On the Curry-Howard Interpretation of a Fragment of Classical Linear Logic with Subexponentials

Chuck Liang¹ and Dale Miller²

¹ Department of Computer Science, Hofstra University, Hempstead, NY, US
   chuck.c.liang@hofstra.edu
² Inria Saclay & LIX/Ecole polytechnique, Palaiseau, France
   dale.miller@inria.fr

Abstract
We construct a partially-ordered hierarchy of delimited control operators similar to those of the CPS hierarchy of Danvy and Filinski [6]. However, instead of relying on nested CPS translations, these operators give directly a Curry-Howard interpretation of a fragment of linear logic extended with subexponentials, i.e., multiple pairs of ! and ?. We show how the fundamental problem of delimited control, from the perspective of logic, is the combination of classical and non-classical inference within one system and how subexponentials give a new approach to combining classical and intuitionistic logics. A natural deduction system called MC (multi-colored classical logic) is formulated with proof terms that include indexed control operators. We then define a call-by-value evaluation strategy for these terms.

1 Introduction
This paper formulates a fragment of classical linear logic with multiple pairs of exponentials, with the intent of extending the Curry-Howard correspondence¹ to include multiple levels of delimited control operators similar to those of Danvy and Filinski [6]. The computational interpretation of classical logic that began with Griffin [11] and Parigot [20] can already explain undelimited control operators such as call/cc. However, there is nothing in classical logic that can explain directly why the capturing of a continuation should be delimited. The fine-grained control over the capture of continuations suggests a combination of classical logic with intuitionistic logic, where escaping scope is restricted. In [12], delimitation is explained by a transition from an intuitionistic to a non-intuitionistic mode of derivation, which necessitates a cut-elimination strategy to deal with these transitions. However, it is known that multiple levels of delimitation may be required in some applications (see [13]). We may even wish to have a partially ordered hierarchy of operators. For example, in the term 
\[(f \ z_i (g \ z_j (\text{control}^1 c. \ldots \text{control}^k c.s)))\]
we can require that control\(^i\) is not delimited by \(z_j\), but is delimited by \(z_i\), if \(i\) is stronger or unrelated to \(j\). Suppose also that an external procedure is then called that contains a control\(^k\) construct. We may wish to specify what, if any, part of the continuation of the calling program can be captured by this operator by designating the relationship of \(i\) and \(j\) to \(k\). Relatively few formal works on delimited continuations go beyond a single level, and fewer still are from the Curry-Howard perspective. Here we will show that higher level delimitations can be explained by restrictions on how cuts can be permuted in the presence of subexponentials (which are introduced in Section 2). We formulate a hierarchy of delimited control operators that bypasses the framework of iterated cps translations by typing such control operators directly in (subexponential) linear logic.

¹ By Curry-Howard we mean propositions as types and proofs as programs.
At the proof-theoretic level, this paper is an example of using linear logic as a deconstructive logic. Strong arguments exist that show the benefits of using linear logic to decompose classical logic in comparison to translations to intuitionistic logic (see [5]). However, relatively few of these arguments are from the Curry-Howard perspective. In fact, most works on Curry-Howard interpretations of linear logic are confined to intuitionistic linear logic, which leaves out ?, involutive negation, and, thus, classical logic. Arguably, however, the most interesting aspect of linear logic is not how it restricts structural rules but how it enables them through the exponential operators ? and !. From its inception, linear logic was based on generalizations of intuitionistic principles. Double-negation translation is replaced by an a priori duality: \( ?A = (?A) \perp = (?!A) \perp \). Every formula of linear logic is already double negated. The operator ? allows for (right side) contractions as in Gentzen’s LK while its dual, !, generalizes the single-conclusion restriction of LJ. The duality between classical and intuitionistic logic is captured directly within linear logic. The most well-known example of reconstructing classical logic using linear logic principles is LC [8], a version of classical sequent calculus that contains distinctively intuitionistic features. LC is based on the concept of polarization and, indirectly, focusing in proofs [1]. Positive formulas are equivalent to those preceded by ! while negatives are equivalent to those preceded by ?. LC can be said to combine intuitionistic and classical logics by allowing a positive “stoup” formula that simulates the properties of a single conclusion. The polarized analysis of logic has lead to important developments in both classical and intuitionistic proof theory, in particular, that focusing extends to these logics [5, 16]. However, the argument for linear logic is less clear when it comes to Curry-Howard interpretations. To capture the \( \lambda\mu \)-calculus it is necessary to restrict LC to its negative-only fragment [10]. The intuitionistic side of LC is lost.

A naive attempt at combining classical and intuitionistic logics can easily result in a collapse into the former, even within the context of linear logic. The single-conclusion characterization of intuitionistic logic was inherited by linear logic. Unfortunately, this characterization leaves little in between intuitionistic and classical logic. Intuitionistic implication is usually translated into \( !A \rightarrow B \) and classical implication can be translated as \( !A \rightarrow ?B \). But consider the sequent \( \vdash !A \rightarrow ?B, !B \rightarrow ?A \), which is provable. The strength of the intuitionistic implication disintegrates as soon as one allows another conclusion. We have only two extremes to choose from: single conclusion or multiple conclusions. There is no middle ground. Since multiple conclusions represent saved continuations in classical computation, we prefer them over the single conclusion extreme.

Fortunately, there are also multiple conclusion characterizations of intuitionistic logic:

\[
\begin{array}{c}
\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta C} & \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A, \Delta W} & \frac{\Gamma \vdash B}{\Gamma \vdash A \rightarrow B, \Delta IL} & \frac{A, \Gamma \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta CL}
\end{array}
\]

Structural rules are allowed and the distinction between classical and intuitionistic logics rests on the \( \rightarrow \) introduction rule: the “IL” rule prevents scope extrusion since it enforces the scope of \( A \) to include only \( B \) and not also the formulas in \( \Delta \). This perspective offers a new opportunity for combining classical and intuitionistic logics, as we have already shown in [17]. Informally, we can hope for something of the form

\[
\frac{A, \Gamma \vdash B, \Delta_2}{\Gamma \vdash A \rightarrow B, \Delta_1 \Delta_2}
\]

Here, the indices 1 and 2 represent different levels of modality. Introducing an implication at level

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2 The case is better with Laurent’s polarized proof nets [15], but these do not correspond to programs in any conventional sense of the term.

3 In most versions of intuitionistic tableaux, there can be more than one formula signed false: these correspond to multiple conclusions.
2 can require forgetting level 1 conclusions (here, the formulas in $\Delta_1$) while keeping those at level $2$ or higher. This is the kind of system that we can build with subexponentials. We will show that subexponentials are not just another “feature” that’s been added to a logic for a specific purpose, but rather a natural generalization of the intuitionistic concept of restricting scope.

2 Subexponential Linear Logic

Subexponential, or multi-colored linear logic was suggested by Girard and first described in [4]. Given a preordered set of indices, there is a $!i$ for each index $i$ with $?i$ as it’s dual ($?iA = (1iA^\bot)^\bot$). $A?i$ does not need to admit contraction or weakening. However, the availability of these structural rules must respect the ordering relation: if $?i$ admits a structural rule and $j \geq i$, then $?j$ must also admit that structural rule.

For all indices $k$, the usual dereliction rule is allowed for $?k$ on the right and $!k$ on the left. The promotion rule is generalized as follows:

\[
\begin{align*}
?n_iA^1,\ldots,A^k, B & \; \vdash \; ?n_iA^1,\ldots,A^k, \land_j B & \; \vdash \; ?n_iA^1,\ldots,A^k, \land_j A^k & \; \vdash \; ?n_iA^1,\ldots,A^k, \land_j B \leq n_1,\ldots,n_k \\
!n_iA^1,\ldots,A^k & \; \vdash \; !n_iA^1,\ldots,A^k, \lor_j B & \; \vdash \; !n_iA^1,\ldots,A^k, \lor_j A^k & \; \vdash \; !n_iA^1,\ldots,A^k, \lor_j B \leq n_1,\ldots,n_k
\end{align*}
\]

The single sided form of the rule is equivalent to the two-sided version with a single conclusion. The second form is what we shall use, and which corresponds directly to the semantic soundness argument below. The promotion rule applies dually on the left-hand side since $!iA$ on one side is equivalent to $?jA^\bot$ on the other side.

The ordering on indices implies that both cuts and non-atomic initials can be eliminated. If $i \leq k$ then $!kA \rightarrow i!kA$ and $?iA \rightarrow ?kA$ are provable. However, it is impossible to prove $i!A \equiv !kA$ if $k \not\leq i$ even if both exponentials satisfy the same structural rules. The term subexponential was introduced in [18] along with a focused proof system for them.

2.1 Phase Semantics

Although phase semantics play no technical role in this paper, we describe a phase semantics for subexponentials here to help motivate their naturalness. Subexponentials can be characterized easily as follows: they restrict facts to submonoids. Specifically, a phase space (W, $\perp$) is formed by a commutative monoid W and an arbitrary subset $\perp$. Formulas are interpreted by subsets called facts, which are fixed points $a = a^{\perp\perp}$ where $a^{\perp\perp} = \{x \in W : xy \in \perp \text{ for all } y \in a\}$: this is the linear logic version of the pseudocomplement. Let $1 = 1^{\perp\perp}$ and $J = \{x \in W : xx = x\}$. Both $J$ and $1$ are submonoids. In [9], the phase space interpretation of $!A$ was defined as $(J \cap 1 \cap A)^{\perp\perp}$. Let $W_i, W_j, \ldots$ be a collection of submonoids which are naturally ordered by inclusion. Define

\[
!kA = (J \cap 1 \cap A \cap W_k)^{\perp\perp} \quad \text{and} \quad ?kA = (1kA^\bot)^\bot = (J \cap 1 \cap A^\bot \cap W_k)^\bot.
\]

This definition forms a pair of exponentials that admit both weakening and contraction (on the right for $?k$ and left for $!k$). For a pair that does not admit contraction, delete $J$ from the definition and for one that does not admit weakening, delete 1 from the definition. A subexponential phase model consists of a mapping of atomic formulas to facts and a mapping from the indices of exponentials to a collection of submonoids that preserves the ordering relation as follows: if $i \leq k$ then $W_k \subseteq W_i$.

In phase semantics it holds that $A \rightarrow B$ is valid (contains the unit) if $A \subseteq B$, and that if $A \subseteq B$ then

\footnote{The orientation of $\leq$ and $\subseteq$ are inverted to accommodate the existing definition of the promotion rule. It is possible that this semantics of subexponentials was not previous known despite its simplicity. We do not wish to create confusion with other papers so we will use the promotion rule as defined.}
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Given these properties it is simple to check that the (two-sided) promotion rule is sound if the submonoid $W_j$ contains each submonoid $W_{n_i}$ and that the structural rules admitted by $!_j$ are also admitted by each $!_{n_i}$ (on the left). These properties can be discovered syntactically (via proof of cut elimination), but this semantic treatment makes them rather obvious.

### 2.2 The Impact of Subexponentials on Proofs

Subexponentials appear to be a simple generalization of linear logic save for one significant fact. In most proof systems for ordinary linear logic, and for classical and intuitionistic logics, weakening can be pushed to the initial rules. There is never a need to force weakening at other points in a proof and, therefore, it can be ignored (by incorporating it into the initial rule). With indexed exponentials, however, weakening cannot be pushed to the initial rules. The sequent $!_1A, !_2B ⊢!_2C$ may only be provable if $!_1A$ is weakened away first (assuming, of course, “$2 \not\leq "1"$”). The consequence of this property is the retention of a form of resource control even without restrictions on contraction. Instead of how many times a resource can be used, the restriction is on where it can be used. If a proof of $!_1A \multimap !_2B \multimap !_2C$ is represented by $\lambda x\lambda y.t$, then $x$ cannot appear free in $t$. While this feature of subexponentials has an impact on ordinary $\lambda$-terms, we wish to do more. When one proof is substituted into another as a result of cut, the restrictions on resource usage must be maintained: this implies the need for programming features that restrict how terms should be evaluated.

### 2.3 Fundamental Equivalences and Focusing

A central goal of this paper is to formulate a refinement of classical logic that is well-behaved and self-contained with respect to cut-elimination: we shall construct this refinement as a fragment of linear logic with subexponentials. In this regard, the principles of focusing (focalization) remain important even though our target here is not a focused sequent calculus for cut-free proofs but a natural deduction system for writing programs with possibly many cuts. In terms of focusing, $?$ builds a negative formula and $!$ builds a positive formula because they have the following properties with respect to the negative and positive binary connectives:

$?(A\otimes B) \equiv ?A\otimes ?B, \quad ?(A\& B) \equiv ?A\& ?B, \quad !(A\& !B) \equiv !A\& !B, \quad !(A\otimes !B) \equiv !A\otimes !B$

These equivalences are the basis for LC, and explain why focusing extends to classical logic: the outer $?$ (!) can be removed as long as the inner connective stays negative (positive). To explain focusing in intuitionistic logic in terms of linear logic [16], we also have the following equivalence:

$!(A \multimap B) \equiv !(A \multimap !B)$

Intuitionistic implication is usually translated as $!A \multimap B$, but it is better to regard it as $!(A \multimap B)$ because of this equivalence: the outer $!$ is excluded because promotion on the right is always possible in simulations of LJ sequents. With subexponentials, however, promotion may not always be possible. One can regard focusing as the application of these equivalences in one direction:

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5 To modify the completeness proof for phase semantics (as defined in [9]), construct a canonical model where the monoid $W$ remains the set of all finite multisets of formulas with multiset union as its operation and with the empty multiset as unit. Let $\perp = \{\Gamma : \Gamma \vdash \Gamma \text{ provable}\}$. Define each submonoid $W_k$ as $W$ restricted to only multisets of formulas of the form $?_jA$ with $r \geq k$ (if we accept contraction on $?_k$, which means $J$ is already present in the intersection, then it suffices to restrict $W_k$ to formulas that do not begin with $?_j$ with $j \not\geq k$). Thus if $i \leq k$ then $W_k \subseteq W_i$. Define $J$ to consist of all finite multisets of the form $\{?_{n_1}A_1, \ldots, ?_{n_m}A_m\}$ such that $(?_{n_1}A_1 \& \cdots \& ?_{n_m}A_m) \multimap ?_{n_i}A$ (equivalently $?_{n_i}A_1 \& \cdots \& ?_{n_m}A$) is provable for each $i$ and $A$ ($J$ implicitly contains the empty multiset). $W$ itself is a submonoid that defines the original $!$ and $?$, and their completeness arguments can now be generalized to other exponentials. Semantic completeness implies the admissibility of cut.
removing as many exponentials as possible. Our usage of them is rather in the other direction: adding more exponentials harmlessly. A contraction-enabled \( ? \) must be present for a programmer to use \texttt{call/cc} or some other control operation. We may wish to write a contraction on an arrow type even though in a cut-free focused proof it is never needed. In the original \( \lambda \mu \)-calculus, this is the only form of contraction that occurs. In our system it is of crucial importance to identify the conditions under which the following equivalences are preserved in the presence of subexponentials:

\[
\begin{align*}
!\nu &\equiv !\nu \otimes !\nu, \\
?\mu &\equiv ?\mu \otimes ?\mu.
\end{align*}
\]

The index names are chosen to correspond to formulas in Section 3. A consideration of the structure of the cut-free proofs of the equivalences shows why the restrictions on indices are necessary and sufficient. Although our use of subexponentials will be effusive, we only admit formulas that satisfy these restrictions. They will allow us to (i) write synthetic introduction rules and (ii) compose cuts by ensuring that promotion on the conclusion of the cut is always possible.

### 3 The Fragment MC: Multi-Colored Classical Logic

For our purpose, we assume that all subexponentials \( ?_i \), admit both weakening and contraction on the right, so \( !_i \) admits them on the left. The aspect of linear logic that prevents resources from being reused is for us an orthogonal issue. For clarity, we assume that subexponential indices form a partial order, although some of our examples will simply use natural numbers as indices. We also assume the existence of finite joins and meets and that there is a maximum index \( \max \) and a minimum \( 0 \). We write \( \min(a, b) \) for meets and \( \max(a, b) \) for joins.

#### 3.1 Selection of Modalities

With a single pair of exponentials there are seven equivalence classes of exponential prefixes or modalities: \( !, !, !, !, !, !, !, ! \) and the empty prefix. This property extends to any pair of subexponentials \( !_i \) and \( ?_k \). For any prefix \( \nu \), \( \nu \nu \equiv \nu \). For example, \( !_i ?_k !_i ?_k A \equiv !_i ?_k A \). Most studies of linear logic consider only a few of these modalities. The LC fragment of linear logic, for example, uses only \( ! \) and \( ? \) although \( ! \) and \( ? \) may appear before atoms. The other prefixes have seldom been explored and for good reason: \( ! \), for example, destroys focus, which means that we cannot generally use it to form synthetic connectives. However, with the restrictions on indices described in Section 2.3, adequate focusing behavior can be recovered.

Since our main connective is implication and we wish to capture (at least) classical computations, let us review how classical implication can be represented in linear logic. The most straightforward translation is \( !A \rightarrow \# \) \( B \). This representation is sufficient for cut-free proofs, but for proofs with cuts it is clearly inadequate: one cannot form a cut with a right-side \( \# B \) and a left-side \( !B \). In the terminology of [3], we require an adequate inductive decoration. Two well-known ones are the T-translation: \( !A \rightarrow \# B \), and the Q-translation \( !A \rightarrow \# B \). In each case one modality is a suffix of the other. The T-translation leads to LKT, which can capture \( \lambda \mu \)-calculus. With subexponentials however, we need a more flexible way to switch between left and right modalities because a promotion of \( ?_k B \) to \( !_i ?_k B \) may not always be possible, especially in modeling structural substitution, which is needed by control operators.

The modalities that we shall use will be of the forms \( !_i ?_k \) and \( ?_k !_i \) (for every pair of \( i \) and \( k \)). Note that since \( !_i ?_k !_i ?_k A \equiv !_i ?_k A \), each modality can be seen as a suffix of the other. Promotion and dereliction will be restricted to forms that render them inverse operations:

\[
\begin{align*}
\frac{!_i ?_k A}{?_k !_i ?_k A} & \text{ derelict} \\
\frac{?_k !_i ?_k A}{!_i ?_k A} & \text{ promote}
\end{align*}
\]
Adding a \( !_{i} \) or \( ?_{k} \) before \( !_{i} ?_{k} \) or \( ?_{k} !_{i} \) will still result in something equivalent to one of the two forms. We only use equivalent classes of modalities and never write \( !_{i} ?_{k} \). It is important to have the flexibility that \( i \) and \( k \) may be distinct: it is impossible for \( !_{i} , A \rightarrow !_{j} ?_{j} A \) to hold if \( i \) and \( j \) are unrelated. Only the form \( !_{i} ?_{k} \) may appear on the left side of \( \rightarrow \) (left-side of sequents), thus we effectively use three modalities for each pair of indices \( i \) and \( k \) (\( ?_{i} !_{k} \) as well).

If we use \( !_{k} ?_{k'} A \rightarrow !_{j} ?_{j'} B \) to represent an implication it will not have one of the three modalities that’s allowed. Furthermore, there will be no guarantee that \( ?_{j} !_{j} B \) can be promoted to \( !_{j'} ?_{j'} B \), thus disqualifying it as an adequate decoration. Thus we shall represent implication \( A \rightarrow B \) as follows:

\[
!_{i'} \ (l_{k} ?_{k} A \rightarrow !_{j} ?_{j} B) \text{ or } ?_{i'} !_{i'} (l_{k} ?_{k} A \rightarrow !_{j} ?_{j} B) \quad \text{such that } i' \leq k, j' \text{ and } j \leq k, i
\]

The index restrictions are certainly not ad-hoc: they correspond precisely to those needed to preserve the equivalences \( ?(? A \& ? B) \equiv ? A \& ? B \) and \( !(A \rightarrow !) B \equiv !(A \rightarrow B) \). Under these restrictions, we can show that the above formula is equivalent to several other forms:

\[
!_{i'} \ (l_{k} ?_{k} A \rightarrow !_{j} ?_{j} B) \equiv !_{i'} !_{i'} (l_{k} ?_{k} A \rightarrow !_{j} ?_{j} B) \equiv !_{i} (l_{k} ?_{k} A \rightarrow !_{j} ?_{j} B) \equiv !_{j} (l_{k} ?_{k} A \rightarrow !_{j} ?_{j} B)
\]

The first condition, \( i' \leq k, j' \), makes \( ?_{i'} \) gratuitous and allows us to write a synthetic introduction rule as long as the rule does not break apart \( l_{k} ?_{k} \) in the premise. The second condition, \( j \leq k, i \), means that a conclusion \( ?_{j} !_{j} B \) can always be promoted to \( !_{j'} ?_{j'} B \) (thus it does not matter if we write \( j' ?_{j'} B \) or \( ?_{j'} ?_{j'} B \) on the right of \( \rightarrow \)). The following also hold:

\[
l_{j'} ?_{j'} (l_{k} ?_{k} A \rightarrow !_{m} ?_{m} B) \rightarrow \neg \ (l_{k} ?_{k} A \rightarrow !_{m} ?_{m} B)
\]

(to construct a proof bottom-up, first promote \( !_{m} \), then derelict \( !_{j} \), then promote \( ?_{j'} \). This will allow us to form an adequate \( \rightarrow \)-elimination rule.

For this presentation we restrict to the adequately decorated arrow as our only connective, although other connectives can be added along similar lines. To simplify the interpretation of sequents, we also use the empty modality, but only for sequents. To formally define the language of MC we use the following grammar:

\[
S \rightarrow F \& S | F \rightarrow \neg S | F | F \rightarrow F
\]

\[
F \rightarrow !_{i} ?_{i} C | ?_{i} !_{i} ?_{i} C
\]

\[
C \rightarrow (l_{k} ?_{k} C \rightarrow !_{j} ?_{j} C) | \neg p
\]

The syntactic variable \( p \) ranges over atomic formulas; \( S \) ranges over formulas that represent sequents; \( F \) ranges over formulas preceded by the modalities \( !_{i} ?_{i} \) or \( ?_{i} !_{i} ?_{i} \) for any \( i, i' \); and \( C \) ranges over formulas that are not preceded by these modalities. It is also required that the restrictions on implication \( i' \leq k, j' \) and \( j \leq k, i \) are imposed recursively on \( !_{i} ?_{i} C \) and on all \( ?_{i} !_{i'} ?_{i'} C \). Furthermore, all end-sequents are of the form \( \neg F \). When we speak of a formula of the MC fragment, we are referring to a \( F \)-generated formula of the above grammar, since such formulas form end-sequents.

### 3.2 A Useful Analogy

Informally speaking, the index of an exponential operator indicates a resource class. A proof of \( l_{k} A \) can only contain resources (free variables) of class \( k \) or higher. One useful analogy from
4 Natural Deduction and Proof Terms in MC

In the following we adopt the convenient notations $!_{ij} = !_i, j$ and $?_{ij} = ?_j, i$. So $?_{ij} A$ promotes to $!_i A$, for example. We revert to the unabbreviated forms for clarity when needed.

Single sided sequents suit classical linear logic; two-sided sequents will sometimes cause confusion. However, since our principal connective is $\rightarrow$, using one-sided sequents will appear awkward. Thus we shall make a small concession to Gentzen style systems and use two sided sequents with at most one formula on the left. This means that instead of $\Gamma \vdash ?_{ki} A, ?_{jm} B$ we write $!_i ?_{m!j} B \perp, \Gamma \vdash ?_{ki} A$. This concession is superficial since negation in linear logic is involutive. The interpretation of a sequent $A_1, A_2, \ldots, A_n \vdash B$ is the formula $A_1^\perp \otimes A_2^\perp \otimes \cdots \otimes A_n^\perp \otimes B$, or just $A_1^\perp \otimes \cdots \otimes A_n^\perp$ if the right side is empty. The concession is only one of notation. In this two-sided scenario the modalities that can appear to the right of $\vdash$ are $!_i ?_k$ and $?_k !_i$ while those that can appear on the left are $!_i ?_k$ and $?_k !_i$ (starting from a valid end-sequent). We store multiple conclusions on the left. Since contractions are allowed, the left-hand side of a sequent can do not cross.

The dereliction rule $\text{Consume}$ can also be simulated by $\text{Name}$ followed immediately by $\text{Unname}$. However, we have included it as a separate rule for convenience. The !$DR$ rule is only needed for atoms. The restrictions on implication means that the $\rightarrow$ in $!_i ?_q (!_{k?i} A \rightarrow !_{j?j} B)$ does not interfere with its introduction rule (i.e., it does not destroy focus, despite appearance). However, $!_i ?_q q$ when $q$ is an atom poses a slight problem. This is the only rule that violates focusing boundaries, but it is required for completeness, by which we mean the following:

**Theorem 1.** A formula of the MC fragment is provable by natural deduction in MC if and only if it is provable in subexponential linear logic.
The soundness of the rules is straightforward. For completeness, first we convert the elimination rules to sequent calculus style left-introduction rules in the usual way. Then using cut-elimination in linear logic and the equivalences induced by the index restrictions, we can show how proofs in linear logic are emulated by the rules of MC. Using a focused proof system for subexponentials simplifies the arguments. A somewhat delicate point is that, using cut, we can assume that \( ! \alpha \) !\text{Intro} \( \alpha \leq n, u \notin \Gamma \) \( \lambda u, s : \Gamma \Gamma(n) \vdash !u \alpha \forall u, !u k \alpha a \)

\( \forall \text{Intro} (i \leq n, u \notin \Gamma) \) \( s : \Gamma \Gamma(n) \vdash !u \alpha !u k \alpha A \)

\( st : \Gamma \Gamma(n) \vdash !u \alpha !u k \alpha A[t/x] \) \( \forall \text{Elim} \)

Consider the cut-free proofs of MC sequents in the host sequent calculus of linear logic re-emulated by the rules of MC. Using a focused proof system for subexponentials simplifies the arguments. A somewhat delicate point is that, using cut, we can assume that \( ! \alpha \) appears in the proof only if \( A \) is atomic, at which point the proof can always be emulated using the \( ! \alpha DR \) rule (see Appendix for further details).

The proofs terms here are referred to as \( \text{bounded } \lambda \mu \text{-terms} \) because we have adopted several aspects of the original \( \lambda \mu \) calculus as presented in [20]. First, we prefer to associate the proof term with the entire subproof, and not just the single formula on the right of a sequent. Secondly, Parigot referred to \( [d] t \) as a \( \text{named term} \), which is then unnamed, or bound, by \( \mu \). If one wishes to make sense
of $\lambda\mu$ calculus in terms of intuitionistic logic, then $\mu$ must be considered a non-logical constant of type $(A \rightarrow \bot) \rightarrow \bot \rightarrow A$, which would of course require a double negation/CPS translation to become intuitionistically admissible. This unfortunate interpretation means that $[d]t$ is really $(d \ t)$ where $d$ is of type $A \rightarrow \bot$. This means that the answer type of a captured continuation can only be $\bot$. But in a logic with involutive negation, such an interpretation becomes unnecessary. The only meaningful operation that $Name$ embeds is a dereliction\(^7\). We have adopted the strategy that neither promotion nor dereliction are reflected in proof terms: they do not appear to serve any purpose. The only extra notation we add are the $\text{bounded } \mu^b$ $\text{binder}$, superscripted by the producer class/level of its type, and the $\text{bounded reset indicator } ^\# n$, which is used to decorate every application term, with its subscript index calculated from the consumer classes of continuations that may be captured by some $\mu^b$. Unlike in other formulations of delimited control operators, $^\# n$ is not an independent operator but is rather a form of type annotation.

To illustrate the system we show a generalized proof of a Peirce-like formula:

\[
\begin{align*}
\vdash & \text{App} \quad \lambda x.\mu^b b \leq a, a' \quad \text{Name} \quad \text{Unname} \\
\vdash & \text{Abs} \quad k \leq a' \quad \text{Produce}, \ b \leq a, a' \\
\ldots \vdash & \text{Abs} \quad i \leq \max \\
\vdash & \text{Name} - \text{Unname} \quad \text{Produce}, \ a \leq j \ (\text{automatic}) \\
\vdash & \text{Abs} \quad i \leq \max
\end{align*}
\]

There are only a few restrictions in addition to those already imposed on formulas ($k \leq a'$ and $b \leq a'$). One can easily choose indices that would make this proof valid, including using the same index everywhere. The proof term here would be $\lambda x.\mu^a d, [d](x \ ^\# m (\lambda y.\mu^e e, [d]y))$ where $m \leq \min(j, a')$. As we shall show, the $\mu^a$ term is guaranteed to be able to catch continuations up to the nearest $^\# n$ with $n \geq a'$, or $n < a$ in a linear ordering. In a linear ordering, if $n = k - 1$, for example, then capturing the continuation beyond $^\# n$ would mean that the promotion to $! k$ (as part of the upper Produce rule) cannot be duplicated in the proof.

5 Intuitionistic Logic in MC

We have presented MC as a form of classical logic. However, the resource control aspect of MC means that it also generalizes the kind of restrictions found in intuitionistic logic. Restrict all formulas to use only the modalities $!_2 ?_1$ and $?_2 ?_1$. Then any copies of $?_2 ?_1$ must be weakened away before an implication can be introduced with the Abs rule. This corresponds to the multiple-conclusion version of intuitionistic sequent calculus, at least when restricted to the $\rightarrow$ fragment (it should be easy to extend this, however, to larger fragments):

\[
\begin{align*}
\Gamma \vdash \Delta & \quad \Gamma \vdash A, A, \Delta \\
\Gamma \vdash A, \Delta & \quad A, \Gamma \vdash B, \Delta
\end{align*}
\]

Intuitionistic contraction is justified by the axiom $A \land A \rightarrow A$. This proof system shares the rare property with subexponential linear logic that weakening is forced beneath initial rules. Peirce’s

\(^7\) When restricted to the modalities $!_2 ?_1$ and $?_2 ?_1$, dereliction can be expressed by the axiom $! A \rightarrow A$: this is an intuitionistic implication, which surely has proof $\lambda x. x$.  

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formula, $((P \rightarrow Q) \rightarrow P) \rightarrow P$, cannot be proved using only $!_{ik}$ and $?_{kk}$ with $i \leq k$:

$$!_{21}((l_{21}(l_{21}(P \rightarrow l_{21}(P \rightarrow Q)) \rightarrow Q) \rightarrow Q) \rightarrow Q)$$

Although promotion is applicable (backwards) on $l_{21}P$, any copy of $P$ created by contraction, which will appear as $?_{1}l_{2}?_{1}P$ (or $l_{1}?_{2}l_{1}P^k$ on the left), must be weakened away upon introduction of $l_{21}(l_{21}P \rightarrow Q)$. The only producers that do not labor in vain are those on the left side of an odd number of occurrences of $\rightarrow$. In terms of the generalized proof of Peirce’s formula shown earlier, the condition $k \leq a'$ would fail. Contracting the entire formula first will similarly fail.

It is simple to verify that the above restriction defines intuitionistic logic as a fragment of MC: soundness follows from the multiple-conclusion proof system and completeness follows from the single-conclusion one (via the normal conversions to natural deduction form). The fragment is defined by a restriction on formulas, not by a restriction on proofs. Only a slight difficulty is caused by the $!DR$ rule, which can change the indices of exponentials. But the indices in the premise of $!DR$ can only represent a higher consumer class and a lower producer class, and a simple inductive proof will show that provability is not affected by such a use of $!DR$: the new proof has the property that the minimum consumer index will still be greater than the maximum producer index among all formulas that may appear on the right-hand side of sequents in the proof.

From this perspective, we see that classical and intuitionistic logics are not just duals of each other, but are at opposite extremes. Intuitionistic logic allows no scope extrusion or escape, whereas classical logic make no such restrictions at all. With the appropriate indexing scheme in MC, we can choose to extrude into the scope of some $\rightarrow$ while keeping others intact. MC is as much a generalization of intuitionistic logic as it is a refinement of classical logic and provides a new way in which the logics can be combined.

## 6 Reductions for Bounded $\lambda\mu$

By Theorem 1, cut-elimination holds in MC. This alone, however, does not tell us which reduction strategy is valid. The potential danger of merging two contexts $\Gamma_1^{(n)}$ and $\Gamma_2^{(m)}$ into $\Gamma_1\Gamma_2^{(n,m)}$ is that the lowering of the maximum promotion level will mean that some promotions can no longer be duplicated. To determine what remains as valid reduction strategies, first we note the following:

> **Lemma 2.** If $\Gamma_1^{(n)} \vdash !_{1} ?_{k} A$ is provable without weakening (all formulas in $\Gamma$ appear as free variables in the proof term), then $i \leq n$.

This holds because all formulas in $\Gamma$ are preceded by some $!_{i}, ?_{k}$. By Theorem 1 it is valid to consider cut free proofs in linear sequent calculus. In such a proof, the first rule (bottom up) must be a right-side promotion of $!_{i}$, for anything else will result in a $?_{k}$ on the left and $!_{i}$ on the right, making any further promotions impossible.

Using this lemma, first we verify that $\beta$-reduction is still a valid strategy:

$$s : l_{nuw}A^v, \Gamma_1^{(n,m)} \vdash l_{irr}B \quad v \leq n \quad t : \Gamma_2^{(m)} \vdash l_{nuw}A$$

$$\lambda x.s : \Gamma_1^{(n)} \vdash l_{nuw}(l_{nuw}A \rightarrow l_{nuw}B) \quad s[t/x] : \Gamma_1\Gamma_2^{(n,m)} \vdash l_{irr}B$$

We are only concerned with those branches of the left subproof where $l_{nuw}A$ persists (has not been weakened away). Clearly in these branches there cannot be any promotion higher than to $l_{i}$. But by Lemma 2, the right subproof either ends in weakening, which permutes easily with cut, or it holds that $u \leq m$, and thus $\text{min}(n, m)$ is not lower than $\text{min}(n, u)$, which means these promotions can still occur after $\Gamma_2^{(m)}$ has been added to the left subproof. Thus $\beta$-reduction is still a valid strategy.
The restriction $n' \leq \min(n, i')$ on the App rule can be strengthened to $n' \leq \min(i, j')$ since $i \leq n$ by Lemma 2.

Next, we examine the possibility of capturing the continuation in the style of the original $\lambda\mu$ calculus. In order to not clash with $\beta$-reduction, the original $\lambda\mu$ calculus only allowed the continuation to the right of $\mu$ to be captured, by which we mean the following scenario:

$$\frac{w : \Gamma' \vdash A}{[d]w : (l_{u'w'}(A_{\circ l_{r'}} B))^d, \Gamma' +}$$

We again are only concerned with branches of the left subproof that contains $!_{u'v'}(A_{\circ l_{r'}} B)^\perp$, which means that there can only be promotions up to level $!_{u'}$. But $u' \leq u$ is required of well-formed formulas and by Lemma 2, $u \leq m$. So once again substituting $\Gamma_2^{(m)}$ into the left subproof will not prevent any promotions from being duplicated. This type of continuation capture is also valid.

However, it was soon recognized (e.g., [19]) that continuation capture need not clash with $\beta$-reduction as long as we define a (call-by-value) reduction strategy carefully. The resulting form of continuation capture can be generalized to the capture of an entire evaluation context:

$$\frac{t : \Gamma' \vdash A}{[d]t : (l_{u'w'}A^\perp)^d, \Gamma' +}$$

In order to permute the cut with $f$ into the right subproof, we need to be able to retain promotions up to level $l_{w'}$. Unlike the two previous cases, however, now it would possible for this condition to be violated if $n$ or $r'$ is smaller than $u'$ (see the proof of Peirce’s formula). Thus here we mark the redex with $k$ where $k \leq \min(n, r')$ (or $k \leq \min(v, r')$). The continuation $f$ can be captured by $\mu^v$ only if $\min(n, r') \geq u'$. We allow $k$ to be less than $\min(n, r')$ because we may decide to force $\beta$-reduction anyway. We can reserve the minimum index 0 for this purpose, and require $k > 0$ in all terms $\mu^k d.t$. Then $f \not\in s \Gamma_2^{(n, m)}$ will always force $\beta$-reduction: we can just write $(f s)$ in that case. If capturing $f$ is legal, then the resulting proof can have one of the structures illustrated in Figure 2. The proof on the left retains the $[d]$ and $\mu^v d.$ annotations so that further continuations can be captured. The proof on the right drops them, which should be used when there are no more continuations that can be captured legally. Recall that merely moving a formula from one side of the sequent to the other is a null operation in linear logic due to involutive negation. It is important to note that the final promotion from $!_{v'}l_{r'} B$ to $l_{w'} A_{\circ l_{r'}} B$ is always valid because it is required of legal formulas that $r \leq u, v$ in $l_{w'}(A_{\circ l_{r'}} B)$. But by Lemma 2, $v \leq n$ and $u \leq m$, and thus $r \leq n, m$ which makes the promotion valid. This is a consequence of the equivalence between $!(A_{\circ \neg} B)$ and $!(A \circ B)$ when generalized to subexponentials.
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\[
\frac{(f \sharp_x t) : \Gamma_1^{(n)} \Gamma' \vdash t}{[d](f \sharp_x t) : !_{r'r'}B^+, \Gamma_1^{(n)} \Gamma' \vdash} \quad \text{Name}
\]

\[
\frac{s\{[d](f \sharp_x w)/[d]u]\} : !_{r'r'}B^+, \Gamma_1^{(n,m)} \vdash}{\Gamma_1^{(n,m)} \vdash ?_{r'r'}B} \quad \text{Eval Contexts}
\]

\[
\frac{(f \sharp_x t) : \Gamma_1^{(n)} \Gamma' \vdash t}{!_{r'r'}B^+, \Gamma_1^{(n)} \Gamma' \vdash} \quad \text{DR}
\]

\[
\frac{\mu d.s\{[d](f \sharp_x w)/[d]u]\} : \Gamma_1^{(n,m)} \vdash !_{r'r'}B}{r \leq m, n} \quad \text{Eval Contexts}
\]

\[
\frac{(f \sharp_x t) : \Gamma_1^{(n)} \Gamma' \vdash t}{!_{r'r'}B^+, \Gamma_1^{(n)} \Gamma' \vdash} \quad \text{null}
\]

\[
\frac{s\{[d](f \sharp_x t)/[d]u]\} : \Gamma_1^{(n,m)} \vdash !_{r'r'}B}{r \leq m, n}
\]

\[\textbf{Figure 2} \text{ proof transformation after continuation capture}\]

## 7 A Call-by-Value Reduction Strategy

In a term such as \((f \sharp_x (g \sharp_x (h \sharp_x \mu d.s)))\), \(\mu^3\) should be able to capture both \(h\) and \(g\) but not \(f\). To formalize an evaluation strategy, we define the following.

### Terms and Values:
- \(\lambda\)-variables \(x, \ldots\) and \(\mu\)-variables \(d, \ldots\)
- Values \(V \rightarrow x \mid \lambda x. T\)
- Terms \(T \rightarrow V \mid (T_1 \sharp_x T_2) \mid \mu^k d.t \mid [d]T\)

### Evaluation Contexts:
- \(F^k \rightarrow [ ] \mid (F^k \sharp_n T) \mid (V \sharp_j F^k) \ (j \geq k)\)
- \(E \rightarrow [ ] \mid (V \sharp_m E) \mid (E \sharp_n T)\)

\(E\) is an evaluation context while \(F^k\) is a level-\(k\) context that represents a continuation that be captured by \(\mu^k d.t\) in the “hole” of the context. Note that in the definition of \(F^k\) there is no restriction on the index \(n\), because forward capture is always allowed. The rules for \(F^k\) implies that terms such as \((\mu^m d.s) \sharp_r (\mu^k d.t)\) will have the form \(F^m[\mu^m d.s]\) where \(F^m = [ ]\) since \(\mu\)-terms are not values: the \(\mu^k\) term will be part of the context captured by \(\mu^m\) regardless of whether \(i \geq k\).

### Evaluation Rules:
- \(E[\lambda x. t \sharp_n V] \rightarrow E[t\{V/x\}]\)
- \(E[V \sharp_i F^k[\mu^i d.t]] \rightarrow E[V \sharp_i \{F^k[u]/[d]u\}] \quad (i \geq k)\)

A term of the form \((\lambda x. u) \sharp_n V\) or \((V \sharp_i F^k[\mu^i d.r])\) with \(i \geq k\) is called a **redex**.

There is no evaluation rule for when \(i \leq k\), which forces \(F^k\) to represent the maximum context that can be captured. If no \(\mu^k\) appears in a term, then the second evaluation rule will never be used, the \(\sharp_n\) labels are universally ignored and standard call-by-value reduction takes place.

All application terms include a \(\sharp_i\), which can act as a delimiter, stopping the capture of continuations by \(\mu^k\) with \(i \geq k\). Instead of a null evaluation rule \(\sharp_i(V) \rightarrow V\), which is found in most other systems, in our system \(\beta\)-reduction simply ignores the symbol.

We are missing an evaluation rule for when the entire term is of the form \(F^n[\mu^n d.t]\). However, a \(\lambda x. x\) can always be added in front of such a term. We can require that the minimum index 0, or some reserved index unrelated to any \(k\) that may appear in \(\mu^k d.t\), is reserved for the purpose of forcing \(\beta\)-reduction. For example, \((\lambda x. x) \sharp_0 \mu^0 d.t\), with \(n > 0\), reduces to \((\lambda x. x) \sharp_0 t\{u/[d]u\}\) (because here \(F^n = [ ]\)). In other words it simply deletes the annotations placed on \(t\).
The following key lemma illustrates the workings of the contexts and evaluation rules. For this result we have also included its proof, because it shows how closed terms are decomposed into values, contexts and redexes. A proof term is closed if all variables are bound by some λ or μ. Our results are for closed terms (which are the only proofs possible for end-sequents of the form \( \vdash F \)), but they can also be generalized.

\textbf{Lemma 3. Decomposition.} For all non-value, closed terms \( T \), either \( T \) is of the form \( F^k[\mu^k d.s] \) or of the form \( E[r] \) where \( r \) is a redex. Furthermore, \( E \) or \( F^k \) is uniquely determined.

\textbf{Proof.} Since \( T \) is closed, we can rule out the cases of \( T \) being a variable or of the form \([d]t\). The proof is by induction of the structure of \( T \).

1. If \( T \) is of the form \( \mu^k d.s \), then let \( F^k = [ ] \). This is clearly the only possible decomposition.

2. If \( T_1 \) and \( T_2 \) are both values, then \( T_1 \) must be some \( \lambda x.s \) since it is closed. Thus \( T \) is a redex so let \( E = [ ] \). This is the only decomposition since \( T_2 \) is a value and terms of the forms \( E[r] \) or \( F^k[\mu^k d.s] \) cannot be values.

3. If \( T_1 \) is a value \( V \) but \( T_2 \) is not a value, then by inductive hypothesis \( T_2 \) is either of the form \( E'[r] \) or of the form \( F^k[\mu^k d.s] \). If \( T_2 \) is \( E'[r] \) then let \( E = V \) \( [ ] \). Otherwise we have two subcases:

   a. If \( i \not< k \): in this case \( T = (V \not\equiv_i F^k[\mu^k d.s]) \) is a redex, so let \( E = [ ] \)

   b. If \( i \geq k \): let \( F^k = (V \not\equiv_{i} F^k) \), then \( T = F^k[\mu^k d.s] \)

The uniqueness of \( E \) or \( T \) also follow from inductive hypotheses.

4. If \( T_1 \) is not a value, then by inductive hypothesis it is either of the form \( E'[r] \) or of the form \( F^k[\mu^k d.s] \). In the first case let \( E = E' \) \( [ ] \). In the second case, let \( F^k = F^k \not\equiv_i T_2 \). Uniqueness follows from inductive hypotheses.

It follows easily from the property established by the lemma that if we placed an extra \( \lambda x.x \) before a term \( t \) then all non-value, closed terms have uniquely determined redexes. If \( t \) has type \( \lambda^k \vdash_A A \), then \( (\lambda x.x) \not\equiv_n t \) is well-typed for all \( n \leq k \).

\textbf{Corollary 4. Progress.} If \( s = (\lambda x.x) \not\equiv_0 t \) is a closed proof term, then \( s = E[r] \) where \( r \) is a redex. Furthermore, \( r \) is unique.

By Lemma 3, \( t \) is either a value, some \( E'[r] \), or some \( F^k[\mu^k d.s] \). In the first case, let \( E = [ ] \) and the redex is the entire term \( s \). In the second case, let \( E = V \not\equiv_0 E' \). In the last case, since \( 0 \not\equiv k \), the result follows because of the leading \( \lambda x.x; \) let \( E = [ ] \) and the redex is the entire term \( s \). The redex is unique because the context is unique. Thus evaluation is deterministic.

The following lemma shows that continuation capture is type-safe, and forms part of the Subject Reduction proof.

\textbf{Lemma 5. Let} \( C \) \textit{represent either a context} \( E \) \textit{or} \( F^k \).

1. If \( s : \Gamma \vdash_A A, s' : \Gamma' \vdash_A A \) \textit{and} \( C[s] : \Gamma' \vdash A' \) \textit{are provable, then} \( C[s'] : \Gamma' \vdash A' \) \textit{is also provable.}

2. If \( F^k[\mu^k d.s] : \Gamma \vdash A \) is provable, then \( s(F^k[w]/[d]w) : \Gamma \vdash A \) is also provable.

Each part is proved by induction on the form of the context. The difference between sequents \( \Gamma \vdash A \) and \( A \downarrow, \Gamma \vdash \) is merely notational in classical linear logic. The central argument is similar to what has been shown in Figure 2. Type soundness then follows:

\textbf{Theorem 6. Subject Reduction.} If \( s : \Gamma \vdash_A A \) is provable and \( s \rightarrow t \) using the evaluation rules, then \( t : \Gamma \vdash_A A \) is also provable.
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In terms of the existing literature on delimited control operators, the behavior of our operators is dynamic as opposed to static: they are closer to the control/promt of [7]. Since we do not interpret $\sharp_k$ as an independent operator, we cannot use it to guarantee a static behavior. How $\mu$-terms in the body of the substitution term $F^k[u]$ is to be delimited would depend on its surrounding context, which is not statically known. It is known that such dynamic, delimited control operators can have non-terminating behavior, even in a typed setting (see [2, 14]). The following term, adopted from [14], confirms this:

$$(\lambda x.x) \sharp_0 ((\lambda z.\mu d.(\lambda y.[d]t) \sharp_i[d]t) \sharp_i(\mu d.(\lambda y.[d]t) \sharp_i[d]t))$$

Here, $t$ can be any value of type $\downarrow_j A$ while $y$ and $z$ are vacuous. Let $V = \lambda z.\mu d.(\lambda y.[d]t) \sharp_i[d]t)$, then $F^k = V \sharp_i[]$ and this term reduces to $(\lambda x.x) \sharp_0 ((\lambda y.V \sharp_i[t]) \sharp_i(V \sharp_i[t])$, but since $z$ is vacuous, this then reduces to $(\lambda x.x) \sharp_0 ((\lambda y.V \sharp_i[t]) \sharp_i(\mu d.(\lambda y.[d]t) \sharp_i[d]t))$, leading to an infinite sequence of continuation captures. However, the term is well-typed. Also, it does have a normal form, namely $t$, but this is not reachable using the call-by-value strategy. This phenomenon does not contradict cut-elimination. The possibility of non-termination is hardly cause for alarm as it is entirely consistent with what we already know to be possible with delimited control operators. A static behavior can be simulated by using $(\lambda x.x)\sharp_0[]$, meaning that we change the continuation capture rule to:

$$E[V \sharp_i F^k[j^{k, d.t.}]] \rightarrow E[V \sharp_i t \{((\lambda x.x)\sharp_0 F^k[u])/[d]u\}] \quad (i \geq k)$$

A call-by-name strategy can likewise avoid non-termination but then we can only capture continuations in the form of the original $\lambda \mu$-calculus, which is very limited for direct style programs. Call-by-value offers a much more general way to capture continuations. From the perspective of pure logic, there appears to be no reason to prefer one evaluation strategy to another. However, what we have shown is that if we choose call-by-value to access the general behavior of control operators, then logically these operators must be of the delimited kind.

8 Conclusion

The casual reader who opens this article to an arbitrary page may become dismayed by the large numbers of $\uparrow_j$,$\downarrow_j$ and $\downarrow_j$,$\uparrow_j$ that appear in formulas. Beneath this apparent chaos, however, are the fundamental principles that allow linear logic to be used as a powerful tool for deconstructing other logics. These include the principles of focusing and adequate inductive decoration. We have extended these principles to subexponentials and used them to identify a fragment that enhances classical logic. In the MC fragment intuitionistic logic is found not as a restriction on proofs but as a restriction on formulas. This represents a new way to combine classical with intuitionistic logic within linear logic which is quite different from the polarization approach of LC and related systems. MC is self-contained, with its own proof system and term representation. These terms include information pertaining to the indices of subexponentials, which represent a new way of restricting resource usage in proofs. An evaluation strategy uses this information to determine when the capturing of a continuation needs to be terminated. The hierarchy of subexponentials naturally leads to a hierarchy of delimited control operators that may coexist within the same system.

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References

Appendix: Proof of Theorem 1

This appendix shows MC in sequent calculus form and proves some essential proof-theoretic properties, especially Theorem 1. The proof of Lemma 5 is a straightforward induction on the form of the context and from this lemma subject reduction easily follows.

MC in sequent calculus form is used in the proof of Theorem 1, and is presented below

\[
\begin{align*}
\Gamma^{(n)} \vdash l_k^? \langle k' \rangle A & \quad \vdash_{j} \langle j \rangle B, \Gamma^{(n)} \vdash l_{m}^? \langle m \rangle C \\
\vdash l_{m}^? \langle m \rangle C & \quad \vdash_{j} \langle j \rangle B, \Gamma^{(n)} \vdash l_{m}^? \langle m \rangle C \\
\Gamma^{(n)} \vdash l_{k}^? \langle k \rangle A & \quad \vdash_{j} \langle j \rangle B, \Gamma^{(n)} \vdash l_{k}^? \langle k \rangle A
\end{align*}
\]

The identity rule becomes an instance of the !DR rule. The Name and Unm name operations have been replaced by bureaucratic rules that move formulas between left and right (except that one of them could embed a contraction on the left). We do not need to consider the case of \( \supset \) \( L \) where the right-hand side formula begins with \( ?_m \) or is empty because the Consume (dereliction) rule easily permute with these introduction rules. Technically, the conditions \( m \leq n, j \) and \( m \leq n, k \) are not required on these rules, but their presence simplify the proof of Theorem 1: we would not have to re-proof Lemma 2 relative to these rules, which also implies we would need to proof cut-elimination independently. For completeness we only need to show that cut-free proofs in (multi-color) linear logic correspond to cut-free proofs in MC sequent calculus, then show that sequent calculus rules can be emulated by natural deduction. Notice that the conditions \( m \leq n, j \) and \( m \leq n, k \) also imply (in both cases) that \( m \leq i \) because of the index restrictions already imposed on formulas. Another important characteristic of the MC sequent calculus is that the left-side context is not split by \( \supset \) \( L \), which is justified by the global admissibility of weakening and contraction on the left.

The formal proof of Theorem 1 will also require the following, which is easily verified:

Lemma 7. Initial Elimination: \( !_{iv} C \vdash !_{iv} C \) is provable.

We display the most interesting case of the inductive proof, which shows that the index restrictions between formula and inference rule correspond rather perfectly:

\[
\begin{align*}
\vdash l_{kk}^? \langle k \rangle A & \quad \vdash_{i} \langle i \rangle B, \vdash_{j} \langle j \rangle B \\
\vdash_{j} \langle j \rangle B & \quad \vdash_{i} \langle i \rangle B \\
\vdash_{j} \langle j \rangle B & \quad \vdash_{j} \langle j \rangle B
\end{align*}
\]

This lemma also means that any formula \( !_{iv} (l_{kk}^? \langle k \rangle A) \) that satisfies the index requirements is also provable since the empty context has unbounded maximum promotion level. It is also possible to generalize the result to show that \( !_{iv} A \vdash !_{iv} \langle i \rangle B \) is provable if \( a \geq c \) and \( b \leq d \).

The proof of the completeness direction of Theorem 1 uses the central argument that each formula \( !_{iv} (l_{kk}^? \langle k \rangle A) \) can be replaced (via cut) by formulas without \( ?_i \) because of the index restrictions placed on formulas. The extension of Andreoli-style focusing to subexponentials helps further. The cases of the completeness proof depend on the form of the principal formula of the inference rule (in multi-colored linear logic), and whether the formula is on the left or right of \( \vdash \).
We enumerate these cases using the following diagram, where some cases are given more then one number as label and some may share labels, in order to correspond to the arguments below.

\[
(1, 3) \vdash \text{C, (4, ) } \vdash \text{C} \uparrow \text{A}, (5) \vdash \text{k}\vdash \text{B}, (6) \vdash \text{q} \vdash \text{C}, (4) \vdash \text{C} \uparrow \text{A} \uparrow \vdash, (6, 7) \vdash \text{c} \vdash \text{q}
\]

The following shows how each of the cases above can be emulated in MC. Some of the cases (including 5) show that the situation cannot appear within the MC fragment.

1. The introduction (dereliction) of \( \vdash \text{C} \) on the left where \( \text{C} \) is non-atomic can be immediately followed (bottom-up) by an introduction rule. This is emulated by the (level-2) Decide rule of focused proofs.

2. The introduction of \( \vdash \text{C} \) on the right where \( \text{C} \) is non-atomic is a promotion followed by an asynchronous connective \(-\circ\), which can be also be introduced immediately following the promotion. The index restrictions of the MC rules are consistent with Lemma 2 for cut-free proofs in multi-color linear logic.

3. A \(-\circ\) left introduction rule does not, technically, have to split the context as we hope in the intuitionistically inspired \( \text{C} \vdash \text{L} \) rule: it may form the premise \( \text{C} \vdash \text{L} \vdash \text{A} \vdash \text{C} \). However, in a focused proof the subproof following (bottom-up) \(-\circ\) left must maintain focus on this sequent. But there is no focused proof of such a sequent because focus can never release. This issue has already been well-studied in the literature on polarization and focusing (e.g., [16]).

4. The introduction of \( \vdash \text{A} \vdash \text{q} \vdash \text{A} \) on the left or of \( \vdash \text{A} \vdash \text{q} \vdash \text{A} \) on the right are emulated by the Consume rule and the managerial rules of moving formulas between left and right. These cases are similarly handled if \( \text{A} \) is atomic and \( \vdash \text{q} \) are dropped.

In a focused proof, the consecutive introduction of \( \vdash \text{q} \vdash \text{A} \vdash \) and \( \vdash \text{q} \vdash \text{A} \vdash \) on the left is only possible if each dereliction is followed immediately by a promotion rule and release of focus, which can both be simulated by Produce, Consume and the managerial rules.

5. The introduction of \( \vdash \text{q} \vdash \text{A} \) on the left need not be considered by the subformula property of cut-free proofs. They will never appear in cut-free subproofs of MC end-sequents of the form \( \vdash \text{F} \) where \( \text{F} \) represents those formulas exclusively restricted to the modalities \( \vdash \text{A} \) and their inverses \( \vdash \text{A} \) with only \( \vdash \text{q} \vdash \text{A} \) allowed on the left-hand side of \(-\circ\).

It remains to deal with the intricate case of \( \vdash \text{q} \vdash \text{A} \) where \( \text{q} \) is atomic.

6. In (multi-color) linear logic the formula \( \vdash \text{q} \vdash \text{A} \) on the left can result by dereliction from \( \vdash \text{A} \). Since the introduction rule of \(-\circ\) leaves formulas \( \vdash \text{A} \) in the sequent, we can assume that \( \vdash \text{A} \) is present only if \( \vdash \text{A} \) is also atomic.

   a. If the right-hand side formula is in the form \( \vdash \text{C} \) then there is no proof of the sequent, for any such proof must eventually require an impossible promotion in the presence of another \( \vdash \text{C} \) on the left or \( \vdash \text{C} \) on the right.

   b. If the right hand side is of the form \( \vdash \text{C} \), we can in fact assume that it is empty, but with \( \vdash \text{C} \) on the left. We can assume that the sequent after (above) dereliction is in the form

\[
\vdash \text{C}, \vdash \text{p}_1, \ldots, \vdash \text{p}_n, \vdash \text{C}_1, \ldots, \vdash \text{C}_m, \vdash \text{C}_1, \ldots, \vdash \text{C}_m, \vdash \text{C}_1, \ldots, \vdash \text{C}_m, \vdash \text{C} \uparrow \vdash
\]

If \( \Gamma \) contains \( \vdash \text{D} \) with \( \text{D} \) non-atomic, then introducing them first will again leave \( \vdash \text{q} \vdash \text{D} \) in the sequent, so we can assume that \( \Gamma \) is empty without loss of generality.

But there is clearly no proof of this sequent, as any further left-dereliction of \( \vdash \text{C} \) or \( \vdash \text{C} \) will leave an unprovable sequent, as will promoting \( \vdash \text{q} \) from \( \text{q} \).

We can conclude therefore, that the dereliction of \( \vdash \text{q} \vdash \text{A} \) on the left from \( \vdash \text{A} \) can only take place if the right-hand side is of the form \( \vdash \text{q} \vdash \text{C} \). We can assume that \( \text{q} \) is atomic since \( \vdash \text{C} \) can be dropped from \( \vdash \text{q} \vdash \text{A} \) otherwise,
However, such a right-side formula in a cut-free proof can only appear as a result of promotion to some \( !_c ?_d q' \), which means that \( c \leq a \): we need to show that this holds even if the (backwards) promotion of \( !_c ?_d q' \) occurred before \( !_a ?_b q \) appeared on the left-hand side. This holds because the index restrictions are preserved by introductions of non-atomic formulas, as shown by the following nested inductive argument:

**a. Right-side introduction of \(-\circ\) can be assumed to result from moving \( !_i ?_i' (_i ?_i' \circ !_j j' B) \perp \) from the left. This means that \( c \leq i' \) must hold for the assumed promotion to take place. But \( i' \leq a \) is required of the formula, and so \( c \leq a \) must also hold.

**b. Left-side introduction of \( !_i (_i A \circ !_j j' B) \perp \) means that \( c \leq i \). We know that \( a \leq i \) but furthermore the \( \supset \) rule also requires that \( c \leq a \), and so the condition is again preserved.

Now we have \(?_b q\) on the left and \(?_d q'\) on the right, while other formulas are of the form \( !_i ?_j X \) on the left, and thus the only possible way to complete this proof is by promotion of \(?_b\), which means \( b \leq d \), followed by dereliction of \(?_d\) and with \( q = q' \).

Thus this proof can be emulated by the \( !DR/ID \) rule.

**7. Finally the promotion of \( !_c ?_d q \) from \(?_d q\) may result in \(?_d q\) being weakened away before being used. This is emulated in MC by the **Consume** rule and the rule to move \(?_d !_c ?_d q\) to the left. These arguments suffice to show that the sequent calculus version of MC is complete with respect to its fragment of multi-color linear logic. What remains to be shown is that these sequent calculus rules can be simulated by the natural deduction rules. We show the most important case:

\[
\frac{\Gamma(n) \vdash !_k k' A \quad !_j j' B, \Gamma(n) \vdash !_m m' C}{\Gamma(n) \vdash !_{min(i,n)} (_i A \circ !_j j' B), \Gamma(n) \vdash !_m m' C} \quad \supset \quad L, \; m \leq n, j
\]

is emulated by:

\[
\frac{\Gamma(n) \vdash !_m m' C \\ \Gamma(n) \vdash !_{min(j,n')} (_j B \circ !_m m' C)}{\Gamma(n) \vdash !_i (_i A \circ !_j j' B) \perp \Gamma(n) \vdash !_{min(j,n')} (_j B), \Gamma(n) \vdash !_m m' C} \quad \frac{\Gamma(n) \vdash !_k k' A}{\Gamma(n) \vdash !_{kk'} A} \quad \frac{\Gamma(n) \vdash !_{kk'} A}{\Gamma(n) \vdash !_i (_i A \circ !_j j' B), \Gamma(n) \vdash !_j j' B} \\
\frac{\Gamma(n) \vdash !_i (_i A \circ !_j j' B), \Gamma(n) \vdash !_m m' C}{\Gamma(n) \vdash !_{kk'} A} \quad \frac{\Gamma(n) \vdash !_k k' A}{\Gamma(n) \vdash !_i (_i A \circ !_j j' B)} \quad \frac{\Gamma(n) \vdash !_k k' A}{\Gamma(n) \vdash !_i (_i A \circ !_j j' B)} \quad \frac{\Gamma(n) \vdash !_k k' A}{\Gamma(n) \vdash !_i (_i A \circ !_j j' B)} \quad \frac{\Gamma(n) \vdash !_k k' A}{\Gamma(n) \vdash !_i (_i A \circ !_j j' B)}
\]

\( \text{App} \)

Once again, the index restrictions correspond perfectly: the left-subproof is guaranteed to introduce a valid formula since \( m \leq j \) and \( m \leq n \) are assumed.

*End of Appendix.*