Directed Algebraic Topology and Concurrency

Emmanuel Haucourt

CEA LIST,
Modeling and Analysing Interaction between Systems Laboratory

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Underlying graph and Category of paths I

graph : 1-dimensional pre-simplicial set

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_1} & A' \\
\downarrow s & & \downarrow t' \\
V & \xrightarrow{\phi_0} & V' \\
\end{array}
\quad \begin{array}{ccc}
A & \xrightarrow{\phi_1} & A' & \xrightarrow{\psi_1} & A'' \\
\downarrow s & & \downarrow t' & & \downarrow t'' \\
V & \xrightarrow{\phi_0} & V' & \xrightarrow{\psi_0} & V'' \\
\end{array}
\]
An example of model of a multi-task program from Edsger Wybe Dijkstra “Pakken/Vrijlaten” language
Another example from Edsger Wybe Dijkstra “Pakken/Vrijlaten” language

18 states and 20 arrows
Path : morphism of graph from $\mathbb{I}_n$ to $\Gamma$

Forgetful functor $U : \text{Cat} \rightarrow \text{Grph}$

“Category of paths” functor $F : \text{Grph} \rightarrow \text{Cat}$

$F \dashv U$
A potential execution

program $T_1 = \text{PaPbVaVb} \mid T_2 = \text{PbPaVbVa}$
Anoter potential execution

program $T_1 = PaPbVaVb \mid T_2 = PbPaVbVa$

<table>
<thead>
<tr>
<th>$T_1$</th>
<th>$T_2$</th>
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</thead>
<tbody>
<tr>
<td>$Pa$</td>
<td>$-$</td>
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<tr>
<td>$Pb$</td>
<td>$-$</td>
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<tr>
<td>$Va$</td>
<td>$-$</td>
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<td>$Vb$</td>
<td>$-$</td>
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<tr>
<td>$-$</td>
<td>$Pb$</td>
</tr>
<tr>
<td>$-$</td>
<td>$Pa$</td>
</tr>
<tr>
<td>$-$</td>
<td>$Vb$</td>
</tr>
<tr>
<td>$-$</td>
<td>$Va$</td>
</tr>
</tbody>
</table>

Termination
\[ F(\Gamma \times \Gamma') \not\cong F(\Gamma) \times F(\Gamma') \]

Transitions Systems, CCS/\(\pi\)-calculus, Mazurkiewicz Traces ...
Partially ordered spaces
The category PoTop

Pospace $\overrightarrow{X}$:

- $X$ topological space
- $\subseteq$ closed in $X \times X$

Morphism $f$ from $\overrightarrow{X}$ to $\overrightarrow{X'}$: continuous and order preserving maps

Diagram:

$$\begin{align*}
\text{PoTop} & \longrightarrow \text{PoSet} \\
\downarrow & \\
\text{Haus} & \longrightarrow \text{Set}
\end{align*}$$
Categorical properties of PoTop

Theorem

- The directed compact unit segment is exponentiable in PoTop
- PoTop is complete and cocomplete
- PoTop is symmetric monoidal closed
- CGPoTop is a Cartesian closed reflective subcategory of PoTop
- CPoTop is a (complete and cocomplete) Cartesian closed reflective subcategory of CGPoTop cogenerated by the directed compact unit segment
- PoTop has no loop
Partially ordered spaces
examples

- Real line with standard order and topology : $\mathbb{R}$
- Subset of a pospace (in particular $[0, 1]$)
- Geometric realization of a graph
- Cartesian Product
- Closed subsets of a metric space together with inclusion
Size reduction
Graph $\Gamma_{\vec{X}}$ of paths on a pospace $\vec{X}$

- paths on $\vec{X}$: morphisms from $[0,1]$ to $\vec{X}$
- arrows of $\Gamma_{\vec{X}}$: paths on $\vec{X}$
- source and target of a path $\gamma$ on, $\vec{X}$: $\gamma(0)$ and $\gamma(1)$
The image of a dipath $\alpha$ on a pospace $\overrightarrow{X}$ is either isomorphic (in PoSpc) to $\overrightarrow{[0, 1]}$ or $\{\ast\}$ (hence no directed Peano curve).

Two dipaths sharing the same image are dihomotopic.
Some paths around a cubic hole

$P(a).V(a) \mid P(a).V(a) \mid P(a).V(a)$ with $\alpha_a = 3$
Two “concatenations”
paths on $\Gamma \vec{X}$ vs paths on $\vec{X}$

- Composition on $F(\Gamma \vec{X})$ denoted by $\circ$

- Given $\gamma = (\gamma_n, \ldots, \gamma_1)$ a path on $\Gamma \vec{X}$, we define the following path on $\vec{X}$

$$ (\nu(\gamma))(t) = \begin{cases} 
\gamma_k(nt - k) & \text{si } t \in \left[\frac{k}{n}, \frac{k+1}{n}\right] \text{ et } k < n-1 \\
\gamma_n(nt - n + 1) & \text{si } t \in \left[\frac{n-1}{n}, 1\right]
\end{cases} $$
Directed homotopy
what it is and looks like

Morphism $h$ from $[0, 1]^2$ to $\overrightarrow{X}$ such that $U(h)$ is a homotopy from $\gamma$ to $\delta$
Directed homotopy
an example

T1 gets a and b before T2 => a=2 and b=4

T2 gets b and a before T1 => a=2 and b=3

Each of T1 and T2 gets a ressource
=> Deadlock with a=2 and b=1
A subtlety

directed homotopy is not classical homotopy
Loop-Free category or small categories without loops (LfCat):
\[ C[x, x] = \{\text{id}_x\} \text{ and } (C[x, y] \times C[y, x] \neq \emptyset \implies x = y) \]

One-Way category (OwCat):
\[ C[x, x] = \{\text{id}_x\} \]

\( C \) one-way \iff sk(\( C \)) is loop-free

\( \text{LfCat} \xrightarrow{\subset} \text{OwCat} \xrightarrow{\subset} \text{Cat} \)
Let $\sim$ be the congruence over $F(\Gamma_{\overrightarrow{X}})$ generated by

$$\left\{ ((\gamma_n, \ldots, \gamma_1), (\delta_p, \ldots, \delta_1)) \mid \text{there is a dihomotopy from } \nu(\gamma) \text{ to } \nu(\delta) \right\}$$

The fundamental category $\overrightarrow{\pi_1(\overrightarrow{X})}$ is $F(\Gamma_{\overrightarrow{X}})/\sim$ and we have

$$\overrightarrow{\pi_1(\overrightarrow{X} \times \overrightarrow{Y})} \cong \overrightarrow{\pi_1(\overrightarrow{X})} \times \overrightarrow{\pi_1(\overrightarrow{Y})}$$

$\overrightarrow{\pi_1(\overrightarrow{X})}$ is loop-free

van Kampen theorem
A detailed example
A detailed example
square with centered hole

<table>
<thead>
<tr>
<th>x ∈</th>
<th>y ∈</th>
<th>$\overrightarrow{\pi_1(X)}[x,y]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>${\sigma_{x,y}}$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$B_1$</td>
<td>${\sigma_{x,y}}$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$B_2$</td>
<td>${\sigma_{x,y}}$</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>${\sigma_{x,y}}$</td>
</tr>
<tr>
<td>A</td>
<td>$B_1$</td>
<td>${r_{x,y}}$</td>
</tr>
<tr>
<td>A</td>
<td>$B_2$</td>
<td>${h_{x,y}}$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>C</td>
<td>${h'_{x,y}}$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>C</td>
<td>${r'_{x,y}}$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$B_2$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$B_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>A</td>
<td>C</td>
<td>${u_{x,y}, d_{x,y}}$</td>
</tr>
</tbody>
</table>

With $r'_{y,z} \circ h_{x,y} = u_{x,z}$, $h'_{y,z} \circ r_{x,y} = d_{x,z}$ and 3 points $x$, $y$, $z$ of the square such that $x \sqsubseteq y \sqsubseteq z$; if $x \not\sqsubseteq y$ then $\overrightarrow{\pi_1(X)}[x,y] = \emptyset$. 
future if $C[y, z] \neq \emptyset$, then $C[y, z] \to C[x, z]$ is a bijection and
$$\gamma \downarrow \to \gamma \circ \sigma$$
past if $C[z, x] \neq \emptyset$, then $C[z, x] \to C[z, y]$ is a bijection
$$\delta \downarrow \to \sigma \circ \delta$$
Yoneda system \( \Sigma \) of a small category \( \mathcal{C} \) preserving the past and the future II

1. \( \Sigma \) is stable under composition,
2. \( \Sigma \) contains all the isomorphisms of \( \mathcal{C} \),
3. all the elements of \( \Sigma \) are Yoneda morphisms and
4. \( \Sigma \) is stable under change and cochange of base.
Yoneda systems

Example
Structure of $\Sigma$-components

$\mathcal{C}$ loop-free category and $\Sigma$ Yoneda system over $\mathcal{C}$

**Theorem**

1. $\exists z \, \Sigma[x, z] \times \Sigma[y, z] \neq \emptyset$ iff $\exists z \, \Sigma[z, x] \times \Sigma[z, y] \neq \emptyset$

2. “$\exists z \, \Sigma[x, z] \times \Sigma[y, z] \neq \emptyset$” defines an equivalence relation $\sim$

3. Given any $\sim$-equivalence class $K$, the full subcategory of $\mathcal{C}$ whose set of objects is $K$ is a non empty lattice

4. If $a \sim b$, then the following square is both a pullback and a pushout in $\mathcal{C}$.

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\Sigma} & z \\
\uparrow & & \uparrow \\
\Sigma & \xrightarrow{\Sigma} & y \\
\end{array}
\quad
\begin{array}{ccc}
x & \xrightarrow{\Sigma} & z \\
\uparrow & & \uparrow \\
x & \xrightarrow{\Sigma} & y \\
\end{array}
\quad
\begin{array}{ccc}
a & \longrightarrow & a \lor b \\
\uparrow & & \uparrow \\
a \land b & \longrightarrow & b \\
\end{array}
$$
Theorem

The collection, ordered by inclusion, of the Yoneda systems of a one-way category, forms a locale whose maximum is denoted $\Sigma$. Beside, its minimum is the collection of all isomorphisms of $C$. 
The category of components of a loop-free category $\mathcal{C}$ is the quotient $\mathcal{C}/\Sigma$.

**Theorem**

A loop-free category $\mathcal{C}$ is a non empty lattice iff its category of components is $\{\ast\}$. 
Theorem

1. the collection $\Sigma$ is pure in $C$ ($\beta \circ \alpha \in \Sigma \Rightarrow \beta, \alpha \in \Sigma$),
2. the category $C/\Sigma$ is loop-free and the category $C[\Sigma^{-1}]$ is one-way
3. the categories $C[\Sigma^{-1}]$ and $C/\Sigma$ are equivalent and
4. the category $C[\Sigma^{-1}]$ is fibered over the base $C/\Sigma$. 
The category of components of the Swiss flag
The category of components

Menger sponge first iteration: \( P(a).V(a) \mid P(a).V(a) \mid P(a).V(a) \) with \( \alpha_a = 2 \)

- Interior of the pospace
- Category of components
- Flattened
The components category
of a 2-semaphore: $P(a).V(a) \mid P(a).V(a) \mid P(a).V(a)$ avec $\alpha_a = 3$

the pospace
its category of components
The monoid of (isomorphism classes of) non empty, connected, finite, loop-free categories is countable and free
Example of product
parallel "independent" composition
The Directed Circle

Objects: $S^1$  
Morphisms: $S^1 \times \mathbb{N} \times S^1$  
Identities: $(x,0,x)$ for $x \in S^1$