Realization of cubical sets

in a cocomplete category $\mathcal{C}$

Let $K \in c\mathcal{S}et$ the category of cubical sets, we have

$$K \cong \underset{\substack{\square^n \to K \\ \text{in } c\mathcal{S}et \downarrow K}}{\operatorname{colim}} \square^n$$

Let $\mathcal{C}$ be a cocomplete category and $F : \square \to \mathcal{C}$, we define the geometric realisation in $\mathcal{C}$ as

$$\upharpoonleft K \downharpoonright_{\mathcal{C}} = \underset{\substack{\square^n \to K \\ \text{in } c\mathcal{S}et \downarrow K}}{\operatorname{colim}} F(\square^n)$$

Directed Geometric Realization
of cubical sets

- Taking $F(\square^n) = (\overrightarrow{\mathbb{I}}_1)^n$ we have $|-|_{\mathcal{S}tr}$ and $|-|_{\mathcal{S}\overline{tr}}$
- Taking $F(\square^n) = (\uparrow\mathbb{I}_1)^n$ we have $|-|_{d\mathcal{T}op}$ and $|-|_{d\overline{\mathcal{T}op}}$

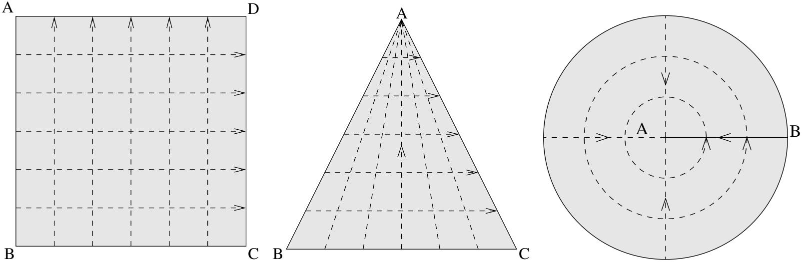
Relations

between the adjunction $S \dashv D$ and the directed geometric realizations

- for all $K \in \mathcal{cSet}$ $\overline{S}\left(\upharpoonleft K \downharpoonleft_{\overline{\mathcal{dTop}}}\right) = \upharpoonleft K \downharpoonleft_{\overline{\mathcal{Str}}}$ and $\overline{D}\left(\upharpoonleft K \downharpoonleft_{\overline{\mathcal{Str}}}\right) = \upharpoonleft K \downharpoonleft_{\overline{\mathcal{dTop}}}$
- for all $K \in \mathcal{cSet}$ $S\left(\upharpoonleft K \downharpoonleft_{\mathcal{dTop}}\right) = \upharpoonleft K \downharpoonleft_{\mathcal{Str}}$ and $\upharpoonleft K \downharpoonleft_{\overline{\mathcal{Str}}} = \upharpoonleft K \downharpoonleft_{\mathcal{Str}}$

$$S\left(\operatorname*{colim}_{\substack{\square^n \to K \\ \text{in } \mathcal{cSet}\downarrow K}} (\uparrow\mathbb{I}_1)^n\right) = \operatorname*{colim}_{\substack{\square^n \to K \\ \text{in } \mathcal{cSet}\downarrow K}} S\left((\uparrow\mathbb{I}_1)^n\right) = \operatorname*{colim}_{\substack{\square^n \to K \\ \text{in } \mathcal{cSet}\downarrow K}} (\overrightarrow{\mathbb{I}}_1)^n$$

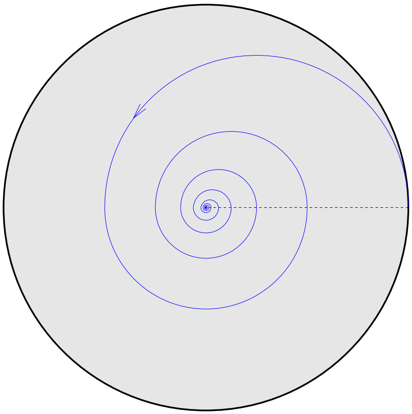
Realizing a vortex
from the directed square



The downward spiral

There may be cubical sets K such that

$D(\upharpoonleft K \downharpoonright_{\mathcal{S}tr}) \neq \upharpoonleft K \downharpoonright_{d\mathcal{T}op}$ and $\upharpoonleft K \downharpoonright_{\overline{d\mathcal{T}op}} \neq \upharpoonleft K \downharpoonright_{d\mathcal{T}op}$



Concrete category over $\mathcal{T}op$

Let $\mathcal{I}$ be the collection of all sub-intervals of $\mathbb{R}$ (including $\emptyset$ and the singletons)

- An adjunction $F \dashv U : \mathcal{C} \to \mathcal{T}op$ with U faithful.
- A family of objects $(\mathbb{I}_\iota)_{\iota \in \mathcal{I}}$ indexed by $\mathcal{I}$,
  for $r \in \mathbb{R}_+$ the notation $\mathbb{I}_r$ stands for $\mathbb{I}_{[0,r]}$.

Axiom 1
Existence of Hypercubes

For all n-uple $(\iota_1, \dots, \iota_n)$ of elements of $\mathcal{I}$ the n-fold product $\mathbb{I}_{\iota_1} \times \cdots \times \mathbb{I}_{\iota_n}$ exists and we suppose that $F(\{0\}) = \mathbb{I}_0$.
By convention the 0-fold product is the terminal object of $\mathcal{C}$.

Axiom 2

Coherence with respect to the product order of $\mathbb{R}^n$

For all continous order[3] preserving $\beta : \iota_1 \times \cdots \times \iota_n \to \iota'_1 \times \cdots \times \iota'_{n'}$ there exists a morphism $\alpha \in \mathcal{C}[\mathbb{I}_{\iota_1} \times \cdots \times \mathbb{I}_{\iota_n}, \mathbb{I}_{\iota'_1} \times \cdots \times \mathbb{I}_{\iota'_{n'}}]$ s.t.

$$U(\alpha) = \beta$$

As a consequence, for all $\iota \in \mathcal{I}$ we have $U(\mathbb{I}_\iota) = \iota$.

Given $x, r, s \in \mathbb{R}_+$ such that $x + r \leqslant s$, $i^s_{x,r} : \mathbb{I}_r \to \mathbb{I}_s$ is the unique morphism of $\mathcal{C}$ such that $U(i^s_{x,r})$ is the following translation.

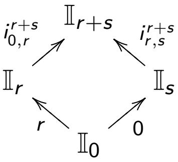
$$\begin{array}{ccc} [0, r] & \longrightarrow & [0, s] \\ t & \longmapsto & x + t \end{array}$$

[3] Here we mean product order.

Axiom 3

Concatenation via Pushout

The following diagram is a pushout square in $\mathcal{C}$



and for all $(\mathbb{I}_{r_1}, \ldots, \mathbb{I}_{r_n})$ and all $i \in \{1, \ldots, n\}$, it is preserved by the following endofunctor of $\mathcal{C}$

$$X \mapsto \mathbb{I}_{r_1} \times \cdots \times \mathbb{I}_{r_{i-1}} \times X \times \mathbb{I}_{r_{i+1}} \times \cdots \times \mathbb{I}_{r_n}$$

A structure satisfying the axioms 1, 2 and 3 is called a framework for fundamental category of fffc

Examples

of frameworks for fundamental categories

The categories $\mathcal{Top}$, $\mathcal{Po}$, $d\mathcal{Top}$, $\mathcal{Str}$, $d\overline{\mathcal{Top}}$ and $\overline{\mathcal{Str}}$ with their obvious forgetful functor and intervals are fffc's.

We associate each object X of a given fffc $\mathcal{C}$ with the following d-space

$$\bigcup_{r \in \mathbb{R}_+} \mathcal{C}[\mathbb{I}_r, X]$$

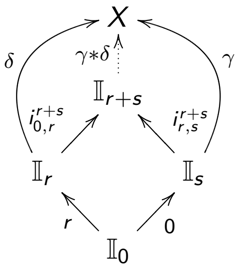
thus defining a faithful functor from $\mathcal{C}$ to $d\mathcal{Top}$

The category of directed paths of an object X of $\mathcal{C}$ denoted by $\overrightarrow{P}(X)$

Objects and Identities : $\mathcal{C}[\mathbb{I}_0, X]$ (points of X)

Morphisms : $\bigcup_{r\in\mathbb{R}_+} \mathcal{C}[\mathbb{I}_r, X]$ (directed paths on X)

Concatenation :

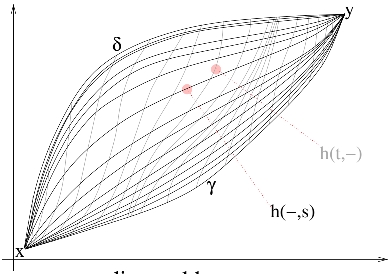


The construction is functorial and there is a natural embeding of $\overrightarrow{P}(X)$ into $P\circ U(X)$

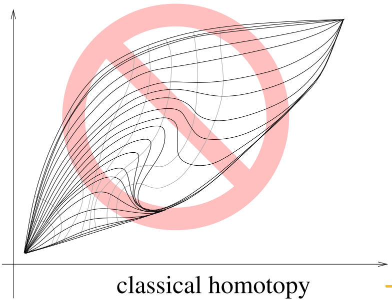
Directed Homotopy

between γ and δ two directed paths on X

Write $\gamma \preccurlyeq \delta$ when there exists two constant paths c_γ, c_δ and some $h \in \mathsf{C}[\mathbb{I}_r \times \mathbb{I}_\rho, X]$ such that $U(h)$ is a usual homotopy from $U(c_\gamma * \gamma)$ to $U(c_\delta * \delta)$ [4]



directed homotopy



classical homotopy

[4] The constant paths are needed so we can relate two directed paths whose domains of definition differ.

$\overrightarrow{\pi_1}(X)$
The Fundamental Category of X

Denote by $\sim$ for the equivalence relation generated by $\preccurlyeq$,
it yields to a congruence over $\overrightarrow{P}(X)$.

Then define the fundamental category of X as the quotient

$$\overrightarrow{\pi_1}(X) := \overrightarrow{P}(X)/\sim$$

The construction is functorial

$$\overrightarrow{\pi_1} : \mathcal{C} \to \mathcal{Cat}$$

and there is a natural morphism from $\overrightarrow{\pi_1}(X)$ to $\pi_1 \circ U(X)$

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The Seifert-Van Kampen Theorem

generic version

We call inclusion any $\alpha \in \mathcal{C}[X, Y]$ s.t. $U(\alpha) = U(X) \hookrightarrow U(Y)$.

Then $U(\mathring{X})$ is the topological interior of $U(X) \subseteq U(Y)$.

Theorem (Seifert - Van Kampen)

A square of inclusions such that $U(\mathring{X}_1)$ and $U(\mathring{X}_2)$ cover $U(X)$ and $U(X_0) = U(X_1) \cap U(X_2)$ is sent to pushout squares of $\mathcal{C}at$ by the functors $\overrightarrow{P}$ and $\overrightarrow{\pi_1}$.

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} \overrightarrow{P}(X_0) & \longrightarrow & \overrightarrow{P}(X_1) \\ \downarrow & & \downarrow \\ \overrightarrow{P}(X_2) & \longrightarrow & \overrightarrow{P}(X) \end{array} \qquad \begin{array}{ccc} \overrightarrow{\pi_1}(X_0) & \longrightarrow & \overrightarrow{\pi_1}(X_1) \\ \downarrow & & \downarrow \\ \overrightarrow{\pi_1}(X_2) & \longrightarrow & \overrightarrow{\pi_1}(X) \end{array}$$

Relations

between $S \dashv D$, $\lceil - \rfloor$ and $\overrightarrow{\pi_1}$

- For all topological spaces X, $\overrightarrow{\pi_1}(X)$ is the fundamental groupoid of X
- For all streams X, $\overrightarrow{\pi_1}(D(X)) = \overrightarrow{\pi_1}(X)$
- For all d-spaces X, if there exists a stream X' such that $X = D(X')$, then $\overrightarrow{\pi_1}(S(X)) = \overrightarrow{\pi_1}(X)$
- For all $X \in \mathcal{S}\overline{tr}$ and all $Y \in d\mathcal{T}\overline{op}$ $\overrightarrow{\pi_1}(\overline{D}(X)) = \overrightarrow{\pi_1}(X)$ and $\overrightarrow{\pi_1}(\overline{S}(Y)) = \overrightarrow{\pi_1}(Y)$
- For all cubical sets K following have the same fundamental category : $D(\lceil K \rfloor_{\mathcal{S}tr})$, $\lceil K \rfloor_{\mathcal{S}tr}$, $\lceil K \rfloor_{\mathcal{S}\overline{tr}}$, $S(\lceil K \rfloor_{d\mathcal{T}\overline{op}})$, $\lceil K \rfloor_{d\mathcal{T}\overline{op}}$
- Question : what about $\lceil K \rfloor_{d\mathcal{T}op}$?

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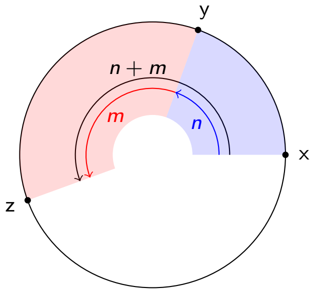
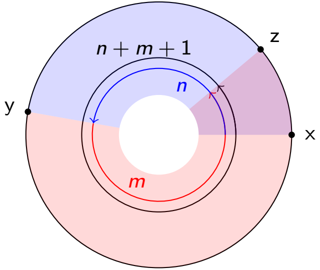
The Fundamental Category of the directed hypercubes

The fundamental category of the directed hypercube $\overrightarrow{\mathbb{I}}_r$ is the product poset $([0, r], \leqslant)^n$.

The Fundamental Category of the Circles

directed or classical





$$\overrightarrow{\pi_1}\left(\overrightarrow{\mathbb{S}^1}\right)[x, y] = \{x\} \times \mathbb{N} \times \{y\}$$

$$\pi_1\left(\mathbb{S}^1\right)[x, y] = \{x\} \times \mathbb{Z} \times \{y\}$$

Define $\omega(x, n, y) := n$

The fundamental category of the directed complex plane

Let $z, z', z'' \in \mathbb{C}$

Define $p : z \in \mathbb{C}\backslash\{0\} \mapsto \frac{z}{|z|} \in \mathbb{S}^1$

$$\overrightarrow{\pi_1}(\overrightarrow{\mathbb{C}})[z, z'] = \begin{cases} \emptyset & \text{if } |z| > |z'| \\ \{\perp_{z'}\} & \text{if } z = 0 \\ \{z\}\times\mathbb{N}\times\{z'\} & \text{if } z \neq 0 \text{ and } |z| \leqslant |z'| \end{cases}$$

$(z, n, z') \circ \perp_z = \perp_{z'}$ i.e. 0 is the initial object of $\overrightarrow{\pi_1}(\overrightarrow{\mathbb{C}})$

$$(z', m, z'') \circ (z, n, z') = \Big(z, \omega\big((pz', m, pz'') \circ (pz, n, pz')\big), z''\Big)$$

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The fundamental category of the directed Riemann sphere

Let $z, z', z'' \in \Sigma$

$$\text{Extend } p : z \in \Sigma \backslash \{0, \infty\} \mapsto \frac{z}{|z|} \in \mathbb{S}^1$$

$$\overrightarrow{\pi_1}(\overrightarrow{\mathbb{C}})[z, z'] = \begin{cases} \emptyset & \text{if } |z| > |z'| \\ \{\bot_{z'}\} & \text{if } z = 0 \\ \{\top_z\} & \text{if } z' = \infty \\ \{z\} \times \mathbb{N} \times \{z'\} & \text{if } z \neq 0 \text{ and } |z| \leqslant |z'| \end{cases}$$

$\bot_\infty = \top_0$

$(z, n, z') \circ \bot_z = \bot_{z'}$ i.e. 0 is the initial object of $\overrightarrow{\pi_1}(\overrightarrow{\Sigma})$

$\top_{z'} \circ (z, n, z') = \top_z$ i.e. ∞ is the terminal object of $\overrightarrow{\pi_1}(\overrightarrow{\Sigma})$

$(z', m, z'') \circ (z, n, z') = \Big(z, \omega\big((pz', m, pz'') \circ (pz, n, pz')\big), z''\Big)$