A construction of Spectra in Algebraic Geometry

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I want to begin by warmly thanking Mathieu Anel, for introducing me to the world of topos theory, and for supporting and encouraging me throughout the last few months despite the distance and the circumstances of the current pandemic.

Introduction

In the 1960's, in order to prove the Weil conjectures, it was apparent that a cohomology theory more refined than that of Zariski sheaves was necessary. In some sense, the Zariski topology of algebraic varieties, or schemes, had "too few" open sets. In the end, instead of finding a finer topology to put on schemes, the very notion of topology had to be refined. *Étale cohomology* was born. The étale cohomology of a scheme X, instead of being defined in terms of sheaves on the topological space X, is defined in terms of sheaves on the *site* $\mathsf{Et}(X)$ of étale morphisms unto X. A site is a category to be thought as generalizing the category of open subsets of a topological space. The objects of the category are "generalized open sets", while the morphisms play the role of inclusion. The additional data of what families of maps $\{f_i : U_i \to X\}$ cover an object X is what makes a category into a site. But a site is just a presentation of a space : different sites S, S' can give rise to equivalent categories of sheaves, and should hence be considered as different presentations for the same space. Spaces that are defined by sites are called (Grothendieck) **topoi**.

In the present text, instead of introducing topoi as categories of sheaves as is usually done in the literature, following the approach of [AJ19], we reserve the term *logos* for such categories. Logoi form a 2-category. A topos is a mathematical object dual to a logos : the 2-category of topoi will be defined as the opposite 2-category of the 2-category of logoi.

Because of this, we first present general concepts from the theory of categorical semantics in section 1, in order to properly introduce logoi in section 2 as categories in which to interpret a certain kind of logic. In section 3, topoi are defined as dual to logoi and their geometrical aspects are explored by comparison with topological spaces. Finally, as an application of the language of topoi and logoi, we define and give a construction for a general notion of *spectra* in algebraic geometry, generalizing the Zariski and étale spectra of rings.

1 Basics of categorical semantics

1.1 Algebraic theories

In this section, we study algebraic theories, also called equational theories. They are first order theories whose only axioms are of the form $s = t$, where s and t are well formed terms of the theory. The specification of how to form terms is given by a signature.

Definition 1.1. A **signature** is a family of sets $\Sigma = \{\Sigma_k\}_{k\in\mathbb{N}}$. The elements of Σ_k are called the k-ary operations. Given a set of variables χ, the **terms** of a signature Σ are defined inductively as :

- any variable $x \in \chi$ is a term,
- If $t_1, \dots, t_n$ are terms and f is an n-ary operation, then $f(t_1, \dots, t_n)$ is a term.

We write $\mathcal{T}(\Sigma, \chi)$ for the set of terms of Σ with variables in χ.

Definition 1.2. An **algebraic theory** T is the data of a signature Σ_T and a set A_T of axioms of the form $s = t$ where t and s are terms of Σ_T (formally, A_T is a set of pairs of terms (s, t)).

Example 1.3. The algebraic theory of abelian groups is given by the signature

$$\Sigma_0 = \{0\}, \Sigma_1 = \{-\}, \Sigma_2 = \{+\}$$

Its axioms are

$$x + 0 = x \qquad x + y = y + x$$
$$x + (-x) = 0 \qquad x + (y + z) = (x + y) + z$$

Example 1.4. The theories of groups, rings, monoids and semigroups are all examples of algebraic theories. The theory of categories is not (at least not evidently) an algebraic theory, since the operation of composing arrows would need to be partially defined.

Models of an algebraic theory can be defined in any category with finite products.

Definition 1.5. An interpretation I of a signature Σ in a category C with finite products is given by an object $|I|$ in C equipped, for every k-ary operation f in Σ, with a morphism $f^I : |I|^k \to |I|$. The object $|I|$ is called the **underlying object** of I.

The interpretation I can be extended to all terms in the following way. A *context* Γ is a finite list of variables $\Gamma = [x_1, \dots, x_n]$ in χ. Given a term t, if all the variables in t appear in a context Γ, we define the interpretation $[\Gamma \mid t]^I$ of t in context Γ by induction on t :

- the interpretation of a variable $x_i \in \Gamma$ is the i-th projection $|I|^n \to |I|$,
- the interpretation of $f(t_1, \dots, t_m)$ is the composition

$$|I|^n \xrightarrow{\left([\Gamma \mid t_1]^I, \dots, [\Gamma \mid t_m]^I\right)} |I|^m \xrightarrow{f^I} |I|$$

Definition 1.6. A **model** of an algebraic theory $T = (\Sigma, A)$ in a category C with finite products is an interpretation I of Σ in C such that, for any axiom $s = t$ in A and any context Γ containing all the variables appearing in s and t, one has $[\Gamma \mid s]^I = [\Gamma \mid t]^I$.

Definition 1.7. A **morphism between models** I, J in C is a morphism $g : |I| \to |J|$ in C such that, for any k-ary operation f in Σ, the following square commutes.

$$\begin{array}{ccc} |I|^k & \xrightarrow{(g\circ\pi_1, \dots, g\circ\pi_k)} & |J|^k \\ {\scriptstyle f^I}\downarrow & & \downarrow{\scriptstyle f^J} \\ |I| & \xrightarrow{\quad g \quad} & |J| \end{array}$$

Models and their morphisms form the category of C-valued models of T : $\mathsf{Mod}(T, \mathsf{C})$. The category of set-valued models will also be written $\mathsf{Mod}(T)$. When the category C is not indicated, we will always mean Set-valued model.

Example 1.8. The category set-valued models of the theory of groups (resp. abelian groups, rings, monoids) is the usual category of groups (resp. abelian groups, rings, monoids). Less trivially, a Group-valued model of the theory of groups is an abelian group (this follows from the Eckmann-Hilton argument).

Let C, C' be categories with finite products, and $F : \mathsf{C} \to \mathsf{C}'$ a finite product-preserving functor. Given an algebraic theory T, the functor F sends models of T in C to models of T in C'. This makes the assignement

$$\begin{aligned} \mathsf{Cat}^\times &\to \mathsf{Cat} \\ \mathsf{C} &\mapsto \mathsf{Mod}(T, \mathsf{C}) \end{aligned}$$

into a 2-functor (where $\mathsf{Cat}^\times$ is 2-category of categories with finite products, finite product-preserving functors and natural transformations). We will see that this 2-functor is representable, by a category called the *syntactic category* of T.

Definition 1.9. The **syntactic category** C_T of an algebraic theory T is defined to have :

- as objects the natural numbers $[n]$, $n \in \mathbb{N}$.
- as morphisms $[m] \to [n]$, sequences of terms in some context :

$$[m] \xrightarrow{([\Gamma \mid t_1, \dots, t_n])} [n]$$

  where Γ is a context of exactly m variables and any variable in one of the t_i appears in Γ. Composition of

$$[l] \xrightarrow{([x_1, \dots, x_l \mid s_1, \dots, s_m])} [m] \xrightarrow{([y_1, \dots, y_m \mid t_1, \dots, t_n])} [n]$$

  is given by $[x_1, \dots, x_l \mid u_1, \dots, u_n]$ where u_i is obtained by replacing all occurences of y_j in t_i by s_j. Morphisms are considered up to equivalence by the congruence relation generated by :

– $[\Gamma \mid s] \sim [\Gamma \mid t]$ if $s = t$ is an axiom of T.

– $[\Gamma \mid t] \sim [\Gamma\sigma \mid t\sigma]$ where σ is a permutation of the variables, $\Gamma\sigma$ is the context obtained from Γ by replacing x_i by $\sigma(x_i)$ for all i and $t\sigma$ is the term obtained by replacing all occurences of x_i by $\sigma(x_i)$ for all i (α-renaming rule).

Proposition 1.10. *The syntactic category of T is a category with finite products. By naming $U := [1]$ in C_T, we have for any $m \in \mathbb{N}$, $[m] = U^m$.*

Proof. Let $m, n \in \mathbb{N}$. We show that $[m+n]$ is the product of $[m]$ and $[n]$ in C_T. The projection maps are defined by

$$\begin{aligned} \pi_1 &= [x_1, \dots, x_{m+n} \mid x_1, \dots, x_m] &&: [m+n] \to [m] \\ \pi_2 &= [x_1, \dots, x_{m+n} \mid x_{m+1}, \dots, x_{m+n}] &&: [m+n] \to [n] \end{aligned}$$

Let $k \in \mathbb{N}$, $p : [k] \to [m]$, and $q : [k] \to [n]$. Using the α-renaming rule, we may assume that $p = [\Gamma \mid t_1, \dots, t_m]$ and $q = [\Gamma \mid t_{m+1}, \dots, t_{m+n}]$ with $\Gamma = (y_1, \dots, y_k)$. We define $h : [k] \to [m+n]$ to be $[\Gamma \mid t_1, \dots, t_{m+n}]$. By construction we have $p = \pi_1 \circ h$, $q = \pi_2 \circ h$, and h is indeed unique with such a property. □

Proposition 1.11. *There is a model U of T in C_T, called the **universal model** of T, defined as follows :*

- *its underlying object is the object $|U| = [1]$*
- *for a k-ary operation f, the interpretation of f is given by*

$$[x_1, \dots, x_k \mid f(x_1, \dots, x_k)] : |U|^k \to |U|$$

Proof. The syntactic category is defined precisely in a way to make this statement hold. □

Theorem 1.12. *The universal model of T is universal, in the following sense : given any category with finite products C and any model M of T in C, there exists a unique functor (up to unique isomorphism) sending the universal model to M.*

*In other words, the pair (C, U) represents the 2-functor $\mathsf{Mod}(\mathsf{T}, -) : \mathsf{Cat}^{\times} \to \mathsf{Cat}$. We say it is the **classifying category** of the theory T.*

Remark 1.13. When M is a model of T seen as a functor $M : \mathsf{C}_T \to \mathsf{C}$, $M(|U|) = |M|$ is the underlying object of M, while $M(|U|^n) = M(|U|)^n = |M|^n$ is its n-th power.

We now turn our attention to another presentation of the classifying category, which will bring us to a remarkable duality between syntax and semantics.

In group theory, there is a notion of free group on n generators, $\mathbb{Z} * \cdots * \mathbb{Z}$, such that for any other group G, the data of n elements in G entirely determines a unique group morphism $\mathbb{Z} * \cdots * \mathbb{Z} \to G$. Similarly, in ring theory, the ring $\mathbb{Z}[X_1, \dots, X_n]$ is the free ring on n generators for the same reason : given any ring R and elements $a_1, \dots, a_n$ in R, there is a unique morphism $\mathbb{Z}[X_1, \dots, X_n] \to R$ sending X_i to a_i for all i. Those are particular cases of a general construction that exists in all algebraic theories.

Proposition 1.14. *Let T be a theory. The forgetful "underlying set" functor $| - | : \mathsf{Mod}(T) \to \mathsf{Set}$ has a left adjoint F, called the **free** functor. The image by F of a finite set with n elements is called the **free** T**-model on** n **generators**.*

Proof. Let Σ be the underlying signature of T. Let X be a set. We define $|F(X)|$ to be the quotient of the set $\mathcal{T}(\Sigma, X)$ by the equivalence relation $\sim$ generated by :

(1) if $s = t$ is an axiom of T and $\sigma : \chi \to X$ is any function, then $s\sigma \sim t\sigma$,

(2) if $f \in \Sigma_k$ and $t_1, \dots, t_n, s_1, \dots, s_n \in \mathcal{T}(\Sigma, X)$ with $t_i \sim s_i$ for all i, then $f(t_1, \dots, t_n) \sim f(s_1, \dots, s_n)$.

For every $f \in \Sigma_k$, we define an interpretation of f in $F(X)$ as :

$$\begin{aligned} f^{F(X)} : |F(X)|^k &\to \quad |F(X)| \\ (t_1, \dots, t_n) &\to \quad f(t_1, \dots, t_n) \end{aligned}$$

This is well-defined because of condition (1), and defines a model of T because of condition (2). We moreover have a unit $\eta_X : X \to |F(X)|$ sending an element $x \in X$ to the term $x \in |F(X)|$ seen as a variable.

Now let R be a model of T and $f : X \to |R|$ be a function. We want to define a morphism $f^\flat : F(X) \to R$ such that $|f^\flat| \circ \eta_X = f$, which determines the value of $|f^\flat|$ on the variables. Moreover, he condition of $f^\flat$ being a morphism imposes the equality

$$|f^\flat|(g(t_1, \dots, t_k)) = g^R(|f^\flat|(t_1), \dots, |f^\flat|(t_n))$$

for all k-ary operation g. By definition of $|F(X)|$, those two conditions entirely determine a unique function $|f^\flat| : |F(X)| \to |R|$, which by construction is a morphism of T-models, hence a unique morphism $f^\flat : F(X) \to R$ such that $|f^\flat| \circ \eta_X = f$.

This construction is natural in R, thus the the assignment $X \mapsto F(X)$ extends to a functor $\mathsf{Set} \to \mathsf{Mod}(T)$, which is a left-adjoint to $| - |$ with unit η. □

Now consider the contravariant Yoneda embedding $c \mapsto \mathsf{Hom}(c, -) : \mathsf{C}_T^{op} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$. For all object c in C_T, the functor $\mathsf{Hom}(c, -)$ preserves all limits that exist in C, and in particular it preserves finite products. So the Yoneda embedding factors through

$$\mathsf{C}_T^{op} \xhookrightarrow{c \mapsto \mathsf{Hom}(c,-)} \mathsf{Fun}^\times(\mathsf{C}, \mathsf{Set}) \xrightarrow{\simeq} \mathsf{Mod}(T)$$

and hence identifies the opposite of the syntactic category as a full subcategory of the category of Set-valued models of T. It turns out the image of this embedding consists exactly of the finitely generated free models of T.

Theorem 1.15 (Lawvere duality)**.** *The Yoneda embedding sends the object U^n in C_T to the free model on n generators in $\mathsf{Mod}(T)$. It hence induces an anti-equivalence of categories between C_T and the full subcategory $\mathsf{Mod}_{fgf}(T)$ of $\mathsf{Mod}(T)$ spanned by finitely generated free models of T, i.e. the image by F of the category of finite sets.*

Proof. Let $M : \mathsf{C}_T \to \mathsf{Set}$ be a model of T, seen as a finite product-preserving functor. Let n be a non-negative integer. The Yoneda lemma states that there is natural a isomorphism

$$\mathsf{Hom}(\mathsf{Hom}_{\mathsf{C}_T}(U^n, -), M) \simeq M(U^n)$$

But by remark 1.13, $M(U^n) \simeq |M|^n$. This is exactly the universal property of the the free model on n generators. □

Remark 1.16. This duality can also be proven by directly showing that the category $\mathsf{Mod}_{fgf}(T)^{op}$ contains a universal model with underlying object $|F(1)|$, and hence by 2-Yoneda this would imply the equivalence $\mathsf{C}_T \simeq \mathsf{Mod}_{fgf}(T)^{op}$. See for instance [AB20].

Example 1.17. The opposite of the category of rings is equivalent to the category of affine schemes. Hence, by Lawvere duality, the syntactic category of the theory of rings is equivalent to the category of free affine schemes of finite presentation of **Spec** $\mathbb{Z}$, that is, the full subcategory of AffSch spanned by the finite powers of the affine line $\mathbb{A}^0, \mathbb{A}^1, \mathbb{A}^2, \dots$.

The fact that the classifying category of T can be described purely in terms of its models makes it a more intrinsic presentation than the usual presentation by a signature and a set of axioms. For instance, a group can also be defined as a set equipped with a constant e and a binary operation $\odot$, satisfying a single axiom [McC93] :

$$(x \odot (((x \odot y) \odot z) \odot (y \odot e))) \odot (e \odot e) = z$$

Usual multiplication and inversion can be recovered through the defining equations

$$x \cdot y = (x \odot e) \odot (y \odot e), \quad x^{-1} = x \odot e$$

Both this presentation of group theory and the usual one give rise to equivalent syntactic categories. Because of this, it is convenient to redefine the notion of algebraic theory in a purely categorical context.

Definition 1.18. A **Lawvere theory** is a category C with finite products equipped with an object c such that any object of C is a finite power of c.

Example 1.19. All algebraic theories give rise to Lawvere theories as their syntactic categories.

Remark 1.20. It can also be useful to consider multi-sorted algebraic theories, which are like algebraic theories but with the additional datum of a set $\mathcal{T}$ of types, and for which any function symbol $f \in \Sigma$ comes equipped with a type-signature. For instance, the theory of monoid morphisms has :

- for types a set with two elements $\mathcal{T} = \{A, B\}$
- for function symbols :

$$
\begin{array}{ccc}
e_A : A, & & e_B : B \\
*_A : A \times A \to A, & & *_B : B \times B \to B \\
& f : A \to B &
\end{array}
$$

- for axioms, equations such that $*_A, *_B$ together with e_A, e_B satisfy monoid axioms, together with
$$f(x *_A y) = f(x) *_B f(y)$$

Models of a multi-sorted algebraic theory may have multiple "underlying sets", one for each type in $\mathcal{T}$. It is still possible to define the syntactic category C_T of multi-sorted algebraic theory T, and in the same way as in the single-sorted case, C_T^{op} will embed as a full subcategory of $\mathsf{Mod}(T)$. But in general there will be no canonical notion of "finitely generated free models", since there is no longer a canonical forgetful functor $\mathsf{C}_T \to \mathsf{Set}$. The categorical notion of multi-sorted algebraic theory, i.e. that of *multi-sorted Lawvere theory*, is then just defined to be a category with finite products.

1.2 Essentially algebraic theories

We saw that categories with finite products are a natural place to define categorical semantics for algebraic theories. But we could have studied, for instance, categories with all finite limits, and functors preserving such limits.

Definition 1.21. A category with finite limits is said to be **left-exact**, or lex for short. A functor from a left-exact category which preserves all finite limits is also said to be **left-exact**. The 2-category of lex categories, lex functors and natural transformations is noted $\mathsf{Cat}^{\mathsf{lex}}$.

The terminology "left-exact" comes from the theory of abelian categories, where left-exact functors are those functors that preserve short exact sequences of the form $0 \to A \to B \to C$. In that context, it is a theorem rather than a definition that all left-exact functors are precisely those that preserve all finite limits.

Definition 1.22. A lex category C is called an **essentially algebraic theory**. The category of models of C in a lex category D is $\mathsf{Fun}^{\mathsf{lex}}(C, D)$.

Essentially algebraic theories can be thought of as a generalization of algebraic theories where operations can be *partially defined*. The most famous example of an essentially algebraic theory ought to be category theory itself.

Indeed, one can define a (small) category internal to a lex category C (also called a *category object in* C)to consist of

- an "objects" object $Ob \in \mathsf{C}$
- a "morphisms" object $Mor \in \mathsf{C}$
- morphisms

$$\begin{aligned} dom &: Mor \to Ob, \\ cod &: Mor \to Ob, \\ id &: Ob \to Mor, \\ comp &: 2Mor \to Mor \end{aligned}$$

  in C, where $2Mor$ is the "pair of composable arrows"-object defined by the pullback square

$$\begin{array}{ccc} 2Mor & \xrightarrow{\pi_2} & Mor \\ \pi_1 \downarrow & \lrcorner & \downarrow dom \\ Mor & \xrightarrow[cod]{} & Ob \end{array}$$

- satisfying the identities :

$$\begin{aligned} dom \circ id &= \mathsf{id}_{Ob} \\ cod \circ id &= \mathsf{id}_{Ob} \\ dom \circ comp &= dom \circ \pi_1 \\ cod \circ comp &= cod \circ \pi_2 \\ comp \circ (\mathsf{id}_{Mor}, id \circ cod) &= \mathsf{id}_{Mor} && \text{(right-composition with the identity is the identity)} \\ comp \circ (id \circ dom, \mathsf{id}_{Mor}) &= \mathsf{id}_{Mor} && \text{(same for left-composition)} \end{aligned}$$

  along with an additional identity expressing the associativity of composition (which is rather cumbersome to write down).

We can also define morphisms of internal categories to be morphisms of the underlying Ob and Mor objects that are natural with respect to dom, cod, id and $comp$.

In summary, for any lex category C, we just defined a category $\mathsf{Mod}(T_{Cat}, \mathsf{C})$ of categories internal to C. Moreover, any lex functor $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ sends any category object in C to a category object in D. Hence the construction $\mathsf{C} \mapsto \mathsf{Mod}(T_{Cat}, \mathsf{C})$ is actually a 2-functor

$$\mathsf{Mod}(T_{Cat}, -) : \mathsf{Cat}^{\mathsf{lex}} \to \mathsf{Cat}$$

Saying that category theory is an essentially algebraic theory amounts to saying that this functor is 2-representable, i.e. there exists a lex category $\mathsf{C}_{\mathsf{Cat}}$ and equivalences

$$\mathsf{Mod}(T_{Cat}, \mathsf{C}) \simeq \mathsf{Fun}^{\mathsf{lex}}(\mathsf{C}_{\mathsf{Cat}}, \mathsf{C})$$

for any lex category C, 2-natural in C. This is indeed the case, but a proof would go beyong the scope of this text. The idea is to take the "free lex category with 2 objects Ob and Mor, morphisms $dom, cod, id, comp$ and satisfying the above identities".

Given a lex category D and a (possibly multisorted) Lawvere theory C, we can consider D-valued models of C. There is a forgetful 2-functor

$$\mathsf{U} : \mathsf{Cat}^{\mathsf{lex}} \to \mathsf{Cat}^{\times}$$

from the 2-category of left-exact categories to the 2-category of categories with finite products and finite-product-preserving functors. This functor admits a left 2-adjoint FC (for Finite Completion). . The adjunction tells us that given a category C with finite products and a lex category D, there is an equivalence of categories

$$\mathsf{Fun}^{\times}(\mathsf{C}, \mathsf{D}) \simeq \mathsf{Fun}^{\mathsf{lex}}(\mathsf{FC}(\mathsf{C}), \mathsf{D})$$

In terms of models, this precisely states that the D-valued models of C seen as a (multisorted) Lawvere theory are the same as the D-valued models of $\mathsf{FC}(\mathsf{C})$ seen as an essentially algebraic theory. So the 2-functor FC "preserves models". Hence it allows us to say that "every algebraic theory is also essentially algebraic", in that FC identifies $\mathsf{Cat}^{\times}$ as a sub-2-category of $\mathsf{Cat}^{\mathsf{lex}}$ in a model-preserving way.

Example 1.23. The Lawvere theory of rings is the opposite of the category of free rings of finite presentation over $\mathbb{Z}$. To get the corresponding essentially algebraic theory, it suffices to freely add equalizers to $\mathsf{Ring}^{op}_{pf,free}$ while preserving the already existing finite products. It turns out the resulting category is equivalent to the opposite of Ring_{pf}, the category of rings of finite presentation over $\mathbb{Z}$. Alternatively, the essentially algebraic theory of rings is equivalent to the category of affine schemes of finite type over **Spec** $\mathbb{Z}$: AffSch_{pf}.

Categorical semantics can be further developped by considering categories with even more internal constructions, allowing an internal interpretation of logical conjunction, disjunction, or even existential/universal quantification and higher order logic. Every additional logical notion we'd like to interpret will give rise to a new type of category : finite product categories (allows the interpretation of equality = and logical conjuction $\wedge$), regular categories ($=, \wedge, \exists$), coherent categories ($=, \wedge, \vee, \exists$), etc.

The categories we study in the next section are logoi. Logoi are categories in which there is a natural interpretation for, among other things :

- finite logical conjuction $\wedge$,
- infinite logical disjunction $\bigvee$,
- existential quantification $\exists$.

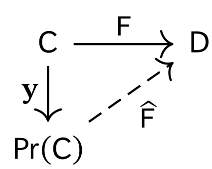
It turns out that infinite conjunction and universal quantification can also be expressed in logoi, but they need not be preserved by morphisms of logoi.

2 Logoi and logic

Definition 2.1. A **logos** (plural *logoi*) is a left-exact localization (see definition A.3) of a presheaf category $\mathsf{Pr(C)}$ for some small category C. A **morphism of logoi** between C and D is a left exact cocontinuous functor $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ (i.e. F preserves finite limits and arbitrary (small) colimits). The category of such morphisms is written $\mathsf{Fun}^{\mathsf{lex}}_{\mathsf{cc}}(\mathsf{C}, \mathsf{D})$. The 2-category of logoi, logoi morphisms and natural transformations is written Logos.

Given a small category C, the identity functor $id : \mathsf{Pr(C)} \to \mathsf{Pr(C)}$ is always a left-exact localization. Hence every presheaf category is a logos. The Yoneda embedding identifies $\mathsf{Pr(C)}$ as the free cocompletion of C, and is the first step towards characterizing logos morphisms from the presheaf logos $\mathsf{Pr(C)}$ to an arbitrary logos D.

Theorem 2.2 (Yoneda lemma)**.** *Given a small category* C*, a cocomplete category* D *and a functor* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$*, there is a unique cocontinuous functor* $\widehat{\mathsf{F}} : \mathsf{Pr(C)} \to \mathsf{D}$ *up to natural isomorphism, such that* $\widehat{\mathsf{F}}$ *extends* F *along the Yoneda embedding* $\mathbf{y}$*, i.e. such that the following diagram commutes.* $\widehat{\mathsf{F}}$ *is called the* Yoneda extension *of* F.



Theorem 2.3. *Given a small lex category* C*, a logos* L *and a lex functor* $\mathsf{F} : \mathsf{C} \to \mathsf{L}$*, the Yoneda extension* $\widehat{\mathsf{F}} : \mathsf{Pr(C)} \to \mathsf{L}$ *is left-exact. Since it is cocontinuous, it is a morphism of logoi.*

By restricting along $\mathbf{y}$*, we furthermore get an equivalence of categories :*

$$\begin{aligned} \mathsf{Fun}^{\mathsf{lex}}(\mathsf{C},\mathsf{L}) &\rightleftarrows \mathsf{Fun}^{\mathsf{lex}}_{\mathsf{cc}}(\mathsf{Pr}(\mathsf{C}),\mathsf{L}) \\ \mathsf{F} &\mapsto \widehat{\mathsf{F}} \\ \mathsf{G}\circ\mathbf{y} &\leftarrow\!\shortmid \mathsf{G} \end{aligned}$$

Remark 2.4. Logoi can actually be characterised as those categories L such that for any small lex catgegory C and lex functor $\mathsf{F} : \mathsf{C} \to \mathsf{L}$, the yoneda extension $\widehat{\mathsf{F}}$ stays lex [GL12].

Similarly to the relation between finite product categories and lex categories, this theorem lets us identify $\mathsf{Cat}^{\mathsf{lex}}$ as a sub-2-category of Logos in a model-preserving way.

2.1 Examples of logoi

Theorem 2.5. *Given a topological space* X*, the category* $\mathsf{Sh}(X)$ *of sheaves of sets over* X *is a logos.*

This theorem will be a corollary of a more general fact about categories of sheaves on a *site* (see theorem 2.13).

Definition 2.6. Given a small lex category C, a **Grothendieck pretopology** on C is the data, for any object c in C, of a collection $K(c)$ of families of morphisms with codomain c, satisfying the following conditions :

(i) for any isomorphism $f : c \to c'$ in C, $\{f : c \to c'\} \in K(c')$,

(ii) if $\{f_i : c_i \to c \mid i \in I\} \in K(c)$ and $g : d \to c$ is any morphism in C, then the family obtained by pullbacks along g, i.e. $\{c_i \times_c d \to d \mid i \in I\}$, is in $K(d)$,

(iii) given a family $\{f_i : c_i \to c \mid i \in I\} \in K(c)$ and for every $i \in I$, a family $\{g_{ij} : d_{ij} \to c_i \mid j \in J_i\} \in K(c_i)$, then the family of composites $\{f_i \circ g_{ij} : d_{ij} \to c \mid i \in I, j \in J_i\}$ is in $K(c)$.

The elements R of $K(c)$ are called *covers* or *covering families* of c. The data (C, K) of a small lex category C together with a Grothendieck pretopology K is called a **site**.

Remark 2.7. In general, sites are not required to be left-exact categories. But all the examples we will consider are left-exact, and it turns out that any logos admits a presentation by a left-exact site (see for instance the appendix of [MM92]), so we do not lose too much generality with that additional assumption.

Remark 2.8. A site is usually defined as a small category C equipped with a Grothendieck *topology* (defined in terms of sieves on C) rather than a Grothendieck pretopology, the latter being a non-canonical presentation for the former. But since sites are themselves non-canonical presentations for logoi (at least for our purposes), we stick with this definition.

Example 2.9. Given a topological space X, its set of open subsets $\mathcal{O}(X)$ is a poset under inclusion, hence a category. Open subsets are closed under finite intersection, hence the category $\mathcal{O}(X)$ is left-exact. A family of morphisms $\{f_i : U_i \to U \mid i \in I\}$ with same codomain U in $\mathcal{O}(x)$ is just a set of open subsets of U. Let $K(U)$ be the set of families $\{U_i \subseteq U \mid i \in I\}$ of open subsets of U such that $\bigcup_{i\in I} U_i = U$. This defines a Grothendieck pretopology on $\mathcal{O}(X)$. Covering families are then covering families in the usual sense of topology.

Example 2.10. The motivating example of a site that does not come frome a topological space comes from algebraic geometry, where the Zariski topology on schemes had "too few" open sets for the purposes of cohomology. In the 1960's, Grothendieck had the idea of considering not only coverings by open subsets $U \hookrightarrow X$, but by arbitrary étale maps $Y \to X$. This idea is what actually gave birth to the theory of sites and Grothendieck topoi, and eventually led to the resolution of the Weil conjectures.

Definition 2.11. Given a site (C, K), a presheaf $\mathsf{F} : \mathsf{C}^{op} \to \mathsf{Set}$ is called a **sheaf** if, for every object c in C and every covering family $\{f_i : c_i \to c \mid i \in I\} \in K(c)$, the following diagram is an equalizer in Set :

$$\mathsf{F}(c) \xrightarrow{\prod \mathsf{F}(f_i)} \prod_{i\in I} \mathsf{F}(c_i) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \prod_{i,j\in I} \mathsf{F}(c_i \times_c c_j)$$

In other words, given any family $(x_i)_{i\in I}$ with $x_i \in \mathsf{F}(c_i)$ for all i and such that for all $i, j \in I$, the images of x_i and x_j in $\mathsf{F}(c_i \times_c c_j)$ coincide, there exists a unique $x \in \mathsf{F}(c)$ such that for all $i \in I$, $\mathsf{F}(f_i)(x) = x_i$.

The full subcategory of $\mathsf{Pr}(\mathsf{C})$ spanned by sheaves for K is noted $\mathsf{Sh}(\mathsf{C}, K)$, or simply $\mathsf{Sh}(\mathsf{C})$ if K is evident from the context.

Example 2.12. In the case where $(\mathcal{O}(X), K)$ is the site associated to a topological space X, then a sheaf for K is just a sheaf of sets on X in the usual sense.

Theorem 2.13. *Given a site* (C, K)*, the category* $\mathsf{Sh}(\mathsf{C}, K)$ *is a left-exact localization of* $\mathsf{Pr}(\mathsf{C})$*, and hence a logos.*

Proof. See [AGV71][II.3]. □

The left adjoint to the inclusion $i : \mathsf{Sh}(\mathsf{C}, K) \hookrightarrow \mathsf{Pr}(C)$ will be written $\mathbf{a}$, and is called the **sheafification** functor, or **associated sheaf** functor.

Remark 2.14. The converse is also true : for any logos L, there exists a small site (C, K) such that L is equivalent to $\mathsf{Sh}(\mathsf{C}, K)$ [MM92].

The sheafification functor $\mathbf{a} : \mathsf{Pr}(C) \to \mathsf{Sh}(\mathsf{C}, K)$ preserves finite limits, and as a left adjoint, it also preserves small colimits. Hence it is a morphism of logoi. Pre-composition with $\mathbf{a}$ thus induces a functor

$$-\circ\mathbf{a} : \mathsf{Logos}(\mathsf{Sh}(\mathsf{C}, K), \mathsf{E}) \to \mathsf{Logos}(\mathsf{Pr}(\mathsf{C}), \mathsf{E})$$

for any logos E. Since $\mathsf{Sh}(\mathsf{C}, K)$ is a reflective localization of $\mathsf{Pr}(\mathsf{C})$ (see proposition A.2), we have the following.

Proposition 2.15. *For all logos E, pre-composition with* $\mathbf{a}$ *identifies* $\mathsf{Logos}(\mathsf{Sh}(\mathsf{C}, K), \mathsf{E})$ *as the full subcategory of* $\mathsf{Logos}(\mathsf{Pr}(\mathsf{C}), \mathsf{E})$ *spanned by functors that factor through* $i : \mathsf{Sh}(\mathsf{C}, K) \hookrightarrow \mathsf{Pr}(\mathsf{C})$.

Theorem 2.3 characterizes morphisms of logoi $\mathsf{Pr}(\mathsf{C}) \to \mathsf{E}$ as left kan extensions of lex functors $\mathsf{C} \to \mathsf{E}$. This equivalence can somewhat be extended to morphisms $\mathsf{Sh}(\mathsf{C}, K) \to \mathsf{E}$, by taking into account the pretopology K.

Definition 2.16. A lex functor $F : \mathsf{C} \to \mathsf{E}$ is said to be **continuous** with respect to the pretopology K if, for every cover $\{f_i : U_i \to U\}$ in K, the induced morphism $\coprod_i F(U_i) \to F(U)$ is an effective epimorphism (see definition A.5) in E.

Theorem 2.17. *A morphism of logoi* $F : \mathsf{Pr}(\mathsf{C}) \to \mathsf{E}$ *factors through the reflection* $\mathsf{Pr}(\mathsf{C}) \to \mathsf{Sh}(\mathsf{C}, K)$ *if and only if its restriction along the Yoneda embedding* $F \circ \mathbf{y} : \mathsf{C} \to \mathsf{E}$ *is a lex functor continuous with respect to* K.

Proof. See [MM92]. □

2.2 General properties of logoi

In this section, we prove general properties about logoi that will be useful later on. The rule of thumb here is that logoi "behave like Set" in many different ways, especially regarding the behaviour of finite limits and small colimits.

Proposition 2.18 (Stability of colimits)**.** *Let* E *be a logos,* D *a small category, and* $F : \mathsf{D} \to \mathsf{E}$ *a diagram in* E*. Then, for every pullback diagram,*

$$\begin{array}{ccc} \mathrm{colim}_{\mathsf{D}} F \times_X Y & \longrightarrow & \mathrm{colim}_{\mathsf{D}} F \\ \downarrow & \lrcorner & \downarrow f \\ Y & \xrightarrow[g]{} & X \end{array}$$

the canonical map $h : \mathrm{colim}_{d\in D}(F(d)\times_X Y) \to (\mathrm{colim}_D F)\times_X Y$ *is an isomorphism. In other words, colimits in a logos are stable under pullback.*

Proof. **1.** First we consider the case $\mathsf{E} = \mathsf{Set}$. If the colimit is a disjoint union (i.e. if D is discrete), we have $\mathrm{colim}_{\mathsf{D}} F = \{(d,x) \mid d \in \mathsf{D}, x \in F(d)\}$, and the canonical map h is the obvious rearranging of triplets

$$\begin{array}{c}\{(d,(x,y)) \mid d \in \mathsf{D}, x \in F(d), y \in Y, f_d(x) = g(y)\} \\ \big\downarrow h \\ \{((d,x),y) \mid d \in \mathsf{D}, x \in F(d), y \in Y, f_d(x) = g(y)\}\end{array}$$

which is clearly an isomorphism (where f_d is the the composite of $F(d) \to \mathrm{colim}_D F \to Y$). The case of quotients by equivalence relations is similar. Coproducts and quotients (coequalizers) generate all colimits, so the result is true in Set.
2. Let C be a small category. The limits and colimits in $\mathsf{Pr}(\mathsf{C})$ are computed objectwise in Set, hence the result is also true in any presheaf category.
3. Now let E be a logos. By definition, E is a left-exact localization of a presheaf category $\mathsf{Pr}(\mathsf{C})$ for some small C, so we have

$$\mathsf{E} \underset{i}{\overset{r}{\underset{\hookrightarrow}{\overset{\leftarrow}{\bot}}}} \mathsf{Pr}(\mathsf{C})$$

where r is left-exact and $r \circ i \simeq \mathrm{id}$. Hence colimits and finite limits are computed in E by taking the image by r of their computation in $\mathsf{Pr}(\mathsf{C})$. Since the result holds in $\mathsf{Pr}(\mathsf{C})$, it then also holds in E. □

Corollary 2.19. *In any logos, effective epimorphisms are stable under pullbacks.*

Proof. An effective epimorphism is a morphism that is the coequalizer of its kernel pair (A.5). Kernel pairs are stable under pullback, and by proposition 2.18, coequalizers are too. □

Proposition 2.20. *In any logos, all epimorphisms are effective epimorphisms.*

Proof. First, notice that all epimorphisms in Set are effective. Since limits and colimits in presheaf categories are computed objectwise, this is also the case in any presheaf category. The result extends to any logos analogously to the proof of proposition 2.18. □

Proposition 2.21. *In a logos* E*, a morphism* f *that is a monomorphism and an epimorphism is also an isomorphism.*

Proof. By proposition 2.20, any epimorphism is effective, and hence is a coequalizer. We show that, more generally, any monic coequalizer is an isomorphism. Let A, B be objects of E, $f, g : A \to B$ morphisms in E, and $h : B \to C$ be a coequalizer of f and g, such that h is a monomorphism. Then since $h \circ f = h \circ g$, we have $f = g$, and $\mathrm{id} : B \to B$ is a coequalizer of f and g. Hence h is an isomorphism. □

2.3 The Zariski logos

We have seen that the lex category AffSch_{pf} is the essentially algebraic theory of rings : for any lex category C, a ring-object in C is the same as the data of a lex functor $\mathsf{F} : \mathsf{AffSch}_{pf} \to \mathsf{C}$. Because of theorem 2.3, if E is a logos, we can also see ring-objects in E as morphisms of logoi $\mathsf{F} : \mathsf{Pr}(\mathsf{AffSch}_{pf}) \to \mathsf{E}$. We say that the logos $\mathsf{Pr}(\mathsf{AffSch}_{pf})$ **classifies** ring-objects, or is the **classifying logos** for the theory of rings. Classifying logoi will be written with the letter B followed by the name of the theory they classify. For instance we write $\mathsf{BRing} := \mathsf{Pr}(\mathsf{AffSch}_{pf})$.

Of particular interest in algebraic geometry is the notion of local rings : a local ring is a ring A with a unique maximal ideal $\mathbf{m}$, or equivalently, such that for every $x \in A$, one has

$$\exists y \in A,\ (x \star y = 1) \vee ((1 - x) \star y = 1).$$

This last characterization has the advantage of being written as a first order formula using only existential quantification and logical disjunction. Hence it can be interpreted in any logos. We will not enter the details of how to interpret any such first-order formula in a logos, but we will do it for this particular example.

Let's consider a logos E and a ring object R in E. The formulas $x \star y = 1$ and $(1 - x) \star y = 1$ can be interpreted as subobjects U and V of $|R| \times |R|$ by the pullbacks :

$$\begin{array}{ccc} U & \longrightarrow & * \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle 1_R} \\ |R| \times |R| & \xrightarrow[u]{} & |R| \end{array} \qquad\qquad \begin{array}{ccc} V & \longrightarrow & * \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle 1_R} \\ |R| \times |R| & \xrightarrow[v]{} & |R| \end{array}$$

where

- $*$ is a terminal element of E,
- $1_R : * \hookrightarrow R$ defines the "neutral element" for the multiplication of R,
- u is the multiplication of R,
- v is defined by $v(x, y) = (1 - x) \star y$ (see definition 1.5 on how to interpret this formula as a morphism in E).

In the case $\mathsf{E} = \mathrm{Set}$, U would be the subset $\{(x, y) \in |R| \times |R| \mid x \star y = 1\}$ of $|R| \times |R|$, and similarly, V would be $\{(x, y) \in |R| \times |R| \mid (1 - x) \star y = 1\}$.

Definition 2.22. A ring object R in a logos E is said to be local if the pair of morphisms

$$\begin{array}{c} U \hookrightarrow |R|^2 \xrightarrow{\pi_1} |R| \\ \\ V \hookrightarrow |R|^2 \xrightarrow{\pi_1} |R| \end{array} \tag{1}$$

previously defined form an epimorphic family in E.

Remark 2.23. In the case of $\mathsf{E} = \mathrm{Set}$, this definition recovers the usual notion of local ring.

Proposition 2.24. *A ring object R in a logos E, defined by a left-exact functor ϕ_R : $\mathsf{AffSch}_{pf} \to \mathsf{E}$, is local if and only if the pair of morphisms*

$$\begin{array}{ll} \mathbb{Z}[X] & \to \mathbb{Z}[X, Y]/(X \star Y - 1) \\ \mathbb{Z}[X] & \to \mathbb{Z}[X, Y]/((1 - X) \star Y - 1) \end{array} \tag{2}$$

is sent to an epimorphic family in E (with codomain $\phi_R(\mathbb{A}) = |R|$).

Proof. The functor ϕ_R is left exact, hence given any affine scheme of finite presentation $X = \textbf{Spec}\ \mathbb{Z}[X_1, \dots, X_n]/(P_1, \dots, P_k)$, $\phi_R(X)$ can be defined as the following defined as the following equalizer in E :

$$\phi_R(X) \hookrightarrow |R|^k \underset{(0, \dots, 0)}{\overset{(P_1, \dots, P_k)}{\rightrightarrows}} |R|$$

The previous monomorphisms $U \hookrightarrow |R|^2$ and $V \hookrightarrow |R|^2$ can also be defined as the following equalizers :

$$U \hookrightarrow |R|^2 \underset{0}{\overset{X \star Y - 1}{\rightrightarrows}} |R|$$

$$V \hookrightarrow |R|^2 \underset{0}{\overset{(1 - X) \star Y - 1}{\rightrightarrows}} |R|$$

Hence the morphisms (1) are just the image by the functor ϕ_R of the morphisms (2). □

Definition 2.25. The **Zariski pretopology** Zar on AffSch_{pf} is defined by

$$Zar(\mathbf{Spec}\ A) = \{\{\mathbf{Spec}\ A[a_i^{-1}] \to \mathbf{Spec}\ A \mid 1 \le i \le n\} \mid a_1, \dots, a_n \in A, \\ \text{the } (a_i)_i \text{ generate the unit ideal of } A\}$$

i.e. covers for the Zarisky pretopology are covers by open affine subschemes in the usual sense of algebraic geometry. Zar defines a Grothendieck pretopology on AffSch_{pf}, and $(\mathsf{AffSch}_{pf}, Zar)$ is called the *Zariski site*.

Proposition 2.26. *Let R be a ring object in a logos E, characterized by a left exact functor $\phi_R : \mathsf{AffSch}_{pf} \to \mathsf{E}$. Then ϕ_R sends Zariski coverings to epimorphic families in E if and only if it sends the pair of morphisms (2) to an epimorphic family.*

Proof. $\boxed{\Longrightarrow}$ The pair of morphisms (2) is a Zariski covering.
$\boxed{\Longleftarrow}$ Now suppose ϕ_R sends the pair of morphisms (2) to an epimorphic family in E. Let A be a ring in Ring_{pf} and $a_1, \dots, a_n$ elements of A such that the $(a_i)_i$ generate the unit ideal of A. We show by induction on n that ϕ_R sends the family

$$\{A \to A[a_i^{-1}] \mid 1 \le i \le n\}$$

to an epimorphic family.

The case $n = 1$ is the case where a_1 is invertible, so the morphism $A \to A[a_1^{-1}]$ is an isomorphism and is thus sent to an epimorphism.

Now suppose $n \ge 2$ and for all $k < n$, all A' in Ring_{pf}, and all $a'_1, \dots a'_k \in A'$ such that the $(a'_i)_i$ generate the unit ideal in A', ϕ_R sends $\{A' \to A'[(a'_i)^{-1}] \mid 1 \le i \le k\}$ to an epimorphic family in E.

Let $b_1, \dots, b_n \in A$ such that $\sum_{i=1}^n a_i b_i = 1_A$. Consider the following pushout diagrams in Ring_{pf}.

$$\begin{array}{ccccc} Z[X,Y]/(XY-1) & \longleftarrow & \mathbb{Z}[X] & \longrightarrow & Z[X,Y]/((1-X)Y-1) \\ \downarrow & \lrcorner & \downarrow {\scriptstyle X \mapsto a_n b_n} & \ulcorner & \downarrow \\ A[(a_n b_n)^{-1}] & \longleftarrow & A & \longrightarrow & A[(\sum_{i<n} a_i b_i)^{-1}] \end{array} \tag{3}$$

Their image by ϕ_R are hence two pullbacks in E. Since ϕ_R sends (2) to an epimorphic family, and epimorphisms are stable under pullbacks in a logos (corollary 2.19 and proposition 2.20), ϕ_R sends the bottom line of the diagram (3) to an epimorphic family in E.

Write $b = \sum_{i<n} a_i b_i$. Using the induction hypothesis with $A' = A[b^{-1}]$ and $a'_i = a_i$ for $i < n$, we deduce that the following family is sent to an epimorphic family in E.

$$\{A \to A\left[b^{-1}, (a_i b_i)^{-1}\right] \mid i < n\} \cup \{A \to A\left[(a_n b_n)^{-1}\right]\}$$

But this family of morphisms factors through the family

$$\{A \to A[a_i^{-1}] \mid 1 \leq i \leq n\}$$

Hence this last family is also sent to an epimorphic family in E. □

Theorem 2.27. *The logos of sheaves on the Zariski site classifies local rings. We write* $\mathsf{BLocRing} := \mathsf{Sh}(\mathsf{AffSch}_{pf}, Zar)$.

Proof. Combine theorem 2.17 with propositions 2.24 and 2.26. □

3 Topoi and logoi

As we have seen in section 2, logoi are categories in which one can interpret some fragments of first-order logic. Their structure is rather algebraic (construction of limits, colimits, etc. and functors that preserve such constructions). In this section however, we will explore geometric aspects of logoi, or rather objects of the 2-category opposite to Logos.

Definition 3.1. The 2-category Topos is the opposite of the 2-catgory Logos. A topos is an object of Topos.

3.1 Frames and locales

In order to understand the geometric aspects of topoi, we first take a detour through the realm of topology.

The topological spaces of algebraic geometry are sober spaces : any irreducible closed subset admits a unique generic point. This is also the case of any Hausdorff (T_2) space. Sober spaces have the nice property that they are entirely determined by their algebras of open sets. More precisely, given any topological space X, its set of open subsets $\mathcal{O}(X)$ equipped with the operations of arbitrary union and finite intersection forms a *frame*.

Definition 3.2. A **frame** is a poset O with arbitrary joins $\bigvee$ and finite meets $\wedge$, satisfying the following *infinite distributive law* : for any $x, (y_i)_i \in O$,

$$x \wedge (\bigvee_i y_i) = \bigvee_i (x \wedge y_i)$$

A **morphism of frames** is an order-preserving map that furthermore preserves finite meets and arbitrary joins.

Given topological spaces X, Y and a continuous map $f : X \to Y$, we get a frame homomorphism $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$. We hence have a contravariant functor from topological spaces to frames $\mathcal{O} : \mathsf{Top}^{op} \to \mathsf{Frames}$. The previous statement that sober spaces are entirely determined by their algebras of open spaces can now be reformulated as follows :

Proposition 3.3. *The induced functor from sober topological spaces to frames*

$$\mathsf{Sober}^{op} \hookrightarrow \mathsf{Top}^{op} \xrightarrow{\mathcal{O}} \mathsf{Frames}$$

is fully faithful.

The contravariance of the functor $\mathcal{O}$ motivates the definition of *locales*, which are just frames viewed in their opposite category.

Definition 3.4. The category of **locales** is defined to be the opposite of the category of frames.

$$\mathsf{Locales} := \mathsf{Frames}^{op}$$

Locales should be thought of as spaces that might not have *enough points*. More precisely, one can define a **point** x of a locale X to be a morphism from the terminal locale to X (the terminal locale being the image of the singleton space by the inclusion $\mathsf{Sober} \hookrightarrow \mathsf{Locales}$). Given a locale X, there is always a canonical morphism $\mathsf{Pts}(X) \to X$, where $\mathsf{Pts}(X)$ is the discrete locale of points of X. X comes from a topological space if and only if that map is an epimorphism, in which case, X is said to **have enough points**, and its set of points as a topological space is precisely $\mathsf{Pts}(X)$.

Remark 3.5. Any locale in the essential image of $\mathsf{Top} \to \mathsf{Locales}$ is also in the essential image of $\mathsf{Sober} \hookrightarrow \mathsf{Locales}$. This is because any topological X space admits a *soberification* X^{sob} with the same underlying frame of open sets (furthermore, the soberification operation actually makes Sober into a reflexive subcategory of Top).

Remark 3.6. The category of topological spaces can actually be recovered from the category of locales. Indeed, a topological space X can be described as a set E together with a subframe of its power set : $\mathcal{O}(X) \subset \mathcal{P}(E)$. Frames of the form $\mathcal{P}(E)$ for some set E are said to be *discrete*, since they come from the discrete topology on E. With this definition, one can prove (basically by definition) that Top is equivalent to the full subcategory of $\mathsf{Locales}^{\to}$ spanned by the epimorphisms $O \twoheadrightarrow O'$ whose domain is discrete (where $\mathsf{C}^{\to}$ is the arrow category of C).

Remark 3.7. Here we already start to see an analogy with logoi and topoi. Logoi, like frames, are algebraic structures. The existence of finite limits and small colimits can be seen as a generalization of the existence of finite meets and arbitrary joins. The infinite distributive law of frames generalize to proposition 2.18 in logoi. The fact that topoi can be understood geometrically is motivated by the relationship between frames and topological spaces, or rather, locales. Actually, topoi are *categorified* locales.

Let X be a topological space. Any open subset U of X is uniquely determined by its characteristic function with values into the Sierpinski space $\mathbf{S} := \{0, 1\}$ where 0 is a closed point and 1 is an open point.

$$\begin{aligned} \mathcal{O}(X) &\rightleftarrows \mathcal{C}(X, \mathbf{S}) \\ U &\mapsto \left(x \mapsto \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases} \right) \\ \{x \in X / f(x) = 1\} &\mapsfrom f \end{aligned}$$

Now more generally, given a locale X, we write $\mathcal{O}(X)$ for the frame corresponding to X. $\mathcal{O}(X)$ needs to be understood as the frame of "opens" of the locale X. Just like in the case of topological spaces, an "open" $U \in \mathcal{O}(X)$ of a locale X is entirely determined by its characteristic function $\phi_U : X \to \mathbf{S}$. Indeed, $\mathcal{O}(\mathbf{S})$ is the frame $\{\emptyset, \{1\}, \{0, 1\}\}$. Hence any frame morphism $\mathcal{O}(\mathbf{S}) \to \mathcal{O}(X)$ is entirely determined by the image of $\{1\}$, which can be any element of $\mathcal{O}(X)$.

This apparent tautology tells us that the Sierpinski space *classifies* frames of open sets : the contravariant identity functor from locales to frames is represented by $\mathbf{S}$, i.e. there is a natural isomorphism of functors

$$\mathcal{O}(-) \simeq \mathsf{Hom}_{\mathsf{Locales}}(-, \mathbf{S}). \tag{4}$$

The frame structure on $\mathsf{Hom}_{\mathsf{Locales}}(X, \mathbf{S})$ comes from the fact that $\mathbf{S}$ has a frame structure internal to the category of locales. It is said to be a *dualizing* object in the categories of frames and locales.

The frame associated to a locale X is interpreted as the structure of open subspaces of X. We now need to understand how to interpret the logos associated to a topos $\mathcal{X}$.

3.2 Topoi and sheaves

Let X, Y be a topological spaces, and $f : Y \to X$ a continuous map. From f, we can construct a sheaf on X : the sheaf Γ_f of local sections of f. It is defined by $\Gamma_f(U) = \{s : U \to Y \mid f \circ s = \mathrm{id}_U\}$. A function $s : U \to Y$ such that $f \circ s = \mathrm{id}_U$ is called a section of f on U.

Reciprocally, given any sheaf F on X, it is possible to construct a space ΛF and a function $\Lambda F \to X$ such that its sheaf of local sections is isomorphic to F.

Definition 3.8. Let X be a topological space, F a sheaf of sets on X and x a point of X. The **fiber** of F at x, written F_x is defined to be

$$\mathrm{colim}_{U/x\in U} F(U)$$

If F is the sheaf of local sections of a continuous map $f : Y \to X$, then for every $x \in X$, one has $F_x \simeq f^{-1}(x)$. This suggests the definition $\Lambda F = \{(x, s) \mid x \in X, s \in F_x\}$. It is indeed possible to give a topology on ΛF such that the first projection $\Lambda F \to X$ is continuous and $\Gamma\Lambda F \simeq F$ [MM92].

The topology on ΛF makes the projection unto X a *local homeomorphism* : for every $y \in \Lambda F$, there exists an open neighbourhood U of x mapped homeomorphically unto an open subset of X. Local homeomorphisms are also called **étale spaces** : one says that ΛF is étale over X.

Proposition 3.9. *The functors Γ and Λ induce an equivalence of categories between the category of sheaves of sets on X and the category of étale spaces over X.*

Proof. See [MM92]. □

Because of this, the logos $\mathsf{Sh}(X)$ can be understood as the "algebra of étale spaces over X".

Proposition 3.10. *Let X be a topological space and U an open subset of X. The inclusion $U \hookrightarrow X$ is an étale space over X.*

Proof. Open inclusions are homeomorphisms unto their image. □

Even more precisely, an étale space $f : Y \to X$ is the inclusion of an open subset of X if and only if the fiber of f at every point of X contains at most one element. In some sense, open sets of X are "continuous at-most-singletons over X", while étale spaces over X are "continuous sets over X".

It is in this regard that topoi generalize topological spaces and locales : instead of being characterized by their frames of open sets, they are defined by their logoi of étale maps (or equivalenty, logoi of sheaves). Because of this, for every topos $\mathcal{X}$, we write $\mathsf{Sh}(\mathcal{X})$ for its associated logos.

3.3 Points of a topos, examples of topoi

The logos Set is initial in the 2-category of logoi. Its corresponding topos is thus the initial topos, and will be written **1** (notice that Set is indeed the category of sheaves on the one-point space, even in the case of topological spaces).

Definition 3.11. A point of a topos $\mathcal{X}$ is a morphism of topoi $\mathbf{1} \to \mathcal{X}$. The 2-category structure of Topos makes the points of $\mathcal{X}$ into a category, written $\mathsf{Pts}(\mathcal{X})$.

Example 3.12. Let X be a sober topological space (or a locale), and $\mathcal{X}$ the topos corresponding to the logos $\mathsf{Sh}(X)$. Then the category of points of $\mathcal{X}$ is equivalent to the set of points of X, ordered under the specialization relation.

Let T be a logos seen as a theory (i.e. logoi morphisms from T to a logos E are seen as models of T in E), and $\mathcal{T}$ its corresponding topos. Then $\mathsf{Pts}(\mathcal{T})$ is by definition the category of Set-valued models of T.

Example 3.13. The theory of sets is the first-order algebraic theory with empty signature and no axioms. As a Lawvere theory, it is the subcategory of Set spanned by finite sets. As an essentially algebraic theory, it's also the category FinSet of finite sets. Hence, as a logos, by theorem 2.3, it's the category Pr(FinSet). Models of Pr(FinSet) in Set are sets with no structure, i.e., sets. We write $\mathcal{S}et$ for the corresponding topos. The category of points of $\mathcal{S}et$ is Set, hence $\mathcal{S}et$ is the "space of sets", which makes sense in the context of topoi, but not in topology.

Given any topos $\mathcal{X}$, a morphism of topoi $\mathcal{X} \to \mathcal{S}et$ corresponds to an element of $\mathsf{Sh}(\mathcal{X})$. Because of this, in the category of topoi, the idea that "a sheaf of sets on $\mathcal{X}$ is a set varying continuously over $\mathcal{X}$" is made precise by the following equivalence of categories.

$$\mathsf{Sh}(\mathcal{X}) \simeq \mathsf{Hom}_{\mathsf{Topos}}(\mathcal{X}, \mathcal{S}et) \tag{5}$$

Remark 3.14. Notice that equation (5) is very similar to equation (4). Indeed, $\mathcal{S}et$ plays the same role in topoi as the Sierpinski space $\mathbf{S}$ played in the theory of locales and frames : it is a dualizing object in topoi and logoi. The notion of dualizing object is very natural in algebraic geometry, the simplest example being the affine line $\mathbb{A} = \mathbf{Spec}\mathbb{Z}[X]$. Indeed, the global sections functor Γ : Schemes $\to$ Ring is represented by $\mathbb{A}$. The ring structure on $\mathsf{Hom}(X, \mathbb{A})$ for a scheme X comes from the ring structure on the scheme $\mathbb{A}$, or, dually, from the *co-ring* structure on $\mathbb{Z}[X]$ given by

$$\begin{aligned} + : \mathbb{Z}[X] &\to \mathbb{Z}[Y, Z] \\ X &\mapsto Y + Z \\ \times : \mathbb{Z}[X] &\to \mathbb{Z}[Y, Z] \\ X &\mapsto Y \times Z \end{aligned}$$

Example 3.15. Let $\mathcal{R}ing$ be the topos corresponding to the logos $\mathsf{Pr}(\mathsf{AffSch}_{pf})$, and $\mathcal{L}oc\mathcal{R}ing$ be the topos corresponding to the logos $\mathsf{Sh}(\mathsf{AffSch}_{pf}, Zar)$. Then $\mathsf{Pts}(\mathcal{R}ing)$ is the category Ring of rings, and $\mathsf{Pts}(\mathcal{L}oc\mathcal{R}ing)$ is the full subcategory of Ring spanned by local rings. The left-exact localization $\mathsf{Pr}(\mathsf{AffSch}_{pf}) \to \mathsf{Sh}(\mathsf{AffSch}_{pf}, Zar)$ induces a morphism of topoi $\mathcal{LocRing} \hookrightarrow \mathcal{Ring}$, identifying the space of local rings as a subspace of the space of rings.

Let $\mathcal{X}$ be a topos. A ring-object in the logos $\mathsf{Sh}(\mathcal{X})$ is the same thing as a morphism of topoi $\mathcal{X} \to \mathcal{Ring}$. By analogy with example 3.13, such a morphism is called a *sheaf of rings* on $\mathcal{X}$. More generally, when a topos $\mathcal{T}$ is thought of as classifying some algebraic structures, a morphism $\mathcal{X} \to \mathcal{T}$ is said to be a $\mathcal{T}$-valued sheaf on $\mathcal{X}$, or a sheaf of $\mathcal{T}$-models (e.g. sheaf of groups, sheaf of rings, sheaf of local rings). This notion of sheaf generalizes the notion of C-valued sheaf in ordinary topology (where C is a category of algebraic objects).

4 Topoi in geometry and logic

In this section, we consider structured topoi. A structured topos is a topos equipped with a $\mathcal{K}$-valued sheaf for a topos $\mathcal{K}$. The topos $\mathcal{T}$ is equipped with an additional structure that allows to distinguish a certain class of morphisms between structured topoi. Think of standard algebraic geometry, where we consider topological spaces equipped with sheaf of local rings, but are only interested in *local morphisms*, i.e. morphisms that preserve the maximal ideals of the stalks. The notion of structured topoi generalizes this in two ways : replacing ordinary topological spaces by arbitrary topoi, and rings by a general notion of *geometric structure*.

Definition 4.1 ([Lur11]). Given a (small) category C, an admissibility structure on C is the data of :

- a Grothendieck pretopology τ on C.
- a class $\mathcal{A}$ of morphisms of C, elements of which will be called *admissible morphisms*.

satisfying the following conditions :

(i) for every object c in C and every cover $\{f_i : c_i \to c \mid i \in I\} \in \tau(c)$, every f_i is in $\mathcal{A}$.

(ii) for every commutative triangle

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{\quad h \quad} & & Z \end{array}$$

in C where g is in $\mathcal{A}$, then $f \in \mathcal{A} \iff h \in \mathcal{A}$.

(iii) for every admissible morphism $f : U \to X$ and any morphism $g : X' \to X$ in C, there exists a pullback square

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

such that f' is admissible.

(iv) every retract of an admissible morphism is admissible.

Remark 4.2. Let $(\mathsf{C}, \tau, \mathcal{A})$ be a category with an admissibility structure. Let $f : c' \xrightarrow{\sim} c$ be an isomorphism in C. Since $\{f\}$ cover of c, then because of (i), f is admissible. Now if $f : e \to d$ and $g : d \to c$ are admissible, then by (ii), $g \circ f$ is admissible. Hence, admissible morphisms form a subcategory C^{ad} of C. The pretopology τ is also automatically a pretopology on C^{ad} by virtue of (i).

Remark 4.3. Condition (iii) states that every admissible morphism $f : U \to X$ admits an admissible pullback $f' : U' \to X'$ along any $g : X' \to X$. If f'' is another such pullback, then it is a retract of f', and hence by (iv) is admissible. So any pullback of an admissible morphism is admissible.

Definition 4.4 ([Lur11]). A **geometry** is a small lex category G together with an admissibility structure.

Example 4.5. The category $\mathtt{AffSch}_{pf}$ together with the Zariski pretopology, and as admissible morphisms the open immersions **Spec** $R[a^{-1}] \to$ **Spec** R, is a geometry. It will be written G_{Zar}.

Proposition 4.6. *Let* G *be a small lex category. We can define a trivial admissiblity structure on* G*, as follows : The Grothendieck pretopology on* G *is the trivial pretopology, where the covers of every element c are the isomorphisms* $f : c' \xrightarrow{\sim} c$*. The admissible morphisms are the isomorphisms. This makes* G *into a geometry, called the* ***discrete geometry*** *on* G.

Definition 4.7. Let G and G' be geometries. A functor $f : \mathsf{G} \to \mathsf{G}'$ is said to be a **transformation of geometries** if :

- f is left exact.
- f sends admissible morphisms in G to admissible morphisms in G'.
- f sends covering families in G to covering families in G'.

Example 4.8. Let G be a geometry. The underlying category of G can be endowed with the discrete geometry, making another geometry $\mathsf{G}_{\mathrm{disc}}$. Then the identity functor $\mathrm{id} : \mathsf{G}_{\mathrm{disc}} \to \mathsf{G}$ is a transformation of geometries.

Example 4.9. Let G be a geometry with $\mathcal{A}$ its set of admissible morphisms. The underlying category of G together with the discrete pretopology and the set of admissible morphisms $\mathcal{A}$ form a geometry $\mathsf{G}_{\mathrm{mix}}$. The identity functor then gives a chain of transformations of geometries :

$$\mathsf{G}_{\mathrm{disc}} \to \mathsf{G}_{\mathrm{mix}} \to \mathsf{G}$$

Definition 4.10. Given a geometry G and a topos $\mathcal{X}$, a G-**structure** on $\mathcal{X}$ is a logos morphism $\mathcal{O} : \mathsf{Sh}(\mathsf{G}) \to \mathsf{Sh}(\mathcal{X})$ (where G is seen as a site). Given G-structures $\mathcal{O}$ and $\mathcal{O}'$ on $\mathcal{X}$, a natural transformation $\alpha : \mathcal{O} \to \mathcal{O}'$ is said to be a **conservative (or local) transformation of** G-**structures** if, for every admissible morphism $f : V \to U$ in G, the induced diagram

$$\begin{array}{ccc} \mathcal{O}(V) & \xrightarrow{\alpha_U} & \mathcal{O}'(V) \\ {\scriptstyle \mathcal{O}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{O}'(f)} \\ \mathcal{O}(U) & \xrightarrow[\alpha_X]{} & \mathcal{O}'(U) \end{array}$$

is a pullback square.

The category of G-structures on $\mathcal{X}$ and natural transformations is written $\mathsf{Str}_{\mathsf{G}}(\mathcal{X})$. Its subcategory whose natural transformations are conservative is written $\mathsf{Str}_{\mathsf{G}}^{cons}(\mathcal{X})$.

Remark 4.11. What we call conservative morphisms are more often called local morphisms by analogy with the case of the Zariski geometry (see example 4.12). The term "conservative" is chosen by analogy with conservative functors, i.e. functors F such that $F(f)$ is an isomorphism if and *only if* f is an isomorphism. We prefer the term "conservative" and the abbrevation *cons* over the term "local" and its abbreviation *loc*, which can be confused with *localizing* morphisms (i.e. morphisms that are localizations).

Example 4.12. Given a topological space X, a G_{Zar}-structure on X is precisely a sheaf of local rings on X, i.e. a sheaf of rings O_X whose stalks $O_{X,x}$ are local rings for every $x \in X$. A morphism of sheaves of local rings $\alpha : O_X \to O'_X$ is local (i.e. conservative) precisely when the induced morphism on stalks $\alpha_x : O_{X,x} \to O'_{X,x}$ is a local morphism of local rings (in the usual sense) for every $x \in X$.

Remark 4.13. By theorem 2.17, a G-structure on a topos $\mathcal{X}$ can equivalently be defined as a lex functor $\mathcal{O} : \mathsf{G} \to \mathsf{Sh}(\mathcal{X})$ such that for every covering family $\{f_i : U_i \to U\}$ in G, the induced map $\coprod_i \mathcal{O}(U_i) \to \mathcal{O}(U)$ is an effective epimorphism in $\mathsf{Sh}(\mathcal{X})$. From this point of view, a $\mathsf{G}_{\mathrm{disc}}$-structure is a just a lex functor $\mathcal{O} : \mathsf{G} \to \mathsf{Sh}(\mathcal{X})$. G-structures are then just $\mathsf{G}_{\mathrm{disc}}$-structures satisfying an additional property. Hence $\mathsf{Str}_{\mathsf{G}}(\mathcal{X})$ can be seen as a full subcategory of $\mathsf{Str}_{\mathsf{G}_{\mathrm{disc}}}$. A similar idea identifies $\mathsf{Str}_{\mathsf{G}}^{cons}(\mathcal{X})$ with the full subcategory of $\mathsf{Str}_{\mathsf{G}_{\mathrm{mix}}}^{cons}(\mathcal{X})$ whose objects are G-structures. We even have the following pullback diagram :

$$\begin{array}{ccccc} \mathsf{Str}_{\mathsf{G}}^{cons}(\mathcal{X}) & \hookrightarrow & \mathsf{Str}_{\mathsf{G}_{\mathrm{mix}}}^{cons}(\mathcal{X}) & & \\ \downarrow & \lrcorner & \downarrow & & \\ \mathsf{Str}_{\mathsf{G}}(\mathcal{X}) & \hookrightarrow & \mathsf{Str}_{\mathsf{G}_{\mathrm{mix}}}(\mathcal{X}) & = & \mathsf{Str}_{\mathsf{G}_{\mathrm{disc}}}(\mathcal{X}) \end{array}$$

Theorem 4.14. *(1) Let* G *be a geometry and* $\mathcal{X}$ *a topos. There exists a factorization system* $(S_L^{\mathcal{X}}, S_R^{\mathcal{X}})$ *on* $\mathsf{Str}_{\mathsf{G}}(\mathcal{X})$*, where* $S_R^{\mathcal{X}}$ *is precisely the class of conservative morphisms. Morphisms in* $S_L^{\mathcal{X}}$ *are called* ***localizations****.*

(2) This factorization system is functorial in $\mathcal{X}$*. In other words, given a geometric morphism* $f : \mathcal{X} \to \mathcal{Y}$*, the induced functor* $f^* \circ - : \mathsf{Str}_{\mathsf{G}}(\mathcal{Y}) \to \mathsf{Str}_{\mathsf{G}}(\mathcal{X})$ *carries* $S_L^{\mathcal{Y}}$ *to* $S_L^{\mathcal{X}}$ *and* $S_R^{\mathcal{Y}}$ *to* $S_R^{\mathcal{X}}$*.*

We write $\mathsf{Str}_{\mathsf{G}}^{loc}(\mathcal{X})$ *for the subcategory of* $\mathsf{Str}_{\mathsf{G}}(\mathcal{X})$ *whose morphisms are localizations.*

Remark 4.15. A class of morphisms in a category C can be seen as a full subcategory of $\mathsf{C}^{\to}$. When seen this way, S_L^{-} and S_R^{-} define sub-2-functors of the 2-functor

$$\begin{array}{rcl} \mathsf{Topos}^{op} & \to & \mathsf{Cat} \\ \mathcal{X} & \mapsto & \mathsf{Str}_{\mathsf{G}}(\mathcal{X}) \end{array}$$

Example 4.16. Taking G to be the Zariski geometry and $\mathcal{X} = \mathbf{1}$ to be the one-point topos, this recovers the usual (localization,local) factorization system on $\mathsf{LocRing}$.

Definition 4.17. Let G be a geometry. We define a 2-category $\mathcal{T}op(\mathsf{G})$ as follows.

- Objects of $\mathcal{T}op(\mathsf{G})$ are pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{X}$ is a topos and $\mathcal{O}_{\mathcal{X}} : \mathsf{Sh}(\mathsf{G}) \to \mathsf{Sh}(\mathcal{X})$ is a G-structure on $\mathcal{X}$.
- A morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is the data of a topos morphisms $f : \mathcal{X} \to \mathcal{Y}$ together with a *local* transformation $\alpha : f^*\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ of G-structures on $\mathcal{X}$.
- A 2-isomorphism

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \underset{(g,\beta)}{\overset{(f,\alpha)}{\rightrightarrows}} (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \qquad \gamma : (f,\alpha) \overset{\sim}{\Rightarrow} (g,\beta)$$

is a natural isomorphism $\gamma : f^* \to g^*$ such that horizontal composition with γ induces a commutative triangle :

$$\begin{array}{ccc} f^*\mathcal{O}_\mathcal{Y} & \xrightarrow[\sim]{\gamma\mathcal{O}_\mathcal{Y}} & g^*\mathcal{O}_\mathcal{Y} \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & \mathcal{O}_\mathcal{X} & \end{array}$$

Proposition 4.18. *Let* G *and* G' *be geometries, and* $f : \mathsf{G} \to \mathsf{G}'$ *be a transformation of geometries. Then f induces a functor* $\mathcal{T}\mathrm{op}(\mathsf{G}') \to \mathcal{T}\mathrm{op}(\mathsf{G})$ *by taking* $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ *to* $(\mathcal{X}, \mathcal{O}_\mathcal{X} \circ f)$.

Remark 4.19. The pullback diagram of remark 4.13 also holds when replacing *cons* by *loc*.

Example 4.20. Take $\mathsf{G}' = \mathsf{G}_{Zar}$ and G to be the discrete geometry on $\mathtt{AffSch}_{pf}$. Then the identity $\mathrm{id} : \mathsf{G} \to \mathsf{G}'$ induces the forgetful functor from locally ringed topoi to ringed topoi. We will show that it admits a right adjoint, which will generalize the usual notion of *spectrum* from algebraic geometry.

Theorem 4.21. *Let* G *be a geometry, and* $\mathsf{G}_{\mathrm{disc}}$ *its discrete counterpart. The induced functor* $\mathcal{T}\mathrm{op}(\mathsf{G}_{\mathrm{disc}}) \to \mathcal{T}\mathrm{op}(\mathsf{G})$ *admits a right adjoint, called the* ***spectrum*** *functor.*

$$\mathbf{Spec} : \mathcal{T}\mathrm{op}(\mathsf{G}_{\mathrm{disc}}) \to \mathcal{T}\mathrm{op}(\mathsf{G})$$

Before proving this theorem, we prove a few results about geometries and factorization systems.

Lemma 4.22. *Let* G *be a geometry,* $\mathcal{X}$ *a topos,* $\mathcal{O} : \mathsf{G}_{\mathrm{mix}} \to \mathsf{Sh}(\mathcal{X})$ *a* $\mathsf{G}_{\mathrm{mix}}$*-structure,* $\mathcal{O}' : \mathsf{G} \to \mathsf{Sh}(\mathcal{X})$ *a* G*-structure, and* $\alpha : \mathcal{O} \to \mathcal{O}'$ *a conservative morphism (of* $\mathsf{G}_{\mathrm{mix}}$*-structures). Then* $\mathcal{O}$ *is in fact a* G*-structure.*

Proof. Let $\{f_i : U_i \to U\}$ be a covering family in G. We must show that the induced map $\coprod_i \mathcal{O}(U_i) \to \mathcal{O}(U)$ is an effective epimorphism in $\mathsf{Sh}(\mathcal{X})$. We know that α is conservative, and by proposition 2.18, coproducts are stable under pullbacks in the logos $\mathsf{Sh}(\mathcal{X})$, hence the following square is cartesian.

$$\begin{array}{ccc} \coprod_i \mathcal{O}(U_i) & \xrightarrow{\coprod_i \alpha_{U_i}} & \coprod_i \mathcal{O}'(U_i) \\ {\scriptstyle \coprod_i \mathcal{O}(f_i)}\downarrow & & \downarrow{\scriptstyle \coprod_i \mathcal{O}'(f_i)} \\ \mathcal{O}(U) & \xrightarrow{\alpha_U} & \mathcal{O}'(U) \end{array}$$

Since $\mathcal{O}'$ is a G-structure, the arrow $\coprod_i \mathcal{O}'(f_i)$ is an effective epimorphism. But effective epimorphisms are also stable under pullbacks in $\mathsf{Sh}(\mathcal{X})$ (by corollary 2.19). Hence $\mathcal{O}$ is a G-structure. □

Definition 4.23. A 2-functor $F : \mathsf{Topos}^{op} \to \mathsf{Cat}$ is said to be representable if it is equivalent to a 2-functor of the form $\mathsf{Geo}(-, \mathcal{K})$ for some topos $\mathcal{K}$. That is, there are equivalences of categories $F(\mathcal{X}) \simeq \mathsf{Geo}(\mathcal{X}, \mathcal{K})$ for all topos $\mathcal{X}$, natural in $\mathcal{X}$.

Theorem 4.24. *Let* G *be a geometry. The* 2*-functors*

$$\begin{aligned} \mathcal{X} &\mapsto \mathsf{Str}_{\mathsf{G}}(\mathcal{X})^{\to} \\ \mathcal{X} &\mapsto S_L^{\mathcal{X}} \\ \mathcal{X} &\mapsto S_R^{\mathcal{X}} \end{aligned}$$

are representable. If $\mathcal{K}$ *denotes the topos associated to* $\mathsf{Sh}(\mathsf{G})$*, the notations for representants of the previous three* 2*-functors are respectively* $\mathcal{K}^{\to}$, $\mathcal{K}^{loc}$ *and* $\mathcal{K}^{cons}$.

Proposition 4.25. *The functors* $\mathsf{dom}, \mathsf{codom} : \mathsf{Str}_{\mathsf{G}}(\mathcal{X})^{\to} \to \mathsf{Str}_{\mathsf{G}}(\mathcal{X})$ *are represented by geometric morphisms* $\mathrm{dom}, \mathrm{codom} : \mathcal{K}^{\to} \to \mathcal{K}$ *(and similarly for* $\mathcal{K}^{loc}$ *and* $\mathcal{K}^{cons}$*).*
Similarly, the functors sending an arrow in $\mathsf{Str}_G(\mathcal{X})$ *respectively to the left and right part of its unique* $(S_L^{\mathcal{X}}, S_R^{\mathcal{X}})$ *factorization are represented by functors*

$$\begin{aligned} \mathrm{locFact} :& \quad \mathcal{K}^{\to} \to \mathcal{K}^{loc} \\ \mathrm{consFact} :& \quad \mathcal{K}^{\to} \to \mathcal{K}^{cons} \end{aligned}$$

Proof. This is a consequence of the Yoneda lemma for 2-categories. Or more precisely, of the fact that the 2-Yoneda embedding is fully faithful. □

Proposition 4.26. *With the notations of theorem 4.24, the following square is* 2*-cartesian.*

$$\begin{array}{ccc} \mathcal{K}^{\to} & \xrightarrow{\mathrm{factCons}} & \mathcal{K}^{cons} \\ {\scriptstyle \mathrm{factLoc}}\downarrow & & \downarrow{\scriptstyle \mathrm{dom}} \\ \mathcal{K}^{loc} & \xrightarrow[\mathrm{codom}]{} & \mathcal{K} \end{array}$$

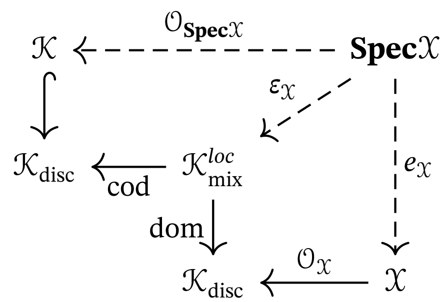
Proof. We have the following equivalences :

$$\begin{aligned} \mathsf{Geo}(\mathcal{X}, \mathcal{K}^{\to}) &\simeq \mathsf{Str}_{\mathsf{G}}(\mathcal{X})^{\to} && \text{(by theorem 4.24)} \\ &\simeq S_L^{\mathcal{X}} \underset{\mathsf{Str}_{\mathsf{G}}(\mathcal{X})}{\times} S_R^{\mathcal{X}} && (*) \\ &\simeq \mathsf{Geo}(\mathcal{X}, \mathcal{K}^{loc}) \underset{\mathsf{Geo}(\mathcal{X},\mathcal{K})}{\times} \mathsf{Geo}(\mathcal{X}, \mathcal{K}^{cons}) && \text{(by theorem 4.24)} \end{aligned}$$

Where the equivalence $(*)$ is precisely the statement that $(S_L^{\mathcal{X}}, S_R^{\mathcal{X}})$ is a functorial unique factorization system.
All these equivalences are natural in $\mathcal{X}$, hence $\mathcal{K}^{\to}$ verifies the universal property of the pullback. □

Proof of Theorem 4.21. Let G be a geometry and $\mathcal{K}, \mathcal{K}_{\text{mix}}, \mathcal{K}_{\text{disc}}$ the topoi associated respectively to $\mathsf{Sh}(\mathsf{G})$, $\mathsf{Sh}(\mathsf{G}_{\text{mix}})$ and $\mathsf{Sh}(\mathsf{G}_{\text{disc}})$.

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ a G_{disc}-structured topos. We define $(\mathbf{Spec}\mathcal{X}, \mathcal{O}_{\mathbf{Spec}\mathcal{X}})$ by the following 2-limit diagram in Topos :



(6)

Now we show that $(\mathbf{Spec}\mathcal{X}, \mathcal{O}_{\mathbf{Spec}\mathcal{X}})$ represents the following 2-functor :

$$\begin{aligned} \mathcal{T}\mathrm{op}(\mathsf{G})^{op} &\to \mathsf{Cat} \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) &\mapsto \mathsf{Hom}_{\mathcal{T}\mathrm{op}(\mathsf{G}_{\text{disc}})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \end{aligned}$$

The fact that **Spec** extends to a fully defined 2-functor $\mathcal{T}\mathrm{op}(\mathsf{G}_{\text{disc}}) \to \mathcal{T}\mathrm{op}(\mathsf{G})$, and that the induced equivalences natural in $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$

$$\mathsf{Hom}_{\mathcal{T}\mathrm{op}(\mathsf{G}_{\text{disc}})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \simeq \mathsf{Hom}_{\mathcal{T}\mathrm{op}(\mathsf{G})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathbf{Spec}\mathcal{X}, \mathcal{O}_{\mathbf{Spec}\mathcal{X}}))$$

are also natural in $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, will then be the consequence of a general lemma on the construction of 2-adjoints (see proposition 4.3.4 of [Rie17] for the 1-categorical case). Let $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a G-structured topos.

We first construct a 2-functor :

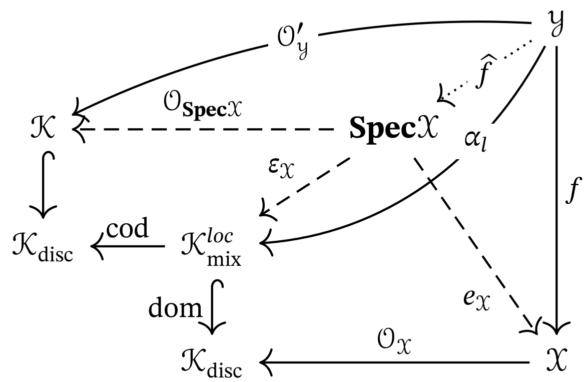
$$\mathsf{Hom}_{\mathcal{T}\mathrm{op}(\mathsf{G}_{\text{disc}})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \to \mathsf{Hom}_{\mathcal{T}\mathrm{op}(\mathsf{G})}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathbf{Spec}\mathcal{X}, \mathcal{O}_{\mathbf{Spec}\mathcal{X}}))$$

Let $(f, \alpha) : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a morphism of G_{disc}-structured topoi. α factors as :

$$f^*\mathcal{O}_{\mathcal{X}} \xrightarrow{\alpha_l} \mathcal{O}'_{\mathcal{Y}} \xrightarrow{\alpha_c} \mathcal{O}_{\mathcal{Y}}$$

where α_l is a localization of G_{mix}-structures and α_c is a conservative morphisms of G_{mix}-structures.

Since $\mathcal{O}_{\mathcal{Y}}$ is a G-structure and α_c is conservative, by lemma 4.22, $\mathcal{O}'_{\mathcal{Y}}$ is also a G-structure, and hence factors through $\mathcal{K} \hookrightarrow \mathcal{K}_{\text{disc}}$. Thus, $\mathcal{O}'_{\mathcal{Y}}$, α_l and f form a cone for the diagram (6). Hence we have a morphism $\widehat{f} : \mathcal{Y} \to \mathcal{X}$ making the following diagram commute up to isomorphism.



(7)

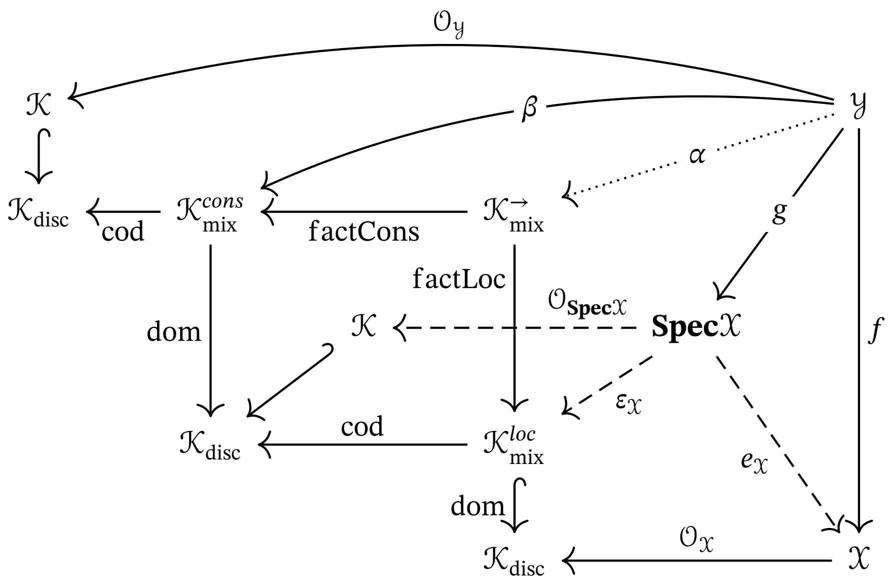
In particular, we have an isomorphism $\eta : g^*\mathcal{O}_{\mathbf{Spec}\mathcal{X}} \to \mathcal{O}'_{\mathcal{Y}}$. Since isomorphisms are conservative, we have a conservative morphism $\alpha_c \circ \eta : f^*\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}}$, and hence a morphism $(\widehat{f}, \alpha_c \circ \eta) : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathbf{Spec}\mathcal{X}, \mathcal{O}_{\mathbf{Spec}\mathcal{X}})$ in $\mathcal{T}\mathrm{op}(\mathsf{G})$.

Now, let $(g, \beta) : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathbf{Spec}\mathcal{X}, \mathcal{O}_{\mathbf{Spec}\mathcal{X}})$ be a morphism of G-structured topoi. Composition with $(e_{\mathcal{X}}, \varepsilon_{\mathcal{X}})$ induces a morphism of $\mathsf{G}_{\mathrm{disc}}$-structures

$$(e_{\mathcal{X}} \circ g, \beta \circ g^*\varepsilon_{\mathcal{X}}) : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

We must now show that these constructions are reciprocal to each other. Let $(f, \alpha) : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a morphism of $\mathsf{G}_{\mathrm{disc}}$-structured topoi. Diagram (7) gives isomorphisms $f \simeq e_{\mathcal{X}} \circ \widehat{f}$ and $\alpha_l \simeq \widehat{f}^*\varepsilon_{\mathcal{X}}$, hence $\alpha \simeq \alpha_c \circ \alpha_l \simeq \alpha_c \circ \widehat{f}^*\varepsilon_{\mathcal{X}}$.

Reciprocally, let $(g, \beta) : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathbf{Spec}\mathcal{X}, \mathcal{O}_{\mathbf{Spec}\mathcal{X}})$ be a morphism of G-structured topoi. Let $f := e_{\mathcal{X}} \circ g$ and $\alpha := \beta \circ g^*\varepsilon_{\mathcal{X}}$. We have the following commutative (up to isomorphism) diagram :



Proposition 4.26 (specifically, the uniqueness of the factorization) implies that $\beta \simeq \alpha_c$. □

Conclusion

In the end, the most important results used for the construction of the spectrum were theorem 4.14 and lemma 4.22. The representability of the factorization system (theorem 4.24) is actually not necessary to construct the spectrum (see for instance [Joh77][Th 6.58]), but it allows for a more diagrammatic proof, so we decided to include it in this text. The treatement of spectra given in [Joh77] actually relies on a minimalistic axiomatization, which more or less reduces to those few properties. The theory of geometries presented here (due to Lurie) is one example of how to generate a wide varieties of examples. Another approach has been given in [Ane09].

The main example we used throughout this text is that of the Zariski site (based on localizations of rings), but we could also have used the étale site (based on henselian maps of rings). In contrary to the Zariski case, the étale spectrum of a ring is generally not a topological space.

Both the approaches of [Lur11] and [Ane09] recover the Zariski and étale pretopology as special cases, among many others.

A Reminders from category theory

A.1 Reflective subcategories, localizations

Definition A.1. A full subcategory C of a category D is said to be a **reflective subcategory** if the inclusion functor $i : \mathsf{C} \hookrightarrow \mathsf{D}$ admits a left adjoint $r : \mathsf{D} \to \mathsf{C}$, called the **reflector**. More generally, any fully faithfull functor $i : \mathsf{C} \hookrightarrow \mathsf{D}$ is said to exhibit C as a full subcategory of D if it admits a left adjoint.

Proposition A.2. *Let $i : \mathsf{C} \hookrightarrow \mathsf{D}$ be a reflective subcategory with reflector r. Then r is a localization functor. More precisely, let $\mathcal{W}$ be the class of morphisms in D sent to isomorphisms in C by r. Then r identifies C as $\mathsf{D}[\mathcal{W}^{-1}]$. Because of this, one also calls C a reflective localization of D.*

Proof. See proposition 3.1 at `https://ncatlab.org/nlab/show/reflective+localization` (visited 12 september 2021). □

Definition A.3. A reflective localization $r : \mathsf{D} \to \mathsf{C}$ is called a **left-exact localization** if it preserves finite limits.

A.2 Equivalence relations, kernels, epimorphisms

Definition A.4. Let C be a category with finite limits and $f : X \to Y$ a morphism in C. The kernel pair of f is the pair (p_1, p_2) defined by the following pullback square.

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Definition A.5. Let C be a category with finite limits and $f : X \to Y$ be a morphism in C. f is said to be an **effective epimorphism** in C if it is the quotient of its own kernal pair, or in other words, if the following diagram is a coequalizer in C.

$$X \times_Y X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{f} Y$$

Proposition A.6. *Any effective epimorphism is an epimorphism in the usual sense.*

Proof. Let $f : X \to Y$ be an epimorphism in a category C with finite limits. Let $g, h : Y \to Z$ be morphisms in C such that $g \circ f = h \circ f$. We have $g \circ f \circ p_1 = g \circ f \circ p_2 = h \circ f \circ p_1 = h \circ f \circ p_2$. Since f is the coequalizer of (p_1, p_2), it follows that $g = h$. □

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