Exam for the graduate course

Proof systems for linear, intuitionistic, and classical logic

Dipartimento di Informatica, Universitá Ca' Foscari di Venezia
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Answers to these questions can be written in English or Italian. In either case, please write clearly. Solutions can be sent by post or fax or scanned and sent as a pdf attachment. Contact information can be found at `http://www.lix.polytechnique.fr/Labo/Dale.Miller/`.

This exam is provided on 24 April 2009. Please return your solutions to Miller by 15 May 2009.

Exercise 1 (Negations) Do Exercise 15 on page 23.

Exercise 2 (Substitution) Do Exercise 40 on page 44.

Exercise 3 (Neg/pos connectives) Do Exercise 45 on page 49.

Exercise 4 (Invertibility) If the linear logic sequent $\Sigma\colon \Gamma \vdash \Delta, B \parr C$ has a proof, say Ξ, then if it has a cut-free proof in which the last inference rule is an introduction rule for this occurrence of $B \parr C$. This can be proved by considering the result of eliminating cut from the following:

$$\dfrac{\dfrac{\overset{\Xi}{\Sigma\colon \Gamma \vdash \Delta, B \parr C} \qquad \overset{\Xi'}{\Sigma\colon B \parr C \vdash B, C}}{\Sigma\colon \Gamma \vdash \Delta, B, C}\ cut}{\Sigma\colon \Gamma \vdash \Delta, B \parr C}\ \parr R$$

Here, Ξ' is the obvious proof of $\Sigma\colon B \parr C \vdash B, C$. Using an argument of this style, prove similar theorems but replace $B \parr C$ by $B \,\&\, C$ and by $\forall x.B$.

Exercise 5 (Getting the focus on ? wrong) Notice that the proof rule in $\mathcal{F}$ (see the proof system in Figure 5.9 on page 59 of the lecture notes) for ?L is unlike the other left rules in that it does not maintain focus on positive subformulas as one moves from the conclusion to a premise. Consider the following variation to that inference rule.

$$\dfrac{\Sigma\colon \Psi; \cdot \xrightarrow{B} \cdot; \Upsilon}{\Sigma\colon \Psi; \cdot \xrightarrow{?B} \cdot; \Upsilon}\ ?\,\mathrm{L}'$$

Do the following two tasks.

1. Build a proof of $?(a \multimap b) \multimap ?(a \multimap b)$ in $\mathcal{F}$.

2. Argue that if we replace ?L with $?\,\mathrm{L}'$, then this formula no longer has a proof.

Exercise 6 (An implications implying its converse) Let P and Q be the par ($\parr$) of atomic formulas. Show that $\vdash P \multimap Q$ implies $\vdash Q \multimap P$.

Exercise 7 (A linear logic program) Let $\mathcal{P}$ be the following set of linear logic program clauses.

$$
\begin{array}{ll}
\mathit{leq}\ z\ N. & \\
\mathit{leq}\ (s\ N)(s\ M) & \multimap\ \mathit{leq}\ N\ M. \\
\mathit{item}\ N \parr \mathit{item}\ M & \multimap\ \mathit{leq}\ N\ M \multimap \mathit{item}\ M. \\
\mathit{item}\ N & \multimap\ \mathit{value}\ N.
\end{array}
$$

Here, we assume the usual Prolog style convention that we capital letter variables are universally quantified around the clause in which it appears. Assume that natural numbers are encoded as using zero z and successor s. Let $m \geq 1$ and let $n_0, n_1, \ldots, n_m$ be natural numbers (encoded as terms). The provability of the sequent

$$\mathcal{P}; \mathit{value}\ n_0 \vdash \mathit{item}\ n_1, \ldots, \mathit{item}\ n_m; \cdot$$

describes a familiar relationship between n_0 and the multiset $n_1, \ldots, n_m$. Describe that relationship and argue carefully (but informally) why provability with this logic program matches that relationship.