Observations in Pure Mathematics

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Part I. Introduction

The true justification of mathematics is aesthetic †, and it is my hope that the essay which follows is to some degree justified. In Part II, I discuss a function which is easily understood in one sense and then describe it in several other ways. The beauty of that section lies in the relationship it has to other branches of mathematics. For this reason I have added a concluding section. The strength of Part II lies in the elementary argument which finds the Riemann *zeta* function for even, positive integers. This approach is contrasted to those found in most other books which use the notion of double series expansion. The concluding theorem in Part III is very intriguing because it joins two very different kinds of primes. This theorem, however, appears to be very limited.

Part II. Functions Concerning the Powers of Two

The following discussion deals with functions which are related to a number's representation in base two. For example, $f(n)$ is defined so that it is the number of ones in the base two form of n. Thus $f(45) = 4$ since $45 = 101101_2$. It is the goal of this part of my paper to give formulas for $f(n)$ and related functions. The ultimate goal is to express these functions in terms of number-theoretic functions, such as $\mu(n)$ and $d(n)$ to be defined later.

We make the following definitions:

(1) $a_n = 1$ if $n = 2^k$ and $a_n = 0$ if $n \neq 2^k$ for some k.
(2) $\theta(n) = k$ if $n = 2^k m$ where $m \equiv 1 (\text{mod } 2)$.
(3) $f(2^k + b) = 1 + f(b)$, if $0 \leq b < 2^k$ and $f(0) = 0$.

Of these functions a_n is the simplest because it assumes only the values 0 and 1. Also, by inspection we notice that

$$\theta(n) = \sum_{d|n} a_d - 1.$$

In order to find a formula for $f(n)$, its generating series is useful. Using only the definition of $f(n)$ it can be shown that

$$\sum_{k=1}^{\infty} f(k)x^k = \frac{1}{1-x} \sum_{k=0}^{\infty} \frac{x^{2^k}}{1+x^{2^k}}. \tag{1}$$

This paper contains a slightly modified version of a paper I sent to the 33^{rd} Westinghouse Science Talent Search in 1974. The principal changes have been the correction of some spelling and grammar errors, and the inclusion of a few extra steps in certain proofs to make them easier to read. The principal reference book for this paper is Hardy and Wright's *An Introduction to the Theory of Numbers.*

† Oh, the simplicity of youth!

Using the following identity for Lambert series

$$\sum_{k=1}^{\infty} b_k \frac{x^k}{1+x^k} = \sum_{k=1}^{\infty} (-1)^{k-1} b_k \frac{x^k}{1-x^k}, \tag{2}$$

we have

$$\sum_{k=0}^{\infty} \frac{x^{2^k}}{1+x^{2^k}} = \frac{1}{1-x} - \sum_{k=1}^{\infty} \frac{x^{2^k}}{1-x^{2^k}}. \tag{3}$$

Now, if we expand and collect terms on the right, we get

$$\sum_{k=1}^{\infty} \frac{x^{2^k}}{1-x^{2^k}} = \sum_{k=1}^{\infty} \theta(k) x^k. \tag{4}$$

Thus using (1), (3), and (4) we can conclude that

$$f(n) = n - \sum_{k=1}^{n} \theta(k).$$

Notice, that

$$\theta(n!) = \sum_{k=1}^{n} \theta(k) \quad \text{and} \quad \theta(n!) = \left\lfloor \frac{n}{2^1} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \cdots.$$

Hence, we can finally conclude that

$$f(n) = n - \left\lfloor \frac{n}{2^1} \right\rfloor - \left\lfloor \frac{n}{2^2} \right\rfloor - \cdots. \tag{6}$$

We now need the following number-theoretic functions:

(1) If n has a square factor then $\mu(n) = 0$. Otherwise $n = p_1 p_2 \cdots p_r$ and $\mu(n) = (-1)^r$, for distinct primes p_i. Also, $\mu(1) = 1$. $\mu(n)$ is called the Möbius function.

(2) Let $d(n) = \sum_{d|n} 1$. $d(n)$ is the number of divisors of n.

We shall also need the following Möbius Inversion Theorems:

(1) $F(n) = \sum_{d|n} G(d)$ if and only if $G(n) = \sum_{d|n} \mu(\frac{n}{d}) F(d)$.

(2) $F(x) = \sum_{k=1}^{\infty} G(x^k)$ if and only if $G(x) = \sum_{k=1}^{\infty} \mu(k) F(x^k)$.

Using the first of these inversion theorems on the definition of $d(n)$, we get:

$$1 = \sum_{d|n} \mu\left(\frac{n}{k}\right) d(k). \tag{7}$$

Let $L(x)$ be the simplest Lambert series, *i.e.*

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} d(n) x^n.$$

Using the second Möbius Inversion theorem, we can determine that

$$\frac{x}{1-x} = \sum_{k=1}^{\infty} \mu(k) l(x^k)$$

and

$$\sum_{k=0}^{\infty} \frac{x^{2^k}}{1-x^{2^k}} = \sum_{k=1}^{\infty} x^k \sum_{\substack{j|k \\ j\equiv 1 \pmod 2}} \mu(j) d\Big(\frac{k}{j}\Big).$$

But with (4) we have

$$\theta(n) = \sum_{\substack{j|n \\ j\equiv 1 \pmod 2}} \mu(j) d\Big(\frac{n}{j}\Big) - 1 \tag{8}$$

and by using (7)

$$\theta(n) = -\sum_{\substack{j|n \\ j\equiv 0 \pmod 2}} \mu(j) d\Big(\frac{n}{j}\Big) - 1 \tag{9}$$

If we let $D(n) = d(1) + d(2) + \cdots + d(n)$ then we can observe that

$$\sum_{k=1}^{n} \theta(k) = -\sum_{\substack{j=2 \\ j\equiv 0 \pmod 2}} \mu(j) D\Big(\Big\lfloor\frac{n}{j}\Big\rfloor\Big)$$

and finally

$$f(n) = n + \sum_{\substack{j=2 \\ j\equiv 0 \pmod 2}} \mu(j) D\Big(\Big\lfloor\frac{n}{j}\Big\rfloor\Big).$$

Related Observations: The connection between the function $f(n)$ and the arithmetic functions $\mu(n)$ and $d(n)$ is very intriguing. The generalization for (9) is: If $|\mu(q)| = 1$ and $n = q^k m$, where $m \not\equiv 0 \pmod q$, then

$$k^h = \mu(q) \sum_{\substack{j|n \\ j\equiv 0 \pmod q}} \mu(j) d\Big(\frac{n}{j}\Big)$$

where h is the number of primes in q.

Equation (3) yields a rather striking identity:

$$\sum_{k=1}^{\infty} \frac{x^{2^k}}{1-x^{2^{k+1}}} = \frac{x}{1-x}.$$

If we define U_n as the n^{th} Fibonacci number, such that $U_0 = 0$, $U_1 = 1$ and $U_n = U_{n-1} + U_{n-2}$ for $n > 1$, then the above identity yields

$$\sum_{k=0}^{\infty} \frac{1}{U_{2^k}} = \frac{7-\sqrt{5}}{2}.$$

Equation (8) and (9) can be combined to give

$$n = \sum_{j=1}^{n} \mu(j) D\Big(\Big\lfloor \frac{n}{j} \Big\rfloor\Big).$$

Employing a form of the Möbius inversion theorems, we have the well-known identity theorem

$$d(1) + d(2) + \ldots + d(n) = \Big\lfloor \frac{n}{1} \Big\rfloor + \Big\lfloor \frac{n}{2} \Big\rfloor + \ldots + \Big\lfloor \frac{n}{n} \Big\rfloor.$$

The function $f(n)$ can answer some interesting questions concerning Pascal's triangle. If can be shown that the number of odd numbers in the n^{th} row of Pascal's triangle is $2^{f(n)}$. Equivalent to this is the fact that the number of odd entries in the n^{th} row of an array of Stirling numbers of the first kind is $2^{f(\lfloor \frac{n}{2} \rfloor)}$.

Part III. Bernoulli Polynomials and Numbers

In this part of my paper I will deal with Bernoulli polynomials $B_n(x)$ and numbers B_n. My goals are to determine a formula for Riemann's zeta function, $\zeta(n)$, for even positive integers, to find a Fourier expansion for $B_n(x)$ without using Fourier Transformations, and to develop a connection between regular primes and Fermat primes.

Bernoulli polynomials are most conveniently defined by the generating series

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x) t^k}{k!},$$

then we may set $B_n = B_n(0)$. Books on infinite series prove that $B_{2n+1} = 0$, for $n > 0$ and $B_n'(x) = nB_{n-1}(x)$ or equivalently $\int B_n(x)dx = \frac{1}{n+1} B_{n+1}(x) + C$.

The answer to the first two goals I proposed lies in studying the series

$$t_n(x) = \sum_{k=1}^{\infty} (2k-1)^{-n} \text{trig}_n (2k-1)\pi x$$

for $0 \le x \le 1$. Here, trig_n is sine if n is odd and cosine if n is even. In order to work with $t_n(x)$, we need two more facts proven in books of analysis

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} (2k-1)^{-1} \sin(2k-1)\pi x$$

for $0 < x < 1$ and that this series is integrable termwise. Thus we can derive the recursive equations for $t_n(x)$

$$t_{2n+1}(x) = \pi \int_0^x t_{2n}(x)dx$$

$$t_{2n}(x) = \beta(2n) - \pi \int_0^x t_{2n-1}(x)dx$$

where $\beta(n) = \sum_{k=1}^{\infty}(2k-1)^{-1}$.

The next step is to find another function which satisfies the above recursive properties. To this end, define $s_n(x)$ as

$$s_n(x) = (-1)^{\lfloor \frac{x}{2} \rfloor - 1}\frac{\pi^n}{2n!}\Big(B_n\Big(\frac{x}{2}\Big)2^n - B_n(x)\Big).$$

We can easily show that s_n has similar recursive properties:

$$s_{2n+1}(x) = \pi \int_0^x s_{2n}(x)dx$$

$$s_{2n}(x) = (-1)^{n-1}\frac{\pi^{2n}(2^{2n}-1)}{2(2n)!}B_{2n} - \pi \int_0^x s_{2n-1}(x)dx$$

In order to show that s_n and t_n are the same functions we need to show that they have the same value for an infinite number of n's. It is easy to show that $t_{2n}(\frac{1}{2}) = 0$. Thus since $t_1(x) = s_1(x) = \frac{\pi}{4}$ and if $s_{2n}(\frac{1}{2}) = 0$, then we would have $t_n(x) = s_n(x)$ for all n and $0 \leq x \leq 1$. It is an easy matter to show that $2^{2n}B_{2n}(\frac{1}{4}) = B_{2n}(\frac{1}{2})$ by examining the series

$$\sum_{k=0}^{\infty}\frac{B_{2k}(x)t^{2k}}{(2n)!} = \frac{t}{e^t - 1}\frac{e^{xt} + e^{(1-x)t}}{2}.$$

Therefore by the definition of $s_n(x)$, $s_{2n}(\frac{1}{2}) = 0$ and hence $t_n(x) = s_n(x)$.

This conclusion is two-fold. First, the infinite trignometric series $t_n(x)$ can be written as a polynomial $s_n(x)$ for $0 \leq x \leq 1$. Secondly,

$$\beta(2n) = (-1)^{n-1}\frac{\pi^{2n}(2^{2n}-1)}{2(2n)!}B_{2n}.$$

However, since $\zeta(2n) = \sum_{k=1}^{\infty}k^{-n}$, we have

$$\zeta(2n) = (-1)^{n-1}\frac{\pi^{2n}2^{2n-1}}{(2n)!}B_{2n}.$$

Using a similar argument, it can be shown that

$$\sum_{k=1}^{\infty} k^{-n}\mathrm{trig}_n k\pi x = (-1)^{\lfloor \frac{n}{2} \rfloor -1}\frac{\pi^n}{n!}2^n B_n\Big(\frac{x}{2}\Big). \tag{10}$$

Thus using the fact that $t_n = s_n$ and (10) we have the Fourier expansion for $B_n(x)$:

$$B_{2n+1} = 2(-1)^{n+1}(2n+1)!\sum_{k=1}^{\infty}(2\pi k)^{-2n-1}\sin 2\pi kx$$

$$B_{2n} = 2(-1)^{n+1}(2n)!\sum_{k=1}^{\infty}(2\pi k)^{-2n}\cos 2\pi kx$$

An interesting integer sequence connected with Bernoulli numbers is the Euler sequence, defined by the series

$$\frac{2}{e^x + e^{-x}} = \sum_{k=o}^{\infty}\frac{E_k x^k}{k!}.$$

Because

$$\frac{4t}{e^{4t}-1} - \frac{2t}{e^{2t}-1} = -t\frac{2}{e^t + e^{-t}}e^{-t}$$

we can show that

$$2^n(2^n-1)B_n = n(E+1)^{n-1} \tag{11}$$

where it is understood that E^n is replaced by E_n.

At this point we need the famous Staudt-Clausen Theorem. In brief, this theorem states that $B_{2n} = \frac{s_{2n}}{r_{2n}}$, where s_{2n} is an integer and r_{2n} is the product of all primes p, such that $p-1|2n$. Also a prime p is said to be regular if $p \not| s_k$ for $k \leq p-3$.

If we let $n = 2^k$ in (11) and let numbers of the form $2^{2^i}+1 = F_n$, then r_{2^n} is the product of the Fermat primes less than or equal to $\log_2 n$, and

$$2^{2^n}F_{n-1}F_{n-2}\cdots F_0\frac{s_{2^n}}{r_{2^n}} = 2^n(E+1)^{2^n-1}.$$

This fact yields an interesting theorem: If p is a regular prime, $p > 2^n+3$ and $p|(E+1)^{2^n-1}$ then $p|F_k$ for some k, $\log_2 n < k \leq n-1$.