Two equivalent ways of directing the spaces

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The *Pakken-Vrijlaten* language
Edsger Wybe Dijkstra (1968)

#mutex a b

The geometric interpretation of the $PV$ language
Scott D. Carson and Paul F. Reynolds (1987)
Partially Ordered Spaces $\mathcal{Po}$
Leopoldo Nachbin (1948, 1965)

\[
\text{pospace } \overrightarrow{X} : \left\{ \begin{array}{l}
X \quad \text{topological space} \\
\subseteq \quad \text{partial order closed in } X \times X
\end{array} \right.
\]

morphism $f$ from $\overrightarrow{X}$ to $\overrightarrow{X'}$: continuous and order preserving maps.

Directed real line $\overrightarrow{\mathbb{R}}$ and the sub-objects of its products.

The directed loops are not allowed in $\mathcal{Po}$. 
Locally Ordered Spaces $\mathcal{Lpo}$

Lisbeth Fajstrup, Eric Goubault and Martin Raußen (1998)

$\overline{X} : \left\{ \begin{array}{ll}
X & \text{topological space} \\
\mathcal{U}_X & \text{open covering\textsuperscript{1} of } X \\
(U, \sqsubseteq_U) & \text{pospace for all } U \in \mathcal{U}_X \\
\end{array} \right.$

$(\sqsubseteq_U)|_{U \cap V} = (\sqsubseteq_V)|_{U \cap V}$ for all $U, V \in \mathcal{U}_X$

$f : \overline{X} \to \overline{X}'$ continuous and locally order preserving maps

i.e. $x \sqsubseteq_U y \Rightarrow f(x) \sqsubseteq_{U'} f(y)$ for all $U \in \mathcal{U}_X$ and $U' \in \mathcal{U}_X'$ such that $U \subseteq f^{-1}(U')$

\textsuperscript{1}Actually one can even suppose that $\mathcal{U}_X$ is a $\subseteq$-ideal.
Morphisms of $Lpo$

\[ x \preceq_{f^{-1}(W)} y \]

\[ f(x) \preceq_W f(y) \]
Locally Ordered Spaces
Directed circle $\overline{S^1}$ and the sub-objects of its products

$x \sqsubseteq y$ and $y \sqsubseteq x$

Problem
Colimits in $Lpo$ are ill-behaved since $Lpo$ does not allow vortex

- $\mathbb{C}\setminus\{|z| < 1\}$ has a local pospace structure such that $(r, \theta) \in [1, +\infty[ \times \mathbb{R} \mapsto re^{i\theta} \in \mathbb{C}\setminus\{|z| < 1\}$ is a morphism of $Lpo$.

- $\mathbb{C}$ has no local pospace structure such that $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R} \mapsto re^{i\theta} \in \mathbb{C}$ is a morphism of $Lpo$.

- The following is a pushout in $Lpo$

\[
\begin{array}{c}
\mathbb{C}\setminus\{|z| < 1\} \\
\bigcup
\end{array}
\xymatrix{
\mathbb{C}\setminus\{|z| < 1\} \ar[r]^{z \mapsto |z|} \ar[d] & \mathbb{R}_+ \ar[d] \\
\{|z| = 1\} \ar[r] & \{0\}
}
\]
A stream is a topological space $X$ equipped with a circulation i.e. a mapping defined over the collection $\Omega_X$ of open subsets of $X$

$$W \in \Omega_X \mapsto \preceq_W \text{ preorder on } W$$

such that for all $W \in \Omega_X$ and all open coverings $(O_i)_{i \in I}$ of $W$

$$(W, \preceq_W) = \bigvee_{i \in I} (O_i, \preceq_{O_i})$$

$f : \overset{\rightarrow}{X} \to \overset{\rightarrow}{X}'$ continuous and locally order preserving maps

i.e. $x \preceq_{f^{-1}(W')} y \Rightarrow f(x) \preceq_{W'} f(y)$ for all $W' \in \Omega_{X'}$
The stream condition
Moore paths and Concatenation on a topological space $X$

A Moore path is a continuous mapping $\delta : [0, r] \to X \ (r \in \mathbb{R}_+)$

Its source $s(\delta)$ and its target $t(\delta)$ are $\delta(0)$ and $\delta(r)$

A subpath of $\delta$ is a path $\delta \circ \theta$ where $\theta : [0, r] \to [0, r']$ is increasing

Given a path $\gamma : [0, s] \to X$ such that $s(\gamma) = t(\delta)$ we have the concatenation of $\delta$ followed by $\gamma$

$$\gamma \ast \delta : [0, r + s] \longrightarrow X$$

$$t \quad \begin{cases} 
\delta(t) & \text{if } t \in [0, r] \\
\gamma(t - r) & \text{if } t \in [r, r + s]
\end{cases}$$
The path category functor from $\text{Top}$ to $\text{Cat}$

- The points of $X$ together with the Moore paths of $X$ and their concatenation form a category $P(X)$ whose identities are the paths defined on $\{0\}$
- This construction is functorial $P : \text{Top} \rightarrow \text{Cat}$
A topological space $X$ and a collection $dX$ of paths on $X$ s.t.
- $dX$ contains all constant paths
- $dX$ is stable under concatenation
- $dX$ is stable under subpath

$f : \vec{X} \to \vec{X}'$ continuous and $f \circ \delta \in dX'$ for all $\delta \in dX$
Examples of d-spaces

- the compact interval \([0, r]\) with all the continuous increasing maps on it: denoted by \(\uparrow \mathbb{I}_r\)
- the Euclidean circle with paths \(t \in [0, r] \mapsto e^{i\theta(t)}\) where \(\theta\) is any increasing continuous map to \(\mathbb{R}\): denoted by \(\uparrow \mathbb{S}^1\)
- the directed complex plane \(\uparrow \mathbb{C}\) with paths \(t \in [0, r] \mapsto \rho(t)e^{i\theta(t)}\) where \(\rho\) and \(\theta\) are any increasing continuous map to \(\mathbb{R}_+\) and \(\mathbb{R}\)
Examples of streams

- the compact interval \([0, r]\) with \(x \leq_U x'\) when \(x \leq x'\) and \([x, x'] \subseteq U\) : denoted by \(\overrightarrow{I}_r\)
- the Euclidean circle with \(x \leq_U x'\) when \(x \bowtie x' \subseteq U\) denoted by \(\overrightarrow{S^1}\)

\(^2x \bowtie x'\) denotes the anticlockwise arc from \(x\) to \(x'\).
Alternative approaches

- Enriching small categories in $\mathcal{Top}$ (Philippe Gaucher)
- Completing $\mathcal{Lpo}$ by means of Sheaves and Localization (Krzysztof Worytkiewicz)
- Using locally presentable category methods to obtain a subcategory of $\mathcal{dTop}$ in which the notion of “directed universal covering” makes sense (Lisbeth Fajstrup/jiri Rosicky)
Let \((X, dX)\) be a d-space and put \(x \preceq_U x'\) when there exists \(\delta \in dX\) such that

- \(\exists t, t' \in \text{dom}(\delta)\) s.t. \(t \leq t', \delta(t) = x\) and \(\delta(t') = x'\)
- \(\text{img}(\delta) \subseteq U\)
Let \((X, (\ll u)_{u \in \Omega_X})\) be a stream and consider the following collection of paths on the underlying space of \(X\):

\[
\bigcup_{r \in \mathbb{R}_+} \text{Str}[\overrightarrow{I_r}, X]
\]

**Theorem (Sanjeevi Krishnan)**

\[
(S : \text{dTop} \to \text{Str}) \dashv (D : \text{Str} \to \text{dTop})
\]

Denote the **unit** and the **co-unit** by \(\eta\) and \(\varepsilon\)
The cores of $\textbf{Str}$ and $\textbf{dTop}$

- Let $\textbf{Str}$ be the full subcategory of $\textbf{Str}$ whose collection of objects is
  \[ \{ S(X) \mid X \text{ d-space} \} \]

- Let $\textbf{dTTop}$ be the full subcategory of $\textbf{dTTop}$ whose collection of objects is
  \[ \{ D(X) \mid X \text{ stream} \} \]

By restricting the codomains of $S$ and $D$ we have the functors

\[ S' : \textbf{dTTop} \rightarrow \textbf{Str} \quad \text{and} \quad D' : \textbf{Str} \rightarrow \textbf{dTTop} \]
Some objects of $d\text{Top}$ and $\text{Str}$

Directed versions of some usual spaces

- **Compact Interval** : $S(\uparrow \mathbb{I}_1) = \overrightarrow{\mathbb{I}_1}$ and $\uparrow \mathbb{I}_1 = D(\overrightarrow{\mathbb{I}_1})$

- **Hypercubes** : $S((\uparrow \mathbb{I}_1)^n) = (\overrightarrow{\mathbb{I}_1})^n$ and $D((\overrightarrow{\mathbb{I}_1})^n) = (\uparrow \mathbb{I}_1)^n$
  for all $n \in \mathbb{N}$

- **Euclidean Circle** : $S(\uparrow \mathbb{S}^1) = \overrightarrow{\mathbb{S}^1}$ and $\uparrow \mathbb{S}^1 = D(\overrightarrow{\mathbb{S}^1})$

- **Complex plane** : $S(\uparrow \mathbb{C}) = \overrightarrow{\mathbb{S}^1}$ and $\uparrow \mathbb{S}^1 = D(\overrightarrow{\mathbb{C}})$

- **Riemann Sphere** : $S(\uparrow \Sigma) = \overrightarrow{\Sigma}$ and $\uparrow \Sigma = D(\overrightarrow{\Sigma})$
The natural transformations $\eta^* D$, $S^* \eta$, $D^* \varepsilon$ and $\varepsilon^* S$ are identities ($S \dashv D$ is an idempotent adjunction), in particular $S \circ D \circ S = S$ and $D \circ S \circ D = D$

the adjoint pair $S \dashv D$ induces a pair of isomorphisms $(\overline{S}, \overline{D})$

\[
\overline{S} \circ \overline{D} = id_{\overline{Str}} \quad \overline{D} \circ \overline{S} = id_{\overline{dTop}}
\]
More properties

- $d\text{Top}$ is a mono and epi reflective subcategory of $d\text{Top}$: the reflector being $\overline{D} \circ S'$
- $St\bar{r}$ is a mono and epi coreflective subcategory of $Str$: the coreflector being $\overline{S} \circ D'$
- $d\text{Top}$ and $St\bar{r}$ are complete and cocomplete
- the following diagrams commute

\[\begin{array}{ccc}
Str & \xrightarrow{D} & d\text{Top} \\
\downarrow & & \downarrow \overline{D} \circ S' \\
St\bar{r} & \xrightarrow{D} & d\text{Top}
\end{array}\] \hspace{1cm} \[\begin{array}{ccc}
Str & \xleftarrow{S} & d\text{Top} \\
\downarrow & & \downarrow \overline{S} \circ D' \\
St\bar{r} & \xleftarrow{\overline{S}} & d\text{Top}
\end{array}\]
Describing the coreflector $\overline{S} \circ D'$

Let $X$ be a stream and $UX$ its underlying space.

For all $W \in \Omega_{UX}$ we have $x \preceq_{W}^{(\overline{S} \circ D'(X))} x'$ iff

\[ \exists \delta \in Str[\overrightarrow{1}, X] \text{ s.t. } s(\delta) = x, \ t(\delta) = x' \text{ and } \text{img}(\delta) \subseteq W \]
Describing the reflector $\overline{D} \circ S'$

Let $X$ be a d-space and $UX$ its underlying space.

Given a path $\gamma \in \text{Top}[[0, r], UX]$, $\gamma \in d(\overline{D} \circ S'(X))$ iff

$\forall W \in \Omega_{UX}$, $\forall t \leq t'$ s.t. $[t, t'] \subseteq \gamma^{-1}(W)$, $\exists \delta \in dX$ s.t.

$s(\delta) = \gamma(t)$, $t(\delta) = \gamma(t')$ and $\text{img}(\delta) \subseteq W$
Realization of cubical sets
in a cocomplete category $C$

Let $K \in \mathcal{cSet}$ the category of cubical sets, we have

$$K \cong \colim_{\square^n \to K} \square^n$$

Let $C$ be a cocomplete category and $F : \square \to C$, we define the geometric realisation in $C$ as

$$\uparrow K \downarrow_C \equiv \colim_{\square^n \to K} F(\square^n)$$
Directed Geometric Realization
of cubical sets

- Taking $F(\Box^n) = (\rightarrow 1)^n$ we have $1 \dashv \text{Str}$ and $1 \dashv \text{Str}$
- Taking $F(\Box^n) = (\uparrow 1)^n$ we have $1 \dashv \text{dTop}$ and $1 \dashv \text{dTop}$
Relations between the adjunction $S \dashv D$ and the directed geometric realizations

- for all $K \in cSet$ $\overline{S}(\downarrow K \downarrow \text{dTop}) = \downarrow K \downarrow \text{St}$ and $\overline{D}(\downarrow K \downarrow \text{St}) = \downarrow K \downarrow \text{dTop}$
- for all $K \in cSet$ $S(\downarrow K \downarrow \text{dTop}) = \downarrow K \downarrow \text{Str}$ and $\downarrow K \downarrow \text{St}=\downarrow K \downarrow \text{Str}$

\[
S\left(\begin{array}{c}
\Box^n \rightarrow K \\
in cSet \downarrow K
\end{array}\right) = \begin{array}{c}
\text{colim} \\
in cSet \downarrow K
\end{array} (\uparrow \Pi_1)^n = \begin{array}{c}
\text{colim} \\
in cSet \downarrow K
\end{array} S((\uparrow \Pi_1)^n) = \begin{array}{c}
\text{colim} \\
in cSet \downarrow K
\end{array} (\overrightarrow{\Pi}_1)^n
\]
Realizing a vortex
from the directed square
The downward spiral
There may be cubical sets $K$ such that

\[ D(1K|_{Str}) \neq 1K|_{dTop} \text{ and } 1K|_{dTop} \neq 1K|_{dTop} \]
Concrete category over $\textit{Top}$

Let $\mathcal{I}$ be the collection of all sub-intervals of $\mathbb{R}$ (including $\emptyset$ and the singletons).

- An adjunction $F \dashv U : C \to \textit{Top}$ with $U$ faithful.
- A family of objects $(I_\iota)_{\iota \in \mathcal{I}}$ indexed by $\mathcal{I}$, for $r \in \mathbb{R}_+$ the notation $I_r$ stands for $I_{[0,r]}$. 
Axiom 1
Existence of Hypercubes

For all $n$-uple $(\iota_1, \ldots, \iota_n)$ of elements of $\mathcal{I}$ the $n$-fold product
$$\prod_{\iota_1} \times \cdots \times \prod_{\iota_n}$$
exists and we suppose that $F(\{0\}) = \mathbb{I}_0$.

By convention the 0-fold product is the terminal object of $\mathcal{C}$. 
Axiom 2
Coherence with respect to the product order of $\mathbb{R}^n$

For all continuous order-preserving $\beta : \iota_1 \times \cdots \times \iota_n \to \iota'_1 \times \cdots \times \iota'_n$, there exists a morphism $\alpha \in C[\mathbb{I}_{\iota_1} \times \cdots \times \mathbb{I}_{\iota_n}, \mathbb{I}_{\iota'_1} \times \cdots \times \mathbb{I}_{\iota'_n}]$ s.t.

$$U(\alpha) = \beta$$

As a consequence, for all $\iota \in \mathcal{I}$ we have $U(\mathbb{I}_\iota) = \iota$.

Given $x, r, s \in \mathbb{R}_+$ such that $x + r \leq s$, $i^s_{x,r} : \mathbb{I}_r \to \mathbb{I}_s$ is the unique morphism of $C$ such that $U(i^s_{x,r})$ is the following translation.

$$[0, r] \xrightarrow{t} [0, s], \quad x + t$$

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$^3$Here we mean product order.
Axiom 3
Concatenation via Pushout

The following diagram is a pushout square in $C$

\[ \begin{array}{ccc}
  I_{r+1} & \xrightarrow{i_{r+1}} & I_{r} \\
  \downarrow i_{0,r} & & \downarrow i_{r,s} \\
  II_{r} & \xrightarrow{r} & II_{s} \\
  \downarrow r & & \downarrow s \\
  I_{0} & \xrightarrow{0} & II_{0}
\end{array} \]

and for all $(I_{r_1}, \ldots, I_{r_n})$ and all $i \in \{1, \ldots, n\}$, it is preserved by the following endofunctor of $C$

\[ X \mapsto I_{r_1} \times \cdots \times I_{r_i-1} \times X \times I_{r_i+1} \times \cdots \times I_{r_n} \]

A structure satisfying the axioms 1, 2 and 3 is called a framework for fundamental category of fFFC.
The categories $\text{Top}$, $\text{Po}$, $d\text{Top}$, $\text{Str}$, $d\overline{\text{Top}}$ and $\overline{\text{Str}}$ with their obvious forgetful functor and intervals are fffc’s.

We associate each object $X$ of a given fffc $C$ with the following d-space

$$\bigcup_{r \in \mathbb{R}_+} C[I_r, X]$$

thus defining a faithful functor from $C$ to $d\text{Top}$
The category of directed paths of an object $X$ of $\mathcal{C}$ denoted by $\overrightarrow{P}(X)$

Objects and Identities: $\mathcal{C}[\mathbb{I}_0, X]$ (points of $X$)

Morphisms: $\bigcup_{r \in \mathbb{R}^+} \mathcal{C}[\mathbb{I}_r, X]$ (directed paths on $X$)

Concatenation:

The construction is functorial and there is a natural embedding of $\overrightarrow{P}(X)$ into $P \circ U(X)$
Directed Homotopy between $\gamma$ and $\delta$ two directed paths on $X$

Write $\gamma \preceq \delta$ when there exists two constant paths $c_{\gamma}$, $c_{\delta}$ and some $h \in C[\mathbb{I}_r \times \mathbb{I}_\rho, X]$ such that $U(h)$ is a usual homotopy from $U(c_{\gamma} \ast \gamma)$ to $U(c_{\delta} \ast \delta)$ \(^4\)

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\(^4\)The constant paths are needed so we can relate two directed paths whose domains of definition differ.
Denote by \( \sim \) for the equivalence relation generated by \( \asymp \), it yields to a congruence over \( \mathcal{P}(X) \).

Then define the fundamental category of \( X \) as the quotient

\[
\overrightarrow{\pi_1}(X) := \overrightarrow{\mathcal{P}}(X)/\sim
\]

The construction is functorial

\[
\overrightarrow{\pi_1} : C \to \text{Cat}
\]

and there is a natural morphism from \( \overrightarrow{\pi_1}(X) \) to \( \pi_1 \circ U(X) \)
We call inclusion any $\alpha \in \mathcal{C}[X, Y]$ s.t. $U(\alpha) = U(X) \hookrightarrow U(Y)$. Then $U(X)$ is the topological interior of $U(X) \subseteq U(Y)$.

**Theorem (Seifert - Van Kampen)**

A square of inclusions such that $U(X_1)$ and $U(X_2)$ cover $U(X)$ and $U(X_0) = U(X_1) \cap U(X_2)$ is sent to pushout squares of $\mathbf{Cat}$ by the functors $\overrightarrow{P}$ and $\overrightarrow{\pi_1}$.
Relations between $S \dashv D$, $\dashv$ and $\overrightarrow{\pi_1}$

- For all topological spaces $X$, $\overrightarrow{\pi_1}(X)$ is the fundamental groupoid of $X$.
- For all streams $X$, $\overrightarrow{\pi_1}(D(X)) = \overrightarrow{\pi_1}(X)$
- For all d-spaces $X$, if there exists a stream $X'$ such that $X = D(X')$, then $\overrightarrow{\pi_1}(S(X)) = \overrightarrow{\pi_1}(X)$
- For all $X \in Str$ and all $Y \in dTop$
  $\overrightarrow{\pi_1}(D(X)) = \overrightarrow{\pi_1}(X)$ and $\overrightarrow{\pi_1}(S(Y)) = \overrightarrow{\pi_1}(Y)$
- For all cubical sets $K$ following have the same fundamental category: $D(\upharpoonright K\downharpoonright Str)$, $\upharpoonright K\downharpoonright Str$, $\upharpoonright K\downharpoonright St$, $S(\upharpoonright K\downharpoonright dTop)$, $\upharpoonright K\downharpoonright dTop$
- Question: what about $\upharpoonright K\downharpoonright dTop$?
The fundamental category of the directed hypercube $\mathbb{I}_r$ is the product poset $([0, r], \leq)^n$. 
The Fundamental Category of the Circles
directed or classical

\[ \pi_1(\mathbb{S}^1)[x, y] = \{x\} \times \mathbb{N} \times \{y\} \]

\[ \pi_1(\mathbb{S}^1)[x, y] = \{x\} \times \mathbb{Z} \times \{y\} \]

Define \( \omega(x, n, y) := n \)
The fundamental category of the directed complex plane
Let $z, z', z'' \in \mathbb{C}$

Define $p : z \in \mathbb{C}\{0\} \mapsto \frac{z}{|z|} \in S^1$

$$\overrightarrow{\pi_1}(\overrightarrow{\mathbb{C}})[z, z'] = \begin{cases} \emptyset & \text{if } |z| > |z'| \\ \{ \perp z' \} & \text{if } z = 0 \\ \{z\} \times \mathbb{N} \times \{z'\} & \text{if } z \neq 0 \text{ and } |z| \leq |z'| \end{cases}$$

$$(z, n, z') \circ \perp_z = \perp_{z'} \text{ i.e. } 0 \text{ is the initial object of } \overrightarrow{\pi_1}(\overrightarrow{\mathbb{C}})$$

$$(z', m, z'') \circ (z, n, z') = \left( z, \omega((pz', m, pz'') \circ (pz, n, pz')), z'' \right)$$
The fundamental category of the directed Riemann sphere

Let $z, z', z'' \in \Sigma$

Extend $p : z \in \Sigma \setminus \{0, \infty\} \mapsto \frac{z}{|z|} \in S^1$

$$
\overrightarrow{\pi_1}(\overrightarrow{C})[z, z'] = \begin{cases} 
\emptyset & \text{if } |z| > |z'| \\
\{\perp z'\} & \text{if } z = 0 \\
\{\top z\} & \text{if } z' = \infty \\
\{z\} \times \mathbb{N} \times \{z'\} & \text{if } z \neq 0 \text{ and } |z| \leq |z'| 
\end{cases}
$$

$\perp \infty = \top_0$

$(z, n, z') \circ \perp_z = \perp_{z'}$ i.e. $0$ is the initial object of $\overrightarrow{\pi_1}(\overrightarrow{\Sigma})$

$\top_{z'} \circ (z, n, z') = \top_z$ i.e. $\infty$ is the terminal object of $\overrightarrow{\pi_1}(\overrightarrow{\Sigma})$

$(z', m, z'') \circ (z, n, z') = \left(z, \omega((pz', m, pz'') \circ (pz, n, pz')), z''\right)$