Bijective counting of plane bipolar orientations

Éric Fusy ??, Dominique Poulalhon ??, and Gilles Schaeffer ??

[a] LIX, École Polytechnique, 91128 Palaiseau Cedex-France

[b] LIAFA (Université Paris Diderot), 2 place Jussieu 75251 Paris cedex 05

Abstract

We introduce a bijection between plane bipolar orientations with fixed numbers of vertices and faces, and non-intersecting triples of upright lattice paths with some specific extremities. Writing ϑ_{ij} for the number of plane bipolar orientations with $(i+1)$ vertices and $(j+1)$ faces, our bijection provides a combinatorial proof of the following formula due to Baxter:

$$\vartheta_{ij} \;=\; 2\,\frac{(i+j-2)!\;(i+j-1)!\;(i+j)!}{(i-1)!\;i!\;(i+1)!\;(j-1)!\;j!\;(j+1)!}. \tag{1}$$

Keywords: bijection, bipolar orientations, non-intersecting paths.

1 Introduction

A *bipolar orientation* of a graph $G = (V, E)$ is an acyclic orientation of G such that the induced partial order on the vertex set has a unique minimum s called the *source*, and a unique maximum t called the *sink*. Alternative definitions and many interesting properties are described in [?]. Bipolar orientations are a powerful combinatorial structure and prove insightful to solve many algorithmic problems such as planar graph embedding [?] and geometric representations of graphs in various flavours. As a consequence, it is an interesting issue to have a better understanding of their combinatorial properties.

This extended abstract focuses on the enumeration of bipolar orientations in the planar case. Precisely, we consider bipolar orientations on rooted planar maps, where a *planar map* is a connected planar graph embedded in the plane without edge-crossings and up to isotopic deformation, and *rooted* means with a marked oriented edge (called the *root*) having the outer face on its left. A *plane bipolar orientation* is a pair (M, X), where M is a rooted map and X is a bipolar orientation of M with the origin (end) of the root as source (sink, respectively). Let ϑ_{ij} be the number of plane bipolar orientations with i vertices and j faces. Baxter [**?**, Eq 5.3] has proved that ϑ_{ij} satisfies Formular (**??**) using algebraic manipulations on generating functions. The main result of this article is a direct bijective proof of Formula (**??**), exemplified in Figure **??**:

Theorem 1.1 *There is a bijection between plane bipolar orientations with* $(i+2)$ *vertices and* $(j+2)$ *faces, and non-intersecting triples of upright lattice paths on the grid* $\mathbb{Z}^2$ *with respective origins* $(-1, 1)$*,* $(0, 0)$*,* $(1, -1)$*, and respective endpoints* $(i-1, j+1)$*,* (i, j)*,* $(i+1, j-1)$*.*

Formula (**??**) is easily derived from this theorem using Gessel-Viennot Lemma [**?**] (classical determinant-type formula to enumerate non-intersecting paths).

Overview. Our bijection relies on several steps. Counting plane bipolar orientations is first reduced to counting quadrangulations endowed with specific edge-bicolorations. Then these edge-bicolored quadrangulations are bijectively encoded by triples of words with some conditions on the prefixes. The encoding draws its inspiration from a nice bijection by Bernardi and Bonichon [**?**], between Schnyder Woods of triangulations and pairs of non-crossing Dyck words. The final step of our bijection is to translate each binary word of the triple to an upright lattice path; the prefix conditions of the words are equivalent to the property that the three paths are non-intersecting.

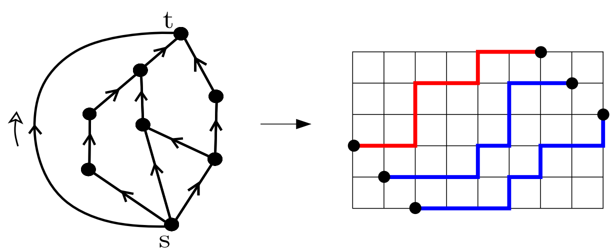


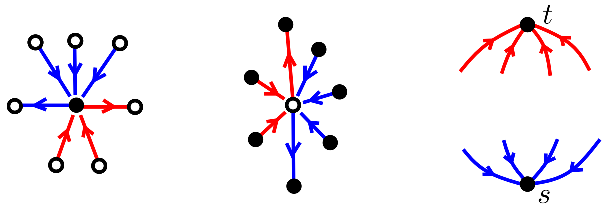
Fig. 1. A plane bipolar orientation and the associated triple of non-intersecting upright lattice paths.

2 The bijection

(I). Reduction to counting edge-bicolored quadrangulations. The *quadrangulation* of a rooted map M is the bipartite map Q with vertex set consisting of vertices and faces of M, and edges corresponding to incidences between these vertices and faces. Q is naturally rooted with the same root vertex as M. From now on, rooted quadrangulations are endowed with their unique bicoloration of vertices in black and white such that the root vertex, called s, is black; the other outer black vertex is denoted by t.

If M is endowed with a bipolar orientation, this classical construction can be enriched in order to transfer the orientation on Q; a rooted quadrangulation is said to be *bicoloriented* if the edges are oriented and partitioned into red and blue edges such that the following conditions are satisfied:

- each inner vertex has exactly two outgoing edges, a red one and a blue one;
- around each inner black (*resp.* white) vertex, ingoing edges in each color follow the outgoing one in clockwise (*resp.* counterclockwise) order;
- edges incident to s are ingoing blue, and edges incident to t are ingoing red.



Theorem 2.1 ([?]) *Plane bipolar orientations with i vertices and j faces are in bijection with rooted bicoloriented quadrangulations with i black vertices and j white vertices.*

(II). Encoding a bicoloriented quadrangulation by a triple of words. Bicoloriented quadrangulations have an interesting property: as shown in [?,?], blue (*resp.* red) edges form two trees spanning all vertices except t (*resp.* s) and oriented toward s (*resp.* t). Let Q be a rooted bicoloriented quadrangulation with $(i+2)$ black vertices and j white vertices, and let T_{blue} be its blue tree. We define the *contour word* W_Q of Q as the word on the alphabet $\{a, \underline{a}, b, \underline{b}, c, \underline{c}\}$ obtained as follows (see Figure **??**). Perform a clockwise traversal of the contour of T_{blue} starting at the root edge, and write a letter a (*resp.* b) each time an edge e of T_{blue} is traversed from a black to a white vertex (*resp.* from a white to a black one). Underline the letter if this is the second traversal of e. Write a letter c each time a red edge is crossed at a white vertex, and underline it if the edge is ingoing.

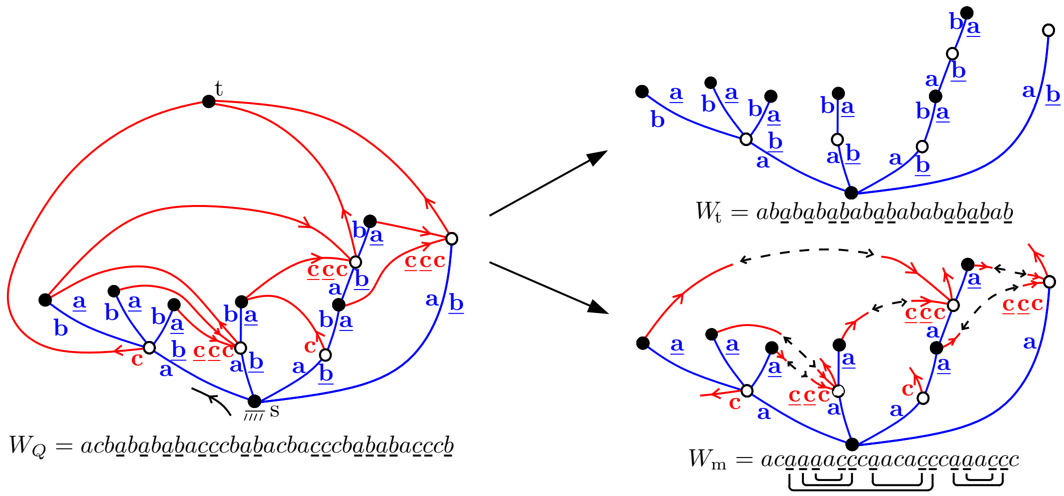


Fig. 2. Contour words of a bicoloriented quadrangulation.

We shall consider three subwords of W_Q: for any l in $\{a, b, c\}$, let W_l denote the word obtained by keeping only the letters in the alphabet $\{l, \underline{l}\}$. In order to describe the properties of these words, we also introduce the *tree-word* W_{t} and the *matching word* W_{m}, that are respectively obtained by keeping the letters in $\{a, \underline{a}, b, \underline{b}\}$, and in $\{a, \underline{a}, c, \underline{c}\}$.

(i): The tree-word encodes the blue tree. Observe that W_{t} corresponds to a classical Dyck encoding of T_{blue}, in which the two alphabets $\{a, \underline{a}\}$ and $\{b, \underline{b}\}$ are used alternatively to encode the bicoloration of vertices. Hence W_{t} is just the shuffle of W_a and W_b at even and odd positions respectively, and each prefix of W_{t} has at least as many non-underlined letters as underlined letters.

It is easily seen that W_a has j occurences of a and i occurences of $\underline{a}$, shortly written $W_a \in \mathfrak{S}(a^j \underline{a}^i)$. Similarly, $W_b \in \mathfrak{S}(b^i \underline{b}^j)$. Moreover, the prefix condition on the Dyck word W_{t} translates to the following property:

Property 1 *For $1 \le k \le i$, the number of a's on the left of the kth occurence of $\underline{a}$ in W_a is strictly larger than the number of $\underline{b}$'s on the left of the kth occurence of b in W_b.*

(ii): The matching word encodes the red edges. Let us now focus on W_c and on the matching word W_{m}. Clearly, any occurence of a c (*resp.* $\underline{c}$) in W_Q corresponds to a red edge with white (*resp.* black) origin, see Figure **??**. Hence $W_c \in \mathfrak{S}(c^j \underline{c}^i)$. Observe also that any occurence of a in W_{m}, which corresponds to the first visit to a white vertex v, is followed by a pattern $\underline{c}^l c$, with l the number of ingoing red edges at v. Hence W_{m} satisfies the regular expression $ac(\underline{a}^* a \underline{c}^* c)^*$, which uniquely defines W_{m} as a shuffle of W_a and W_c.

Let us now consider a red edge with black origin; its origin (encoded by a letter $\underline{a}$) has to be encountered before its endpoint (encoded by a letter $\underline{c}$). Hence planarity ensures that the restriction of W_{m} to the alphabet $\{\underline{a}, \underline{c}\}$ is a parenthesis word, which translates to the following property:

Property 2 *For $1 \leq k \leq j$, the number of $\underline{a}$'s on the left of the kth occurence of a in W_a is at least as large as the number of $\underline{c}$'s on the left of the kth occurence of c in W_c.*

(III). Representation as a triple of non-intersecting paths. Each of the three words (W_a, W_b, W_c) gives rise to an upright lattice path: the binary word is read from left to right and the path goes up or right depending on the letter (letters associated to up steps are a, $\underline{b}$, and c, respectively). Letting the corresponding paths $\mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c$ start at $(0, -1)$, $(1, -1)$, and $(-1, 0)$, properties 1 and 2 easily imply that the triple of paths is non-intersecting. Path steps corresponding to redundant letters (two in each word) are then deleted. This concludes the description of the mapping stated in Theorem **??**.

Finally, each step of the mapping admits an explicit inverse operation, so that the mapping is bijective.

References

[1] R. J. Baxter. Dichromatic polynomials and Potts models summed over rooted maps. *Annals of Combinatorics*, 5:17, 2001.

[2] O. Bernardi and N. Bonichon. Catalan's intervals and realizers of triangulations. In *Proceedings of FPSAC'07*, 2007.

[3] T. Biedl and F. J. Brandenburg. Partitions of graphs into trees. In *Proceedings of Graph Drawing'06 (Karlsruhe)*, volume 4372 of *LNCS*, pages 430–439, 2007.

[4] H. de Fraysseix, P. Ossona de Mendez, and P. Rosenstiehl. Bipolar orientations revisited. *Discrete Appl. Math.*, 56(2-3):157–179, 1995.

[5] S. Felsner, C. Huemer, S. Kappes, and D. Orden. Binary labelings for plane quadrangulations and their relatives. arxiv:math.CO/0612021.

[6] I. Gessel and X. Viennot. Binomial determinants, paths, and hook formulae. *Adv. Math*, 58:300–321, 1985.

[7] A. Lempel, S. Even, and I. Cederbaum. An algorithm for planarity testing of graphs. In *Theory of Graphs, Int. Symp (New York)*, pages 215–232, 1967.