How to explain a counterexample

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A modern issue with using proof assistants

In many modern proof assistants, when the user proposes a theorem to prove, the assistant often searches for a counterexample to that proposed theorem.

- Maybe the empty set was not considered properly or the existence of an even prime number was overlooked.

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If the machine finds a counterexample, the natural question is:

> *How can the machine help the user understand what is wrong with their proposed theorem?*

- Revealing a term—such as the empty set or the prime number 2—might be sufficient.
- Sometimes much more sophistication is needed.

Useful automated tools

Systems that search for counterexamples to proposed theorems.

- Refute and Nitpick are available in Isabelle/HOL.
- QuickChick is implemented in Coq.

Property based testing systems are closely related.

$$\forall L, K \in \mathit{list}.\ \{\mathit{true}\}\ K = \mathit{SortProg}(L)\ \{\mathit{sorted}\ K \wedge \mathit{perm}\ L\ K\}$$

- Quickcheck: tests code with lots of (well selected) examples and checks various proscribed properties of them.
- Originally developed within the Haskell setting, similar tools have been built for ACL2, Agda, Isabelle, and PVS.

In these cases, the machine has a proof that the user is motivated to learn *some* aspects of it.

How can a user learn from a formal proof?

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Print a LaTeX document? Naive for several reasons.

- The result could be very long and difficult to read.
- The line between computation (not needing explanation) and deduction (needing explanation) is often ad hoc.
- Different readers might need different levels of detail.
- The reader might need to consider only small parts of a proof.

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Proof browsing?

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Proof browsing? Seems less naive.

- The user chooses the parts of the proof of interest.
- If more details are needed, they can be selectively unfolded.

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- The user chooses the parts of the proof of interest.
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An *interaction* between the user and the proof seems best.

Two assumptions underlying this talk

1. The human user has proposed a conjecture and the machine has found a counterexample.
   - The explanation is provided to a *motivated* user and someone *familiar with the proof assistant*.
2. The theorems and conjectures will be based on items of computational interest instead of general mathematical interest.
   - This explains the kinds of examples I will be using.

We shall refer to the human as the *user* and the machine holding the proof as the *oracle*.

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This project is only getting started.

- Many technical issues remain.
- Lifting this project to a broader setting is certainly of interest.

Proof evidence as sequent calculus proofs

Some proof assistants build natural deduction style proofs, often encoded as dependently typed λ-terms.

Proof evidence appears in many other formats: resolution refutations, Herbrand disjunctions, tableaux proofs, etc.

Most proof evidence can be presented as sequent calculus proofs.

For example, *Foundational Proof Certificates* can be used to translate a wide range of proof evidence into sequent calculus proofs [Chihani, M, and Renaud, 2017].

Sequents and the search for proofs

Assume that I have a several assumptions $H_1, \ldots, H_n$ written at the top of sheet of paper and one conclusion B at the bottom of the proof.

In the middle of the sheet is blank space that needs to be filled with a proof.

This state of affairs is encoded in sequent calculus as:

$$\frac{\vdots}{H_1, \ldots, H_n \vdash B}$$

The sheet of paper is encoded as the sequent $H_1, \ldots, H_n \vdash B$ and the empty space corresponds to $\vdots$.

We usually read inference rules from conclusion to premises.

Sequent calculus inference rules

Structural rules: weakening and contraction

$$\frac{\Gamma \vdash \Delta}{\Gamma, B \vdash \Delta}\ wL \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, B}\ wR \quad \frac{\Gamma, B, B \vdash \Delta}{\Gamma, B \vdash \Delta}\ cL \quad \frac{\Gamma \vdash \Delta, B, B}{\Gamma \vdash \Delta, B}\ cR$$

Identity rules: initial and cut

$$\frac{}{B \vdash B}\ initial \qquad \frac{\Gamma \vdash \Delta, B \qquad B, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

Introduction rules: collections of left and right rules for every logical connective

We introduce these as we need them.

Cut-elimination and consistency

The central result in proof theory is *cut-elimination*: a sequent provable using the cut rule can be proved without the cut rule.

This result is usually proved by permuting cut rules up into the proof until they disappear.

Consistency of the underlying logic follows immediately: Assume that Ξ_1 is a proof of $\vdash B$ and Ξ_2 is a proof of $\vdash \neg B$ (equivalently, $B \vdash$). Thus, we have the following proof.

$$\dfrac{\begin{array}{c}\Xi_1\\ \cdot \vdash B\end{array} \qquad \begin{array}{c}\Xi_2\\ B \vdash \cdot\end{array}}{\cdot \vdash \cdot}\ \mathit{cut}$$

This empty sequent must also have a cut-free proof, but this is impossible.

A partial proof of the empty sequent

Assume that the user is convinced that sequent $\vdash B$ is provable (in, say, first-order intuitionistic or classical arithmetic).

Also assume that the proof assistant (the oracle) has constructed a proof Ξ of the negation of B, i.e., of the sequent $B \vdash \cdot$.

We can write the *partial* proof structure

$$\dfrac{\cdot \vdash B \qquad \overset{\Xi}{B \vdash \cdot}}{\cdot \vdash \cdot}\ \mathit{cut}$$

Thus, there can be no proof of the left premise.

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Main design choice: If the user is still convinced of the provability of B, we take advantage of that state of mind and allow the user to continue building a proof of B.

A conjunctive conjecture: the additive case

If B is $B_1 \wedge B_2$, the user is convinced that B_1 and B_2 are provable. If the oracle's proof uses the *additive* rule for conjunction, then it knows that one of these cases has a counterexample.

$$\dfrac{\dfrac{\cdot \vdash B_1 \qquad \cdot \vdash B_2}{\cdot \vdash B_1 \wedge B_2} \qquad \dfrac{\begin{matrix}\Xi_i \\ B_i \vdash \cdot\end{matrix}}{B_1 \wedge B_2 \vdash \cdot}}{\cdot \vdash \cdot}\ cut.$$

Permute the cut rule upward in this partial proof.

$$\dfrac{\cdot \vdash B_i \qquad \begin{matrix}\Xi_i \\ B_i \vdash \cdot\end{matrix}}{\cdot \vdash \cdot}\ cut$$

Thus the oracle should instruct the user to try to prove B_i. The user is informed which of these two cases should be pursued to discover a problem in the formulation of the theorem.

A disjunctive conjecture

If B is $B_1 \vee B_2$, the user is convinced that B_1 or B_2 is provable.

$$\dfrac{\dfrac{\cdot \vdash B_i}{\cdot \vdash B_1 \vee B_2} \qquad \dfrac{\begin{array}{c}\Xi_1\\ B_1 \vdash \cdot\end{array} \qquad \begin{array}{c}\Xi_2\\ B_2 \vdash \cdot\end{array}}{B_1 \vee B_2 \vdash \cdot}}{\cdot \vdash \cdot}\ \mathit{cut}.$$

Permute the cut rule upward in this partial proof.

$$\dfrac{\cdot \vdash B_i \qquad \begin{array}{c}\Xi_i\\ B_i \vdash \cdot\end{array}}{\cdot \vdash \cdot}\ \mathit{cut}$$

Thus the oracle is prepared to respond to either case that the user wants to explore.

A universally quantified conjecture

If the B is the universally quantified formula $\forall x.B'$, the interaction would provide an actual instance of that quantifier that would lead to a dead-end in the proof attempt.

$$\dfrac{\dfrac{\cdot \vdash B'x}{\cdot \vdash \forall x.B'x} \qquad \dfrac{\begin{array}{c}\Xi \\ B't \vdash \cdot\end{array}}{\forall x.B'x \vdash \cdot}}{\cdot \vdash \cdot}\ cut$$

In this case, permuting the cut rule upwards causes the term t to be substituted for the eigenvariable x, yielding

$$\dfrac{\cdot \vdash B't \qquad \begin{array}{c}\Xi \\ B't \vdash \cdot\end{array}}{\cdot \vdash \cdot}\ cut.$$

The user is asked to focus on one particular instance of the universal quantifier they believe should be true.

A conjunctive conjecture: the multiplicative case

If the oracle's proof uses the *multiplicative* rule for conjunction, then it knows only that both conjunctions cannot be proved.

$$\dfrac{\dfrac{\cdot \vdash B_1 \qquad \cdot \vdash B_2}{\cdot \vdash B_1 \wedge B_2} \quad \dfrac{\overset{\Xi}{B_1, B_2 \vdash \cdot}}{B_1 \wedge B_2 \vdash \cdot}}{\cdot \vdash \cdot}\ \mathit{cut}.$$

Permute the cut rule upward in this partial proof.

$$\dfrac{\cdot \vdash B_1 \qquad \cdot \vdash B_2 \qquad \overset{\Xi}{B_1, B_2 \vdash \cdot}}{\cdot \vdash \cdot}\ \mathit{multicut}$$

The oracle can claim that if one of B_1 or B_2 can be proved then the other one cannot be proved.

The user should attempt to prove the easier or these two first.

Three disciplines: Game theory, ludics, proof theory

- There seems to be a strong connection here between *dialogue games for proofs* [Hintikka, Lorenzen, etc]. In the interaction between the user and the oracle, the oracle has a winning strategy that is derived from its a formal proof.
- We use cut-elimination on non-proof objects: they necessarily have open premises. Such objects have been called *paraproofs*. This observation suggests connections also with *Ludics* [Girard 2001].
- Proof theory, especially, the theory of *focused proof systems*, can be used to extend these examples.

Focused Proof Systems

Andreoli gave a focused proof system for linear logic in 1991.

Focusing is ambiguous when applied to classical and intuitionistic logics. Liang & M [2009, 2011] have described a general framework for obtaining focused proof systems for those two logics.

Proofs are constructed using phases of inference rules: the *invertible* (negative) phase and the *non-invertible* (positive) phase.

These two phases can be related to the moves in a two players game. A precise connection between the cut-free proofs in MALL and winning strategies is given in [Delande, M, & Saurin, 2010].

Synthetic inference rules

Focused proof systems can be used to build *synthetic inference rules*. Cut-elimination automatically holds for such synthetic inference rules [Marin, M, Pimentel, Volpe 2020].

Consider defining a path in graph with adjacency give by $adj(\cdot, \cdot)$.

$$\forall x\ [path(x, x)]$$
$$\forall x, y, z\ [adj(x, y) \wedge path(y, z) \supset path(x, z)]$$

$$\frac{}{\cdot \vdash path(x, x)} \qquad \frac{\cdot \vdash adj(x, y) \qquad \cdot \vdash path(y, z)}{\cdot \vdash path(x, z)}$$

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To provide left-rules, we move beyond logic towards arithmetic.

Unfolding fixed points

The predicate $path(\cdot, \cdot)$ can be defined as a fixed point using techniques described by Schroeder-Heister (definitional reflection) [1993] and Girard [1992]. In that setting, there is no least or greatest fixed points: this is arithmetic without induction.

When the underlying graph is a finite DAG (directed acyclic graph), the least and greatest fixed points coincide.

If we make equality and $adj(\cdot, \cdot)$ into side conditions, we have the following right and left introduction rules for $path(\cdot, \cdot)$.

$$\frac{}{\cdot \vdash path(x, x)} \qquad \frac{\cdot \vdash path(y, z)}{\cdot \vdash path(x, z)} \;\text{ provided } adj(x, y)$$

$$\frac{\{\; path(y, z) \vdash \cdot \;\mid\; adj(x, y) \;\}}{path(x, z) \vdash \cdot} \;\text{ provided } x \neq z$$

Path or no path in a DAG

Assume that the oracle and user agree on equality of nodes and adjacency in the graph.

Assume that a, w, $b_1, \ldots, b_n$ ($n \geq 0$) are all distinct nodes and that a is adjacent to exactly $b_1, \ldots, b_n$.

$$\dfrac{\dfrac{\cdot \vdash path(b_j, w)}{\cdot \vdash path(a, w)} \qquad \dfrac{\{\ path(b_i, w) \vdash \cdot\ \}_{i=1}^{n}}{path(a, w) \vdash \cdot}}{\cdot \vdash \cdot}\ cut$$

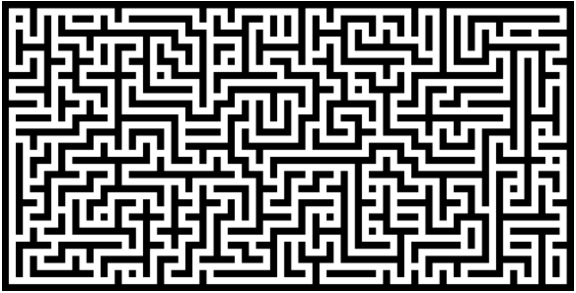
$$\dfrac{\cdot \vdash path(b_j, w) \qquad path(b_j, w) \vdash \cdot}{\cdot \vdash \cdot}\ cut$$

If the oracle has a proof of $path(a, w) \vdash \cdot$, it seems to have no useful information to give to the user.

"You can't prove a negative"

The meaning of this questionable expression might be rephrased:

> *You claim that there is no treasure in this maze. Since I don't trust you, I will conduct my own search.*



In the finite DAG situation, this means that interacting with the oracle provides no information to convince the skeptic.

Information content of focused proofs

A collection of invertible rules (called a *negative* or *asynchronous phase*) contains no useful proof information beyond the direct computation of its premises from its conclusion.

E.g. the left-introduction rule for $path(\cdot, \cdot)$ is *invertible*.

A collection of non-invertible rules (called a *positive* or *synchronous phase*) contains useful proof information that an oracle can communicate.

E.g. the right-introduction rule for $path(\cdot, \cdot)$ is *not invertible*.

Least and greatest fixed points

The proof theory of "generic fixed points" has been extended to include least fixed points (induction) and greatest fixed points (co-induction) within intuitionistic and linear logics [Baelde, McDowell, M, Momigliano, Tiu 2000-2012].

These extensions yield Heyting arithmetic and "linearized" arithmetic.

$$\frac{\Gamma, St \vdash C \qquad BSx \vdash Sx}{\Gamma, \mu Bt \vdash C}\ \mathit{Induction}$$

- μBt is the least fixed point of the predicate operator B applied then to term t.
- S is the *invariant* of this rule.
- While this rule breaks the subformula property, cut-elimination results can still be proved.

No-path example

One way to prove that there is no path from a to b is to find a collection $\mathcal{C}$ of nodes such that

$$\frac{\cdot \vdash a \in \mathcal{C} \qquad x \in \mathcal{C} \wedge adj(x, y) \vdash y \in \mathcal{C} \qquad b \in \mathcal{C} \vdash \cdot}{path(a, b) \vdash \cdot}$$

The skeptical user can attempt a proof.

$$\cfrac{\cfrac{\cfrac{\vdots}{\cdot \vdash path(a_2, b)} \; adj(a_1, a_2)}{\cdot \vdash path(a_1, b)} \; adj(a, a_1,)}{\cdot \vdash path(a, b)}$$

Continuing in this way, we have $\{a, a_1, a_2, \ldots\} \subseteq \mathcal{C}$.

But there is no information to guide the skeptic user.

Generalizations: Simulation and Bisimulation

In the study of concurrent processes, simulation and bisimulation are defined using greatest fixed points.

$$sim\ P\ Q := \forall A, P'\ [P \xrightarrow{A} P' \supset \exists Q'[Q \xrightarrow{A} Q' \wedge\ sim\ P'\ Q']]$$

This is a bipole, flipping from negative and positive polarity.

The game that arises from examining the winning strategies associated to focused proofs of this formula match exactly Stirling's games for simulation.

When the transition system (the $\cdot \xrightarrow{\cdot} \cdot$ relation) is a finite DAG, then the interaction between the user and oracle can proceed as expected.

$$\frac{\cdot \vdash sim\ P\ Q \qquad sim\ P\ Q \vdash \cdot}{\cdot \vdash \cdot}$$

Further directions

- Better treatment of multiplicative inference rules and of induction.
- What if the oracle also has a partial proof? Maybe that has value if it has enough proof evidence to convince the user.
- More generally, if someone proposes to pay anyone for a proof of B, there should also be a value for a proof of $\neg B$.
- Possible implementations
  - Abella: a small proof system that does not yet have a fixed notion of proof-as-a -value: it only has proof scripts.
  - Coq: with the addition of a plugin that implements λProlog, rather sophisticated interactions should be natural to write.

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Thank you for
your attention

Art by Nadia Miller