Abstracting sequent calculus proofs to get canonical proof structures

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Outline

Unity in (Computational) Logic

Focused proof systems

Maximal Multifocusing

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Unity in (Computational) Logic

There are many logics and they are based on many styles of formulas.

Church's STT provided an unifying framework for a rich set of formulas

- propositional logics
- first-order (multisort) logics
- higher-order logic
- also: ϵ-terms, modal operators, new quantifiers, etc.

STT also provides a unified approach to binding and substitution.

There are many formats for proofs

- Hilbert style
- resolution
- natural deduction, sequent calculus, tableaux, etc.
- model checking, winning strategies, etc

ProofCert: a five year ERC funded project given to my team to develop a notion of "proof certificate" for all of these kinds of (formal) proofs. A single "simple" checker will check all of these certificates.

Today: to describe a more modest proposal to connect many proof formats.

Sequent calculus and the revolutions against it

The sequent calculus *obscure the essence of a proof* by burying it under possibly vast quantities of low-level detail.

Individual inference rules are tiny.

Sequent proofs are *sequential* and proof steps that are non-interfering and permutable must be written a some order.

The sequent calculus lacks a notion of *canonicity*: proofs that should be considered the same may not have a common syntactic form.

As a result, many researchers *revolt* against the sequent calculus and replace it with proof structures that are more parallel or geometric.

Examples: Proof-nets, matings, and atomic flows.

Sequent calculus and evolution with it

We describe here an *evolutionary* approach to recover canonicity within the sequent calculus.

We use a *multi-focused* sequent system as our means of abstracting away the details from classical sequent proofs.

We then show that, among the focused sequent proofs, the *maximally multi-focused* proofs, which make the foci as parallel as possible, are canonical.

Moreover, such proofs are isomorphic to *expansion tree proofs*—simple and parallel generalization of Herbrand disjunctions—for classical first-order logic.

We thus provide a systematic method of recovering the essence of any sequent proof without abandoning the sequent calculus.

Outline

Classical logic and one-sided sequents

Two conventions for dealing with classical logic.

- Formulas are in *negation normal form*.
  - $B \supset C$ is replaced with $\neg B \vee C$,
  - negations are pushed to the atoms
- Sequents will be one-sided. In particular, the two sided sequent

$$B_1, \ldots, B_n \vdash C_1, \ldots, C_m$$

will be converted to

$$\vdash \neg B_1, \ldots, \neg B_n, C_1, \ldots, C_m.$$

LKF: Focusing for Classical Logic

The connectives are *polarized*: $\wedge^-$, $\wedge^+$, $\vee^-$, $\vee^+$, t^-, t^+, f^-, f^+.

A formula is *positive* if it is a top-level $\wedge^+$, $\vee^+$, t^+, f^+ or an atom.
A formula is *negative* if it is a top-level $\wedge^-$,$\vee^-$, t^-, f^- or a negated atom.

LKF is a focused, one-sided sequent calculus with the sequents

$$\vdash \Theta \Uparrow \Gamma \qquad \text{and} \qquad \vdash \Theta \Downarrow \Gamma$$

Here, Γ is a multiset of formulas and Θ is a multiset of positive formulas and negated atoms.

LKF : focused proof systems for classical logic

$$\frac{}{\vdash \Theta \Uparrow \Gamma, t^-} \quad \frac{\vdash \Theta \Uparrow \Gamma, B \quad \vdash \Theta \Uparrow \Gamma, C}{\vdash \Theta \Uparrow \Gamma, B \wedge^- C} \quad \frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, f^-} \quad \frac{\vdash \Theta \Uparrow \Gamma, B, C}{\vdash \Theta \Uparrow \Gamma, B \vee^- C}$$

LKF : focused proof systems for classical logic

$$\frac{}{\vdash \Theta \Uparrow \Gamma, t^-} \quad \frac{\vdash \Theta \Uparrow \Gamma, B \qquad \vdash \Theta \Uparrow \Gamma, C}{\vdash \Theta \Uparrow \Gamma, B \wedge^- C} \quad \frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, f^-} \quad \frac{\vdash \Theta \Uparrow \Gamma, B, C}{\vdash \Theta \Uparrow \Gamma, B \vee^- C}$$

$$\frac{}{\vdash \Theta \Downarrow t^+} \quad \frac{\vdash \Theta \Downarrow \Gamma_1, B_1 \qquad \vdash \Theta \Downarrow \Gamma_2, B_2}{\vdash \Theta \Downarrow \Gamma_1, \Gamma_2, B_1 \wedge^+ B_2} \quad \frac{\vdash \Theta \Downarrow \Gamma, B_i}{\vdash \Theta \Downarrow \Gamma, B_1 \vee^+ B_2}$$

LKF : focused proof systems for classical logic

$$\frac{}{\vdash \Theta \Uparrow \Gamma, t^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, B \qquad \vdash \Theta \Uparrow \Gamma, C}{\vdash \Theta \Uparrow \Gamma, B \wedge^- C} \qquad \frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, f^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, B, C}{\vdash \Theta \Uparrow \Gamma, B \vee^- C}$$

$$\frac{}{\vdash \Theta \Downarrow t^+} \qquad \frac{\vdash \Theta \Downarrow \Gamma_1, B_1 \qquad \vdash \Theta \Downarrow \Gamma_2, B_2}{\vdash \Theta \Downarrow \Gamma_1, \Gamma_2, B_1 \wedge^+ B_2} \qquad \frac{\vdash \Theta \Downarrow \Gamma, B_i}{\vdash \Theta \Downarrow \Gamma, B_1 \vee^+ B_2}$$

Init	Store	Release	Decide
$\dfrac{}{\vdash \neg A, \Theta \Downarrow A}$	$\dfrac{\vdash \Theta, C \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, C}$	$\dfrac{\vdash \Theta \Uparrow \mathcal{N}}{\vdash \Theta \Downarrow \mathcal{N}}$	$\dfrac{\vdash \mathcal{P}, \Theta \Downarrow \mathcal{P}}{\vdash \mathcal{P}, \Theta \Uparrow \cdot}$

$\mathcal{P}$ multiset of positives; $\mathcal{N}$ multiset of negatives;
A atomic; C positive formula or negated atom

Results about LKF

Let B be a first-order logic formula and let $\hat{B}$ result from B by placing $+$ or $-$ on t, f, $\wedge$, and $\vee$ (there are exponentially many such placements).

Theorem. B is a first-order theorem if and only if $\hat{B}$ has an LKF proof. [Liang & M, TCS 2009]

Thus the different polarizations do not change *provability* but can radically change the *proofs*.

One can easy move from a *linear-sized* proof to an *exponentially-sized* proof simply by changing the polarity of connectives.

Immediate by inspection of LKF

The only form of *contraction* is in the Decide rule.

$$\frac{\vdash \mathcal{P}, \Theta \Downarrow \mathcal{P}}{\vdash \mathcal{P}, \Theta \Uparrow \cdot}$$

Thus: only positive formulas are contracted.

The only occurrence of *weakening* is in the Init rule.

$$\overline{\vdash \neg A, \Theta \Downarrow A}$$

Thus formulas that are top-level $\wedge^-$,$\vee^-$, t^-, f^- are treated *linearly* (in the sense of linear logic).

The abstraction behind focused proofs

If we ignore the internal structure of phases and consider only their boundaries, we move from *micro-rules* (the atoms of inference) to *macro-rules* (pos or neg phases, the molecules of inference).

$$\frac{\vdash \Theta_1 \Uparrow \cdot \qquad \cdots \qquad \vdash \Theta_n \Uparrow \cdot}{\vdash \Theta \Uparrow \cdot}$$

An example: single focus

Let a, b, c be atoms and let Θ contain the formula $a \wedge^+ b \wedge^+ \neg c$.

$$
\dfrac{\dfrac{\dfrac{}{\vdash \Theta \Downarrow a}\;\textit{Init} \qquad \dfrac{}{\vdash \Theta \Downarrow b}\;\textit{Init} \qquad \dfrac{\dfrac{\vdash \Theta, \neg c \Uparrow \cdot}{\vdash \Theta \Uparrow \neg c}\;\textit{Store}}{\vdash \Theta \Downarrow \neg c}\;\textit{Release}}{\vdash \Theta \Downarrow a \wedge^+ b \wedge^+ \neg c}}{\vdash \Theta \Uparrow \cdot}\;\textit{Decide}
$$

This derivation is possible iff Θ is of the form $\neg a, \neg b, \Theta'$. Thus, the “macro-rule” is

$$
\dfrac{\vdash \neg a, \neg b, \neg c, \Theta' \Uparrow \cdot}{\vdash \neg a, \neg b, \Theta' \Uparrow \cdot}
$$

An example: multifocus

Let a, b, c, d, e, f be atoms and let Θ contain the formulas $a \wedge^+ b \wedge^+ \neg c$ and $d \wedge^+ e \wedge^+ \neg f$.

$$\frac{\vdash \Theta \Downarrow a \wedge^+ b \wedge^+ \neg c, d \wedge^+ e \wedge^+ \neg f}{\vdash \Theta \Uparrow \cdot} \ \textit{Decide}$$

This phase can be completed iff Θ is of the form $\neg a, \neg b, \neg d, \neg e, \Theta'$ and the resulting "macro-rule" is

$$\frac{\vdash \neg a, \neg b, \neg c, \neg d, \neg e, \neg f, \Theta' \Uparrow \cdot}{\vdash \neg a, \neg b, \neg d, \neg e, \Theta' \Uparrow \cdot}$$

A multifocus decide rule can initiate a parallel inference step.

Outline

Maximal Multifocusing

Let Π be a focused proof that ends in an instance of the Decide rule.

Denote by $\mathrm{foci}\,(\pi)$ the foci in the positive sequent that is the premise of that instance of the Decide rule.

We say that that instance is *maximal* iff for every Π′ that is "permutation equivalence" to Π, it is the case that

$$\mathrm{foci}\,(\pi') \subseteq \mathrm{foci}\,(\pi) \qquad \text{(as multisets).}$$

A focused proof is *maximal* iff every instance of Decide in it is maximal.

Expansion trees as MMF proof in LKF

Let $\hat{B}$ be the polarized version of B where

- all propositional connectives are the negative ones. Thus, the existential is the only positive connective.
- Add "negative delays" around positive literals.

Theorems

- Among the focused sequent proofs, the *maximally multi-focused* proofs are canonical.
- Such proofs are isomorphic to *expansion tree proofs*, a known, simple, and parallel generalization of Herbrand disjunctions for classical first-order logic.

From: "A Systematic Approach to Canonicity in the Classical Sequent Calculus" by Chaudhuri, Hetzl, and M.

Proof Nets as MMF proofs in MALL

A completely analogous connection exists in multiplicative additive linear logic between

maximal multifocused proofs

and

proof nets

was given in "Canonical Sequent Proofs via Multi-Focusing" by Chaudhuri, M, and Saurin. IFIP TCS, September 2008.

Conclusion

Focusing provides some large-scale structure to sequent calculus proofs.

Multifocusing allows for parallel inference steps.

Maximal multifocuing captures *all the parallelism* in a given proofs.

MMF proofs can be canonical and are likely to be isomorphic to proof structures already in existence.