Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science 230 (2009) 39–69
Electronic Notes in Theoretical Computer Science
www.elsevier.com/locate/entcs

A Framework for Component Categories

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Abstract

This paper provides further developments in the study of the component categories which have been introduced in [7]. In particular, the component category functor is seen as a left adjoint hence preserves the pushouts. This property is applied to prove a Van Kampen like theorem for component categories. This last point is very important to make effective calculations. The original purpose of component categories is to suitably reduce the size of the fundamental categories which are the directed counterpart of classical fundamental groupoids (see [15]). In concrete examples, the fundamental category is as "big" as $\mathbb{R}$ while the component category is "finitely generated". We take advantage of this fact to define the cohomology of a directed geometrical shape as the cohomology of its component category. The cohomology of small categories is defined in [2] and [1]. Still, in the recent paper [18], the homology of small categories is defined in a very similar way and applied to the study of asynchronous transition systems.

Keywords: algebraic topology, directed algebraic topology, partially ordered spaces, pospaces, local pospaces, Yoneda inversible morphisms, inessential morphisms, weak equivalence subcategory, loop-free category, fundamental category, component category, Van Kampen theorem, fundamental monoid, directed (co)homology

1 Introduction

Given a small category $\mathcal{C}$ and a subcategory Σ of $\mathcal{C}$, we define the **quotient category** $\mathcal{C}/_{\Sigma}$ applying the results developed in [3]. Indeed, the size of $\overrightarrow{\pi_1}(\overrightarrow{X})/_{\Sigma}$ decreases as the one of Σ increases. As one can expect, if $\Sigma = \overrightarrow{\pi_1}(\overrightarrow{X})$ then $\overrightarrow{\pi_1}(\overrightarrow{X})/_{\Sigma}$ is $\{*\}$. Then the **component category** of a pospace $\overrightarrow{X}$ is defined as $\overrightarrow{\pi_1}(\overrightarrow{X})/_{\Sigma}$ where Σ is the **greatest weak-equivalences subcategory** of $\mathcal{C}$ and $\overrightarrow{\pi_1}(\overrightarrow{X})$ the fundamental category of $\overrightarrow{X}$. We have in mind that Σ is made of the dipaths [4] of $\overrightarrow{X}$ [5] along which "no choice is made" so we do not lose information removing them [6].

[1] Thanks to everyone who should be thanked
[2] Email: haucourt@cea.fr
[3] Email: couserid@codept.coinst.coedu
[4] or execution traces from a computer scientist point of view.
[5] seen as the space of states of a computer on which a program runs.
[6] precisely, they are not removed but turned into identities.

doi:10.1016/j.entcs.2009.02.016

The previous construction can be done in a category whose objects are taken in the class of pairs $(\mathcal{C}, \Sigma)$, such a pair is called a system over $\mathcal{C}$, where $\mathcal{C}$ is an object of a subcategory of CAT and $\Sigma \subseteq \mathcal{C}$. The idea is to equip the objects $\mathcal{C}$ of a sub-category of CAT with a sub-poset of the poset of all subcategories of $\mathcal{C}$. Then we define the quotient functor sending $(\mathcal{C}, \Sigma)$ on $\mathcal{C}/_{\Sigma}$. The component category is obtained when Σ is optimal i.e. when the size of $\mathcal{C}/_{\Sigma}$ is minimal without loss of relevant information. Several examples are given, involving different subcategories of CAT, and we define component categories of **pospaces**, **local pospaces** and **d-spaces**. Some proofs of technical points are skipped and the paper is organized in the following way:

(i) Pospaces, local pospaces and d-spaces are defined. Concrete but informal examples are given to make the reader understand what component categories should be.

(ii) Generalized congruences and some related tools are described.

(iii) A general theorem describes a framework in which the component category functor can be defined. As we shall see, this theorem makes the component category functor a left adjoint.

(iv) The previous theorem is applied to define the component category of pospaces, local pospaces and d-spaces. We check that we have obtained what was expected.

(v) Preservation of colimits by component category functor is applied to prove Van Kampen like theorems for component category (instead of fundamental category). Examples are given.

(vi) A form of directed cohomology is defined as the cohomology of the component category.

2 Geometrical intuition of component categories through examples

Component categories first appear in [7] in order to reduce the size of the fundamental category. Pospaces are certainly the simplest model of directed topology one may find.

Definition 2.1 [Pospaces] A **pospace** is a triple $(X, \tau_X, \leq_X)$ where (X, τ_X) is a topological space, $(X, \leq_X)$ is a poset and $\leq_X$ is a closed subset of $(X, \tau_X) \times (X, \tau_X)$. A **dimap** from a pospace $(X, \tau_X, \leq_X)$ to a pospace $(Y, \tau_Y, \leq_Y)$ is a set-theoretic function f from X to Y inducing a continuous map from (X, τ_X) to (Y, τ_Y) and an increasing map from $(X, \leq_X)$ to $(Y, \leq_Y)$. The collection of po-spaces together with dimaps between them form a category denoted POSPC. Isomorphisms of POSPC are called **dihomeomorphisms** and are bijective (one-to-one, onto) dimaps whose inverse is also a dimap. Monomorphisms are one-to-one dimaps.

Note that the caracterization of epimorphisms is much more complicated. The unit segment $\overrightarrow{[0,1]}$ with classical topology and order is a pospace as well as all its

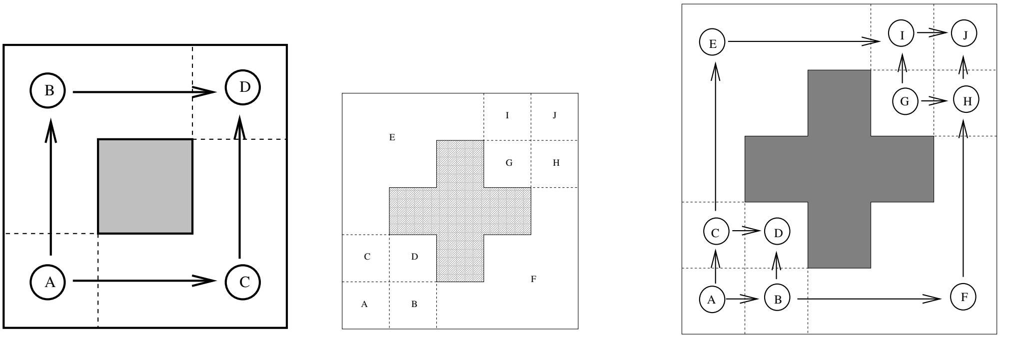


Fig. 1. Square with centered hole and Swiss flag

products with product topology and order. $\overrightarrow{[0,1]}$ is in fact the "standard" example in the sense that it is the cogenerator [7] of the category of compact pospaces [8]. The examples of figure 1 are built up from the unit square with classical topology and order in which "holes" have been dug. In each case the underlying space is divided into "components" which give the set of objects of the component category, their borders are drawn with the dashed lines. Two components sharing a frontier are "neighbours" and we put a unique "prime arrow" between neighbour components, the source component being the left most bottom most one. The morphisms of the component category are "generated" by those "prime arrows". In the first example, the component category is free, in the "swiss flag" example (figure 1) it is not the case any more because we have $\xrightarrow{BD} \circ \xrightarrow{AB} = \xrightarrow{CD} \circ \xrightarrow{AC}$.

The two examples of figure 2 are not dihomeomorphic since their component categories are clearly not isomorphic, it suffices to compare how many morphisms go from the left most bottom most object to the right most upper most one.

Before going further in the study of examples, let me emphasize the fact that we "read" the dipaths of the pospace in its component category. In mathematical terms, we have a lifting property which says that any morphism of the fundamental category is represented by a unique morphism of the component category, conversely, any morphism of the component category represents a morphism of the fundamental one. This property can be found in [7], it is also given for free provided we define the component category by means of generalized congruences, see [3] and the description of the component category given in the rest of the paper. Next examples are 3−dimensional, the unit cube (with classical topology and order) with a centered hole is shown on the left side of figure 3. The right side picture depicts its components, whose border are represented by "walls":

In figure 5, the blue parallelepipeds are holes and the red cube is a deadlock

[7] see [4] for the definition.
[8] i.e. the underlying topological space is compact.

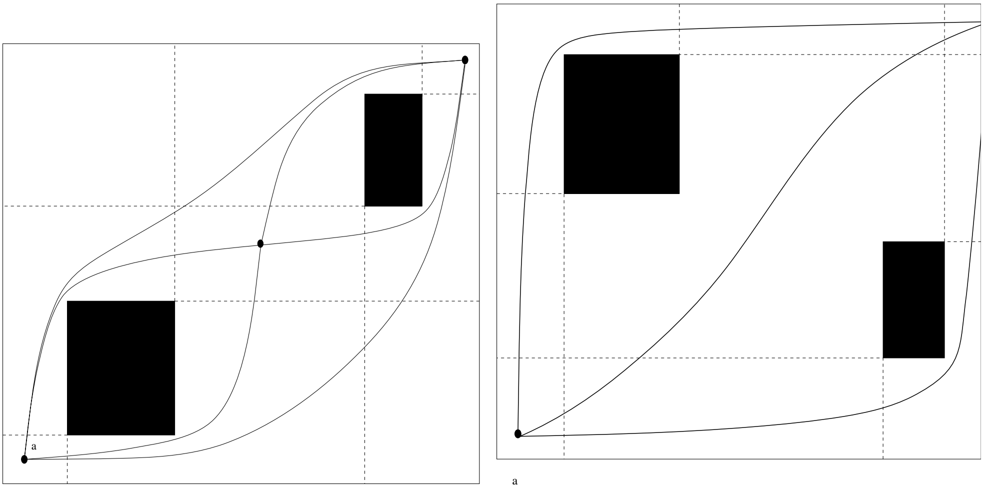


Fig. 2. Two possible configurations of two holes in a square

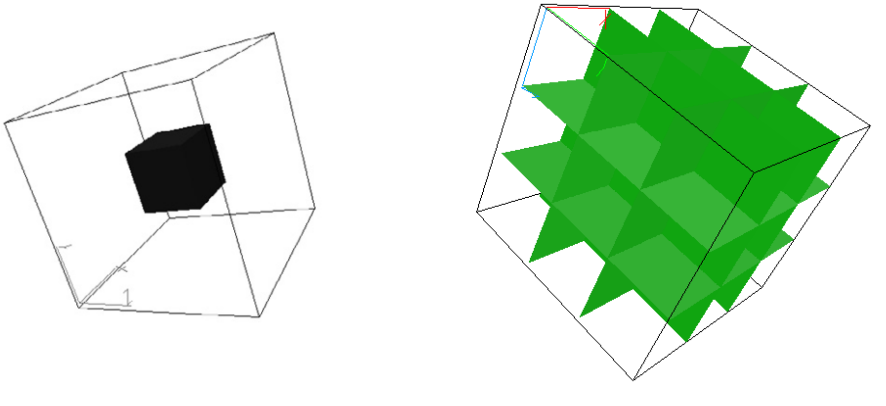
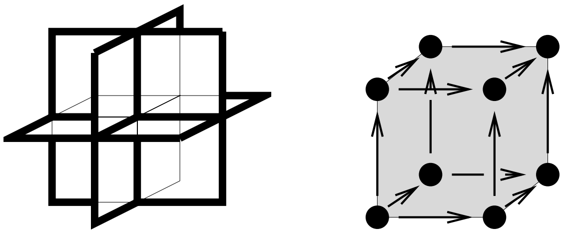


Fig. 3. The cube with a centered hole



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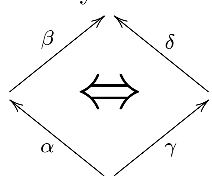
The point at the center of the left side figure is represented by a 3−morphism (the grey filling on the right side figure) in a suitable 3−category built up from the component category.

Fig. 4. Dimensional duality

area, i.e. any dipath entering in it will not go beyond the deep right upper corner of the red cube. On the right side, the corresponding component category is depicted, but the conventions of representation are different, vertices are components, edges are elementary arrows and faces represent relations between morphisms. By the way, this convention of representation induces a "dimensional duality", components are 3−dimensional subpospaces of the cube and they are represented by points, which are 0−dimensional. Faces of the components are 2−dimensional subpospaces and they are represented by "elementary arrows" (hence $1-\textit{dimensional}$) from component to the neighbour it shares the face with. A segment shared by four faces is a 1− dimensional subpospace and is represented by a relation



between the four "elementary arrows" representing the four faces. This relation can be seen as a $2-\textit{dimensional}$ arrow provided we turn the component category into a 2−category adding a trivial groupoid between $\beta \circ \alpha$ and $\delta \circ \gamma$. One can even go further with a point of the pospace shared by six segments all of them being shared by four faces, which makes us reach 3−categories, see figure 4.
This "duality" property has been practically applied by Eric Goubault to write a program which provides a 3-dimensional "view" of the component category of the 3-dimensional pospaces. A detailed description of the method is available in [11]. The right side picture of figure 5 has been produced by this program.

3 Generalized congruences

This section is devoted to generalized congruences which have been formalized in [3].

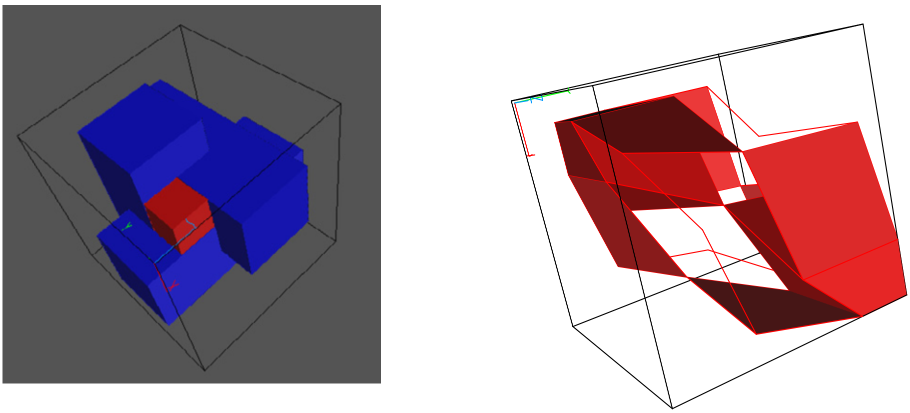


Fig. 5. Three philosophers diner

Definition 3.1 [Generalized Congruences [3]] A generalized congruence on a small category $\mathcal{C}$, is an equivalence relation $\sim_o$ on $Ob(\mathcal{C})$ and a partial equivalence relation $\sim_m$ on $Mo(\mathcal{C})^+$ (the set of all non-empty finite sequences of morphisms of $\mathcal{C}$) satisfying the following conditions ($\cdot$ is the usual concatenation, the α's, β's and γ's range over $Mo(\mathcal{C})$):

- $(\beta_n, ..., \beta_0) \cdot (\alpha_p, ..., \alpha_0) \sim_m (\gamma_q, ..., \gamma_0) \Rightarrow tgt(\alpha_p) \sim_o src(\beta_0)$
- $(\beta_n, ..., \beta_0) \sim_m (\alpha_p, ..., \alpha_0) \Rightarrow tgt(\beta_n) \sim_o tgt(\alpha_p)$ and $src(\beta_0) \sim_o src(\alpha_0)$
- $x \sim_o y \Rightarrow id_x \sim_m id_y$
- $(\beta_n, ..., \beta_0) \sim_m (\alpha_p, ..., \alpha_0)$ and $(\delta_q, ..., \delta_0) \sim_m (\gamma_r, ..., \gamma_0)$ and $tgt(\beta_n) \sim_o src(\delta_0) \Rightarrow$
  $(\delta_q, ..., \delta_0) \cdot (\beta_n, ..., \beta_0) \sim_m (\gamma_r, ..., \gamma_0) \cdot (\alpha_p, ..., \alpha_0)$
- $src(\beta) = tgt(\alpha) \Rightarrow (\beta \circ \alpha) \sim_m (\beta, \alpha)$

Given a functor $f \in \mathsf{CAT}[\mathcal{C}_1, \mathcal{C}_2]$, in other words a morphism of CAT from $\mathcal{C}_1$ to $\mathcal{C}_2$, the generalized congruence $\sim_f := (\sim_o, \sim_m)$ is defined by $x \sim_o y$ iff $f(x) = f(y)$ and given $\sim_o$-composable sequences $(\beta_m, ..., \beta_0)$ and $(\alpha_n, ..., \alpha_0)$, $(\beta_m, ..., \beta_0) \sim_m (\alpha_n, ..., \alpha_0)$ iff $f(\beta_m) \circ ... \circ f(\beta_0) = f(\alpha_n) \circ ... \circ f(\alpha_0)$. For any generalized congruence $\sim$, $(\alpha_n, ..., \alpha_0)$ is a $\sim_o$-composable sequence iff $\forall k \in \{0, ..., n-1\}$, $src(\alpha_{k+1}) \sim_o tgt(\alpha_k)$.

Theorem 3.2 (Quotient Category [3]) *Given $(\sim_o, \sim_m)$ a generalized congruence on a small category $\mathcal{C}$, we define the* **quotient category** *$\mathcal{C}/_\sim$ by*

- $Ob(\mathcal{C}_{/\sim}) := \{[x]_{\sim_o} / x \in Ob(\mathcal{C})\}$
- $src([(\gamma_n, ..., \gamma_0)]_{\sim_m}) = [src(\gamma_0)]_{\sim_o}$ *and* $tgt([(\gamma_n, ..., \gamma_0)]_{\sim_m}) = [tgt(\gamma_n)]_{\sim_o}$
- $[(\beta_n, ..., \beta_0)]_{\sim_m} \circ [(\alpha_p, ..., \alpha_0)]_{\sim_m} = [(\beta_n, ..., \beta_0) \cdot (\alpha_p, ..., \alpha_0)]_{\sim_m}$

Moreover, there is a **quotient** *functor* $Q_\sim : \mathcal{C} \to \mathcal{C}_\sim$*, defined by* $Q_\sim(x) = [x]_{\sim_o}$ *and* $Q_\sim(\gamma) = [\gamma]_{\sim_m}$*.* $Q_\sim$ *enjoys the following universal property, for any functor* $f : \mathcal{C} \to \mathcal{C}_2$*, if* $\sim \subseteq \sim_f$ *then* $\exists! g : \mathcal{C}_{/\sim} \to \mathcal{C}_2$ *making the following diagram commutes*

$$\begin{tikzcd} & \mathcal{C}_{/\sim} \arrow[dr, dashed, "g"] \arrow[d, phantom, "="] & \\ \mathcal{C} \arrow[ur, "Q_\sim"] \arrow[rr, "f"'] & & \mathcal{C}_2 \end{tikzcd}$$

Still, we have the following facts :

- g *is a monomorphism iff* $\sim_f = \sim$
- $\sim_{Q_\sim} = \sim$
- $Q_\sim$ *is an extremal epimorphism*

Lemma 3.3 ([3]) *Generalized congruences on a given small category, ordered by componentwise inclusion form a complete lattice whose meets are componentwise intersections. The total relation which identifies all objects and all non-empty finite sequences of morphisms is a generalized congruence, precisely* $\top$ *of the lattice, while* $(=_{Ob(\mathcal{C})}, =_{Mo(\mathcal{C})^+_\wr})$ *is* $\bot$*.* $Mo(\mathcal{C})^+_\wr$ *is the set of non empty* $\sim_o$*-composable sequences. Thus, for an arbitrary pair of relations* R_o *on* $Ob(\mathcal{C})$ *and* R_m *on* $Mo(\mathcal{C})^+$*, there is a least generalized congruence containing* (R_o, R_m)*.*

4 The Component Category functor

4.1 Loop Free, One Way and Directed categories

Pureness first appears in [7] and is an unavoidable technical tool to study component categories, indeed, good properties of $\mathcal{C}/_\Sigma$ directly depend on pureness of Σ. In ideas, if Σ consists of execution paths [9] along which nothing happens then if $\beta \circ \alpha \in \Sigma$ it is expectable that $\beta, \alpha \in \Sigma$ too. It is also a convenient way to define loop free, one way and directed categories.

Definition 4.1 A sub-category $\mathcal{B}$ of $\mathcal{C}$ is **pure** in $\mathcal{C}$ iff $\forall f, g$ morphisms of $\mathcal{C}$ with $src(g) = tgt(f)$, $g \circ f \in \mathcal{B} \Rightarrow f, g \in \mathcal{B}$.

Pureness is a kind of generalization of convexity in poset framework, indeed, a subposet $(A, \leq_A)$ of a poset $(X, \leq_X)$ is convex iff A' is a pure subcategory of X', where A' and X' are the small categories coresponding to A and X.

[9] it is a computer science point of view.

Definition 4.2 A **loop free category** is a category whose subcategory of endomorphisms is **pure** and **discrete** [10].
A **one way category** is a category whose subcategory of isomorphisms is **pure** and **discrete**. [11]
A **directed category** or **d-category** is a category whose subcategory of isomorphisms is **pure**. Loop free, one way and directed small categories respectively form epi-reflective subcategories of CAT respectively denoted LFCAT, OWCAT, dCAT, see proposition 4.3.

The fundamental category of a pospace is obviously loop free, the one of a local pospace is one way, but it is much harder to prove, and I conjecture that the one of a d-space is directed, it is in fact the reason why I called them "directed", roughly speaking, it comes from the fact that dX is stable under direparametrization (see definition 7.2).

We say that $\mathcal{A}$ is a **reflective** subcategory of $\mathcal{B}$ when

- $\mathcal{A}$ is a full subcategory of $\mathcal{B}$
- for all $A \in Ob(\mathcal{A})$ and $B \in Ob(\mathcal{B})$, if A and B are isomorphic in $\mathcal{A}$ then $A \in Ob(\mathcal{B})$.
- the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{B}$ has a left adjoint.

When the two first points are satisfied, we say that $\mathcal{A}$ is a **replete** subcategory of $\mathcal{B}$. The left adjoint is called the **reflection**. Moreover, if all the elements of the counit of the adjunction are epimorphisms, we say that $\mathcal{A}$ is an **epireflective** subcategory of $\mathcal{B}$.

Proposition 4.3
We have the inclusion functors

$$\mathsf{LFCAT} \hookrightarrow \mathsf{OWCAT} \hookrightarrow \mathsf{dCAT} \hookrightarrow \mathsf{CAT}$$

and the domain of each inclusion depicted on the previous diagram is an epireflective subcategory of its source.

For further details about reflective subcategories, see [4].

Conjecture 4.4 *We have the following commutative diagram*

$$\begin{tikzcd}
\mathsf{POSPC} \arrow[r, hook] \arrow[d, "\pi_1"] & \mathsf{LPOSPC} \arrow[r, hook] \arrow[d, "\pi_1"] & \mathsf{dSPC} \arrow[d, "\pi_1"] \\
\mathsf{LFCAT} \arrow[r, hook] & \mathsf{OWCAT} \arrow[r, hook] & \mathsf{dCAT}
\end{tikzcd}$$

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[10] i.e. for all diagram $\begin{tikzcd} \cdot \arrow[r, bend left, "\alpha"] & \cdot \arrow[l, bend left, "\beta"] \end{tikzcd}$ *in* $\mathcal{C}$, α and β are identities. Hence $\mathcal{C}$ has no "loops", whence the name.

[11] A one way category might have loops, but each loop is either clockwise or anticlockwise never both at the time.

4.2 Weak Equivalences Subcategory

Next materials are directly related to the choice of a Σ such that $\mathcal{C}/_\Sigma$ is the component category of $\mathcal{C}$. As we shall see, all the rest of the subsection, in particular the existence of a non empty weak equivalences subcategory, holds for any directed category $\mathcal{C}$. Then the component category of a pospace/local pospace/directed space, is defined as the component category of its fundamental category.

4.2.1 Yoneda inversible morphisms, Left/Right extension properties and Weak Equivalences Subcategories

Definition 4.5 [[7]] Let $\mathcal{C}$ be a category. A morphism σ of $\mathcal{C}$ is said to be **Yoneda revertible** iff $\forall x \in Ob(\mathcal{C}), (\mathcal{C}[x, src(\sigma)] \neq \emptyset \Rightarrow \gamma \in \mathcal{C}[x, src(\sigma)] \mapsto \sigma \circ \gamma \in \mathcal{C}[x, tgt(\sigma)])$ is bijective and $\forall y \in Ob(\mathcal{C}), (\mathcal{C}[tgt(\sigma), y] \neq \emptyset \Rightarrow \gamma \in \mathcal{C}[tgt(\sigma), y] \mapsto \gamma \circ \sigma \in \mathcal{C}[src(\sigma), y])$ is bijective.

Definition 4.5 is closely related to representable functors of $\mathcal{C}$ and Yoneda's lemma (see [4]), however, the restriction $\forall x \in Ob(\mathcal{C}), (\mathcal{C}[x, src(\sigma)] \neq \emptyset$... and $\forall y \in Ob(\mathcal{C}), (\mathcal{C}[tgt(\sigma), y] \neq \emptyset$... cannot be removed, otherwise, a Yoneda inversible morphism would necessarily be an isomorphism which is silly for loop free and one way categories whose only isomorphisms are identities. From a computer science point of view, the subtle difference between Yoneda inversible morphisms and isomorphisms give a theoretical method for deadlock detection, but we will not develop this remark here. In all examples given in section 2, any dipath joining two points of the same component gives rise to a Yoneda inversible morphism of the fundamental category.

Lemma 4.6 *Let $\mathcal{C}$ be any (small) category, x, y objects of $\mathcal{C}$ and $\sigma_1, \sigma_2 \in \mathcal{C}[x, y]$ Yoneda inversible, $\exists! f_1, f_2 \in Iso(\mathcal{C})[y, y], \sigma_2 = f_1 \circ \sigma_1, \sigma_1 = f_2 \circ \sigma_2$ and $\exists! g_1, g_2 \in Iso(\mathcal{C})[x, x], \sigma_2 = \sigma_1 \circ g_1, \sigma_1 = \sigma_2 \circ g_2$.*

Proof. $\mathcal{C}[y, y] \neq \emptyset$ hence by definition of Yoneda inversible applied to σ_1, $\exists! f_1 \in \mathcal{C}[y, y]$ such that $\sigma_2 = f_1 \circ \sigma_1$. Exchanging σ_1 and σ_2, $\exists! f_2 \in \mathcal{C}[y, y]$ such that $\sigma_1 = f_2 \circ \sigma_2$. In particular, $\sigma_2 = f_1 \circ (f_2 \circ \sigma_2) = (f_1 \circ f_2) \circ \sigma_2$ and $\sigma_1 = f_2 \circ (f_1 \circ \sigma_1) = (f_2 \circ f_1) \circ \sigma_1$, but, by definition of Yoneda inversible, id_y is the only morphism of $h \in \mathcal{C}[y, y]$ such that $\sigma_2 = h \circ \sigma_2$. It is also only morphism of $h \in \mathcal{C}[y, y]$ such that $\sigma_1 = h \circ \sigma_1$. It follows that $f_1 \circ f_2 = f_2 \circ f_1 = id_y$ i.e. $f_1, f_2 \in Iso(Mo(\mathcal{C}))$. It works the same way for the g's. □

Corollary 4.7 *Let $\mathcal{C}$ be a (small) category such that $Iso(\mathcal{C})$ is dicrete, then given x, y objects of $\mathcal{C}$, $\mathcal{C}[x, y] \cap \{Yoneda\ inversibles\}$ is either $\emptyset$ or a singleton.*

Remark 4.8 Any isomorphism is Yoneda inversible morphism, and a composition of Yoneda inversible morphisms is a Yoneda inversible morphism. Moreover, if $\mathcal{L}$ is loop-free and σ is a Yoneda inversible morphism of $\mathcal{L}$ then $\mathcal{L}[src(\sigma), tgt(\sigma)] = \{\sigma\}$.

To prove the last point, note that $\gamma \in \mathcal{L}[src(\sigma), src(\sigma)] \longmapsto \sigma \circ \gamma \in \mathcal{L}[src(\sigma), tgt(\sigma)]$ is a bijection. Up to now, this definition has only proved its relevance in loop-free cases. First, we recall from [7] that **the Σ-zigzag connected component of x in $\mathcal{L}$** denoted C_x is the subcategory of $\mathcal{L}$ whose objects are those connected to x by a zigzag of morphisms of Σ and satisfying for all objects y, z of C_x, $C_x[y,z] = \mathcal{L}[y,z] \cap \Sigma$.

Definition 4.9 **R**ight **E**xtension **P**roperty
Σ has the right extension property with respect to $\mathcal{C}$ iff $\forall \gamma : y' \longrightarrow x'$, $\forall \sigma : x \longrightarrow x' \in \Sigma$, $\exists \sigma' : y \longrightarrow y' \in \Sigma$, $\exists \gamma' : y \longrightarrow x$ such that $\sigma \circ \gamma' = \gamma \circ \sigma'$, i.e. the following diagram is commutative:

$$\begin{tikzcd}
 & y \arrow[dl, dashed, "\exists\sigma'\in\Sigma"'] \arrow[dr, dashed, "\exists\gamma'"] & \\
y' \arrow[dr, "\forall\gamma"'] & & x \arrow[dl, "\forall\sigma\in\Sigma"] \\
 & x' &
\end{tikzcd}$$

Left **E**xtension **P**roperty is obtained "dualizing" definition 4.9

Definition 4.10 [Eric Goubault] [12] Let $\mathcal{C}$ be a small category, $\Sigma \subseteq Mo(\mathcal{C})$ is a **WE-subcategory** iff (by definition) Σ is stable under composition (of $\mathcal{C}$) and satisfies

1 $Iso(\mathcal{C}) \subseteq \Sigma \subseteq Yoneda(\mathcal{C})$ [13]

2 Σ is stable under pushouts and pullbacks (with any morphism in $\mathcal{C}$). It means that Σ has both **REP** and **LEP** with respect to $\mathcal{C}$ and further the commutative squares provided by **REP** and **LEP** can be chosen in order to be respectively **pullback** and **pushout** squares in $\mathcal{C}$.

Eric Goubault, in [12], has changed the definition of Weak Equivalences subcategory of [7] [14] replacing left and right extension axiom by pushout/pull back stability axiom, providing an extremely handy tool. Indeed, any WE-subcategory of any small category $\mathcal{C}$ is pure (it will be proved later) and has left and right extension properties (it is obvious). Moreover, if $Iso(\mathcal{C})$ is pure in $\mathcal{C}$ (i.e. $\mathcal{C}$ is directed) then $\mathcal{C}$ has a $\subseteq$-biggest WE-subcategory.

4.2.2 Locale of the Weak Equivalences of a small category

We give several results which will be combined to prove that the collection of WE-subcategories of a small category $\mathcal{C}$ such that $Iso(\mathcal{C})$ is pure in $\mathcal{C}$ forms a <u>locale</u>. We recall that a **locale** is a poset $(L, \leq_L)$ such that $\forall U \subseteq L, U$ has a least upper bound and a greatest lower bound (it is a complete lattice) and $\forall (b_j)_{j \in J} \in L^J\ \forall a \in L,\ a \wedge$

[12] in directed categories framework, this definition is equivalent to Eric Goubault's one.

[13] $Iso(\mathcal{C})$ and $Yoneda(\mathcal{C})$ are subcategories of $\mathcal{C}$ respectively generated by isomorphisms and Yoneda inversible morphisms of $\mathcal{C}$.

[14] definition of [7] was itself inspired by the notion of calculus of fractions, see [8] and [4].

$\left(\bigvee_{j\in J} b_j\right) = \bigvee_{j\in J}(a \wedge b_j)$ (see [5], [19], [9] or [10]). Lemma 4.11 is due to Eric Goubault, it is the reason for definition 12. Indeed, in [7], we had to enforce the pureness of Σ by an axiom, unfortunately, the resulting definition was not "stable" in the sense that the subcategory generated by two pure subcategory is not, in general, pure.

Lemma 4.11 *Let $\mathcal{C}$ be a small category such that $Iso(\mathcal{C})$ is pure in $\mathcal{C}$. Then any WE-subcategory of $\mathcal{C}$ is pure in $\mathcal{C}$.*

Proof. Take $\sigma \in \Sigma$ and $f_1, f_2 \in Mo(\mathcal{C})$ such that $\sigma = f_2 \circ f_1$. By 2^{nd} point of definition 12, we have a $\sigma' \in \Sigma$ and f_1' which form a pushout square and a unique $g \in Mo(\mathcal{C})$ making the push-out diagram

$$\begin{tikzcd}
& \bullet & \\
& \bullet \arrow[u, "g"] & \\
\bullet \arrow[ur, dashed, "f_1'"] \arrow[uur, bend left, "id"] \arrow[rr, phantom, "pushout"] & & \bullet \arrow[ul, dashed, "\sigma'"'] \arrow[uul, bend right, "f_2"'] \\
& \bullet \arrow[ul, "\sigma"] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

commutative. By pureness of $Iso(\mathcal{C})$ in $\mathcal{C}$, f_1' and g are isomorphisms, hence by 1^{st} point of definition 12, belongs to Σ. So, by stability under composition of Σ (definition 12), $f_2 = g \circ \sigma' \in \Sigma$. The same way, using the pull-back (instead of push-out) extension property, one proves that $f_1 \in \Sigma$. Thus Σ is pure in $\mathcal{C}$. □

Lemma 4.12 *Let $\mathcal{C}$ be a small category. If $Iso(\mathcal{C})$ is pure in $\mathcal{C}$ then $Iso(\mathcal{C})$ is a WE-subcategory of $\mathcal{C}$.*

Proof. Stability under composition and 1^{st} point of definition 12 are obviously satisfied since, any isomorphism is Yoneda inversible (remark 4.8) and isomorphisms compose. Let $\sigma \in Iso(\mathcal{C})$ and $f \in Mo(\mathcal{C})$ be, then we have

$$\begin{tikzcd}
& \bullet & \\
\bullet \arrow[ur, dashed, "f\circ\sigma^{-1}"] \arrow[rr, phantom, "pushout"] & & \bullet \arrow[ul, dashed, "id"'] \\
& \bullet \arrow[ul, "\sigma"] \arrow[ur, "f"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \bullet & \\
\bullet \arrow[ur, "\sigma"] \arrow[rr, phantom, "pullback"] & & \bullet \arrow[ul, "f"'] \\
& \bullet \arrow[ul, dashed, "\sigma^{-1}\circ f"] \arrow[ur, dashed, "id"'] &
\end{tikzcd}$$

so the 2^{nd} point is also satisfied. □

Lemma 4.13 *If $(\Sigma_j)_{j\in J}$ is a non empty family of WE subcategories of a small category $\mathcal{C}$ then $\bigcap_{j\in J}\Sigma_j$ is a WE-subcategory of $\mathcal{C}$.*

Proof. $\bigcap_{j\in J}\Sigma_j$ obviously enjoys stability under compose and the 1^{st} point of definition 12. Suppose $\sigma \in \bigcap_{j\in J}\Sigma_j$ and $f \in Mo(\mathcal{C})$ with $src(f) = src(\sigma)$. Take

$j_1, j_2 \in J$, since $\sigma \in \Sigma_{j_1}$ we have a push out square

$$\begin{tikzcd}[column sep=small] & x_1 & \\ {} \arrow[ur, dashed, "f'_1"] & \textit{pushout} & {} \arrow[ul, dashed, "\sigma'_1\in\Sigma_{j_1}"'] \\ & {} \arrow[ul, "\sigma"] \arrow[ur, "f"'] & \end{tikzcd} \quad \text{and also} \quad \begin{tikzcd}[column sep=small] & x_2 & \\ {} \arrow[ur, dashed, "f'_2"] & \textit{pushout} & {} \arrow[ul, dashed, "\sigma'_2\in\Sigma_{j_2}"'] \\ & {} \arrow[ul, "\sigma"] \arrow[ur, "f"'] & \end{tikzcd}$$

because $\sigma \in \Sigma_{j_2}$. By uniqueness (up to isomorphism) of the pushout, we have an isomorphism τ from x_2 to x_1 such that $\sigma'_1 = \tau \circ \sigma'_2$. By 1^{st} point of definition 12 and for τ is an isomorphism, $\tau \in \Sigma_{j_2}$ which is stable under composition (1^{st} point), thus $\sigma'_1 = \tau \circ \sigma'_2 \in \Sigma_{j_2}$. By the same argument, $\forall j \in J, \sigma'_1 \in \Sigma_j$ i.e. $\sigma'_1 \in \bigcap_{j\in J} \Sigma_j$

and we have $\begin{tikzcd}[column sep=small] & {} & \\ {} \arrow[ur, dashed, "f'_1"] & \textit{pushout} & {} \arrow[ul, dashed, "\sigma'_1\in\bigcap_{j\in J}\Sigma_j"'] \\ & {} \arrow[ul, "\sigma\in\bigcap_{j\in J}\Sigma_j"] \arrow[ur, "f"'] & \end{tikzcd}$. The same proof holds for pull-backs. □

Lemma 4.14 *If $(\Sigma_j)_{j\in J}$ is a non empty family of WE subcategories of a small category $\mathcal{C}$ then $\biguplus_{j\in J} \Sigma_j$ is a WE-subcategory of $\mathcal{C}$. Where $\biguplus_{j\in J} \Sigma_j$ is the least sub-category of $\mathcal{C}$ including all the Σ_j's.*

Proof. By definition, $\biguplus_{j\in J} \Sigma_j = \{\sigma_n \circ ... \circ \sigma_1 / for\ n \in \mathbb{N}^*\ \{j_1, ..., j_n\} \subseteq J\ and\ \forall k \in \{1, ..., n\}\sigma_k \in \Sigma_{j_k}\}$, stable under composition and 1^{st} point of definition 12 immediately follows since a composition of Yoneda inversible morphisms (respectively isomorphism) is Yoneda inversible (respectively isomorphism) (see remark 4.8). Take $\sigma_n \circ ... \circ \sigma_1 \in \biguplus_{j\in J} \Sigma_j$ with $n \in \mathbb{N}^*$, $\{j_1, ..., j_n\} \subseteq J$, $\forall k \in \{1, ..., n\}\sigma_k \in \Sigma_{j_k}$

and $f \in Mo(\mathcal{C})$ with $src(\sigma_1) = src(f)$. We have $\begin{tikzcd} {} & & & \\ {} \arrow[u, "f"] \arrow[r, "\sigma_1\in\Sigma_{j_1}"'] & {} \arrow[r, dotted, no head] & {} \arrow[r, "\sigma_n\in\Sigma_{j_n}"'] & {} \end{tikzcd}$. With a finite induction (apply consecutively the 2^{nd} point of definition 12 for $\Sigma_{j_1}, ..., \Sigma_{j_n}$), we

have $\begin{tikzcd} {} \arrow[r, dashed, "\sigma'_1\in\Sigma_{j_1}"] & {} & {} \arrow[r, dashed, "\sigma'_n\in\Sigma_{j_n}"] & {} \\ {} \arrow[u, "f"] \arrow[r, "\sigma_1\in\Sigma_{j_1}"'] \arrow[ur, phantom, "\textit{pushout}"] & {} \arrow[u, dashed, "f_1"'] \arrow[r, dotted, no head] & {} \arrow[u, dashed, "f_{n-1}"] \arrow[r, "\sigma_n\in\Sigma_{j_n}"'] \arrow[ur, phantom, "\textit{pushout}"] & {} \arrow[u, dashed, "f_n"'] \end{tikzcd}$. Now, it is a general fact that a "composition" of

push-out squares is a push-out square (see [4]) hence $\begin{tikzcd}[column sep=huge] {} \arrow[r, dashed, "\sigma'_n\circ...\circ\sigma'_1\in\biguplus_{j\in J}\Sigma_j"] & {} \\ {} \arrow[u, "f"] \arrow[r, "\sigma_n\circ...\circ\sigma_1\in\biguplus_{j\in J}\Sigma_j"'] \arrow[ur, phantom, "\textit{pushout}"] & {} \arrow[u, dashed, "f_n"'] \end{tikzcd}$.

It works analoguously for pull-backs, thus the 2^{rd} point of definition 12 is satisfied.□

Lemma 4.15 *Let $\mathcal{C}$ be a (small) category. If $\mathcal{A}$ is a pure subcategory of $\mathcal{C}$ then for all families $(\mathcal{C}_j)_{j\in J}$ of subcategories of $\mathcal{C}$, $\mathcal{A} \cap \left(\biguplus_{j\in J} \mathcal{C}_j\right) = \biguplus_{j\in J}(\mathcal{A} \cap \mathcal{C}_j)$*

Proof. The inclusion $\mathcal{A} \cap \left(\biguplus_{j\in J} \mathcal{C}_j\right) \supseteq \biguplus_{j\in J}(\mathcal{A} \cap \mathcal{C}_j)$ is always satisfied. Indeed, if f is an element of the right member, then one has $n \in \mathbb{N}^*$, $\{j_1, ..., j_n\} \subseteq J$,

$\forall k \in \{1, ..., n\}\sigma_k \in \mathcal{A} \cap \Sigma_{j_k}$ and $f = \sigma_n \circ ... \circ \sigma_1$. Now $\mathcal{A}$ is a subcategory of $\mathcal{C}$ and in particular $\forall k \in \{1, ..., n\}\sigma_k \in \mathcal{A}$, hence $f \in Mo(\mathcal{A})$. Conversely, suppose that we have $n \in \mathbb{N}^*$, $\{j_1, ..., j_n\} \subseteq J$, $\forall k \in \{1, ..., n\}\sigma_k \in \Sigma_{j_k}$ and $f = \sigma_n \circ ... \circ \sigma_1 \in Mo(\mathcal{A})$, by pureness of $\mathcal{A}$, $\sigma_n, ..., \sigma_1 \in Mo(\mathcal{A})$. Then $\forall k \in \{1, ..., n\}\sigma_k \in \mathcal{A} \cap \Sigma_{j_k}$ and f is an element of the left member. □

Remark 4.16 If $\mathcal{C}$ satisfies the following property: $\forall \gamma_1, \gamma_2 \in Mo(\mathcal{C}), \gamma_2 \circ \gamma_1 = \gamma_2 \Rightarrow \gamma_1 = id$ and $\gamma_2 \circ \gamma_1 = \gamma_1 \Rightarrow \gamma_2 = id$, then the converse of lemma 4.15 is true.

Proof. Take $\gamma_2 \circ \gamma_1 \in Mo(\mathcal{A})$ where $\gamma_2, \gamma_1 \in Mo(\mathcal{C})$. Set $\mathcal{C}_1 := \{\gamma_1\}$, $\mathcal{C}_2 := \{\gamma_2\}$ [15] and apply the distributivity for the family $\{\mathcal{C}_1, \mathcal{C}_2\}$. If $\gamma_1 \notin Mo(\mathcal{A})$ and $\gamma_2 \notin Mo(\mathcal{A})$ then $(\mathcal{A} \cap \mathcal{C}_1) \uplus (\mathcal{A} \cap \mathcal{C}_2) = \emptyset$ while $\mathcal{A} \cap (\mathcal{C}_1 \uplus \mathcal{C}_2) = \{\gamma_2 \circ \gamma_1\}$. If $\gamma_1 \notin Mo(\mathcal{A})$ and $\gamma_2 \in Mo(\mathcal{A})$ then $(\mathcal{A} \cap \mathcal{C}_1) \uplus (\mathcal{A} \cap \mathcal{C}_2) = \{\gamma_2\}$ while $\mathcal{A} \cap (\mathcal{C}_1 \uplus \mathcal{C}_2) = \{\gamma_2, \gamma_2 \circ \gamma_1\}$ and $\gamma_2 \neq \gamma_2 \circ \gamma_1$ by the property of $\mathcal{C}$, precisely, if we had $\gamma_2 = \gamma_2 \circ \gamma_1$, we would have $\gamma_1 = id_{src(\gamma_1)}$ hence $id_{src(\gamma_1)} \in Mo(\mathcal{A})$ because $\mathcal{A}$ is a subcategory of $\mathcal{C}$.
□

The required property is true if $\mathcal{C}$ is a groupoid or a loop-free category. In fact, having $\mathcal{A} \cap \left(\biguplus_{j \in J} \mathcal{C}_j\right) = \biguplus_{j \in J}(\mathcal{A} \cap \mathcal{C}_j)$ is equivalent to the existence of the right adjoint of the functor $\mathcal{A}\cap\_$: $(\{\text{subcategories of } \mathcal{C}\}, \subseteq) \longrightarrow (\{\text{subcategories of } \mathcal{C}\}, \subseteq)$, where the continuous lattice $(\{\text{subcategories of } \mathcal{C}\}, \subseteq)$ is seen as a complete and co-complete small category. The equivalence directly comes from the special adjoint functor theorem . This equivalence is related to the link between locales and complete **Heyting algebras**, see [5] for further details.

Corollary 4.17 *Let $(\Sigma_j)_{j \in J}$ be a family of WE-subcategories of a (small) category $\mathcal{C}$ such that $Iso(\mathcal{C})$ is pure in $\mathcal{C}$ and Σ a WE-subcategory of $\mathcal{C}$. Then $\Sigma \cap \left(\biguplus_{j \in J} \Sigma_j\right) = \biguplus_{j \in J}(\Sigma \cap \Sigma_j)$.*

Proof. By lemma 4.11, Σ is pure in $\mathcal{C}$, the result follows by lemma 4.15. Note that the hypothesis that all the Σ_j's are WE-subcategories is not used in the proof. □

Remark 4.18 $\bigcap$ and $\biguplus$ are associative over the family of subcategories of a small category $\mathcal{C}$.

Theorem 4.19 *Let $\mathcal{C}$ be a small category such that $Iso(\mathcal{C})$ is pure in $\mathcal{C}$ (i.e. $\mathcal{C}$ is directed). Then, the family of WE-subcategories of $\mathcal{C}$ is not empty and, together with $\subseteq$ it forms a <u>locale</u> whose l.u.b. operator is $\biguplus$ and g.l.b operator is $\bigcap$. Moreover, the least element of this locale ("bottom") is $Iso(\mathcal{C})$.*

Proof. Axioms of a locale are given by lemmas 4.13, 4.14 and corollary 4.17.
□

As it is explained in [5] and [19], the notion of locale generalizes the notion of family of open subsets of a topological space, thus, theorem 4.19 gives us a kind of topology over $\mathcal{C}$ as soon as $Iso(\mathcal{C})$ is pure in $\mathcal{C}$. This pureness hypothesis is

[15] an abuse of notation to say that $\mathcal{C}_1$ and $\mathcal{C}_1$ are subcategories of $\mathcal{C}$ respectively generated by γ_1 and γ_2

actually very "natural". Ideologically, if we want to consider an isomorphism of $\mathcal{C}$ as a path that can be run forward, which is the case when $\mathcal{C}$ is a fundamental category, it geometrically makes sense to expect that all its subpaths can also be run forward i.e. are isomorphisms. When this "geometrical" assumption is fulfilled by a small category $\mathcal{C}$, roughly speaking, $\mathcal{C}$ describes the arc-wise connectedness of a "geometrical shape".

4.3 Quotient of a small category by one of its subcategory : $\mathcal{C}/_\Sigma$

Given Σ a subcategory of a small category $\mathcal{C}$, we can define $\mathcal{C}/_\Sigma := \mathcal{C}/_\sim$ where $\sim$ is the least generalized congruence on $\mathcal{C}$ containing

$$(\emptyset, \{(id_{tgt(\sigma)}, \sigma), (\sigma, src_{src(\sigma)})/\sigma \in Mo(\Sigma)\})$$

This definition holds by lemma 3.3.

Theorem 4.20 (Description and universal property of $\mathcal{C}/\Sigma$)
Given a small category $\mathcal{C}$ and $\Sigma \subseteq Mo(\mathcal{C})$, closed under composition (in fact, take Σ a subcategory of $\mathcal{C}$). Let $(\sim_{o,\Sigma}, \sim_{m,\Sigma})$ be the least generalized congruence containing $(\emptyset, \{(id_{tgt(\sigma)}, \sigma), (\sigma, id_{src(\sigma)})/\sigma \in \Sigma\})$. Then $\forall x, y \in Ob(\mathcal{C}), x \sim_{o,\Sigma}$ iff there is a Σ-zig-zag between x and y. $\forall (\beta_n, ..., \beta_0), (\alpha_m, ..., \alpha_0)$ $\sim_{o,\Sigma}$-composable sequences (i.e. $src(\alpha_{i+1}) \sim_{o,\Sigma} tgt(\alpha_i)$ and $src(\beta_{i+1}) \sim_{o,\Sigma} tgt(\beta_i)$), we have $(\beta_n, ..., \beta_0) \sim_{m,\Sigma} (\alpha_m, ..., \alpha_0)$ iff there is a finite sequence of "elementary transformation" from $(\alpha_m, ..., \alpha_0)$ to $(\beta_n, ..., \beta_0)$, where an "elementary transformation" is either

- $(\alpha_n, ..., \alpha_{i+1}, \sigma, \alpha_{i-1}, ..., \alpha_0) \sim^1_{m,\Sigma} (\alpha_n, ..., \alpha_{i+1}, id_{src(\sigma)}$ *or* $id_{tgt(\sigma)}, \alpha_{i-1}, ..., \alpha_0)$ *if* $\sigma \in \Sigma$

  or

- $(\alpha_n, ..., \alpha_{i+2}, \alpha_{i+1}, \alpha_i, \alpha_{i-1}, ..., \alpha_0) \sim^1_{m,\Sigma} (\alpha_n, ..., \alpha_{i+2}, \alpha_{i+1} \circ \alpha_i, \alpha_{i-1}, ..., \alpha_0)$ *if* $src(\alpha_{i+1}) = tgt(\alpha_i)$.

$\mathcal{C}/\Sigma$ is caracterized by the following universal property, $\forall f \in \mathsf{CAT}[\mathcal{C}, \mathcal{C}']$, if $\forall \sigma \in \Sigma, f(\sigma) = id$ then $\exists! g \in \mathsf{CAT}[\mathcal{C}, \mathcal{C}/\Sigma]$ such that

$$\begin{tikzcd}
 & \mathcal{C}/\Sigma \arrow[dr, "g"] \arrow[d, phantom, "commutes"] & \\
\mathcal{C} \arrow[ur, "Q_\Sigma"] \arrow[rr, "f"'] & {} & \mathcal{C}'
\end{tikzcd}$$

Moreover, if $\mathcal{C}_1 \xrightarrow{f} \mathcal{C}_2$ satisfies $f(\Sigma_1) \subseteq \Sigma_2$ then $\exists!\, \mathcal{C}_1/\Sigma_1 \xrightarrow{h} \mathcal{C}_2/\Sigma_2$ making the *following diagram commutes*

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "f"] \arrow[d, "Q_{\Sigma_1}"'] \arrow[dr, phantom, "="] & \mathcal{C}_2 \arrow[d, "Q_{\Sigma_2}"] \\ \mathcal{C}_1/\Sigma_1 \arrow[r, "h"'] & \mathcal{C}_2/\Sigma_2 \end{tikzcd}$$

Where Q_Σ is the quotient functor (refer to theorem 3.2) associated to the generalized congruence induced by Σ. g is also denoted $f_{/\Sigma}$, and in the same stream of notation h is denoted $f_{/\Sigma_1,\Sigma_2}$.

Definition 4.21 (Definition of the Component Category) The **component category of a directed category** $\mathcal{C}$ is defined as $\mathcal{C}/_{\top_{WE(\mathcal{C})}}$ where $\top_{WE(\mathcal{C})}$ is the biggest weak equivalence subcategory of $\mathcal{C}$.Given a pospace/local pospace/directed space X, the component category of X is defined as the component category of $\pi_1(X)$, the fundamental category of X.

It makes sense by theorem 4.19 and 4.3. Remark we have not the functoriality yet. Next theorem establishes a relation between connectedness [16] and component category of the fundamental groupoid of a topological space.

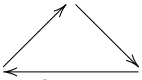
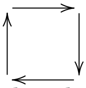
Theorem 4.22 *Let $\mathcal{G}$ be a groupoid, then $Mo(\mathcal{G})$ is the $\subseteq$-biggest WE-subcategory of $\mathcal{G}$. Moreover $\mathcal{G}/_{Mo(\mathcal{G})}$ is (isomorphic to) the set (prcisely a discrete category seen as its set of objects) of zigzag connected components of $\mathcal{G}$. If $\mathcal{G} := \Gamma_1(X,\tau_X)$ the fundamental groupoid of topological space (X,τ_X), then $\mathcal{G}/_{Mo(\mathcal{G})}$ is the set of arc-wise connected components of (X,τ_X).*

Proof. Any morphism of $Mo(\mathcal{G})$ is an isomorphism of $Mo(\mathcal{G})$, thus, by 1^{st} point of definition 12, if $\mathcal{G}$ has a WE-subcategory, it is necessarily $Mo(\mathcal{G})$ which is stable under composition. By remark 4.8, each morphism of a groupoid is Yoneda inversible hence 1^{st} point of definition 12 is satisfied. Finally, it is a general fact that if σ is an isomorphism, then any morphism f such that $src(f) = src(\sigma)$ has a push-out along σ and any morphism g such that $tgt(g) = tgt(\sigma)$ has a pull-back along σ, thus we have the 2^{nd} point of definition 12.

Then each morphism of $\mathcal{G}$ is identified with the identity of its source and target. Two objects x, y of $\mathcal{G}$ are identified iff there is a zigzag between them (note that, since $\mathcal{G}$ is a groupoid, it is equivalent to $\mathcal{G}[x,y] \neq \emptyset$). □

Remark 4.23 Any free category is obviously a one-way category, so we can always define the component category of a free category. For example, the component category of the monoid $(\mathbb{N},+)$ seen as a small category is $(\mathbb{N},+)$. However, $(\mathbb{N},+)$ is not the component category of the free categories generated by the following graphs

[16] in the classical algebraic topology sense.

 ,  ,  ... indeed, there are there own component category. This is a rather severe drawback for the study of directed shapes with loops since, ideologically, we expected that they admit $(\mathbb{N}, +)$ as component category. The last section of the paper is devoted to some ways of fixing this up. Also note that $\overrightarrow{S^1}$ can be seen as a continuous generalization of the previous examples. [17] .

Theorem 4.24 *Let $\mathcal{C}$ be a small category and Σ a wide subcategory of $\mathcal{C}$.* **If** *$\mathcal{C}$ is loop-free and Σ is a pure subcategory of Yoneda inversible morphisms admitting left and right extension properties* **then** *$\mathcal{C}/\Sigma$ is loop-free.* **If** *Σ is pure in $\mathcal{C}$* **then** *$\mathcal{C}/\Sigma$ is one-way.*

Proof. Omitted. □

Theorem 4.25 *For any small category $\mathcal{C}$, $\mathcal{C}/_{\Sigma_{loop}}$ is loop-free. Where Σ_{loop} is the wide subcategory of $\mathcal{C}$ generated by morphisms σ such that $\exists \alpha, \beta, \gamma \in Mo(\mathcal{C})$ forming a loop as follows*

$$\begin{tikzcd} \bullet \arrow[dr, "\alpha"'] & & & \bullet \arrow[lll, bend right, "\gamma"'] \\ & \bullet \arrow[r, "\sigma"'] & \bullet \arrow[ur, "\beta"'] & \end{tikzcd}$$

the above diagram is not required to be commutative.

Proof. Omitted. □

Note that Σ_{loop} is a pure subcategory of $\mathcal{C}$

Definition 4.26 A category is **thin** iff its biggest weak equivalences subcategory is discrete. TLFCAT, TOWCAT and TdCAT are the full sub categories of thin loop-free categories of LFCAT, OWCAT and dCAT.

Conjecture 4.27 *Let $\mathcal{L}$ be a small loop-free category and $\Sigma_{\mathcal{L}}$ the biggest WE-subcategory of $\mathcal{L}$. Then $\mathcal{L}/\Sigma_{\mathcal{L}}$ is thin (see definition 4.26).*

4.4 Functoriality of component categories

Next theorem gives the general framework in which the notion of component category becomes functorial. As pointed out in the abstract, the idea is to equip any small category $\mathcal{C}$ in our scope of interest with a subcategory of distinguished morphisms (called "inessential" in [7]) which are unformally those along which "nothing happens".

Theorem 4.28 (General framework for component category functor) *Let K be a subcategory of CAT and Φ be an "assignment" which gives to each $\mathcal{C}$ object of K a subposet of $(Sb(\mathcal{C}), \subseteq)$ (which is the complete partial order of subcategories of $\mathcal{C}$) with "top" and "bottom" elements. Then we define $\mathsf{K}\Phi$ the category whose objects*

[17] Still, note that the fundamental category of $\pi_1(\overrightarrow{S^1})$ is not free as described in section 2.

are pairs $(\mathcal{C},\Sigma)$ *where* $\mathcal{C}$ *is an object of* K *and* $\Sigma \in \Phi(\mathcal{C})$ *and* $\mathsf{K}\Phi[(\mathcal{C}_1,\Sigma_1),(\mathcal{C}_2,\Sigma_2)] := \{f \in \mathsf{K}[\mathcal{C}_1,\mathcal{C}_2] / \forall \sigma \in \Sigma_1,\ f(\sigma) \neq id \Rightarrow f(\sigma) \in \Sigma_2\}$.

(i) $\forall f \in \mathsf{K}[\mathcal{C}_1,\mathcal{C}_2] \forall \sigma \in \top_{\Phi(\mathcal{C}_1)},\ f(\sigma) \neq id \Rightarrow f(\sigma) \in \top_{\Phi(\mathcal{C}_2)}$

(ii) $\forall f \in \mathsf{K}[\mathcal{C}_1,\mathcal{C}_2] \forall \sigma \in \bot_{\Phi(\mathcal{C}_1)},\ f(\sigma) \neq id \Rightarrow f(\sigma) \in \bot_{\Phi(\mathcal{C}_2)}$

(iii) *For all* $\mathcal{C}$ *object of* K, $\forall \Sigma \in \Phi(\mathcal{C})$
  - (a) $\bot_{\Phi(\mathcal{C})} \subseteq id(\mathcal{C})$
  - (b) $Q_\Sigma : \mathcal{C} \longrightarrow \mathcal{C}/\Sigma$ *is a morphism of* K *(hence* $\mathcal{C}/\Sigma$ *is an object of* K*)*
  - (c) $\forall f \in \mathsf{K}\Phi[(\mathcal{C},\Sigma),(\mathcal{C}',\Sigma')],\ f/_\Sigma : \mathcal{C}/\Sigma \longrightarrow \mathcal{C}'$ *and* $f/_{\Sigma,\Sigma'} : \mathcal{C}/\Sigma \longrightarrow \mathcal{C}'/\Sigma'$ [18] *are morphisms of* K

Then we have

- $(iiia) \Rightarrow (ii)$
- *If* (i) *is satisfied then* R *is well defined and* $U \dashv R$
- *If* (ii) *is satisfied then* L *is well defined and* $L \dashv U$
- *If* (iii) *is satisfied then* $Comp\Phi$ *is well defined and* $Comp\Phi \dashv L$

Where

$$\begin{tikzcd}
\mathsf{K}\Phi \arrow[r, bend left, "U"] \arrow[r, bend right=70, "Comp\Phi"'] & \mathsf{K} \arrow[l, dashed, bend right=70, "R"'] \arrow[l, dashed, bend left, "L"]
\end{tikzcd}$$

U *is the obvious forgetful functor.*

Given $\mathcal{C} \in \mathsf{K}$, $L(\mathcal{C}) := (\mathcal{C}, \bot_{\Phi(\mathcal{C})})$, $R(\mathcal{C}) := (\mathcal{C}, \top_{\Phi(\mathcal{C})})$, $Comp\Phi(\mathcal{C},\Sigma) := \mathcal{C}/_\Sigma$.

Given $f : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$, $R(f)$ *is the induced morphism from* $(\mathcal{C}_1, \top_{\Phi(\mathcal{C}_1)})$ *to* $(\mathcal{C}_2, \top_{\Phi(\mathcal{C}_2)})$ *(i.e.* $U(R(f)) = f$*)* $L(f)$ *is the induced morphism from* $(\mathcal{C}_1, \bot_{\Phi(\mathcal{C}_1)})$ *to* $(\mathcal{C}_2, \bot_{\Phi(\mathcal{C}_2)})$ *(i.e.* $U(L(f)) = f$*) and for all* $f \in \mathsf{K}\Phi[(\mathcal{C}_1\Sigma_1),(\mathcal{C}_2,\Sigma_2)]$, $Comp\Phi(f) := f_{\Sigma_1,\Sigma_2}$.

Proof. $\underline{(iiia) \Rightarrow (ii)}$: Take $f \in \mathsf{K}[\mathcal{C}_1,\mathcal{C}_2]$ and $\sigma \in \bot_{\Phi(\mathcal{C}_1)}$ by (ii), σ is an identity so necessarily $f(\sigma)$ is an identity.

$\underline{(i) \Rightarrow U \dashv R}$: R is well defined because the object part does not raise any problem and (i) is exactly the assumption we need to ensure that morphism of K from $\mathcal{C}_1$ to $\mathcal{C}_2$ induces a morphism of $\mathsf{K}\Phi$ from $(\mathcal{C}_1, \top_{\Phi(\mathcal{C}_1)})$ to $(\mathcal{C}_2, \top_{\Phi(\mathcal{C}_2)})$. The unit of the ajunction is $\eta_{(\mathcal{C},\Sigma)} : (\mathcal{C},\Sigma) \longrightarrow (\mathcal{C}, \top_{\Phi(\mathcal{C})})$, which is a morphism of $\mathsf{K}\Phi$ since $\Sigma \subseteq \top_{\Phi(\mathcal{C})}$. The co-unit is $\varepsilon_\mathcal{C} := id_\mathcal{C}$. Given $f : (\mathcal{C},\Sigma) \longrightarrow (\mathcal{C}',\Sigma')$, put $g := U(f)$, it is clearly the only morphism of K such that $f = g \circ id_\mathcal{C}$ and $f = R(g) \circ \eta_{(\mathcal{C},\Sigma)}$. The naturality of η is obvious.

$\underline{(ii) \Rightarrow L \dashv U}$: L is well defined because the object part does not raise any problem

[18] see lemma 4.20 for notations $f/_\Sigma$ and $f/_{\Sigma,\Sigma'}$.

and (ii) is exactly the assumption we need to ensure that morphism of K from $\mathcal{C}_1$ to $\mathcal{C}_2$ induces a morphism of $\mathsf{K}\Phi$ from $(\mathcal{C}_1, \perp_{\Phi(\mathcal{C}_1)})$ to $(\mathcal{C}_2, \perp_{\Phi(\mathcal{C}_2)})$. The unit of the ajunction is $\eta_{\mathcal{C}} = id_{\mathcal{C}}$ the co-unit is $\varepsilon_{(\mathcal{C},\Sigma)} : (\mathcal{C}, \perp_{\Phi(\mathcal{C})}) \longrightarrow (\mathcal{C}, \Sigma)$ which is a morphism of $\mathsf{K}\Phi$ because $\perp_{\Phi(\mathcal{C})} \subseteq \Sigma$. Given a morphism $f \in \mathsf{K}[\mathcal{C}_1, \mathcal{C}_2]$, setting $g := \varepsilon_{(\mathcal{C},\Sigma)} \circ L(f)$, we have $f = \eta_{\mathcal{C}} \circ L(g)$ i.e. $f = L(g)$.

$\underline{(iii) \Rightarrow CC\Phi \dashv L}$: The object part of $CC\Phi$ is well defined by $(iiib)$, the morphism part of $CC\Phi$ is well defined by $(iiic)$ ($f/_{\Sigma,\Sigma'} : \mathcal{C}/\Sigma \longrightarrow \mathcal{C}'/\Sigma'$ is a morphism of K). The unit of the adjunction is the only morphism $\eta_{(\mathcal{C},\Sigma)} : (\mathcal{C}, \Sigma) \longrightarrow (\mathcal{C}/\Sigma, \perp_{\Phi(\mathcal{C}/\Sigma)})$ such that $U(\eta_{(\mathcal{C},\Sigma)}) = Q_\Sigma$, Q_Σ is in K by $(iiib)$, moreover $\forall \sigma \in \Sigma$, $Q_\Sigma(\sigma)$ is an identity, hence by definition of $\mathsf{K}\Phi$, $\eta_{(\mathcal{C},\Sigma)}$ is in $\mathsf{K}\Phi$. Let $f : (\mathcal{C}, \Sigma) \longrightarrow (\mathcal{C}', \perp_{\Phi(\mathcal{C}')})$ morphism of $\mathsf{K}\Phi$, it follows that $\forall \sigma \in \Sigma$, $f(\sigma) \neq id \Rightarrow f(\sigma) \in \perp_{\Phi(\mathcal{C}')}$, however, by $(iiia)$, $\perp_{\Phi(\mathcal{C}')} \subseteq id(\mathcal{C}')$ then $\forall \sigma \in \Sigma$, $f(\sigma)$ is an identity. So we can apply lemma 4.20, $f/_\Sigma$ is the only morphism of small categories from $\mathcal{C}/\Sigma$ to $\mathcal{C}'$ such that $U(f) = f/_\Sigma \circ Q_\Sigma$. It follows that $f/_\Sigma$ is the only morphism of K (cf $(iiic)$) such that $f = L(f/_\Sigma) \circ \eta_{(\mathcal{C},\Sigma)}$. Naturality of $\eta_{(\mathcal{C},\Sigma)}$ is a consequence of uniqueness property of lemma 4.20. □

Definition of the Component Category Functor by Means of Theorem 4.28

It suffices to set K:=LFCAT and $\Phi(\mathcal{C}) := WE(\mathcal{C})$, (ii) and $(iiia)$ are satisfied because $\perp_{\Phi(\mathcal{C})} := \{id_x / x \in Ob(\mathcal{C})\}$. By theorem 4.24, $\forall \Sigma \in WP(\mathcal{C})$, $\mathcal{C}/\Sigma$ is a loop-free category, since LFCAT is a full sub-category of CAT, $(iiib)$ and $(iiic)$ are also satisfied. Note that (i) is not necessarily satisfied, hence we do not have, in general, the functor R.

We can do the same setting K:=OWCAT and $\Phi(\mathcal{C}) := WE(\mathcal{C})$, (ii) and $(iiia)$ are satisfied because $\perp_{\Phi(\mathcal{C})} := \{id_x / x \in Ob(\mathcal{C})\}$. By theorem 4.24, $\forall \Sigma \in WP(\mathcal{C})$, $\mathcal{C}/\Sigma$ is a one-way category, since OWCAT is a full sub-category of CAT, $(iiib)$ and $(iiic)$ are also satisfied. Once again, (i) is not necessarily satisfied, hence we do not have, in general, the functor R.

For directed categories, things are slightly more intricate, the reason is that the least weak equivalences subcategory of a directed category $\mathcal{C}$ might contain isomorphisms which are not identities, hence $(iiia)$ of theorem 4.28 is not necessarily satisfied. However, by theorem 4.3, OWCAT is a reflective subcategory of dCAT hence, if $L \dashv (\mathsf{dCAT} \hookrightarrow \mathsf{OWCAT})$ we define the compoent categroy functor as $Compo_{OW} \circ L$ where $Compo_{OW}$ is the component category functor defined in the case of one way categories. It is natural, isomorphisms are Yoneda inversible so they have to be turned into identities, the fact that we have to identify them before applying theorem 4.28 is just a technical twist which does not change the underlying philosophy of the method.

4.5 Comments and examples

4.5.1 Is there any relation with weak equivalences in model categories ?

In our context, morphisms of the weak eaquivalence subcategory of $\mathcal{C}$ are to be called weak equivalences. However, these weak equivalences are far from model categories ones. There is a slight analogy between them, due to the pushout/pullback stability property but it does not really go further. In fact, the main difference is that, in model categories, the weak equivalences are (almost) always given by an intrisic property of the morphisms, for example in SPC the category of topological spaces, weak equivalences are continuous maps giving rise to isomorphisms between homotopy groups in all dimensions. This definition just depends on the map and its domains and codomains, in some sense, it is local. On the other hand, weak equivalences in our context are defined as part of a subcategory which is defined in a global way. Let us consider $\overrightarrow{T} := \{(x, y)/0 \le x, y; x + y \le 1\}$ and $\overrightarrow{C} := \{(x, y)/0 \le x, y \le 1\}$ with classical topology and order. It is easy to check that in $\pi_1(\overrightarrow{T})$ as well as in $\pi_1(\overrightarrow{C})$, all morphisms are Yoneda inversible. $\pi_1(\overrightarrow{C})$ clearly has all pushouts and pullbacks hence any morphism of $\pi_1(\overrightarrow{C})$ is a weak equivalence while the only weak equivalences of $\pi_1(\overrightarrow{T})$ are identities. The reason is that for any non identical morphism σ of $\pi_1(\overrightarrow{T})$ one can find a morphism γ so that the (right) extension property is not fulfilled. The last example emphasizes the global and geometric aspect of our weak equivalence definition.

4.5.2 Detailed calculation of the component category of the "L" pospace

The idea is to find morphisms that are "obviously" not weak equivalences and to check the remaining ones form a weak equivalence subcategory. Let L be the pospace depicted in figure 6 with classical topology and order. Given $(x, y) \le (x', y')$ there is, up to dihomotopy, a unique morphism from (x, y) to (x', y'), hence any morphism is Yoneda inversible. Now suppose that a morphism σ crosses the vertical dotted segment, then take γ a morphism which crosses the horizontal one. Clearly, the right extension property is not satisfied by σ. Now it is easy to check that the subcategory made of the morphisms of $\pi_1(L)$ which do not cross any dotted segments are weak equivalences. By the way, note that if a morphism has its source or target <u>exactly</u> on the dotted line, it is still a weak equivalence. This is due to topological properties of components which have been deeper studied in [7].

5 Tool for calculation of component categories

The presentation given above could let the reader think that theorem 4.28 is useless to define component categories, and forgeting the functoriality question, he is right! The point is that, in concrete case, we want to be able to calculate component categories and, in order to do so, we need efficient tools. One of the most classical results towards calculation of fundamental groups, groupoids and categories are Van

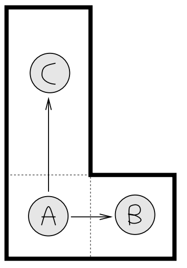


Fig. 6. The "L" pospace

Kampen theorem [19]. The idea of the theorem is as follows, given a geometrical shape X (classical or directed), instead of directly calculating the fundamental object of X, split X into two parts, say A and B whose fundamental objects are known (or at least easier to calculate) then "glue" the fundamental objects of A and B to have the fundamental object of X. If you see a geometrical shape as a program and its fundamental object as an abstract interpretation (see [6]) of this program, then Van Kampen theorem becomes a kind of "compositionality" result. Technical details of Van Kampen theorem are of out of the scope of this paper, so we just give an unformal statement.
In theorems 5.1 and 5.2, $\star$SPC are $\star$CAT are taken by pair according to the following table
Table 19

$\star$SPC	$\star$CAT
POSPC	LFCAT
LPOSPC	OWCAT
dSPC	dCAT

where $\star$SPC is the domain of the fundamental category functor π_1 and $\star$CAT its codomain.

Theorem 5.1 (Van Kampen for fundamental category) *Let $\overrightarrow{X}_1, \overrightarrow{X}_2$ be sub-objects of $\overrightarrow{X}$ (object of $\star$SPC) such that the underlying topological space of $\overrightarrow{X}$ is the union of the interiors*[20] *of the underlying topological spaces of $\overrightarrow{X}_1$ and $\overrightarrow{X}_2$ and let $\overrightarrow{X}_0 := \overrightarrow{X}_1 \cap \overrightarrow{X}_2$. $i_1 : \overrightarrow{X}_0 \hookrightarrow \overrightarrow{X}_1, i_2 : \overrightarrow{X}_0 \hookrightarrow \overrightarrow{X}_2, j_1 : \overrightarrow{X}_1 \hookrightarrow \overrightarrow{X}$ and $j_2 : \overrightarrow{X}_2 \hookrightarrow \overrightarrow{X}$*

[19] there are several versions depending on the framework : see [22] and [24] for groups, [15] for groupoids, [14] for categories.
[20] with respect to the underlying topology of $\overrightarrow{X}$.

the inclusion maps. Then we have the following push-out squares

$$\begin{tikzcd} & \overrightarrow{X} & \\ \overrightarrow{X}_1 \arrow[ur, "j_1"] & \text{push-out} & \overrightarrow{X}_2 \arrow[ul, "j_2"'] \\ & \overrightarrow{X}_0 \arrow[ul, "i_1"] \arrow[ur, "i_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(\overrightarrow{X}) & \\ \pi_1(\overrightarrow{X}_1) \arrow[ur, "\pi_1(j_1)"] & \text{push-out} & \pi_1(\overrightarrow{X}_2) \arrow[ul, "\pi_1(j_2)"'] \\ & \pi_1(\overrightarrow{X}_0) \arrow[ul, "\pi_1(i_1)"] \arrow[ur, "\pi_1(i_2)"'] & \end{tikzcd}$$

respectively in $\star$SPC *and* $\star$CAT*.*

Theorem 5.2 (Van Kampen for component category)
Let $\overrightarrow{X}_1$, $\overrightarrow{X}_2$ be sub-objects of $\overrightarrow{X}_3$ (object of $\star$SPC) such that the underlying topological space of $\overrightarrow{X_3}$ is the union of the interiors of the underlying topological spaces of $\overrightarrow{X}_1$ and $\overrightarrow{X}_2$ and let $\overrightarrow{X}_0 := \overrightarrow{X}_1 \cap \overrightarrow{X}_2$. $i_1 : \overrightarrow{X}_0 \hookrightarrow \overrightarrow{X}_1, i_2 : \overrightarrow{X}_0 \hookrightarrow \overrightarrow{X}_2, j_1 : \overrightarrow{X}_1 \hookrightarrow \overrightarrow{X_3}$ and $j_2 : \overrightarrow{X}_2 \hookrightarrow \overrightarrow{X_3}$ the inclusion maps.

Moreover, we suppose that Σ_1, Σ_2 are WE-subcategories of $\pi_1(\overrightarrow{X}_1)$, $\pi_1(\overrightarrow{X}_2)$, $\pi_1(j_1)(\Sigma_1) \biguplus \pi_1(j_2)(\Sigma_2)$ (also denoted Σ_3) is a WE-subcategory of $\pi_1(\overrightarrow{X}_3)$, $\pi_1(i_1)(\Sigma_0) \subseteq (\Sigma_1)$ and $\pi_1(i_2)(\Sigma_0) \subseteq (\Sigma_2)$ (i.e. $\pi_1(i_1), \pi_1(i_2)$ are morphisms of $\star$CATΦ).

Then i_1, i_2, j_1 and j_2 give rise to i'_1, i'_2, j'_1 and j'_2 morphisms of $\star$CATΦ and we have

$$\begin{tikzcd} & (\pi_1(\overrightarrow{X}_3), \Sigma_3) & \\ (\pi_1(\overrightarrow{X}_1), \Sigma_1) \arrow[ur, "j'_1"] & \begin{array}{c}\text{push out in}\\ \star CAT\Phi\end{array} & (\pi_1(\overrightarrow{X}_2), \Sigma_2) \arrow[ul, "j'_2"'] \\ & (\pi_1(\overrightarrow{X}_0), \Sigma_0) \arrow[ul, "i'_1"] \arrow[ur, "i'_2"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} & Comp\Phi(\pi_1(\overrightarrow{X}_3), \Sigma_3) & \\ Comp\Phi(\pi_1(\overrightarrow{X}_1), \Sigma_1) \arrow[ur, "Comp\Phi(j'_1)"] & \begin{array}{c}\text{push out in}\\ \star CAT\end{array} & Comp\Phi(\pi_1(\overrightarrow{X}_2), \Sigma_2) \arrow[ul, "Comp\Phi(j'_2)"'] \\ & Comp\Phi(\pi_1(\overrightarrow{X}_0), \Sigma_0) \arrow[ul, "Comp\Phi(i'_1)"] \arrow[ur, "Comp\Phi(i'_2)"'] & \end{tikzcd}$$

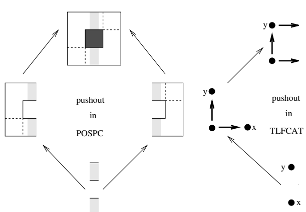
The proof of theorem 5.1 requires three cases, one for each line of table 19. POSPC/LFCAT case can be found in [13]. dSPC/dCAT is available in [14]. In all the cases one might define the fundamental category of a local pospace as the fundamental category of its corresponding directed space see theorem 4.3.

Proof. Theorem 5.1 gives us pushout squares in $\star$SPC and $\star$CAT:

$$\begin{tikzcd} & \overrightarrow{X} & \\ \overrightarrow{X}_1 \arrow[ur, "j_1"] & push{-}out & \overrightarrow{X}_2 \arrow[ul, "j_2"'] \\ & \overrightarrow{X}_0 \arrow[ul, "i_1"] \arrow[ur, "i_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(\overrightarrow{X}) & \\ \pi_1(\overrightarrow{X}_1) \arrow[ur, "\pi_1(j_1)"] & push{-}out & \pi_1(\overrightarrow{X}_2) \arrow[ul, "\pi_1(j_2)"'] \\ & \pi_1(\overrightarrow{X}_0) \arrow[ul, "\pi_1(i_1)"] \arrow[ur, "\pi_1(i_2)"'] & \end{tikzcd}$$

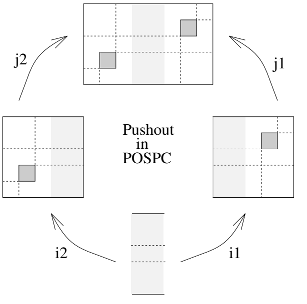
We have to prove that $\pi_1(\overrightarrow{X}_0)$, $\pi_1(\overrightarrow{X}_1)$, $\pi_1(\overrightarrow{X}_2)$ and $\pi_1(\overrightarrow{X}_3)$ respectively equiped with Σ_0, Σ_1, Σ_2 and Σ_3 give rise to a pushout square in $\star\mathsf{CAT}\Phi$. Given $f_1 : (\pi_1(\overrightarrow{X}_1), \Sigma_1) \longrightarrow (\mathcal{L}, \Sigma)$ and $f_1 : (\pi_1(\overrightarrow{X}_2), \Sigma_2) \longrightarrow (\mathcal{L}, \Sigma)$ morphisms of $\star\mathsf{CAT}\Phi$ such that $f_1 \circ i_1 = f_2 \circ i_2$, by hypothesis, $\exists! h : \pi_1(\overrightarrow{X}_3) \longrightarrow \mathcal{L}$ (morphism of $\star$CAT) such that $f_1 = h \circ j_1$ and $f_2 = h \circ j_2$. It remains to see that h gives rise to a morphism of $\star\mathsf{CAT}\Phi$ i.e. $h(\Sigma_3) \subseteq \Sigma$. By hypothesis, $\Sigma_3 = j_1(\Sigma_1) \biguplus j_2(\Sigma_2)$ so any element of Σ_3 can be written $j_2(\alpha_{2n+1}) \cdot j_1(\alpha_{2n}) \cdot \ldots \cdot j_2(\alpha_1) \cdot j_1(\alpha_0)$ where $\forall k \in \{0, ..., n\}, \alpha_{2k} \in \Sigma_1$ and $\alpha_{2k+1} \in \Sigma_2$, so $h(j_2(\alpha_{2n+1}) \cdot j_1(\alpha_{2n}) \cdot \ldots \cdot j_2(\alpha_1) \cdot j_1(\alpha_0)) = (h \circ j_2)(\alpha_{2n+1}) \cdot (h \circ j_1)(\alpha_{2n}) \cdot \ldots \cdot (h \circ j_2)(\alpha_1) \cdot (h \circ j_1)(\alpha_0) = f_2(\alpha_{2n+1}) \cdot f_1(\alpha_{2n}) \cdot \ldots \cdot f_2(\alpha_1) \cdot f_1(\alpha_0) \in \Sigma$ since f_1, f_2 are morphisms of $\star\mathsf{CAT}\Phi$, hence h gives rise to a morphism of $\star\mathsf{CAT}\Phi$ from $(\pi_1(\overrightarrow{X}_3), \Sigma_3)$ to $(\mathcal{L}, \Sigma)$. Thus we have a pushout square in $\star\mathsf{CAT}\Phi$. Now by theorem 4.28, we know that $Comp\Phi$ is a left adjoint hence [21] preserves colimits and, in particular, pushouts. $\square$

Theorem 5.2 does not necessarily give the biggest WE-subcategory of $\pi_1(\overrightarrow{X}_3)$, so one has to guess what this biggest WE-subcategory is in order to choose appropriate Σ_1 and Σ_2, the choice of Σ_0 is not as important, and once Σ_1 and Σ_2 are given, it might be possible to take Σ_0 as the biggest WE-subcategory of $\pi_1(\overrightarrow{X}_0)$ satisfying $\pi_1(i_1)(\Sigma_0) \subseteq (\Sigma_1)$ and $\pi_1(i_2)(\Sigma_0) \subseteq (\Sigma_2)$. A very simple application of theorem 5.2 to calculate the component category of the first example given in section 2.

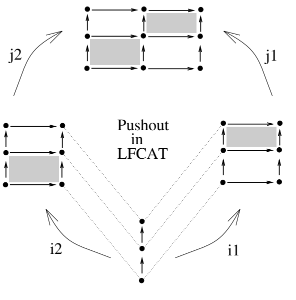


Let us come back to the example of the rectangle with two holes:

[21] general facts of category theory see [4].



which gives, by theorem 5.2



In this figure, rectangle filled with grey color are not commutative. The holes of the geometrical shape are represented by non-commutative squares in the component category.

Applying theorem 5.2 we can also prove that the component category of the cube with a centered cubical hole has 26 objects[22]. It can be represented in $\mathbb{R}^3$ putting an object in the "center" of each vertex, edge and face ($8\ vertices + 12\ edges + 6\ faces = 26\ objects$). Morphisms are generated by arrows from a point to its "closer neighbours in the future", for example those of $(0,0,0)$ are $(0,0,\frac{1}{2}),(0,\frac{1}{2},0)$ and $(\frac{1}{2},0,0)$ while $(1,1,1)$ has no such neighbours. In order to have the hypothesis of theorem 5.2 satisfied, we split the cube into two parts so that, following notation of theorem 5.1, $X_0 :=]\frac{1}{2}-\epsilon,\frac{1}{2}+\epsilon[\times[0,1]\times[0,1]$. It is the analog of the previous example in three dimensions.

[22] geometrically, picture the Rubik's cube, the interior cube is the hole, all other cubes give an object.

6 Towards directed cohomology

In [2] and [1] a cohomology of small categories is presented by means of natural systems of factorization. The idea would be to define the cohomology of a directed geometrical object $\overrightarrow{X}$ as the cohomology of its fundamental category [23]. However, as we have already pointed it out, the fundamental category has often as many objects as $\mathbb{R}$. Still, there is only few of them which is relevant, and finding them amounts to calculate the component category. Thus, the cohomology of $\overrightarrow{X}$ could be defined as the cohomology of the component category of $\overrightarrow{X}$. For example, with this definition, the fourth and fifth examples given in section 2 are distinguished by their first cohomology groups.

In this paragraph, "cubical" pospace means a disjoint union of unit cubes of dimension n in which finitely many parallelepipeds [24] have been dug out. As we have remarked in the previous paragraph, the choice of a "good" natural system is influenced by "good" properties of the small category we want to calculate the cohomology groups.

Definition 6.1 • A morphism γ is said **prime** iff for any morphisms $\gamma_n, ..., \gamma_0$ such that $\gamma = \gamma_n \circ ... \circ \gamma_0, \exists ! i \in \{1, ..., n\}, \gamma_i \neq id$.

- A category $\mathcal{C}$ is **generated by primes** iff any non trivial morphism of $\mathcal{C}$ can be written as a finite composition of prime morphisms.
- A category $\mathcal{C}$ is **homogeneous** iff $\mathcal{C}$ is generated by primes and for all composable sequences of prime morphisms $(\gamma_n, ..., \gamma_0)$ and $({\gamma'}_{n'}, ..., {\gamma'}_0)$ we have $(\gamma_n \circ ... \circ \gamma_0) = ({\gamma'}_{n'} \circ ... \circ {\gamma'}_0) \Rightarrow n = n'$. $n+1$ is the **length** of $\gamma_n \circ ... \circ \gamma_0$.
- A category $\mathcal{C}$ is said **strongly homogeneous** iff $\mathcal{C}$ is generated by primes and $\forall x, y \in Ob(\mathcal{C}) \exists N_{x,y}$ such that for all composable sequences of prime morphisms $(\gamma_n, ..., \gamma_0)$ with $src(\gamma_0) = x$ and $tgt(\gamma_n) = y$ we have $n = N_{x,y}$. In this case, length depends only on src and tgt.
- A category $\mathcal{C}$ is said **bounded** iff the length of the composable sequences of $\mathcal{C}$ whose elements are not trivial are bounded, i.e. $\exists N_{\mathcal{C}} \in \mathbb{N}$ such that for all composable sequences $(\gamma_n, ..., \gamma_0)$ satisfying $\gamma_i \neq id$, we have $n \leq N_{\mathcal{C}}$.
- A category $\mathcal{C}$ is said **weakly bounded** iff $\forall \gamma \in Mo(\mathcal{C}), Max(\{n \in \mathbb{N} / \exists (\alpha_n, ..., \alpha_0) \textit{ such that } \alpha_n \circ ... \circ \alpha_0\}) < +\infty$

The relations existing between these properties are given in the following dia-

[23] it is abusive to write "the" cohomology of a small category because, as far as I know, it depends on the natural system one has put on the small category one wants to calculate "the" cohomology. Hence, it becomes a part of the art to choose a good natural system. In partical cases, the component category of $\overrightarrow{X}$ has good properties which induce an "obviously" interesting natural system.

[24] with faces parallel to the faces of the unit cube.

gram

$$\begin{tikzcd}
 & \text{Free} \arrow[d, Rightarrow] \\
\text{Strongly Homogeneous} \arrow[r, Rightarrow] \arrow[ddd, Rightarrow, bend right] & \text{Homogeneous} \arrow[d, Rightarrow] \\
\text{Bounded} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Weakly Bounded} \arrow[d, Rightarrow] \\
\text{Direct} \arrow[d, Rightarrow] & \text{generated by Primes} \arrow[d, Rightarrow] \\
\text{Loop-free} \arrow[r, Rightarrow] & \text{Iso is discrete} \arrow[d, Rightarrow] \\
 & \text{Skeletal}
\end{tikzcd}$$

Prime morphisms generalize prime numbers, indeed, the monoid $(\mathbb{N}, +)$ seen as a small category has prime morphisms which are exactly the prime numbers. In fact, it is homogeneous by the famous prime number decomposition theorem. In particular, $\pi_1(\overrightarrow{S^1})$ is homogeneous. The notion of direct categories is related to model category theory, see [17] or [16] for further details.

Definition 6.2 A **linear extension** of a small category $\mathcal{C}$ is a functor $f : \mathcal{C} \to \lambda$ such that $\forall \gamma \in Mo(\mathcal{C}), f(\gamma) = id \Rightarrow \gamma = id$ and where λ is an **ordinal** [25] i.e. a poset whose any non empty subset has a minimum.
A **direct category** is a small category having a linear extension. An **inverse category** is a small category whose dual is direct.

Conjecture 6.3 *The component category of a cubical pospace is homogeneous. Moreover, if its underlying space is connected, the component category is bounded.*

In general it is not strongly homogeneous as shown by the right side of figure 2 nor bounded because it is always possible to have a infinite disjoint union of connected cubical pospaces $C_0 \sqcup ... \sqcup C_n \sqcup ...$ such that $\forall n \in \mathbb{N}$, C_n has a composable sequence of prime morphisms of length n. Being homogeneous induces a natural system as follows. Given a small category $\mathcal{C}$, the **category of factorizations** of $\mathcal{C}$ (denoted $F\mathcal{C}$) is given by $Ob(F\mathcal{C}) = Mo(\mathcal{C})$ and $F\mathcal{C}[\alpha, \beta]$ is the collection of pairs $(\gamma_2, \gamma_1) \in \mathcal{C}[tgt(\alpha), tgt(\beta)] \times \mathcal{C}[src(\beta), src(\alpha)]$ such that $\beta = \gamma_2 \circ \alpha \circ \gamma_1$ [26]. Given a small category $\mathcal{C}$, a **natural system (of abelian groups)** [27] on $\mathcal{C}$ is a functor $D : F\mathcal{C} \to \mathsf{Ab}$, where Ab is the category of abelian groups and group morphisms between them.

[25] see [17] or [20] or any set theory textbook for the definition.
[26] If $\mathcal{C}$ is small then so is $F\mathcal{C}$. Moreover, if $\mathcal{C}$ is loop-free then so is $F\mathcal{C}$.
[27] see [2] and [1] for further details.

Lemma 6.4 *Let $\mathcal{C}$ be a homogeneous small category. We define a natural system on $\mathcal{C}$ setting $D(\gamma) := Z^{length(\gamma)}$ and for*

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] \arrow[dr, phantom, "="] & \cdot \arrow[d, bend left, "\gamma_2"] \\ \cdot \arrow[u, bend left, "\gamma_1"] \arrow[r, "\beta"'] & \cdot \end{tikzcd}$$

$D(\gamma_2,\gamma_1) : (x_n, ..., x_1) \in Z^{length(\alpha)} \hookrightarrow (\underbrace{0, ..., 0}_{length(\gamma_2)\ times}, x_n, ..., x_1, \underbrace{0, ..., 0}_{length(\gamma_1)\ times}) \in Z^{length(\beta)}$*, with $length(\gamma_2)$ zeros on the left side of x_n and $length(\gamma_1)$ zeros on the right side of x_1.*

Instead of a (boring and) formal proof that we actually have a functor, observe the following example, suppose $length(\gamma_1) = 1$, $length(\gamma_2) = 2$, $length(\beta) = 6$, then necesarily, $length(\alpha) = 3$ and $D(\gamma_2, \gamma_1)$ is an abelian group embedding pictured by the following diagram:

$$\begin{tikzcd} \{0\} \arrow[d, "0"] & \times & \{0\} \arrow[d, "0"] & \times & Z \arrow[d, "id"] & \times & Z \arrow[d, "id"] & \times & Z \arrow[d, "id"] & \times & \{0\} \arrow[d, "0"] \\ Z & \times & Z & \times & Z & \times & Z & \times & Z & \times & Z \end{tikzcd}$$

It is important to notice that the image of a morphism of $F\mathcal{C}$ only depends on the length of γ_1, γ_2 and α.

7 Dealing with loops: the fundamental monoid

As one can notice, the category POSPC does not contain any satifactory model of the directed circle "$\overrightarrow{S^1}$". Indeed, the only authorized paths of $\overrightarrow{S^1}$ are the clockwise ones [28] . The problem is to modelize this idea. What order relation should equip S^1 in order to make it a pospaces whose dipaths are exactly the clockwise ones? Suppose that such a relation $\leq$ exists, in particular, $t \in \overrightarrow{I} \longmapsto (\cos(-2\pi t), \sin(-2\pi t))$ is clockwise, so we should have $\forall t \in [0, 1],\ (0, 1) \leq (\cos(-2\pi t), \sin(-2\pi t)) \leq (0, 1)$ hence, by antisymmetry, $(\cos(-2\pi t), \sin(-2\pi t)) = (0, 1)$ which is a contradiction. A naive solution consists of weakening the definition of a pospace asking $\leq$ be a preorder instead of an order relation. But then, by transitivity, $\forall t, t' \in [0, 1],\ (\cos(2\pi t), \sin(2\pi t)) \leq (\cos(2\pi t'), \sin(2\pi t'))$ so $t \in \overrightarrow{I} \longmapsto (\cos(2\pi t), \sin(2\ \pi t))$ which is anticlockwise would also be directed. Marco Grandis approach consists of equiping a topological space X with a set of distinguished paths denoted dX and submited to some conditions. The elements of dX are naturaly called the directed paths. Then it suffices to equip S^1 with the set of all clockwise paths to obtain a model of the directed circle. It is also possible to have a model of directed circle by covering S^1 with open subsets, each of which being suitably equiped with an order relation $\leq$ that <u>locally</u> makes S^1 a pospace.

[28] obviously we could have chosen the anticlokwise ones

Besides, the fact that a pospace does not have loops makes its fundamental category loop-free, in particular it has no endomorphisms. As a direct consequence, trying to define the "fundamental monoid" of a pospace $\overrightarrow{X}$ as $\overrightarrow{\pi_1}(\overrightarrow{X})[x,x]$ is sound but pointless because $\overrightarrow{\pi_1}(\overrightarrow{X})[x,x] = \{id_x\}$. Introducing loops in our models, the "fundamental monoid" becomes relevant.

Ideas related to the definition of local pospaces are borrowed from the ones of differential geometry and smooth manifold theory, for a deeper analogy see [21] and [23].

Definition 7.1 [Local Pospaces] A **local pospace** is a triple $(X, \tau_X, \leq_X)$ such that (X, τ_X) is a topological space, $\leq_X$ a relation on X and $\forall x \in X \exists U$ an open neighbourhood of x such that $(U, \tau_U, \leq_U)$ is a pospace. τ_U and $\leq_U$ are respectively the restriction of τ_X and $\leq_X$ to U. An **atlas** of $(X, \tau_X, \leq_X)$ is an open covering $(U_i)_{i\in I}$ of (X, τ_X) such that $\forall i \in I, (U_i, \tau_{U_i}, \leq_{U_i})$ is a pospace. A **local dimap** $f : (X, \tau_X, \leq_X) \to (Y, \tau_Y, \leq_Y)$ is a continuous map between underlying topological spaces such that $\exists (U_j)_{j\in J}$ atlas of $(X, \tau_X, \leq_X)$ $\exists (V_j)_{j\in J}$ atlas of $(Y, \tau_Y, \leq_Y)$ satisfying $\forall j \in J,\ f_{U_j \to V_j} : x \in U_j \longmapsto f(x) \in V_j$ is a dimap (i.e. a morphism of POSPC). Local pospaces and local dimaps organize themselves in a category denoted LPOSPC.

As $\overrightarrow{[0,1]}$ is the standard example of pospace, the directed circle $\overrightarrow{S^1}$ is the standard example of local pospace, its relation is described by means of maps $\theta_0 : x \in]0, 2\pi[\longmapsto (\cos(x), \sin(x)) \in S^1$ and $\theta_1 : x \in] -\pi, \pi[\longmapsto (\cos(x), \sin(x)) \in S^1$ setting $\forall x, y \in]0, 2\pi[,\ \theta_0(x) < \theta_0(y)$ if $x < y$ and $\forall x, y \in] -\pi, \pi[,\ \theta_1(x) < \theta_1(y)$ if $x < y$.

The next definition is due to Marco Grandis in [14] [29]

Definition 7.2 [d-spaces] A **directed space** or **d-space** is a triple (X, τ_X, dX) where (X, τ_X) is a topological space and $dX \subseteq \{\text{paths of } (X, \tau_X)\}$ with the following conditions

(i) $\{\text{constant paths}\} \subseteq dX$

(ii) for all $\theta : [0,1] \to [0,1]$ continuous and <u>increasing</u>, for all $\gamma \in dX$, $\gamma \circ \theta \in dX$ (dX is stable under di-reparametrization)

(iii) for all $\gamma_1, \gamma_2 \in dX, \gamma_2 \circ \gamma_1 \in dX$ (dX is stable under concatenation)

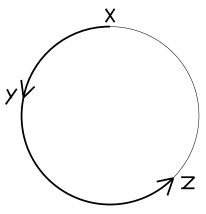
A d-map from (X, τ_X, dX) to (Y, τ_Y, dY) is a continuous map f from (X, τ_X) to (Y, τ_Y) such that $\forall \gamma \in dX\ f \circ \gamma \in dY$ d-spaces and d-maps organize themselves in a category denoted dSPC.

Remark that we have the "obvious" inclusion functors

$$\mathsf{POSPC} \hookrightarrow \mathsf{LPOSPC} \hookrightarrow \mathsf{dSPC}$$

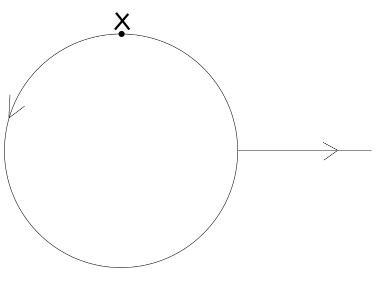
Now let us focus on two examples:

[29] [14] also contains a definition of local pospace which differs from the presently given one.



Denoting $\pi_1(\overrightarrow{S^1})$ the fundamental category of $\overrightarrow{S^1}$, we have $\forall x \in \overrightarrow{S^1},\ \pi_1(\overrightarrow{S^1})[x,x]$ isomorphic to $\mathbb{N}$. Compare $\mathbb{N}$ to the fundamental group of the circle. Precisely, $\pi_1(\overrightarrow{S^1})$ can be described the following way, for each $x, y \in S^1$ there is a distinguished arrow $\alpha_{x,y}$ and the family of distinguished arrows is submitted to the following axiom, $\forall x, y, z \in S^1,\ \alpha_{y,z} \circ \alpha_{x,y} = \alpha_{x,z}$, where $y \in (x, z)$. Here, (x, z) is the clockwise open arc from x to z. Intuitively, the distinguished arrow from $x \in \overrightarrow{S^1}$ to $y \in \overrightarrow{S^1}$ is the clockwise path from x to y on the directed circle, see the left side figure. Then $\forall \gamma \in \pi_1(\overrightarrow{S^1})[x, y]\ \exists! n \in \mathbb{N}$ such that $\gamma = (\alpha_{y,y})^n \circ \alpha_{x,y}$ and $\forall \gamma \in \pi_1(\overrightarrow{S^1})[x, x]\ \exists! n \in \mathbb{N}$ such that $\gamma = (\alpha_{x,x})^n$. Hence we could define the fundamental monoid of $\overrightarrow{S^1}$ as $(\mathbb{N}, +)$.

The idea of the fundamental monoid is attractive but does not work because, in general, it depends on the base point x:



The left side picture can easily be described as a local pospace or a directed space denoted $\overrightarrow{X}$ in both cases. Adapting the description of the fundamental category of $\overrightarrow{S^1}$, it is easy to describe the one of $\overrightarrow{X}$. Then we observe that $\vec{\pi_1}(\overrightarrow{X})[x, x] \cong (\mathbb{N}, +)$ in MON - the category of monoids - while $\vec{\pi_1}(\overrightarrow{X})[y, y] \cong \{\bullet\}$. The base point dependence makes impossible to define the fundamental monoid of $\overrightarrow{X}$ as the straightforward generalization of the fundamental group.

In addition, the component category of $\overrightarrow{S^1}$ is its fundamental one. Indeed, none of the morphisms $\alpha_{x,y}$ of $\pi_1(\overrightarrow{S^1})$ is Yoneda inversible. By definition, if $\alpha_{x,y}$ were Yoneda inversible then, since $\pi_1(\overrightarrow{S^1})[y, x] \neq \emptyset$, we would have a morphism g from y to x such that $g \circ \alpha_{x,y} = id_x$, which is impossible. Hence, as any arrow of $\pi_1(\overrightarrow{S^1})$ can be written as a composition of α's, none of them is Yoneda inversible.
In particular the component category, as it has been defined previously, does not efficiently reduce the size of the fundamental category of a local pospace or a directed space which "contains" $\overrightarrow{S^1}$. Still, the next result may provide a way to solve this problem:

Proposition 7.3 *Let $\mathcal{C}$ be a small category. Suppose that $\sigma : x \longrightarrow y$ is a morphism* *of* $\mathcal{C}$*:*

(i) **If** $\forall\delta \in \mathcal{C}[x,x]\ \exists!\gamma \in \mathcal{C}[y,y]$ *such that* $\sigma\circ\delta = \gamma\circ\sigma$ **then** *the map* $\Phi_\sigma : \delta \in \mathcal{C}[x,x] \longmapsto \gamma \in \mathcal{C}[y,y]$ *is a morphism of monoids*

(ii) **If** $\forall\gamma \in \mathcal{C}[y,y]\ \exists!\delta \in \mathcal{C}[x,x]$ *such that* $\gamma\circ\sigma = \sigma\circ\delta$ **then** *the map* $\Psi_\sigma : \gamma \in \mathcal{C}[y,y] \longmapsto \delta \in \mathcal{C}[x,x]$ *is a morphism of monoids*

(iii) **If** $\forall\delta \in \mathcal{C}[x,x]\ \exists!\gamma \in \mathcal{C}[y,y]$ *such that* $\sigma\circ\delta = \gamma\circ\sigma$ *and* $\forall\gamma \in \mathcal{C}[y,y]\ \exists!\delta \in \mathcal{C}[x,x]$ *such that* $\gamma\circ\sigma = \sigma\circ\delta$ **then** $\Psi_\sigma\circ\Phi_\sigma = Id_{\mathcal{C}[x,x]}$ *and* $\Phi_\sigma\circ\Psi_\sigma = Id_{\mathcal{C}[y,y]}$

Proof. $\sigma\circ id_x = id_y\circ\sigma$, thus $\Phi_\sigma(id_x) = id_y$. Moreover, $\sigma\circ(\delta_2\circ\delta_1) = \Big(\Phi_\sigma(\delta_2\circ\delta_1)\Big)\circ\sigma$ and $\sigma\circ(\delta_2\circ\delta_1) = (\sigma\circ\delta_2)\circ\delta_1 = \Big(\Phi_\sigma(\delta_2)\circ\sigma\Big)\circ\delta_1 = \Phi_\sigma(\delta_2)\circ(\sigma\circ\delta_1) = \Big(\Phi_\sigma(\delta_2)\Big)\circ\Big(\Phi_\sigma(\delta_1)\Big)\circ\sigma$. By uniqueness, $\Phi_\sigma(\delta_2\circ\delta_1) = \Phi_\sigma(\delta_2)\circ\Phi_\sigma(\delta_1)$, hence Φ_σ is a morphism of monoids. The same holds for Ψ_σ dualizing everything. Suppose we have the hypothesis of the third point, then $\sigma\circ\delta = \Phi_\sigma(\delta)\circ\sigma = \sigma\circ\Psi_\sigma(\Phi_\sigma(\delta))$, hence, by uniqueness, $\Psi_\sigma(\Phi_\sigma(\delta)) = \delta$. The same way, $\Phi_\sigma(\Psi_\sigma(\gamma)) = \gamma$. □

Proposition 7.4 *Let* $\mathcal{C}$ *be a small category. Suppose that* $\sigma : x \longrightarrow y$ *is a morphism of* $\mathcal{C}$ *such that* $f_\sigma : \delta \in \mathcal{C}[x,x] \longmapsto \sigma\circ\delta \in \mathcal{C}[x,y]$ *and* $g_\sigma : \gamma \in \mathcal{C}[x,x] \longmapsto \gamma\circ\sigma \in \mathcal{C}[x,y]$ *are bijective. Then the hypothesis of the third point of proposition 7.3 are satisfied.*

Proof. Given $\delta \in \mathcal{C}[x,x]$, by definition of the bijections f and g, $\gamma := g_\sigma^{-1}(\sigma\circ\delta)$ is the only element of $\mathcal{C}[y,y]$ such that $\sigma\circ\delta = \gamma\circ\sigma$. Of course, given $\gamma \in \mathcal{C}[y,y]$, by defintion of the bijections f and g, $\delta := f_\sigma^{-1}(\gamma\circ\sigma)$ is the only element of $\mathcal{C}[x,x]$ such that $\gamma\circ\sigma = \sigma\circ\delta$. In particular, $\Phi_\sigma = g_\sigma^{-1}\circ f_\sigma$ and $\Psi_\sigma = f_\sigma^{-1}\circ g_\sigma$. □

Corollary 7.5 *Any Yoneda inversible morphism satisfies the hypothesis of proposition 7.4*

For example, we remark that $\forall x,y \in S^1$, $\alpha_{x,y}$ satisfies the hypothesis of the third point of proposition 7.3 and of proposition 7.4, nevertheless, as we have already seen, they are not Yoneda inversible. This remark leads to settle some definitions, the former Yoneda inversible morphisms are, from now, called **strong Yoneda inversible**, the morphisms satisfying the hypothesis of proposition 7.4 are called **Yoneda inversible** and the ones enjoying the hypothesis of the third point of proposition 7.3 are called **weak Yoneda inversible**. The next step consists of adapting the definition of inessential system, putting Yoneda or weak Yoneda instead of strong Yoneda. In the new context, there is no doubt that the definition of the component category will require some changes too.

7.1 Component category of the directed torus with a hole

Take the directed square with a hole (see figure 1) then identify $[0,1]\times\{0\} \approx [0,1]\times\{1\}$ and $\{0\}\times[0,1] \approx \{1\}\times[0,1]$. We obtain a local pospace whose underlying topological space is a torus with a hole and where the local order is clockwise on the “small” and “large” generators, denote T this local pospace. Figure 7 represents T with the identifications described above. No morphism is

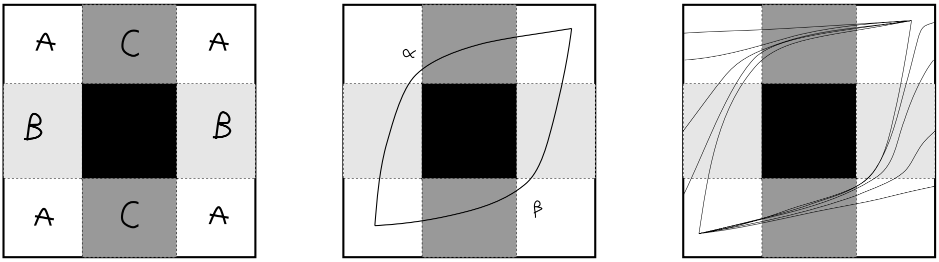


Fig. 7. Directed torus T^2 -unfold representation-

Yoneda inversible, for the same reason as in the directed circle case. Still, all the morphisms of the fundamental category they induce are both monic and epic making the correponding set theoretic maps are one-to-one but not onto (see definition 4.5). For example, the dipaths α and β on figure 7 are not dihomotopic, the right hand part shows the "only" dihomotopy one could image. In fact, α and β are not even homotopic, it is a classical algebraic topology problem. The consequence is that $\forall\delta$, morphism of $\pi_1(\overrightarrow{T^2})$, $\circlearrowleft^B \circ (C \to A) \neq (C \to A) \circ \delta$. Where $(C \to A)$ is an arrow whose beginning is in the interior of C and the end in the interior of A, and $\circlearrowleft^B$ is a loop starting from A going through B and coming back to its initial point. In particular, any dipath crossing the fronteer of A, B or C, cannot be weakly Yoneda inversible. Conversely, it seems that any dipath staying in the same "component" is weakly Yoneda inversible. Then, ideologically, we "should" have three components[30] and the component category "should" be the

free category generated by
$$\begin{tikzcd} B \arrow[r, bend right, "g_B"'] & A \arrow[l, bend right, "f_B"'] \arrow[r, bend left, "f_C"] & C \arrow[l, bend left, "g_C"] \end{tikzcd}.$$

These last statements are just prospective thoughts and conjectures.

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