Focusing Gentzen’s LK proof system

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Draft: March 31, 2021

Abstract  Gentzen’s sequent calculi LK and LJ are landmark proof systems. They identify the structural rules of weakening and contraction as notable inference rules, and they allow for an elegant statement and proof of both cut elimination and consistency for classical and intuitionistic logics. Among the undesirable features of those sequent calculi is that their inferences rules are low-level and frequently permute over each other. As a result, large-scale structures within sequent calculus proofs are hard to identify. In this paper, we present a different approach to designing a sequent calculus for classical logic. Starting with Gentzen’s LK proof system, we examine the proof search meaning of his inference rules and classify those rules as involving either don’t care nondeterminism or don’t know nondeterminism. Based on that classification, we design the focused proof system LKF in which inference rules belong to one of two phases of proof construction depending on which flavor of nondeterminism they involve. We then prove that the cut rule and the general form of the initial rule are admissible in LKF. Finally, by showing that the inference rules for LK are all admissible in LKF, we can give a relative completeness proof for LKF provability with respect to LK provability. We shall also apply these properties of the LKF proof system to establish other meta-theoretic properties of classical logic, including Herbrand’s theorem.

Key words: Sequent calculus, Gentzen’s LK, focused proof system, LKF, polarization, cut elimination

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1 Introduction

In his attempt to prove the *Hauptsatz* (cut elimination) for both intuitionistic and classical logics, Gentzen (1935) moved away from natural deduction to the sequent calculus. The sequent calculus allowed him to introduce the structural rules of weakening and contraction: their use on the right-hand side of sequents was fundamental to capturing both classical and intuitionistic logics in his one framework. If we are only interested in proving cut elimination and consistency, then the sequent calculus, as Gentzen presented it, is a great tool. If, however, we wish to apply logic and proof theory to, say, computation, then Gentzen’s sequent calculus has some significant problems: we discuss four such problems in Section 2.

In earlier work (Liang and Miller 2009), we have presented the *focused proof system* LJF as an improved version of Gentzen’s sequent system LJ for intuitionistic logic. Such focused proof systems have been used to give a foundation to logic programming (Miller 1989, Miller et al. 1991), model checking (Heath and Miller 2019), and term representation (Herbelin 1995, Scherer 2016).

This paper examines a *focused* version of the LK sequent calculus proof system, called LKF. The key properties of LKF—cut elimination and relative completeness for LK—have been proved elsewhere (Liang and Miller 2009; 2011) by using complex and indirect arguments involving linear logic (Girard 1987), a focused proof system for linear logic due to Andreoli (1992), and the focused proof system LJF. Here, we present LKF from first principles: we make no use of intuitionistic or linear logics nor the meta-theory of other proof systems. Additionally, proof transformations here, including those for cut elimination, should be more immediately formalized in, say, proof assistants than the more abstract arguments used elsewhere (including in Liang and Miller (2011)).

After presenting the LK inference rules, we describe some of the shortcomings of that proof system in Section 2. In Section 3, that criticism of LK motivates the design of LKF. We then prove the following results about LKF.

1. The cut rule in LKF is admissible in (cut-free) LKF (Section 4).
2. While the initial rule in LKF is limited to atomic formulas, the general form of the initial rule is admissible (Section 5).
3. The rules of LK are admissible in LKF (Section 7).

Taken together, these results prove that LKF is complete for LK. A similar proof outline for proving the relative completeness of focused proof systems has been used by Laurent (2004) for linear logic, by Chaudhuri et al. (2008b) for an intuitionistic version of linear logic, and by Simmons (2014) for a propositional intuitionistic logic. The proofs of these meta-theoretic results for LKF rely almost exclusively on tedious arguments about the permutability of inference rules. One of the design goals for LKF has been to build a calculus that can be used directly to prove other proof-theoretic results without the need to involve such tedious permutation arguments. We illustrate this principle by proving the admissibility of cut in cut-free LK (Section 9.1) and by proving Herbrand’s theorem (Section 9.3): both proofs do not explicitly involve permutation arguments.
2 The LK proof system

Formulas for first-order classical logic are defined as follows. Atomic formulas are of the form \( P(t_1, \ldots, t_n) \), where \( n \geq 0 \), \( P \) is a predicate of arity \( n \), and \( t_1, \ldots, t_n \) is a list of first-order terms. Formulas are built from atomic formulas using both the logical connectives \( \land, \lor, \top, \perp, \supset \) as well as the two first-order quantifiers \( \forall \) and \( \exists \). We shall assume the usual treatment of bound variables and substitutions: in particular, the expression \( \left[ B/x \right] \) denotes the result of performing a capture-avoiding substitution of term \( s \) for all free occurrences of the variable \( x \) in the formula \( B \).

Figure 1 presents the LK sequent proof calculus of (Gentzen 1935). Inference rules are between sequents which are pairs of multisets of formula, formally written with an infix \( \vdash \). The rules there are divided into introduction rules, structural rules, and identity rules. Note that the rules in this latter group, namely the \( \text{init} \) and the \( \text{cut} \) rules, require checking that two occurrences of a formula, here \( B \), on different sides of a sequent or in different sequents, have the same identity (e.g., are equal). Note also that the first important results about the LK sequent calculus imply that completeness is maintained if almost all of the identity rules are eliminated: one must only retain occurrences of the \( \text{init} \) rule where \( B \) is atomic.

The main differences between the proof system in Figure 1 and Gentzen’s presentation of LK are the following.
1. In Gentzen’s system, contexts are lists of formulas, and the exchange rule, which allowed two adjacent formulas to be swapped, was used. In Figure 1, contexts (Γ and Δ) are multisets of formulas, and the exchange rule is not used.

2. Gentzen did not have the logical units for true and false while here they are explicitly written as \( t \) and \( f \); they also have associated inference rules.

3. Gentzen’s system contained negation as a primitive connective while we shall treat it as an abbreviation: in particular, \( \neg B \) is defined to be \( B \supset f \).

For this paper, we shall make the following distinction between proof and derivation. By *proof*, we mean a tree structure of inference rules and sequents such that all premises are closed, in the sense that the inference rules at the leaves have zero premises (such as the initial rule). By *derivation*, we mean a similar tree structure of inference rules and sequents, but we do not assume that all leaves are closed: derivations can have unproved premises.

Gentzen’s sequent calculus was designed to support the proof of cut elimination (for both classical and intuitionistic logics). As we suggested in the introduction, sequent calculus is difficult to apply in a number of application areas. We describe four major shortcomings of the LK sequent calculus.

### 2.1 The collision of cut and the structural rules

Consider the following instance of the cut rule.

\[
\frac{\Gamma \vdash C \quad \Gamma', C \vdash B}{\Gamma, \Gamma' \vdash B} \text{ cut} \quad (\dagger)
\]

If the right premise is proved by a left-contraction rule from \( \Gamma', C, C \vdash B \), then cut elimination proceeds by permuting the cut rule to the right premises, yielding the derivation

\[
\frac{\Gamma \vdash C \quad \Gamma', C, C \vdash B}{\Gamma, \Gamma' \vdash B} \text{ cut} \quad \frac{\Gamma, \Gamma', C, C \vdash B}{\Gamma, \Gamma' \vdash B} \text{ cut} \quad \frac{\Gamma, \Gamma' \vdash B}{\Gamma, \Gamma' \vdash B} \text{ cL.}
\]

(An inference figure written with double lines indicates possibly several applications of the rules listed as its justification.) In the intuitionistic variant of the sequent calculus, it is not possible for the occurrence of \( C \) in the left premise of (\( \dagger \)) to be contracted since two formulas are not allowed on the right of the sequent arrow. If the cut inference in (\( \dagger \)) takes place in the classical proof system LK, it is possible that the left premise is the conclusion of a contraction applied to \( \Gamma \vdash C, C \). In that case, cut elimination can also proceed by permuting the cut rule to the left premise.
Thus, in LK, it is possible for both occurrences of \( C \) in (†) to be contracted and, hence, the elimination of cut is nondeterministic since the cut rule can move to both the left and right premises.

Such nondeterminism in cut elimination is even more pronounced when we consider the collision of the cut rule with weakening. Consider the derivation (taken from (Girard et al. 1989; Appendix B)).

\[
\begin{align*}
\Xi_1 & \vdash B \\
\vdash C, B & \overset{wR}{\rightarrow} C \vdash B \\
\Xi_2 & \vdash B \\
C \vdash B & \overset{wL}{\rightarrow} \vdash B, B \\
\vdash B, B & \overset{cR}{\rightarrow} \vdash B \\
\end{align*}
\]

cut elimination here can yield either \( \Xi_1 \) or \( \Xi_2 \); thus, nondeterminism arising from weakening can lead to completely different proofs of \( B \). This kind of example does not occur in the intuitionistic (single-sided) version of the sequent calculus.

These problems with cut elimination and the structural rules were noted in (Danos et al. 1997) and by Lafont in (Girard et al. 1989). Lafont concludes that in order to avoid this problem with cut elimination, one can choose from among two solutions: either make the sequent calculus asymmetric (leading to intuitionistic logic where the structural rules are not available on the right) or forbid all structural rules (leading to linear logic where structural rules are not available on the left and right). It is possible, however, to remain in classical logic by employing a third solution that uses both polarization and focused proof systems. Such an approach was proposed by Girard (1991) in his LC proof system and by Danos et al. (1997) in their LK\(^n\) proof system. In this paper, we present the LKF proof system, which is also based on the notions of polarization and focusing. As we shall see, the problems with the nondeterminism in cut elimination caused by the use of structural rules in classical logic disappear in LKF for two reasons. First, weakening will be allowed only in the initial rules of LKF where it cannot cause problems with cut elimination. Second, a cut takes place between two polarized formulas of opposite polarity and, in LKF, contraction is only applied to positive formulas.

**2.2 Permutations of inference rules**

A dominating feature of sequent calculus proofs in LK is that many pairs of inference rules permute over each other (Kleene 1952). For example, when an occurrence of \( \forall L \) is below \( \forall R \), as in the derivation
the order of these two rules can be switched to form the derivation

$$
\begin{align*}
\Gamma_1 \vdash B, \Delta_1 & \quad \frac{\Gamma_2, C \vdash \{y/x\}D, \Delta_2}{\forall R} \\
\Gamma_1, \Gamma_2, B \supset C \vdash \forall x.D, \Delta_1, \Delta_2 & \supset L,
\end{align*}
$$

Similarly, the following two deviations are such that permuting the inference rules in one derivation yields the other derivation.

$$
\begin{align*}
\Gamma_1 \vdash B, \Delta_1 & \quad \frac{\Gamma_2, C \vdash \{y/x\}D, \Delta_2}{\forall R} \\
\Gamma_1, \Gamma_2, B \supset C \vdash \forall x.D, \Delta_1, \Delta_2 & \supset L.
\end{align*}
$$

If one is trying to find structure in sequent calculus proofs, then it is likely that both of these pairs of derivations should be identified in some way.

The existence of such permutations of inference rules suggests that uncovering structures in proofs will always be disturbed by the possibilities of such shallow rearrangements of inference rules. For such reasons, people have often argued that the “essence” of proof structures is better captured in some radically different proof systems, such as, for example, expansion trees (Miller 1987), proof nets (Girard 1987, Laurent 2011), and atomic flows (Guglielmi and Gundersen 2008). In this paper, we also replace Gentzen-style sequent calculus with something else, namely \textbf{LKF}, but this time, that replacement will still resemble sequent calculus but with more structure added to both sequents and inference rules.

An introduction rule of \textbf{LK} is invertible if whenever there is an \textbf{LK} proof of its conclusion, there are \textbf{LK} proofs of the premises. When attempting to build a proof from the bottom-up, invertible rules can always be applied without losing provability. If an introduction rule is not invertible, it is non-invertible. The \textbf{LK} introduction rules can be classified as follows: the invertible rules are $\land R, tR, \lor L, fL, \supset R, \forall R, \exists L$ while the non-invertible rules are $\land L, \lor R, \supset L, \forall L, \exists R$. Note that every connection has an invertible introduction rule on one side of the $\vdash$, and every occurrence of the corresponding introduction rule on the other side is non-invertible. (This last statement is vacuously true for $t$ and $f$ since they have zero introduction rules on the left and right, respectively.) Observing the invertibility of introduction rules allows us to give some structure to the permutation of inference rules. In particular, an invertible rule above any other rule can always be permuted down. Furthermore, two non-invertible rules, one above the other, can always be permuted as well.

We make one additional observation: if an occurrence of a non-atomic formula on the left or right of a sequent can be the consequence of an invertible rule, that formula occurrence never needs to have a structural rule applied to it. For example, the contraction-left rule never needs to be applied to a disjunction since the disjunction-left rule is invertible.
These three observations about invertible and non-invertible rules—the left-right duality regarding invertibility; the permutations involving invertible and non-invertible rules; and the connection between invertible rules and the structural rules—will all be made explicit in the design of the LKF proof system.

2.3 Additive and multiplicative rules and connectives

The LK rules that have two premises can be classified as either additive, in which case the side formulas \((\Gamma, \Delta)\) are the same in the conclusion as well as in both premises, or multiplicative, in which case the side formulas in the premises \((\Gamma, \Delta\) and \(\Gamma', \Delta')\) are accumulated to form the side formulas in the conclusion. Of the four inference rules in Figure 1 with two premises, the cut rule and the implication-left rule are multiplicative while the disjunction-left rule and the conjunction-right rule are additive.

Consider the alternative inference rules in Figure 2 for conjunction and disjunction. The rules in that figure with two premises are multiplicative. We can make the following observations.

1. The rules in Figure 2 are inter-admissible, in the sense of preserving the provability of sequents, with those for the same connectives given in Figure 1. Establishing that fact requires using the structural rules of weakening and contraction.
2. The \(\land R\) rule in Figure 1 is invertible while the corresponding \(\land R\) rule in Figure 2 is not invertible. Similarly, the \(\lor R\) rule in Figure 1 are not invertible while the corresponding \(\lor R\) rule in Figure 2 is invertible.
3. If we are keen to separate the roles of structural rules from cut elimination, then we should not mix the various rules in Figures 1 and 2. For example, if we replace the \(\land L\) rule in Figure 1 with \(\land L\) in Figure 2, then the proof that a cut of a conjunction can be eliminated will necessarily use a structural rule.

Although Gentzen used the additive rules for conjunction and disjunction, there are reasons to admit other choices. For example, it is a popular choice to select the invertible right introduction rules for both conjunction and disjunction, which means selecting the additive conjunction and the multiplicative disjunction. Ketonen
introduced such a variant of Gentzen’s original calculus and used it to give “a strikingly elegant proof of completeness” (von Plato 2012). People working in automated theorem proving often use the invertible rules since it simplifies implementations of proof search. In particular, it is possible to define one-sided sequent systems for classical logic in such a way that all (right) introduction rules are invertible except for the existential introduction rule. As a result, proof search algorithms can limit backtracking to only the treatment of existential quantifiers.

The LKF proof system contains both the additive and multiplicative versions of conjunction and disjunction (and their units).

2.4 The need for synthetic inference rules

Our final criticism of LK is that its inference rules are too small, especially for applications involving theories. For example, assume that we are working with a theory (a set of assumptions) that has an axiom that declares that the binary predicate path is transitive: that is, that the theory contains the formula

$$\forall x \forall y \forall z \ (\text{path}(x, y) \supset \text{path}(y, z) \supset \text{path}(x, z)).$$

If that formula is invoked in an LK proof, there will be a minimum of five introduction rules involved in that invocation. That seems unfortunate since it is more natural to view that formula as denoting one of the following inference rules.

$$\frac{\Gamma \vdash \Delta, \text{path}(x, y) \quad \Gamma \vdash \Delta, \text{path}(y, z)}{\Gamma \vdash \Delta, \text{path}(x, z)} \quad \text{or} \quad \frac{\text{path}(x, y), \text{path}(y, z), \text{path}(x, z), \Gamma \vdash \Delta}{\text{path}(x, y), \text{path}(y, z), \Gamma \vdash \Delta}$$

These synthetic rules would be a more appropriate way to invoke the transitivity axiom. Such synthetic rules have been addressed before in the literature, particularly as a back-chaining inference rule (Hallnäs and Schroeder-Heister 1990, Miller et al. 1991) or as a forward-chaining inference rule (Negri and von Plato 1998). One of the immediate applications of LKF is as a formal framework for computing and justifying the addition of such synthetic inference rules to LK.

3 The LKF proof system

The LKF proof system does not deal with formulas but with polarized formulas: these are built from atomic formulas and negated atomic formulas (collectively called literals), and polarized logical connectives as well as the first-order quantifiers $\forall$ and $\exists$. The polarized logical connectives come in two flavors: the positive connectives are $f^+$, $\lor^+$, $t^+$, $\land^+$, and $\exists$ while the negative connectives are $f^-$, $\land^-$, $\lor^-$, $\forall^-$, and $\forall$.

Literals are also assigned a polarity as follows. An atomic bias assignment is a function $\delta(\cdot)$ that maps atomic formulas to the set of two tokens $\{+,-\}$: if $\delta(A)$ is
+ then that atomic formula is positive and if \( \delta(A) \) is \( \neg(\cdot) \) then that atomic formula is negative. We extend \( \delta(\cdot) \) to literals by setting \( \delta(\neg A) \) to be the opposite polarity of \( \delta(A) \). We may ask that all atomic formulas are positive, that they are all negative, or we can mix polarity assignments. In particular, the atomic bias assignment \( \delta^+(\cdot) \) assigns all atoms the positive polarity while \( \delta^-(\cdot) \) assigns all atoms the negative polarity. We shall often suppress explicit reference to atomic bias assignments, assuming that they have been specified and fixed. The only restriction we impose on atomic bias assignments is that they are stable under substitution: that is, for all atomic formulas \( A \) and every first-order variable \( x \) and term \( s \), then \( \delta'(A) = \delta'(x/s/A) \). This restriction is equivalent to saying that the value of \( \delta(\cdot) \) is determined by the predicate that is the top-level symbol of \( A \): that is, if \( A \) and \( A' \) are two atoms formed with the same predicate, then \( \delta(A) = \delta(A') \).

A polarized formula is positive if it is a positive literal or its top-level connective or quantifier is positive (i.e., it is of the form \( A \land B, A \lor B, \exists x.A, \land^* \) or \( \lor^* \)); similarly, a polarized formula is negative it is a negative literal or its top-level connective or quantifier is negative (i.e., it is of the form \( A \land B, A \lor B, \forall x.A, \land^- \) or \( \lor^- \)).

Polarized formulas are in negation normal form (nnf), meaning that there are no occurrences of implication \( \Rightarrow \) and that the negation symbol \( \neg \) has only atomic scope. When the negation symbol \( \neg \) is used with the non-atomic polarized formulas of \textbf{LKF}, we shall view it as the following function that transforms that polarized formula to its De Morgan dual.

\textbf{Definition 1} The negation symbol \( \neg \) is defined as the following function when applied to non-atomic polarized formulas.

- \( \neg \neg a = a \) for atomic formula \( a \)
- \( \neg(\neg A \land B) = \neg A \lor \neg B, \quad \neg(\neg A \lor B) = \neg A \land \neg B \)
- \( \neg(\neg A \land B) = \neg A \lor \neg B, \quad \neg(\neg A \lor B) = \neg A \land \neg B \)
- \( \neg\exists x.A = \forall x.\neg A, \quad \neg\forall x.A = \exists x.\neg A \)

It is easily shown that \( \neg A = A \) for all polarized formulas \( A \). Clearly, negation is treated differently between unpolarized formulas (where it is an abbreviation for “implies false”) and polarized formulas (where it computes the De Morgan dual).

The sequent calculus \textbf{LKF} for polarized formulas is presented in Figure 3: this presentation is a simplification of our original presentation given in (Liang and Miller 2009). This proof system uses one-sided sequents, but of two varieties, namely, \( \vdash \Gamma \uparrow \Theta \) and \( \vdash A \downarrow \Theta \), where \( \Gamma \) is a multiset of polarized formulas, \( \Theta \) is a set of polarized formulas, and \( A \) is a single polarized formula. The up and down arrows separate sequents into two zones: the zone on the right of the arrows (written using \( \Theta \)) is called the storage for that sequent. In notation such as \( \vdash \Gamma, \Gamma' \uparrow \Theta, \Theta' \), the multiset \( \Gamma, \Gamma' \) represents the multiset sum of \( \Gamma \) and \( \Gamma' \) while the set \( \Theta, \Theta' \) represents the union of the two sets \( \Theta \) and \( \Theta' \): it is, of course, possible for \( \Theta \) and \( \Theta' \) to share a non-empty intersection. When moving a collection of polarized formulas from the left of the \( \uparrow \) into storage, we coerce multisets into sets in the obvious way. Note that by inspection, the storage of the sequent in the conclusion of an inference rule is always a subset of storage of the sequents in the premises. We say that the polarized
Asynchronous introduction rules

\[ \vdash r^-, \Gamma \uparrow \Theta \quad \vdash A, \Gamma \uparrow \Theta \quad \vdash B, \Gamma \uparrow \Theta \quad \vdash \gamma, \Gamma \uparrow \Theta \quad \vdash f^-, \Gamma \uparrow \Theta \quad \vdash A, B, \Gamma \uparrow \Theta \quad \vdash A \lor B, \Gamma \uparrow \Theta \quad \vdash \left[ y/x \right] B, \Gamma \uparrow \Theta \quad \vdash \forall x, B, \Gamma \uparrow \Theta \]

Synchronous introduction rules

\[ \vdash r^+, \Theta \quad \vdash A \uparrow \Theta \quad \vdash B \uparrow \Theta \quad \vdash \gamma' \quad \vdash B_1 \uparrow \Theta \quad \vdash B_1 \lor B_2 \uparrow \Theta \quad \forall^*, \quad i \in \{1, 2\} \quad \vdash \left[ s/x \right] B \uparrow \Theta \quad \vdash \exists x, B \uparrow \Theta \quad \exists \]

Initial, store, release, and decide rules

\[ \vdash p \downarrow \lnot p, \Theta \quad \vdash \Gamma \uparrow Q, \Theta \quad \vdash N \uparrow \Theta \quad \vdash P \uparrow P, \Theta \quad \vdash \Gamma \uparrow \Theta \quad \vdash N \uparrow \Theta \quad \vdash P \uparrow P, \Theta \quad \text{store} \quad \text{release} \quad \text{decide} \]

Fig. 3 The inference rules for LKF. Here, \( P \) is a positive polarized formula and \( p \) is a positive literal; \( N \) is a negative polarized formula and \( Q \) is a positive polarized formula or negative literal. The rule for \( \forall \) has the usual eigenvariable restriction: \( y \) is not free in any polarized formula in the concluding sequent.

A formula \( B \) has an LKF proof if the sequent \( \vdash B \uparrow \cdot \) has a proof using the inference rules from Figure 3.

Before completing the details of the LKF proof system, we informally describe its relationship to Gentzen’s LK proof system. Just as polarized formulas are essentially regular formulas with some additional structure added (the + and − annotations), LKF sequents are essentially LK sequents with additional structure. That extra structure is the establishment of two zones within a sequent, namely, the storage zone and the non-storage zone. Additionally, LKF sequents come in two kinds, as is witnessed by the use of either \( \uparrow \) or \( \downarrow \). Thus, if one wishes to relate LKF sequents to Gentzen’s original sequents, one only needs to forget this additional structure. In particular, the arrows \( \uparrow \) and \( \downarrow \) can be replaced by a comma and all the polarization annotations on polarized formulas can be deleted.

We borrow the terminology asynchronous and synchronous rules from Andreoli (1992). A derivation composed only of asynchronous introduction rules (see Figure 3) and the store rule will be called an asynchronous phase, and a derivation composed only of synchronous introduction rules and the init rule will be called a synchronous phase. The sequents in an asynchronous phase all involve \( \uparrow \)-sequents while the sequents in a synchronous phase all involve \( \downarrow \)-sequents. An LKF proof is composed of alternations of these two kinds of phases. In particular, the decide rule connects a synchronous phase above its premise with an asynchronous phase below its conclusion, and the release rule connects an asynchronous phase above its premise with a synchronous phase below its conclusion.
The asynchronous phase can be used to encapsulate what is often called *don't care nondeterminism*. That is, if we consider the asynchronous phase as a large scale inference rule having a sequent of the form \( \vdash N \upharpoonright \Theta \) as its conclusion and sequents of the form \( \vdash \cdot \upharpoonright \Theta' \) as its premises, then that large scale rule is independent of the sequence of rule applications within the asynchronous phase (see Lemma 2). On the other hand, the synchronous phase is a sequence of applications of inference rules with choices (particularly for the \( \lor^+ \) and \( \exists \) introduction rules), and different choices will yield different synchronous phases: such phases, therefore, capture *don't know nondeterminism*.

While the weakening and contraction rules are not explicitly given in LKF, both of these rules occur implicitly. The *decide* rule does an implicit contraction on the polarized formula \( P \): hence, the only polarized formulas contracted in an LKF proof are positive polarized formulas. The *init* and the \( r^+ \) rules do implicit weakening on the polarized formulas in \( \Theta \): thus weakening is available for positive polarized formulas and negative literals. Thus, a negative, non-literal polarized formula is never weakened nor contracted: such polarized formulas are treated *linearly*, in the sense of linear logic (Girard 1987).

Polarized formulas in the storage zone play two different roles in proof search. With the *decide* rule, a positive polarized formula in the storage is *simultaneously* contracted and made available to introduction rules. On the other hand, with the *init* rule, a negative literal in storage is available to end the proof. No other kind of polarized formula will occur in storage.

The four binary logical connectives of LKF—\( \lor^+, \lor^-, \land^+, \land^- \)—can be classified using three different attributes: positive or negative; additive or multiplicative; and conjunctive or disjunctive. By fixing any two of these attributes, the third attribute is uniquely determined. For example, a connective that is both additive and positive must be the disjunction \( \lor^+ \). Note also that the De Morgan dual of a logical connective (in the sense of Definition 1) flips its polarity and conjunctive/disjunctive status but does not change its additive/multiplicative status. The introduction rule for \( \land^+ \) looks additive since the storage \( \Theta \) in the conclusion and the premises are all the same. The essential multiplicative character of \( \land^+ \) is not apparent in this proof system in which there can be only one focused polarized formula in a sequent. In Section 10, we present a *multifocused* version of LKF, and in that enlarged setting, it will be clear that \( \land^+ \) is, in fact, a multiplicative connective.

The proof system for LKF given in Figure 3 has no cut rule; thus the proofs built using the rules in Figure 3 are cut-free proofs. Cut-rules for LKF and a cut-elimination theorem will be presented in the next section.

Let \( B \) be a polarized formula and let \( \tilde{B} \) be the *depolarized* version of \( B \): that is, \( \tilde{B} \) is the unpolarized formula that results from removing the superscript + and – from the logical connectives in \( B \). Since \( B \) is in negation normal form, the formula \( \tilde{B} \) might have occurrences of negated atomic formulas, say \( \neg A \), and these should be seen as abbreviations for \( A \supset f \). Depolarizing a multiset or set of polarized formulas \( \Gamma \) is the set \( \tilde{\Gamma} \) resulting from depolarized the formulas in \( \Gamma \).

**Theorem 1 (Soundness of LKF)** Let \( B \) be a polarized formula and let \( \Gamma \) and \( \Theta \) be a multiset and set, respectively, of polarized formulas. If \( \vdash \Gamma \upharpoonright \Theta \) is provable in LKF
then $\vdash \Gamma, \Theta$ is provable in LK. If $\vdash B \bowtie \Theta$ is provable in LKF then $\vdash B, \Theta$ is provable in LK.

**Proof** This theorem can be proved by a straightforward mutual induction on the structure of (cut-free) LKF proofs. Most cases of this mutual induction are straightforward. For example, the introduction rule for $\lor^+$ in LKF corresponds to the introduction rule for $\lor$ in LK, while the introduction rule for $\lor^-$ in LKF corresponds to the multiplicative version of the introduction rule for $\lor$ in Figure 2. The *init* rule in LKF corresponds, however, to the following LK derivation.

$$
\begin{array}{c}
p \vdash p \\
\hline
p \vdash p, f, \Theta \\
\vdash p, p \bowtie f, \Theta
\end{array}
$$

Finally, decide in LKF corresponds to the cR rule, and store and release do not contribute to the LK proof. $\square$

The converse of this soundness theorem is more challenging to prove: we shall state and prove such completeness as Theorem 8 in Section 8. (Every time we mention completeness theorems in this paper, we shall mean *relative completeness* with respect to another proof system: we will not use the model theory notion of validity in this paper.) In anticipation of that result, we state a version of that completeness theorem here. Let $B$ be a first-order polarized formula, let $\delta(\cdot)$ be any atomic bias assignment, and let $C$ be the unpolarized formula $\hat{B}$. If $C$ is provable in LK (in the sense that $\vdash C$ is provable in LK) then $B$ is provable in LKF. A consequence of this completeness theorem is the following: if $C$ be an unpolarized formula that is provable in LK, then for every polarized formula $B$ (and atomic bias assignment) such that $\hat{B} = C$, then $B$ has an LKF proof. Note that if there are $n$ occurrences of propositional connectives in $C$, there are $2^n$ polarized formulas $B$ such that $\hat{B} = C$. Clearly, polarization does not affect provability, but it can have a large impact on the structure of (focused) proofs.

We now state two properties about (cut-free) LKF proofs.

**Lemma 1 (Admissibility of Weakening)** If $\vdash A \bowtie \Theta$ and $\vdash A \vdash \Theta$ are (cut-free) provable and if $\Theta'$ is a set of positive polarized formulas and negative literals then $\vdash \Gamma \vdash \Theta, \Theta'$ and $\vdash A \vdash \Theta, \Theta'$ are also provable.

This lemma is proved easily by induction on the structure of proofs. The proof further shows that weakening also does not affect the structure of proofs in that the same inference rules are applied at each step.

The following lemma captures the fact that the asynchronous phase of inference rules can deal with don’t-care-nondeterminism: any polarized formula to the left of the $\vdash$ can be selected to be processed first.

**Lemma 2** If there is a (cut-free) proof of $\vdash A, \Gamma \vdash \Theta$ then there is a (cut-free) proof that ends with either an introduction of $A$ or a store rule on $A$. 
Proof  This lemma holds because the asynchronous introduction rules permute over each other in such a way that the same premises remain. The formal proof of this lemma is by induction on the sum of the sizes of formulas in $\Gamma$. The size of a formula is the number of occurrences of literals, connectives, and quantifiers in the formula. In particular, $A$ and $\neg A$ are of the same size. In the base case, $\Gamma$ is empty, and the result is trivial. For the inductive case, let $\Gamma = B_1, \Gamma'$ and assume that the sequent $\vdash A, B, \Gamma' \upharpoonright \Theta$ is the conclusion of an inference rule $\rho$ which is either an introduction or store on $B$. We then proceed to show that the $\rho$ rule can be permuted above the introduction or store of $A$. There are several cases to consider.

Case: $A$ and $B$ are both either positive formulas or negative literals. In this case, $\rho$ is store on $B$ with premise $\vdash A, \Gamma' \upharpoonright \Theta, B$. By inductive hypothesis on the smaller $\Gamma'$, the next rule above must be a store on $A$, with premise $\vdash \Gamma' \upharpoonright \Theta, A, B$. But clearly we can switch the order of the two store rules:

$$
\begin{align*}
\vdash \Gamma' \upharpoonright \Theta, A, B \\
\vdash B, \Gamma' \upharpoonright \Theta, A \\
\vdash A, B, \Gamma' \upharpoonright \Theta & \text{ store}
\end{align*}
$$

Case: $A$ is a positive formula or negative literal and $B$ is a non-literal negative formula. In this case, we consider the structure of $B$. For example, if $B$ is $B_1 \lor B_2$, then the premise of $\rho$ is $\vdash A, B_1, B_2, \Gamma' \upharpoonright \Theta$. Since the size of $B_1, B_2, \Gamma'$ is smaller than the size of $B_1 \lor B_2, \Gamma'$, the inductive hypothesis provides a proof where the rule above $\rho$ is the store rule applied to $A$ with premise $\vdash B_1, B_2, \Gamma' \upharpoonright \Theta, A$. Starting from that sequent, we can switch the store and $\lor$ rules, resulting in

$$
\begin{align*}
\vdash B_1, B_2, \Gamma' \upharpoonright \Theta, A \\
\vdash B_1 \lor B_2, \Gamma' \upharpoonright \Theta, A \\
\vdash A, B_1 \lor B_2, \Gamma' \upharpoonright \Theta & \text{ store}
\end{align*}
$$

The cases of $B$ is $t^-, B_1 \land B_2, \forall x. B'$ and $f^-$ are similar.

Case: $B$ is a positive formula or negative literal and $A$ is a non-literal negative formula. This case is analogous to the above case. We illustrate with the case that $A$ is $A_1 \land A_2$. Since the $\rho$ rule is store on $B$, its premise is $\vdash A_1 \land A_2, \Gamma' \upharpoonright \Theta, B$. By the inductive hypothesis, the next rule above is the introduction for $\land$:

$$
\begin{align*}
\vdash A_1, \Gamma' \upharpoonright \Theta, B \\
\vdash A_2, \Gamma' \upharpoonright \Theta, B \\
\vdash A_1 \land A_2, \Gamma' \upharpoonright \Theta & \text{ store}. \quad \land^{-}
\end{align*}
$$

These rules can be permuted to yield the desired form

$$
\begin{align*}
\vdash A_1, \Gamma' \upharpoonright \Theta, B \\
\vdash A_1, B, \Gamma' \upharpoonright \Theta & \text{ store} \\
\vdash A_2, \Gamma' \upharpoonright \Theta, B \\
\vdash A_2, B, \Gamma' \upharpoonright \Theta & \text{ store} \\
\vdash A_1 \land A_2, B, \Gamma' \upharpoonright \Theta & \text{ store.} \quad \land^{-}
\end{align*}
$$
Case: A and B are both non-literal negative polarized formulas. There are several cases to consider, but they are all similar. For example, if A and B are \( A_1 \lor A_2 \) and \( B = B_1 \lor B_2 \), respectively, and the last rule introduces B, we just need to show that the two \( \lor \)-introductions permute over each other, which follows easily from the fact that both proofs can be constructed from the common premise of \( \vdash A_1, A_2, B_1, B_2, \Gamma' \upharpoonright \Theta \).

In the case where \( A = A_1 \lor A_2 \) and \( B = B_1 \land B_2 \), introducing \( A_1 \land B_2 \) results in the premises \( \vdash A_1 \lor A_2, B_1, \Gamma' \upharpoonright \Theta \) and \( \vdash A_1 \lor A_2, B_2, \Gamma' \upharpoonright \Theta \), both of which have a smaller inductive measure, which allows us to assume that the next rule above will introduce \( A_1 \lor A_2 \) and we can therefore build the proof

\[
\vdash A_1, A_2, B_1, \Gamma' \upharpoonright \Theta \quad \vdash A_1, A_2, B_2, \Gamma' \upharpoonright \Theta \\
\vdash A_1 \lor A_2, B_1 \land B_2, \Gamma' \upharpoonright \Theta \quad \lor^n
\]

The remaining cases are treated similarly. □

Definition 2 We say that a (cut-free) proof of \( \vdash A, \Gamma \upharpoonright \Theta \) is eager with respect to A if the last inference rule introduces A or is a store rule on A. We say that the proof is delayed with respect to A if either
1. \( \Gamma \) is empty, or
2. the last inference rule does not introduce A, is not a store rule on A, and each immediate subproof above \( \vdash A, \Gamma \upharpoonright \Theta \) is also delayed with respect to A.

In other words, a proof is delayed with respect to A if A is only subject to an introduction or store rule on A when it appears in a conclusion of the form \( \vdash A \upharpoonright \Theta \). Note also that a proof of \( \vdash A \upharpoonright \Theta \) is both eager and delayed with respect to A.

Lemma 2 implies that a proof can be transformed into either the eager or the delayed form.

4 Cut Elimination for LKF

Given that \( \text{LKF} \) has two kinds of sequents and each of these has two zones for holding polarized formulas, we introduce in Figure 4 a total of four cut rules in order to state and prove the cut-elimination theorem for \( \text{LKF} \). The cut rule (called the unfocused cut rule) applies only to \( \upharpoonright \)-sequents while the cut rule (called the focused cut rule) involves one \( \downarrow \)-sequent. Both of those cut rules also have a “delayed” version in which one of the occurrences of the polarized cut formula is “locked” in storage.

It is important to note that in the delayed cuts, the polarized cut formula \( P \) is positive and not a negative literal: in particular if \( P \) were a negative literal in the \( \text{dcut}_f \) rule and if \( B = \neg P \) then \( \text{dcut}_f \) is not admissible since focusing on a positive literal requires the proof to end in an initial rule.

A simple observation shows that the cut-rules in Figure 4 do not suffer the collision problems mentioned in Section 2.1. As we noted in the previous section, only positive polarized formulas are contracted (by the decide rule) in \( \text{LKF} \) proofs:
as a result, exactly one of the pair of polarized formulas $A$ and $\neg A$ involved in a cut rule will be positive, and only one of them can be contracted. Similarly, weakening only appears within the \textit{init} rule in LKF proofs and, as a result, the problematic case involving weakening also disappears.

The general strategy for proving cut elimination in LKF extended with these cut rules is familiar: we reduce cuts to “key cases” in which the polarized cut formula is principal in both premises. The proof proceeds by simultaneous induction over the permutabilities of all four cuts. The inductive measure is the lexicographical ordering consisting of the size of the polarized cut formula followed by the sum of the heights of the subproofs above the cut. We apply the procedure to the topmost cuts first, thus assuming that the cuts to be reduced have cut-free subproofs.

Lemma 2 is used to simplify the cut-elimination proof. However, the application of this lemma for proof transformation may affect the height of proofs (because of the $C^{-}$ rule). These transformations must be applied carefully to preserve the inductive measure. For the cut-elimination proof, we further require that the following conditions be placed on the cut rules.

1. In $\text{cut}_u$, the subproof of the premise with the positive cut formula must be \textit{eager} with respect to the cut formula; the subproof of the premise with the negative cut formula must be \textit{delayed} with respect to the cut formula.
2. In $\text{dcut}_u$, the subproof of the premise with the negative cut formula must be \textit{delayed} with respect to the cut formula.
3. In $\text{cut}_{5}$, the subproof of the sequent $\vdash \neg A, \Gamma' \parallel \Theta'$, where $\neg A$ is the cut formula, must be \textit{eager} with respect to $\neg A$ regardless of the polarity of $A$.

The third requirement may appear inconsistent with the others when $\neg A$ is negative in $\text{cut}_{5}$: however, the transition from $\text{cut}_u$ or $\text{dcut}_u$ to a $\text{cut}_{5}$ only occurs when the cut formula is decomposed into subformulas, which reduces the stronger inductive measure. For the $\text{dcut}_{5}$ rule, the subproof above the negative cut formula $\neg P$ can be considered both eager and delayed with respect to $\neg P$ because it is the only formula to the left of $\parallel$. By Lemma 2, any proof can be transformed into the required forms so that the reducibility of the restricted cuts also implies the reducibility of the unrestricted versions. In other words, before the application of any cut, we can always apply Lemma 2 to assume that the subproofs are in the required forms. The
cut-elimination arguments will show that all restrictions are preserved when any of
the four cut rules are permuted to other cut rules.

We detail the permutation of each of the four cuts. We sometimes do not repeat
cases that are obvious, and we generally ignore the quantifiers as the first-order
quantifiers add nothing to the argument: their treatment is completely standard.

4.1 Permutations of $\text{cut}_u$

The $\text{cut}_u$ rule has the general form, repeated here for convenience:

$$
\frac{\Gamma, \Theta \vdash \Theta', \Theta'}{\Gamma, \Theta \vdash \Theta'} \quad (\text{cut}_u)
$$

Assume without loss of generality that $A$ is positive and, therefore, $\neg A$ is negative. It
is also required that the left subproof above $\text{cut}_u$ is eager with respect to the positive
$A$, i.e., it ends in a $\text{store}$ rule on the cut formula $A$. Furthermore, the right subproof
above the negative cut formula $\neg A$, is required to be delayed with respect to $\neg A$.
These assumptions mean that this cut can be transformed immediately into a $\text{dcut}_u$:

$$
\frac{\Gamma, \Theta \vdash \Theta', \Theta'}{\Gamma, \Theta \vdash \Theta'} \quad (\text{cut}_u) \quad \implies \quad \frac{\Gamma, \Theta \vdash \Theta', \Theta'}{\Gamma, \Theta \vdash \Theta'} \quad (\text{dcut}_u)
$$

Clearly the restriction on the delayed form of the subproof above the negative cut
formula $\neg A$ is preserved for the $\text{dcut}_u$ rule. The inductive measure is reduced by the
smaller height of the left subproofs above the cut.

4.2 Permutations of $\text{dcut}_u$

The delayed, unfocused $\text{dcut}_u$ rule has the form

$$
\frac{\Gamma, \Theta \vdash \Theta'}{\Gamma, \Theta \vdash \Theta'} \quad (\text{dcut}_u)
$$

where the cut formula $P$ is positive. It is required that the subproof above the right
premise is delayed with respect to the cut formula $\neg P$. These cuts are permuted to
the point where $P$ is selected for focus, at which point the cut transforms into a
combination of $\text{cut}_f$ and $\text{dcut}_f$. In other words, the “goal” or “target” of all
permutations of $\text{dcut}_u$ is to be able to apply the following transformation when the
left premise of the $\text{dcut}_u$ is the $\text{decide}$ rule.
The "delayed" restriction on the right subproof above \( \text{dcut}_f \) has subproofs of lesser height measure, while the lower \( \text{cut}_u \) is a key case cut where the cut formula is principal in both subproofs. That is, cut-free proofs for \( \vdash P \mid \Theta, \Theta' \) and \( \vdash \neg P \mid \Theta' \) must both end with the cut formulas \( P \) and \( \neg P \) subject to an inference rule. The key-case cuts immediately decompose into cuts on subformulas of a smaller size than \( P \) (or reduces completely by weakening in case of \( P \) being a positive literal). Thus, the inductive measure of both cuts is reduced.

Note that the \( \text{eager} \) restriction on the right subproof above \( \text{cut}_f \) is trivially preserved since \( \neg P \) is the only polarized formula on the left of \( \uparrow \).

All other permutations of \( \text{dcut}_u \) make progress toward this case. We organize these permutations into two stages.

The first stage performs permutations over inference rules in the right subproof of \( \text{dcut}_u \). The right subproof above \( \text{dcut}_u \) ends in \( \vdash \neg P, \Gamma' \mid \Theta' \). We permute \( \text{dcut}_f \) until it has such a right subproof with an empty \( \Gamma' \). The fact that this subproof is delayed with respect to \( \neg P \) means that if it ends in a conclusion \( \vdash \neg P, B, \Gamma' \mid \Theta' \) we can assume that the last rule either introduces \( B \) or is a \( \text{store} \) rule on \( B \) (and not on \( \neg P \)). There are many subcases depending on the form of \( B \):

**Case: \( B \) is a positive polarized formula or negative literal.** In this case, the rule above in a \( \text{store} \) on \( B \), resulting in the following permutation.

\[
\frac{\vdash P, \Gamma' \mid \Theta'}{\vdash \neg P, \Gamma' \mid \Theta'} \quad \frac{\vdash P, \neg P, B, \neg P, \text{store}}{\vdash B, \text{dcut}_u} \quad \frac{\vdash P, \neg P, B, \text{store}}{\vdash B, \text{dcut}_u}
\]

The "delayed" restriction on the right subproof above \( \text{dcut}_u \) is preserved by definition: an immediate subproof of a delayed proof is also delayed. This property applies similarly to all subsequent cases.

**Case: \( B \) is \( B_1 \lor B_2 \).** In this case, we can transform

\[
\frac{\vdash \neg P, B_1, B_2, \Gamma' \mid \Theta'}{\vdash \neg P, B_1 \lor B_2, \Gamma' \mid \Theta'} \quad \frac{\vdash \neg P, B_1, B_2, \text{store}}{\vdash B_1, B_2, \text{dcut}_u} \quad \frac{\vdash \neg P, B_1, B_2, \text{store}}{\vdash B_1, B_2, \text{dcut}_u}
\]

into the following derivation.

\[
\frac{\vdash \neg P, B_1, B_2, \Gamma' \mid \Theta'}{\vdash B_1, B_2, \Gamma' \mid \Theta'} \quad \frac{\vdash B_1, B_2, \text{dcut}_u}{\vdash B_1, B_2, \text{dcut}_u} \quad \frac{\vdash B_1, B_2, \text{dcut}_u}{\vdash B_1, B_2, \text{dcut}_u}
\]

**Case: \( B \) is \( B_1 \land \neg B_2 \).** In this case, we can transform
\[ \vdash \Gamma \upharpoonright \Theta, P \quad \vdash \neg P, B_1, \Gamma' \upharpoonright \Theta' \quad \vdash \neg P, B_1, \Gamma' \upharpoonright \Theta' \quad \vdash B_1 \wedge B_2, \Gamma' \upharpoonright \Theta, \Theta' \quad \text{dcut}_u \]

into the following derivation.

\[ \vdash \Gamma \upharpoonright \Theta, P \quad \vdash \neg P, B_1, \Gamma' \upharpoonright \Theta' \quad \text{dcut}_u \quad \vdash \Gamma \upharpoonright \Theta, P \quad \vdash \neg P, B_2, \Gamma' \upharpoonright \Theta' \quad \vdash B_1 \wedge B_2, \Gamma' \upharpoonright \Theta, \Theta' \quad \text{dcut}_u \]

The other cases for \( B \) are proved similarly. This stage ends when the right subproof concludes with a sequent of the form \( \vdash \neg P \uparrow \Theta' \).

The second stage performs permutation over inference rules in the left subproof of \( \text{dcut}_u \). The cases of asynchronous introduction rules are analogous to the cases demonstrated above and are equally straightforward. Generally speaking, the permutation of cut above introduction rules is always straightforward. The important cases to point out are the \text{decide}, \text{release}, and \text{store} rules. A \text{store} rule ending the left subproof is also a trivial case because it cannot affect the cut formula. The interesting case is when the left subproof ends in the form \( \vdash \cdot \upharpoonright \Theta, P \). The rule above this sequent must be \text{decide}. There are two cases depending on whether or not the polarized formula selected for focus is the cut formula \( P \) or not. If it is not the cut formula but, say, another formula \( Q \), then we can permute inference rules as follow.

\[ \vdash Q \downarrow Q, \Theta, P \quad \text{decide} \quad \vdash \cdot \upharpoonright Q, \Theta, P \quad \text{dcut}_u \quad \vdash Q \downarrow Q, \Theta, P \quad \vdash \neg P \uparrow \Theta' \quad \text{dcut}_f \quad \vdash Q \downarrow Q, \Theta, \Theta' \quad \text{decide} \]

If the polarized formula selected for focus is \( P \), then we have reached the targeted transition to key-case cuts as already described above.

### 4.3 Permutations of \( \text{dcut}_f \)

The general form of \( \text{dcut}_f \) is

\[ \vdash B \downarrow \Theta, P \quad \vdash \neg P \uparrow \Theta' \quad \text{dcut}_f \]

with \( P \) positive. This cut permutes over synchronous introduction rules until reaching an \text{init} or \text{release} rule on its left premise, at which point the cut will transition to a \( \text{dcut}_u \) with lower subproofs:

\[ \vdash B \downarrow \Theta, P \quad \text{release} \quad \vdash B \downarrow \Theta, P \quad \text{dcut}_f \quad \vdash B \downarrow \Theta, P \quad \vdash \neg P \uparrow \Theta' \quad \text{dcut}_f \quad \vdash B \downarrow \Theta, \Theta' \quad \text{dcut}_u \quad \vdash B \downarrow \Theta, \Theta' \quad \text{release} \]
Besides the cases of initial rules, all other permutations of \( \text{dcut}_f \) make progress towards this case. Since \( \neg p \) is the only polarized formula to the left of \( \supset \), the “delayed” requirement of \( \text{dcut}_u \) is trivially met. The right-side subproof with the negative cut formula stays intact during these permutations. We consider two cases where \( B \) is a positive polarized formula: the other cases are treated similarly. If \( B \) is a positive literal, then \( \vdash B \upharpoonright \Theta, P \) must be the conclusion of an initial rule. Since \( P \) is also positive, it must be the case that \( B \in \Theta \). Thus \( \vdash b \upharpoonright \Theta, \Theta' \) is also the conclusion of an initial rule. If \( B \) is \( B_1 \lor B_2 \), then we have the following transformation (here, \( i \) is 1 or 2):

\[
\begin{align*}
\vdash B_i \upharpoonright \Theta, P & \quad \vdash \neg p \upharpoonright \Theta' \\
\vdash B_1 \lor^* B_2 \upharpoonright \Theta, \Theta' & \quad \vdash \neg p \upharpoonright \Theta' \\
\vdash B_1 \lor^* B_2 \upharpoonright \Theta, \Theta' 
\end{align*}
\]

\[\text{dcut}_f \rightarrow \vdash B_i \upharpoonright \Theta, P \quad \vdash \neg p \upharpoonright \Theta' \quad \vdash B_i \upharpoonright \Theta, \Theta' \]

\[\vdash B_1 \lor^* B_2 \upharpoonright \Theta, \Theta' \]

### 4.4 Permutations of \( \text{cut}_f \)

The \( \text{cut}_f \) rule has the general form

\[
\begin{align*}
\vdash A \upharpoonright \Theta & \quad \vdash \neg A, \Gamma' \upharpoonright \Theta' \\
\vdash \Gamma' \upharpoonright \Theta, \Theta' & \quad \text{cut}_f
\end{align*}
\]

It is required that the subproof above the unfocused sequent \( \vdash \neg A, \Gamma' \upharpoonright \Theta' \) is eager with respect to \( \neg A \).

If \( A \) is negative, then the left subproof above \( \text{cut}_f \) must be the conclusion of a release rule, and the cut permutes to a \( \text{cut}_u \) with shorter subproofs:

\[
\begin{align*}
\vdash A \upharpoonright \Theta & \quad \vdash \neg A, \Gamma' \upharpoonright \Theta' \\
\vdash \Gamma' \upharpoonright \Theta, \Theta' & \quad \text{cut}_u
\end{align*}
\]

As for the restrictions on \( \text{cut}_u \), \( \neg A \) must be positive if \( A \) is negative, so the subproof above the positive cut formula stays eager with respect to that polarized formula, and the subproof above \( \vdash A \upharpoonright \Theta \) is trivially delayed above the negative cut formula.

If \( A \) is positive, then the left subproof above \( \text{cut}_f \) must be either init or an introduction of the cut formula \( A \). We illustrate three cases below: the other cases are similar.

1. If \( A \) is a positive literal \( p \) then the left premise of \( \text{cut}_f \), namely \( \vdash p \upharpoonright \Theta \), is the conclusion of an initial rule with \( \neg p \in \Theta \). The other, eager subproof of \( \vdash \neg p, \Gamma' \upharpoonright \Theta' \) must end in a store rule on \( \neg p \), with premise \( \vdash \Gamma' \upharpoonright \Theta', \neg p \). But since \( \neg p \in \Theta \), the provability of \( \vdash \Gamma' \upharpoonright \Theta, \Theta' \) follows from weakening.

2. If \( A \) is \( A_1 \lor A_2 \), then \( \neg A \) is \( \neg A_1 \land \neg A_2 \). This key case requires transforming the derivation
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\[ \vdash A_i \downarrow \emptyset \quad \vdash \neg A_1, \Gamma' \uparrow \Theta' \quad \vdash \neg A_2, \Gamma' \uparrow \Theta' \]
\[ \vdash \neg A_1 \land \neg A_2, \Gamma' \uparrow \Theta' \land \neg \quad \text{cut}_f \]

into the derivation
\[ \vdash A_i \downarrow \emptyset \quad \vdash \neg A_1, \Gamma' \uparrow \Theta' \]
\[ \vdash \Gamma' \uparrow \Theta, \Theta' \quad \text{cut}_f \]

The inductive measure is reduced by the size of the cut formulas. Here we can apply Lemma 2 to the subproof above \vdash \neg A_1, \Gamma' \uparrow \Theta' so that it becomes \textit{eager} with respect to (each) \neg A_i without regard to how the transformation might affect the height of proofs because the lexicographical inductive measure is still reduced. This argument similarly applies to the other key cases.

3. if \( A \) is \( A_1 \land^* A_2 \) then \( \neg A \) is \( \neg A_1 \lor \neg A_2 \) and the proof is transformed as follows:

\[ \vdash A_1 \downarrow \emptyset \quad \vdash A_2 \downarrow \emptyset \quad \vdash \neg A_1, \neg A_2, \Gamma' \uparrow \Theta' \quad \vdash \neg A_1 \lor \neg A_2, \Gamma' \uparrow \Theta' \]
\[ \vdash \Gamma' \uparrow \Theta, \Theta' \quad \text{cut}_f \]

The two cuts introduced are both on smaller cut formulas compared to the original cut: the inductive hypothesis is first applied to the upper cut to obtain a cut-free proof, then to the lower one.

With these permutation results in hand, we can now prove the cut-admissibility theorem for LKF.

**Theorem 2** The rules \( \text{cut}_u \), \( \text{cut}_f \), \( \text{dcut}_u \) and \( \text{dcut}_f \) are admissible in LKF.

**Proof** The formal proof is a nested induction argument: first on the number of cuts in each proof, the second on the lexicographical measure for each cut. The corresponding procedure is: select a top-most cut with cut-free subproofs and apply Lemma 2 so that the subproofs satisfy the requirements concerning the \textit{eager} and \textit{delayed} properties. Then apply the transformations to reduce the cut. Apply this procedure repeatedly until all cuts are eliminated. \( \square \)

5 Admissibility of the general \textit{init} rule

The initial rule of LKF requires \( A \) to be a literal in order to prove the sequent \vdash A \downarrow \neg A, \emptyset. \) Just as important as the admissibility of cut is the admissibility of
the more general form of \textit{init}: that is, the sequent \( \vdash A, \neg A \uparrow \Theta \) is provable for every polarized formula \( A \). For unfocused sequent calculus, the proof of this result is straightforward because of the perfect duality between the introduction rules for dual logical connectives. In particular, assuming that \( A \) is negative, apply its (invertible) introduction rule followed by the introduction rule for \( \neg A \) (reading rules from conclusion to premises). The induction hypothesis can then be applied directly to the premises. In a focused setting, however, the proof becomes more difficult since multiple asynchronous or synchronous connectives are introduced in a single phase. To solve this problem, we introduce the following relation, which was also used in (Liang and Miller 2011).

\textbf{Definition 3} Let \( \uparrow \) be the binary relation between a polarized formula and multisets of polarized formulas defined inductively as follows.

\begin{itemize}
  \item \( A \uparrow \{ A \} \) if \( A \) is a positive polarized formula or negative literal.
  \item \( f \uparrow \{ \} \).
  \item \( A \lor B \uparrow \Phi, \Phi' \) if \( A \uparrow \Phi \) and \( B \uparrow \Phi' \).
  \item \( A \land B \uparrow \Phi \) if \( A \uparrow \Phi \) or \( B \uparrow \Phi \).
  \item \( \forall x. A \uparrow \Phi \) if \( A \uparrow \Phi \).
\end{itemize}

Clearly each such \( \Phi \) contains only positive polarized formulas and negative literals. Note that the polarized formulas \( \neg C \) and \( \lor C \) are not \( \uparrow \)-related to any multiset of polarized formulas.

The following lemmas establish the properties of the asynchronous and synchronous phases in a form that allows us to derive the admissibility of the general \textit{init} rule.

\textbf{Lemma 3} For all polarized formulas \( A \), multisets of polarized formulas \( \Gamma \), and sets of polarized formulas \( \Theta \), if \( \vdash \Phi, \Gamma \uparrow \Theta \) is provable for all \( \Phi \) such that \( \uparrow \Phi \), then \( \vdash A, \Gamma \uparrow \Theta \) is also provable.

\textbf{Proof} The proof is by induction on the size of \( A \). If a polarized formula \( A \) is not \( \uparrow \)-related to any multiset of polarized formulas then we say that \( \uparrow \) is \textit{undefined} for \( A \). Note that if \( \uparrow \) is undefined for \( A \) then the lemma implies that \( \vdash A, \Gamma \uparrow \Theta \) is provable.

\begin{itemize}
  \item If \( A \) is a positive polarized formula or negative literal, the property is trivial since only \( A \uparrow \{ A \} \) holds and \( \Phi \) contains only \( A \).
  \item If \( A \) is the constant \( f \), then the property holds by the \( f \) rule.
  \item If \( A \) is the constant \( r \), then \( r, \Gamma \uparrow \Theta \) is provable by the rule for \( r \).
  \item Let \( A \) be the polarized formula \( B \land C \). If \( \uparrow \) is undefined for \( A \), then it is undefined for \( B \) and for \( C \), and the inductive hypothesis states that \( \vdash B, \Gamma \uparrow \Theta \) and \( \vdash C, \Gamma \uparrow \Theta \) are provable. Otherwise, if \( \vdash \Phi, \Gamma \uparrow \Theta \) is provable for all \( \Phi \) such that \( A \uparrow \Phi \), then it is provable for all \( \Phi \) such that \( B \uparrow \Phi \) or \( C \uparrow \Phi \). The inductive hypothesis yields the provability of both \( \vdash B, \Gamma \uparrow \Theta \) and \( \vdash C, \Gamma \uparrow \Theta \). In either case, the \( \land \) rule yields a proof of \( \vdash B \land C, \Gamma \uparrow \Theta \).
  \item Let \( A \) be the polarized formula \( B \lor C \). Assume that \( \vdash \Phi, \Gamma \uparrow \Theta \) is provable for all \( \Phi \) such that \( B \lor C \uparrow \Phi \). This assumption is equivalent to assuming that \( \vdash \Phi', \Phi'', \Gamma \uparrow \Theta \) is provable for all \( \Phi' \) and \( \Phi'' \) such that \( B \uparrow \Phi' \) and \( C \uparrow \Phi'' \).
\end{itemize}
Now assume that $B \uparrow \Phi'$ and $C \uparrow \Phi''$ hold. By the above hypothesis, we have $\vdash \Phi', \Phi'', \Gamma \uparrow \Theta$ is provable. By the inductive hypothesis applied to $B$, we know that $\vdash B, \Phi'', \Gamma \uparrow \Theta$ is provable and by the inductive hypothesis applied to $C$, we know that $\vdash B, C, \Gamma \uparrow \Theta$ is provable. If $\uparrow$ is undefined for either $B$ or $C$, we reach the same conclusion. The $\lor$ rule thus yields a proof of $\vdash B \lor C, \Gamma \uparrow \Theta$.

• For $A$ be the polarized formula $\forall x . B$, we assume that $x$ is not free in $\Gamma, \Theta$. If $A \uparrow \Phi$ then $B \uparrow \Phi$. If $\uparrow$ is undefined for $A$ then it is also undefined for $B$. In either case the inductive hypothesis states that if $\vdash \Phi, \Gamma \uparrow \Theta$ is provable for all $\Phi$ such that $B \uparrow \Phi$, then $\vdash B, \Gamma \uparrow \Theta$ is provable. The property is established by applying the $\lor$ rule.

$\square$

The next lemma connects the synchronous phase with the $\uparrow$-relation.

**Lemma 4** For all polarized formulas $A$ and multisets of polarized formulas $\Phi$, if $A \uparrow \Phi$ then $\vdash \neg A \uparrow \Phi$ is provable.

**Proof** The proof proceeds by induction on the size of $A$, which is the same as the size of $\neg A$.

• If $A$ is $\top$, then the property holds vacuously.
• If $A$ is a negative literal then the property holds by the initial rule $init$.
• If $A$ is $\bot$, the property holds by the rule for $\bot$.
• If $A$ is $B \land C$ then $\neg A$ is $\neg B \lor \neg C$. Assuming that $A \uparrow \Phi$ then either $B \uparrow \Phi$ or $C \uparrow \Phi$. Assume without loss of generality that $B \uparrow \Phi$; by inductive hypothesis $\vdash \neg B \uparrow \Phi$ is provable. Thus, $\vdash \neg B \lor \neg C \uparrow \Phi$ is provable using the $\lor$ rule.
• If $A$ is $B \lor C$ then $\neg A$ is $\neg B \land \neg C$. Assuming that $A \uparrow \Phi$ then there are multisets $\Phi'$ and $\Phi''$ such that $B \uparrow \Phi'$ and $C \uparrow \Phi''$. By the inductive hypotheses, we know that $\vdash \neg B \uparrow \Phi$ and $\vdash \neg C \uparrow \Phi''$ are provable. Apply weakening (Lemma 1) to both sequents and we get that $\vdash \neg B \uparrow \Phi, \Phi'$ and $\vdash \neg C \uparrow \Phi, \Phi''$ are provable. Thus $\vdash \neg B \land \neg C \uparrow \Phi, \Phi''$ is provable using the $\land$ rule.
• If $A$ is $\forall x . B$ the $\neg A$ is $\exists x . \neg B$. If $A \uparrow \Phi$ then $B \uparrow \Phi$. By inductive hypothesis we have $\vdash \neg B \uparrow \Phi$ and by the $\exists$ rule, we have $\vdash \exists x . \neg B \uparrow \Phi$.
• If $A$ is a positive polarized formula, then the inductive hypothesis also applies to the proper subformulas of $\neg A$, which is negative and of the same size as $A$. Thus if $\neg A \uparrow \Phi$ then the cases above show that $\vdash A \uparrow \Phi$ is provable. By weakening $\vdash A \uparrow A, \Phi$ is also provable, and we can form the derivation

\[
\frac{\vdash A \uparrow A, \Phi}{\vdash \Phi \uparrow A} \text{ decide, store}
\]

is provable where a sequence of $store$ rules are applied to the positive polarized formulas and negative literals in $\Phi$. This holds for all $\Phi$ such that $\neg A \uparrow \Phi$, so by Lemma 3, $\vdash \neg A \uparrow A$ is provable, and by applying the $release$ rule, we have a proof of $\vdash \neg A \uparrow A$. This establishes the property for positive $A$ for which only $A \uparrow \{A\}$ holds.
The following theorem states the admissibility of the general form of the init rule.

**Theorem 3** \( \vdash A, \neg A \vdash \cdot \) is provable for all polarized formulas \( A \).

**Proof** Assume without loss of generality that \( A \) is positive. Then \( A \vdash \{A\} \) and Lemma 4 states that \( \vdash \neg A \vdash A \) is provable. Since \( \neg A \) is negative, this sequent must be the conclusion of a release rule in a cut-free proof, so \( \vdash \neg A \vdash A \) is provable. Applying the store rule on \( A \) to this sequent gives a proof of \( \vdash A, \neg A \vdash \cdot \). □

### 6 Generalized invertibility

The following results about the invertibility of the negative introduction rules is now easily proved using the admissibility of cut. The following corollary is the converse of Lemma 3.

**Corollary 1** If \( \vdash A, \Gamma \vdash \Theta \) is provable and \( A \vdash \Phi \), then \( \vdash \Phi, \Gamma \vdash \Theta \) is provable.

**Proof** Given the assumption \( A \vdash \Phi \), Lemma 4 implies that the sequent \( \vdash \neg A \vdash \Phi \) is provable. Using a cut rule, we therefore have the following proof.

\[
\begin{align*}
\vdash A, \Gamma \vdash \Theta & \quad \vdash \neg A \vdash \Phi \\
\vdash \Gamma \vdash \Phi, \Theta & \quad \text{cut}_{f} \\
\vdash \Phi, \Gamma \vdash \Theta & \quad \text{store}
\end{align*}
\]

The final result follows from the admissibility of cut (Theorem 2). □

From the generalized invertibility property and Lemma 3, we can derive the invertibility of the individual asynchronous introduction rules.

**Lemma 5** The introduction rules for the negative connectives are invertible; i.e., the provability of the conclusion of each rule implies the provability of all of its premises.

**Proof** First, consider the case for \( \lor \). Assume that \( \vdash A, B, \Gamma \vdash \Theta \) is provable and assume that \( A \) is \( \vdash \)-related to exactly the multisets \( \Phi_{A}^{1}, \ldots, \Phi_{A}^{n} \) and that \( B \) is \( \vdash \)-related to exactly \( \Phi_{B}^{1}, \ldots, \Phi_{B}^{m} \), where \( n, m \geq 0 \). By the definition of \( \vdash \), we know that \( A \lor B \vdash \Phi_{A}^{i} \Phi_{B}^{k} \) for each \( i \) and \( k \) such that \( 1 \leq i \leq n \) and \( 1 \leq k \leq m \). (Note that if either \( n \) or \( m \) is 0 then this statement is vacuously true.) Corollary 1 implies that \( \vdash \Phi_{A}^{i} \Phi_{B}^{k}, \Gamma \vdash \Theta \) is provable. By Lemma 3, this means that \( \vdash A, B, \Gamma \vdash \Theta \) is provable.

To consider the case for \( \land \) assume that \( \vdash A \land B, \Gamma \vdash \Theta \) is provable and (as above) \( A \) is \( \vdash \)-related to \( \Phi_{A}^{1}, \ldots, \Phi_{A}^{n} \) and \( B \) is \( \vdash \)-related to \( \Phi_{B}^{1}, \ldots, \Phi_{B}^{m} \), where \( n, m \geq 0 \). Then \( A \land B \vdash \Phi_{A}^{i} \), for each \( i \) such that \( 1 \leq i \leq n \) and \( A \land B \vdash \Phi_{B}^{k} \) for each \( k \) such that \( 1 \leq k \leq m \). By Corollary 1, this implies that \( \vdash \Phi_{A}^{i}, \Gamma \vdash \Theta \) is provable for each \( i \) such that \( 1 \leq i \leq n \) and \( \vdash \Phi_{B}^{k}, \Gamma \vdash \Theta \) is provable for each \( k \) such that \( 1 \leq k \leq m \). By Lemma 3, \( \vdash A, \Gamma \vdash \Theta \) and \( \vdash B, \Gamma \vdash \Theta \) are provable.

The cases for \( \forall \) and \( \exists \) are similar and omitted. □
Given Lemmas 3 and 4, we often use the following argument schema to establish the provability of \( \vdash A_1, \ldots, A_n, \Gamma \uparrow \Theta \): If \( \uparrow \) is undefined for any \( A_i \) then Lemma 3 already shows that the sequent is provable. Otherwise, assume that for each \( i \in \{1, \ldots, n\} \) there is an \( n_i \) greater than or equal to 1 such that \( A_i \) is \( \uparrow \)-related to exactly \( \Phi_1^i, \ldots, \Phi_{n_i}^i \). Show that for each possible selection of \( \Phi_1^i, \ldots, \Phi_{n_i}^i \), the sequent \( \vdash \Gamma \uparrow \Theta, \Phi_1^i, \ldots, \Phi_{n_i}^i \) is provable. Then \( \vdash A_1, \ldots, A_n, \Gamma \uparrow \Theta \) is provable by Lemma 3 plus enough applications of the store rule to move each member of \( \Phi_i^i \) to the left side of \( \uparrow \). Furthermore, if \( \Gamma \) consists of a single positive polarized formula \( P \) (\( P \) can also be in \( \Theta \) with \( \Gamma \) empty) and \( \vdash P \parallel P, \Theta, \Phi_1^i, \ldots, \Phi_{n_i}^i \) is provable, then using the decide rule

\[
\begin{align*}
\vdash P &\parallel P, \Theta, \Phi_1^i, \ldots, \Phi_{n_i}^i \\
\vdash \Theta, \Phi_1^i, \ldots, \Phi_{n_i}^i &\Downarrow \theta \quad \text{decide}
\end{align*}
\]

the provability \( \vdash A_1, \ldots, A_n, P \parallel \Theta \) also follows from Lemma 3 and the store rule. The provability of the focused sequent above decide often follows from Lemma 4.

7 Returning to LK

In this section, we show how the unfocused LK proof system can be faithfully captured within LKF. We do this in three steps: (1) we translate the two-sided proof system LK into a one-sided system; (2) we show that a more general form of contraction is admissible in LKF; and (3) we prove that the unfocused introduction rules of (the one-side version of) LK are admissible in LKF. As a consequence, LKF is complete for LK.

Gentzen’s original version of LK used the additive versions of conjunction and disjunction, namely \( \wedge^+ \) and \( \vee^+ \), while his implication \( \supset \) was multiplicative. Gentzen himself noted (Gentzen (1935), Remark 2.4) that LK is ‘dual’ in the sense that the left and right inference rules are symmetrical except for \( \supset \). In LKF, the multiplicative connective \( \wedge^- \) can be used to encode \( A \supset B \) into \( \neg A \vee B \); hence, \( \neg (A \supset B) \) is encoded as \( A \wedge^- \neg B \). As a result, we can remove implications and negated implications by mapping them to these multiplicative connectives.

**Definition 4** The LK -polarization \((-)^\pm \) of classical formulas is defined as follows (recall that the negation of polarized formulas is given in Definition 1):

- For any atomic polarized formula \( a, a^\pm = a \) and \( \neg a \)^\pm = \neg a.
- \( (A \wedge B)^\pm = A^\pm \wedge^- B^\pm; \ (A \vee B)^\pm = A^\pm \vee^+ B^\pm; \ t^\pm = t^-; \ f^\pm = f^+; \)
- \( (A \supset B)^\pm = \neg A^\pm \vee^- B^\pm \)

We also assume that all atomic polarized formulas are polarized positively.

Figure 5 contains the inference rules for LKi, a sequent calculus intermediate between LK and LKF in the sense that it is a one-sided sequent calculus that contains polarized formulas but it is not focused. An LK sequent \( A_1, \ldots, A_n \vdash B_1, \ldots, B_m \) is represented in this setting as \( \vdash \neg A_1^\pm, \ldots, \neg A_n^\pm, B_1^\pm, \ldots, B_m^\pm \). Each inference rule
Structural rules and Identity rules

\[
\begin{align*}
& \vdash \Delta, B, B' & \xi R \\
& \vdash \Delta, B & \omega R \\
& \vdash B, \neg B & \text{init} \\
& \vdash B, \Delta & \vdash \neg B, \Delta' & \text{cut}
\end{align*}
\]

Introduction rules

\[
\begin{align*}
& \vdash \Gamma, \Delta & t^- \\
& \vdash \Delta, \Theta & \vdash A \land B, \Theta & \text{\land}^- \\
& \vdash \Gamma, \Theta & \vdash A \land B, \Theta & \text{\land}^+ \\
& \vdash \Gamma, \Delta & \vdash B_i, B, \Theta & \text{\lor}^+ \\
& \vdash \Delta, B, \Theta & \vdash A \lor B, \Theta & \text{\land}^- \\
& \vdash \Delta, B, \Theta & \vdash A \lor B, \Theta & \text{\lor}^- \\
& \vdash \Delta, y/x & B & \vdash \Delta, \forall x, B \\
& \vdash \Delta, B & \vdash \Delta, \exists x, B & \text{\exists}
\end{align*}
\]

Fig. 5 The rules for LKi. In the \( \lor \) rule, the variable \( y \) is not free in the conclusion. In the \( \land \) rule, \( i \in \{1, 2\} \).

of LK is translated directly into this setting: replace each sequent in the premises and conclusion of the rule with their one-sided, polarized versions. Left-introductions rules on \( i \) are thus represented as one-sided introduction rules on \( \neg A_i \).

**Theorem 4** Let \( n, m \geq 0 \) and let \( A_1, \ldots, A_n, B_1, \ldots, B_m \) be unpolarized formulas. If the sequent \( \vdash \neg A_1, \ldots, \neg A_n, B_1, \ldots, B_m \vdash \cdot \) is provable in LKF then the sequent \( A_1, \ldots, A_n \vdash B_1, \ldots, B_m \) is provable in LK.

**Proof** Note that an LKF proof of \( \vdash \neg A_1, \ldots, \neg A_n, B_1, \ldots, B_m \vdash \cdot \) can easily be translated to an LKi proof of \( \vdash \neg A_1, \ldots, \neg A_n, B_1, \ldots, B_m \vdash \cdot \). Such an LKi proof can then be converted to a proof of the two-sided sequent \( A_1, \ldots, A_n \vdash B_1, \ldots, B_m \) in LK. In this later transformation, when the multiplicative connectives \( \lor \) and \( \land \) are introduced in the LKi proof, implications are introduced on the right or left in the LK proof.

We shall now proceed to prove that the rules of LKi are admissible in LKF by presenting new admissible LKF rules derived from the LKi rules. When naming the new admissible LKF rules, we will add parentheses around the name of the LKi rule. For example, the \text{init} rule of LKi yields the admissible LKF rule

\[
\vdash B, \neg B, \Gamma \vdash \cdot \quad \text{(init)}.
\]

The admissibility of \text{(init)} follows immediately from Theorem 3. The admissibility of \text{(wR)}, namely,

\[
\vdash \Delta \vdash \Theta \\
\vdash B, \Delta \vdash \Theta, \Theta' \quad \text{(wR)}
\]

follows from Lemma 1 and a simple induction on the structure of \( B \). We delay the proof of the admissibility of the LKi \text{cut} rule until Section 9.1. We now proceed to prove the admissibility of contraction and the introduction rules of LKi.
Unlike \( \text{LK} \) and \( \text{LKi} \), \( \text{LKF} \) does not include explicit rules for contraction. In \( \text{LKF} \), the rule of contraction is only applied to positive polarized formulas and only within the \text{decide} rule. We now show that contraction for all polarized formulas is admissible in \( \text{LKF} \).

**Lemma 6** The following rule is admissible in \( \text{LKF} \) for all polarized formulas \( A \).

\[
\frac{\vdash A, A, \Gamma \uparrow \Theta}{\vdash A, \Gamma \uparrow \Theta} \quad (cR)
\]

**Proof** Assume that \( \vdash A, A, \Gamma \uparrow \Theta \) has an \( \text{LKF} \) proof. Using Lemma 2, we can assume that this proof is eager for the first occurrence of \( A \). If \( A \) is a positive polarized formula or negative literal, then the only rule that can be applied to it is \text{store}, which means that the sequent \( \vdash A, \Gamma \uparrow A, \Theta \) has an \( \text{LKF} \) proof. Again, this sequent has a proof eager for \( A \) and, thus, must be proved by the \text{store} rule, which implies that \( \vdash \Gamma \uparrow A, \Theta \) has an \( \text{LKF} \) proof. By using that sequent as the premise of the \text{store} rule we have an \( \text{LKF} \) proof of \( \vdash A, \Gamma \uparrow \Theta \).

Consider the cases where \( A \) is a non-literal negative polarized formula. The case where \( A \) is \( f^{-} \) is immediate. The case where \( A \) is \( f^{+} \) follows using Lemma 5 twice. If \( A \) is \( B \lor C \) then, using Lemmas 2 and 5 twice, it is the case that \( \vdash B, B, C, C, \Gamma \uparrow \Theta \) is provable. The result follows by using the inductive assumption twice along with the \text{\lor} rule. If \( A \) is \( B \land C \) then, using Lemmas 2 and 5 twice, it is the case that both \( \vdash B, B, \Gamma \uparrow \Theta \) and \( \vdash C, C, \Gamma \uparrow \Theta \) are provable. The result follows by using the inductive assumption twice along with the \text{\land} rule. Finally, the case where \( A \) is universally quantified is similar and omitted here. \( \Box \)

From the results in the preceding sections, we can show the admissibility of the unfocused introduction rules (corresponding to the rules of \( \text{LKi} \)) in \( \text{LKF} \).

**Theorem 5 (Admissibility of unfocused introduction rules)** All the introduction of \( \text{LKi} \) are admissible in \( \text{LKF} \).

**Proof** Throughout this proof, we use the admissibility of cut combined with the argument schema outlined at the end of Section 6.

The \text{\lor} -introduction rule for \( \text{LKi} \) is admissible in \( \text{LKF} \) in the form

\[
\frac{\vdash B_{i}, \Gamma \uparrow \Theta}{\vdash B_{1} \lor^{+} B_{2}, \Gamma \uparrow \Theta} \quad (\text{\lor}^{+})
\]

for \( i \in \{1, 2\} \). Admissibility follows from using the admissibility of the \text{cut}_{\text{au}} rule in the derivation

\[
\frac{\vdash B_{i}, \Gamma \uparrow \Theta \quad \vdash B_{i} \lor^{+} B_{2} \uparrow \Theta}{\vdash B_{1} \lor^{+} B_{2}, \Gamma \uparrow \Theta} \quad \text{cut}_{\text{au}}
\]

To show the provability of the right premise above the cut we apply the argument schema of Section 6. Let \( \vdash \neg B_{1} \uparrow \Phi^{1}, \ldots, \neg B_{i} \uparrow \Phi^{n} \) be an exhaustive list of multisets of polarized formulas \( \vdash \)-related to \( \neg B_{1} \); for \( n \geq 0 \). If \( n = 0 \) then the sequent is provable by Lemma 3. Otherwise, \( n \) is positive. For each \( \Phi^{k} \ (k \in 1 \ldots n) \), construct the following subproof
\[
\frac{\vdash B_1 \bowtie B_1 \uparrow^* B_2, \Phi^k}{\vdash B_1 \uparrow^* B_2 \downarrow B_1 \uparrow^* B_2, \Phi^k} \quad \text{\textit{decide}}
\]
\[
\frac{\vdash B_1 \uparrow^* B_2, \Phi^k \downarrow}{\vdash B_1 \uparrow^* B_2, \Phi^k} \quad \text{\textit{store}}
\]

The provability of the top sequent follows from Lemma 4 and the provability of \(\vdash \neg B_1, \neg \vdash \vdash \neg \vdash \neg B_1 \downarrow \cdot \) follows from all such subproofs by Lemma 3.

The \(\land^+\)-introduction rule for \textbf{LK} is admissible in \textbf{LKF} in the form
\[
\frac{\vdash \land^+, \Phi}{\vdash \land^+, \Phi} \quad (\land^+)
\]

This rule is also justified using the admissibility of \textit{cut}_\text{u} as follows.
\[
\frac{\vdash B, \Gamma \uparrow \Theta}{\vdash A, \Gamma \uparrow \Theta} \quad \vdash A, \Gamma \vdash \Theta \quad \vdash \neg A, \neg B, A \land^+ B \vdash \cdot \quad \vdash \neg B, A \land^+ B, \Gamma \uparrow \Theta \quad \text{\textit{cut}_\text{u}}
\]
\[
\vdash A \land^+ B, \Gamma \uparrow \Theta \quad \text{\textit{cut}_\text{u}}
\]
\[
\vdash A \land^+ B, \Gamma \uparrow \Theta \quad (cR)
\]

The provability of the top right sequent uses the argument schema described above: let \(\neg A \uparrow \Phi_i, \ldots, \neg A \uparrow \Phi_n \) and \(\neg B \uparrow \Phi_{i}, \ldots, \neg B \uparrow \Phi_m \) be exhaustive lists of multiset of set related to \(\neg A\) and \(\neg B\), respectively. If either \(n\) or \(m\) is 0, then the sequent is already provable. Otherwise, for each pair \(\Phi_i, \Phi_j\) construct the subproof
\[
\frac{\vdash A \land^+ B, \Phi_i, \Phi_j}{\vdash A \land^+ B, \Phi_i, \Phi_j} \quad (\forall R)
\]

The provability of the top sequents follows from Lemma 4 and from these subproofs the provability of \(\vdash \neg A, \neg B, A \land^+ B \vdash \cdot \) follows by Lemma 3.

To prove the admissibility of the introduction of \(\exists\), we similarly rewrite
\[
\frac{\vdash A[s/x], \Gamma \uparrow \Theta}{\vdash \exists x. A, \Gamma \uparrow \Theta} \quad (\exists)
\]
\[
\frac{\vdash A[s/x], \Gamma \uparrow \Theta}{\vdash \exists x. A, \Gamma \uparrow \Theta} \quad \vdash \neg A[s/x], \exists x. A \vdash \cdot \quad \text{\textit{cut}_\text{u}}
\]

The provability of the right premise again uses the argument schema of Section 6: let \(\neg A[s/x] \uparrow \Phi_i, \ldots, \neg A[s/x] \uparrow \Phi_n \) be the exhaustive list of multisets that are \(\uparrow\)-related to \(\neg A[s/x]\). If \(n = 0\), then the premise is already provable. Otherwise, for each \(\Phi^i\) we have
\( \vdash A[s/x] \upharpoonright \exists x. A, \Phi' \)

\[ \vdash \exists x. A \upharpoonright \exists x. A, \Phi' \]

\( \text{store} \)

\[ \vdash \exists x. A, \Phi' \upharpoonright \cdot \]

from which the provability of \( \vdash \neg A[s/x], \exists x. A \upharpoonright \cdot \) follows.

The \( \text{LK}^i \) introduction rule for \( t^+ \) yields the following admissible rule, which can be justified by the associated \( \text{LK} \) derivation.

\[ \vdash t^+, \Gamma \upharpoonright \Theta (t^+) \]

\[ \vdash t^+ \downarrow t^+ \]

\( \text{decide} \)

\[ \vdash t^+ \upharpoonright \cdot \]

\( \text{store} \)

\[ \vdash t^+, \Gamma \upharpoonright \Theta \text{ (wR)} \]

\[ \vdash t^+, \Gamma \upharpoonright \Theta \]

\[ \text{decide} \]

\[ \vdash t^+ \upharpoonright \cdot \]

\( \text{store} \)

\[ \vdash t^+, \Gamma \upharpoonright \Theta \]

\[ \text{wR} \]

The negative introduction rules already apply on the left side of \( \upharpoonright \). Thus every unfocused inference rule can be emulated on the left side \( \upharpoonright \), and the completeness of \( \text{LK}^f \) with respect to the intermediate \( \text{LK}^i \), and to the original \( \text{LK} \) is therefore established.

**Theorem 6 (Weak completeness of \( \text{LK}^f \))** If the sequent \( A_1, \ldots, A_n \vdash B_1, \ldots, B_m \) is provable in \( \text{LK} \) then the sequent \( \vdash \neg A_1, \ldots, \neg A_n, B_1, \ldots, B_m \upharpoonright \cdot \) is provable in \( \text{LK}^f \).

We have labeled this theorem as “weak completeness” since it states that if an unpolarized formula is provable in \( \text{LK} \), then there is some polarization of that formula (namely \( (\cdot)^\pm \)) which is provable in \( \text{LK}^f \). Theorem 8 in the next section is a stronger version of the completeness theorem since it states that every polarization of an unpolarized theorem is provable in \( \text{LK}^f \).

### 8 Choosing the polarization of formulas

We are now able to prove that every polarization of a formula provable in \( \text{LK} \) is provable in \( \text{LK}^f \). Formally, we say that the polarized formula \( B \) (together with an atom bias assignment \( \delta(\cdot) \)) is a polarization of \( C \) if \( \tilde{B} \) is \( C \).

We write \( A \equiv B \) to mean that both \( \vdash \neg A, B \upharpoonright \cdot \) and \( \vdash \neg B, A \upharpoonright \cdot \) are provable. We first show that the positive and negative versions of each connective are equivalent.

**Lemma 7** For every pair of polarized formulas \( A \) and \( B \), it is the case that \( A \lor^+ B \equiv A \lor^- B \) and \( A \land^+ B \equiv A \land^- B \).

**Proof** To prove the first equivalence, we need proofs of \( \vdash \neg A \land^- B, A \lor^- B \upharpoonright \cdot \) and \( \vdash \neg A \land^+ B, A \lor^+ B \upharpoonright \cdot \). The first of these is straightforward given the admissibility of the general initial rule. The provability of the second sequent is equally simple.
given the admissibility of the *unfocused* introduction rules shown in Section 7, as demonstrated by the following derivation.

\[
\begin{align*}
\vdash \neg A, \ A \dagger & \blacktriangleright \cdot \\
\vdash \neg A, \ A \lor^+ B \dagger & \blacktriangleright \cdot (\lor^+) \\
\vdash \neg A, \ A \lor^+ B \dagger & \blacktriangleright \cdot (\lor^+) \\
\vdash \neg A \land^+ B, \ A \lor^+ B \dagger & \blacktriangleright \cdot 
\end{align*}
\]

Showing \(A \land^+ B \equiv A \land^- B\) is similar, and the equivalences between the positive and negative versions of the units are straightforward. □

**Definition 5** Let \(\circ\) represent one of the binary connectives \(\lor^-\), \(\lor^+\), \(\land^-\), or \(\land^+\) and let \(F\) be a syntactic variable ranging over arbitrary polarized formulas. Let \(S\) range over *subformula contexts* which are defined inductively by

\[
S = [\cdot] | S \circ F | F \circ S | \exists x.S | \forall x.S.
\]

Here, \([\cdot]\) is a constant denoting a primitive subformula context. The notation \(S[A]\) denotes the polarized formula that results from replacing \([\cdot]\) in \(S\) with \(A\).

**Theorem 7** Let \(S\) be a subformula context. If \(A \equiv B\) then \(S[A] \equiv S[B]\).

**Proof** We prove the general property: if \(\vdash \neg A, B \dagger \cdot\) is provable then for any subformula context \(S\), \(\vdash \neg S[A], S[B] \dagger \cdot\) is also provable.

The proof of this property essentially repeats the arguments for eliminating the generalized initial rule. However, instead of replicating Lemmas 3 and 4, we can take advantage of the admissibility of unfocused rules for the positive connectives.

We proceed by induction on \(S\). In the base case, \(S = [\cdot]\) and the property is immediate. If, instead, \(S = F \lor^- S'\) then \(S[A] = F \lor^- S'[A]\), and \(\neg S[A] = \neg F \land^+ \neg S'[A]\): we construct

\[
\begin{align*}
\vdash \neg F, F, S'[B] \dagger \cdot & \quad \vdash \neg S'[A], S'[B], F \dagger \cdot (\land^+) \\
\vdash \neg F \land^+ \neg S'[A], F \lor^- S'[B] \dagger \cdot & \quad \lor^- 
\end{align*}
\]

The left premise follows from the general initial rule admissibility and the right premise is provable by inductive hypothesis (plus weakening). All the other cases are proved similarly. □

**Theorem 8 (Strong completeness of LKF)** Let \(C\) be an unpolarized formula that is provable in \(\text{LK}\) and let \(B\) be a polarization of \(C\). Then \(B\) is provable in \(\text{LKF}\).

**Proof** Let \(C\) be an unpolarized formula that is provable in \(\text{LK}\) and let \(B\) be a polarized version of \(C\) and let \(\delta(\cdot)\) be any atomic bias assignment. By weak completeness (Theorem 6), we know that \(C^\pm\) is provable in \(\text{LKF}\). Since the only difference between \(C^\pm\) and \(B\) are polarized formulas is that the + and – signs on logical connectives might be different and, by construction, the atoms in \(C\) are all given positive bias. Using the equivalences in Lemma 7 and Theorem 7, we can conclude that \(B\) is provable, assuming that all atoms are positively biased.
What remains to be shown is that provability is preserved by imposing the atomic bias assignment $\delta(\cdot)$. Translating a proof with a negative atom $a$ into one where $a$ is considered positive is the same as translating a proof with $\neg a$ considered positive to one where $\neg a$ is considered negative, so we only need to show one direction of the translation. Assume that $a$ is considered negative in a proof. A strategy for reconstructing the proof where $a$ is considered positive is to use delays together with cut. In particular, we define the polarized formula $B^\delta$ as the result of replacing every occurrence of $a$ in $B$ with $a \lor \neg f^-$ (and therefore every occurrence of $\neg a$ by $\neg a \land t^+$). The strategy is to show that every proof of $\vdash B^\delta \uparrow \cdot$ with $a$ considered negative corresponds to a proof of $\vdash B^\delta \uparrow \cdot$ with $a$ considered positive. Then by the cut rule
\[
\frac{\vdash B^\delta \uparrow \cdot \quad \vdash \neg B^\delta, B \uparrow \cdot}{\vdash B \uparrow \cdot}\quad \text{cut}_u
\]
we derive a proof of $B$ without delays and with $a$ considered positive. We can easily generalized the proof of a single formula to the proof of a sequent since (by invertibility) a multiset $\{B_1, \ldots, B_n\}$ is equivalent to $B_1 \lor B_2 \ldots \lor B_n$.

The rules that may have a literal as principal formula are store, release, decide, and init. We show how each rule is emulated in a proof of $B^\delta$:

- Both $a$ and $\neg a$ can be subject to a store, in which case the emulations are as follow.

\[
+ \Gamma \uparrow a, \Theta \quad \Rightarrow \quad \frac{\vdash \Gamma \uparrow a, \Theta}{\vdash a, \Gamma \uparrow \Theta} \quad \text{store}
\]
\[
\frac{\vdash a, f^-, \Gamma \uparrow \Theta}{\vdash a \lor f^-, \Gamma \uparrow \Theta} \quad \sqrt{\cdot}
\]

\[
+ \Gamma \uparrow \neg a, \Theta \quad \Rightarrow \quad \frac{\vdash \Gamma \uparrow \neg a, \Theta}{\vdash \neg a \land t^+, \Gamma \uparrow \Theta} \quad \text{store}
\]
\[
\frac{\vdash \neg a \land t^+, \Gamma \uparrow \Theta}{\vdash \neg a \land t^+, \Gamma \uparrow \Theta} \quad \sqrt{\cdot}
\]

Thus, in a proof of $B^\delta$, $a$ will appear on the right side of $\uparrow$ and $\parallel$ as $a$ but $\neg a$ will appear as $\neg a \land t^+$.

- The release rule is applicable when $a$ is considered negative and is still applicable to $a \lor f^-$ when $a$ is considered positive. Since $a$ is a literal, the only rule that can apply above release is store.

\[
\vdash \cdot \uparrow a, \Theta \quad \Rightarrow \quad \frac{\vdash \cdot \uparrow a, \Theta}{\vdash a \uparrow \Theta} \quad \text{store}
\]
\[
\frac{\vdash a, f^- \uparrow \Theta}{\vdash a \lor f^- \uparrow \Theta} \quad \sqrt{\cdot}
\]

- In the init rule, $a$ is negative: it is emulated as indicated.
Finally, when \( a \) is considered negative, the \text{decide} \ rule can only be applied to \( \neg a \), and must be preceded from above by an \text{init}, and so is emulated as follows:

\[
\begin{align*}
\vdash & \neg a \downarrow \neg a, \Theta \\
\vdash & \cdot \upoarrow \neg a, \Theta \\
\vdash & \neg a \downarrow \neg a, \Theta \\
\vdash & t \uparrow \vdash \neg a \wedge t^*, a, \Theta \\
\vdash & \neg a \wedge t^* \downarrow a, \Theta \quad \wedge^*
\end{align*}
\]

\[\text{The proof of the remaining premise is easy to find.}\]

Finally, to show that \( \vdash \neg B^\delta, B \uparrow \cdot \) is provable with \( a \) considered positive, we induct on the structure of \( B \):

- If \( B \) is \( a \) or \( \neg a \), consider the following derivations:

\[
\begin{align*}
\vdash & a \downarrow a, \neg a \\
\vdash & \cdot \upoarrow a, \neg a \\
\vdash & a, \neg a \upoarrow \\
\vdash & a, f^-, \neg a \upoarrow \\
\vdash & a \lor f^-, \neg a \upoarrow
\end{align*}
\]

- If \( B \) is \( C \lor D \), we apply the admissible unfocused rules to simplify the proof:

\[
\begin{align*}
\vdash & \neg C^\delta, C, D \upoarrow \cdot \\
\vdash & \neg B^\delta, C, D \upoarrow \cdot \quad \wedge^* \\
\vdash & \neg C^\delta \wedge \neg D^\delta, C, D \upoarrow \cdot \\
\vdash & \neg C^\delta \wedge \neg D^\delta, C \lor D \upoarrow \cdot \quad \lor^*
\end{align*}
\]

The premises are provable by inductive hypotheses and by weakening.

- If \( B \) is \( C \land D \):
Pimentel et al. (2016) give a similar analysis of how changing the polarity of atoms within the intuitionistic focused proof system $\text{LJF}$ (Liang and Miller 2009) affects the structure of such proofs.

9 Four applications of LKF

Part of the motivation for developing the $\text{LKF}$ proof system is that its meta-theory should help in proving other proof-theoretic results about first-order classical logic. To support this claim, we present four applications of $\text{LKF}$.

9.1 The admissibility of cut in LK

We can prove that the admissibility of cut holds for $\text{LK}$ given that we have proved cut-admissibility for the more complex proof system $\text{LKF}$. While it is no surprise that this can be done, it is reassuring to see that that result for $\text{LK}$ follows so directly from the results for $\text{LKF}$.

**Theorem 9** The cut rule for $\text{LK}$ is admissible in the cut-free fragment of $\text{LK}$.

**Proof** Assume that the sequents $\Gamma \vdash \Delta, B$ and $\Gamma', B \vdash \Theta'$ have cut-free $\text{LK}$-proofs. By the weak completeness of $\text{LKF}$ (Theorem 6), the sequents $\vdash \neg(\Gamma)^\pm, (\Delta)^\pm, B^\pm \vdash \cdot$ and $\vdash \neg(\Gamma')^\pm, (\Delta')^\pm \vdash \cdot$ both have (cut-free) $\text{LKF}$ proofs. By the admissibility of cut for $\text{LKF}$ (Theorem 2), we know that $\vdash \neg(\Gamma)^\pm, (\Delta)^\pm, (\Delta')^\pm \vdash \cdot$ has a (cut-free) $\text{LKF}$ proof. Finally, by Theorem 4, we know that $\Gamma, \Gamma' \vdash \Delta, \Delta'$ has a cut-free $\text{LK}$ proof. □

9.2 Synthetic inference rules

Following up on the suggestion in Section 2.4, we show how to define larger-scale, synthetic inference rules using the $\text{LKF}$ proof system.

A sequent of the form $\vdash \cdot \mid \Theta$ is called a border sequent. The only $\text{LKF}$ proof rule that can have a border sequent as a conclusion is the decide rule.

**Definition 6** (Synthetic inference rule) A synthetic inference rule is an inference rule involving only border sequents. They are of the form

$$
\vdash \cdot \mid \Theta_1, \ldots, \vdash \cdot \mid \Theta_n
$$

which is justified by a derivation of the form
Here, $n \geq 0$, and the derivation $\Pi$ contains exactly one occurrence of the \textit{decide} rule and that occurrence is the last inference rule (having the conclusion $\vdash \cdot \upharpoonright \Theta$).

If that \textit{decide} rule selects as its focus the polarized formula $B \in \Theta$, we say that this derivation is a \textit{synthetic inference rule for} $B$.

Consider again using the formula (from Section 2.4)

$$\forall x \forall y \forall z. (\text{path}(x, y) \supset \text{path}(y, z) \supset \text{path}(x, z))$$

as an assumption in a given fixed theory. In the one-sided sequent setting of \textbf{LKF}, consider instead the negation of this assumption, namely,

$$\exists x \exists y \exists z. (\text{path}(x, y) \wedge^+ \text{path}(y, z) \wedge^+ \neg \text{path}(x, z)).$$

Assuming that this positive polarized formula is a member of $\Theta$, then consider the following derivation.

$$\begin{array}{c}
\Xi_1 \vdash \text{path}(r, s) \downharpoonright \Theta \\
\Xi_2 \vdash \text{path}(s, t) \downharpoonright \Theta \\
\Xi_3 \vdash \neg \text{path}(r, t) \downharpoonright \Theta
\end{array} \quad \begin{array}{c}
\wedge^\times 2 \\
\exists \times 3
\end{array} \quad \begin{array}{c}
\Xi_1 \vdash \exists x \exists y \exists z. (\text{path}(x, y) \wedge^+ \text{path}(y, z) \wedge^+ \neg \text{path}(x, z)) \downharpoonright \Theta \\
\vdash \cdot \upharpoonright \Theta
\end{array}$$

In order to determine the shape of the proofs $\Xi_1$, $\Xi_2$, and $\Xi_3$, we must declare the polarization given to atoms with the \textit{path} predicate. If all such atoms have a negative polarity assigned to them, then both $\Xi_1$ and $\Xi_2$ end with the \textit{release} and \textit{store} rules while the proof $\Xi_3$ must be trivial (just containing the \textit{init} rule) and $\text{path}(r, t)$ must be a member of $\Theta$. We can write the synthetic rule justified by the above derivation as

$$\begin{array}{c}
\vdash \cdot \upharpoonright \text{path}(r, s), \Theta \\
\vdash \cdot \upharpoonright \text{path}(s, t), \Theta \\
\vdash \cdot \upharpoonright \text{path}(r, t), \Theta
\end{array}$$

However, if all \textit{path}-atoms have a positive polarity assigned to them, then $\Xi_3$ ends with the \textit{release} and \textit{store} rules while the proof $\Xi_1$ and $\Xi_2$ must be trivial and both $\neg \text{path}(r, s)$ and $\neg \text{path}(s, t)$ must be members of $\Theta$. We can write the synthetic rule justified by the above derivation as

$$\begin{array}{c}
\vdash \cdot \upharpoonright \neg \text{path}(r, s), \neg \text{path}(s, t), \neg \text{path}(r, t), \Theta
\end{array}$$

Note that these synthetic inference rules are the one-sided version of the back-chaining and forward-chaining synthetic inference rules for \textit{path} displayed in Section 2.4.
The paper (Marin et al. 2020) develops the proof theory of synthetic inferences for both classical and intuitionistic logic by using the focused proof systems LKF and LJF. That paper also shows that cut and the general initial rule are both admissible in the LK and LJ proof systems augmented with such synthetic inference rules based on geometric formulas.

### 9.3 Herbrand’s theorem

The completeness of LKF proofs yields a surprisingly simple proof of Herbrand’s theorem, particularly the variant of Herbrand’s theorem based on formulas with only existential quantifiers in prefix position. A richer connection between a more general form of Herbrand’s theorem, based on expansion trees (Miller 1987), and LKF proofs can be found in (Chaudhuri et al. 2016).

**Theorem 10 (Herbrand’s theorem)**

Let $\phi$ be an (unpolarized) quantifier-free formula of first-order classical logic, $n \geq 1$, and $x_1, \ldots, x_n$ be a list of first-order variables containing all the free variables of $\phi$. The formula $\exists x_1 \ldots \exists x_n \phi$ is provable in LK if and only if there is an $1 \leq \ell \leq n$ and substitutions $\theta_1, \ldots, \theta_m$ for the variables $x_1, \ldots, x_n$ such that $B \theta_1 \lor \cdots \lor B \theta_m$ is provable in LK.

**Proof**

Let $\hat{B}$ be a polarized version of $B$ in which all logical connectives and units in $B$ are polarized negatively. (For convenience, we abbreviate $\exists x_1 \ldots \exists x_n$ with $\exists \vec{x}$.) Since $\exists \vec{x}. B$ is provable in LK, the sequent $\vdash \exists \vec{x}. B \upharpoonright \cdot$ must have an LKF proof, say $\Xi$. Clearly, the last inference rule of $\Xi$ is the store rule with premise $\vdash \cdot \upharpoonright \exists \vec{x}. B$. Given our choice of polarization, it is easy to show that every border sequent in $\Xi$ is of the form $\vdash \cdot \upharpoonright \exists \vec{x}. B, L$, where $L$ is a set of literals. Thus, there are only two different ways that the decide rule is applied in $\Xi$. If the decide rule is used with a positive literal, the premise is immediately proved using the init rule. Otherwise, the decide rule starts the synchronous phase with the choice of $\exists \vec{x}. B$ and the subproof determined by that occurrence of the decide rule ends with the following inference rules.

$$
\begin{align*}
\vdash & \exists \vec{x}. B, L \\
\vdash & \exists \vec{x}. B, L \\
\vdash & \exists \vec{x}. B \upharpoonright \exists \vec{x}. B, L
\end{align*}
$$

That is, every non-trivial synchronous phase encodes a substitution. Let $m \geq 1$ be the number of such non-trivial synchronous phases and let $\theta_1, \ldots, \theta_m$ be the substitutions that those phases encode.

Now let $C$ be the polarized formula $C$ equal to $B \theta_1 \lor^* \cdots \lor^* B \theta_m$ and consider building an LKF proof of $\vdash C \upharpoonright \cdot$. In order to ensure that $C$ is polarized positively, if $m = 1$, we take $C$ to be $C \lor^* f^*$ (essentially encoding a unary version of the binary $\lor^*$). It is now a simple matter to convert the proof $\Xi$ of $\vdash \exists \vec{x}. B \upharpoonright \cdot$ into a proof of $\vdash B \theta_1 \lor^* \cdots \lor^* B \theta_m \upharpoonright \cdot$ by copying the asynchronous phases directly and by replacing all the non-trivial synchronous phase in $\Xi$ as follows.
In this way, the phase-by-phase structure of $\Xi$ can be used to build an LKF proof for $\vdash \hat{B}\theta_1 \upharpoonright C, L \vdash \hat{B}\theta_1 \vdash \uparrow \vdash \hat{B}\theta_1 \vdash C$.

### 9.4 Hosting other focused proof systems

Proof systems with focusing-like behaviors can sometimes be hosted inside LKF. Such hosting is usually done by translating unpolarized classical logic formulas into polarized formulas in which delays have been inserted. These delays are written as $\partial_l(B)$ and $\partial_r(B)$ and are such that they are both logically equivalent to the polarized formula $B$ and are such that $\partial_l(B)$ is negative and $\partial_r(B)$ is positive. The expression $\partial_l(B)$ can be defined to be either $f^\prec \lor A, f^\prec \land A$, or $\forall x B$ (where $x$ is not free in $B$). Similarly, the expression $\partial_r(B)$ can be defined to be either $f^\succ \lor A, f^\succ \land A$, or $\exists x B$ (where $x$ is not free in $B$).

The LKQ and LKT proof systems of (Danos et al. 1995) can be seen as LKF proofs in which the following polarization functions are used. Figure 6 defines the left and right translations of unpolarized formulas containing only implications and atoms to polarized formulas. In that figure, $A$ ranges over atomic formulas. It is the case that (cut-free) proofs in LKT of an unpolarized formula $B$ using only implications correspond to LKF proofs of $(B)^\ell$ (using the LKT definition) and (cut-free) proofs in LKQ of an unpolarized formula $B$ using only implications correspond to LKF proofs of $(B)^r$ (using the LKQ definition). LKT focuses only on the left and LKQ only on the right of two-sided sequents. These systems are also examples of “less aggressive” focused systems that designate a “stoup” formula: these systems impose fewer restrictions than the formula under focus in LKF. The delays emulate the one-sided focusing character of these systems as well as adopt the stoup to a strongly focused system.

<table>
<thead>
<tr>
<th>LKT</th>
<th>LKQ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^\ell = \neg A$</td>
<td>$A^r = A$</td>
</tr>
<tr>
<td>$(B \supset C)^\ell = (B)^r \land^r (C)^\ell$</td>
<td>$(B \supset C)^r = \partial_l((B)^r \lor^r (C)^r)$</td>
</tr>
<tr>
<td>$(B \supset C)^r = (B)^\ell \lor^\ell \partial_r((C)^\ell)$</td>
<td>$(B \supset C)^\ell = \partial_r((B)^\ell \land^\ell (C)^\ell)$</td>
</tr>
</tbody>
</table>

Fig. 6 Two different ways to translate classical logic formulas into polarized formulas.
Synchronous introduction rules

\[ \vdash B_1, \Theta_1 \Gamma \quad \vdash B_2, \Theta_2 \Gamma \quad \vdash B_i, \Theta \Gamma \quad i \in \{1, 2\} \quad \vdash [s/x]B, \Theta \Gamma \]

Release and decide rules

\[ \vdash \Delta \Gamma \quad \vdash \Delta \Gamma \quad \vdash \Delta \Gamma \quad \vdash \Delta \Gamma \]

The † proviso requires that \( \Delta \) consists of only negative polarized formulas. In the decide rule, \( \Lambda \) is a non-empty multiset of positive polarized formulas and \( \Lambda \) is its underlying set of polarized formulas. The ‡ proviso is discussed in the text.

Fig. 7 Variations in some of the LKF inference rules.

10 Other variations for focusing in classical logic

There have been several variations on focusing systems studied in the literature. In fact, the general phenomena of focusing for classical logic can be seen as arising from Girard uses of linear logic exponentials to encode classical logic (Girard 1987) and from Andreoli’s discovery of polarity (Andreoli 1992).

The LKF proof system we have given here can be called a strongly focused system: the decide rule can only be invoked after every negative non-atomic polarized formula has been removed from the sequent. If we do not insist that all negative polarized formulas have been removed in this way, the resulting variant is called a weakly focused proof system following (Laurent 2004, Simmons and Pfenning 2011). Girard’s LC proof system is an early example of a weakly focused proof system for classical logic (Girard 1991). A variant on strong focusing is a system where one chooses a predetermined suspension criterion and then allows explicitly suspected negative polarized formulas to remain in the conclusion of the (suitably modified) decide rule: suspensions of this kind have proved useful in a setting where logic contains fixed point expressions (Gérard and Miller 2017).

Let \( \text{LKF}^\text{fm} \) be the proof system that results from replacing the inference rules for LKF with the extended version of the synchronous introduction rules and the release and decide rules given in Figure 7. If the ‡ proviso on the decide rule requires that the multiset \( \Delta \) contains exactly one positive polarized formula, then \( \text{LKF}^\text{fm} \) is the same as LKF. It is for this reason that we say that LKF is single focused: in such proofs, the zone to the left of the \( \Downarrow \) always contains exactly one polarized formula (the focus of that sequent). If the ‡ proviso restricts \( \Delta \) to be just a non-empty set of positive polarized formulas, then the resulting proof system is multifocused and that proof system contains more proofs than the single conclusion system. Multifocused proofs were first considered in (Delande and Miller 2008, Delande et al. 2010) (in the context of linear logic) and the notion of maximal multifocused proofs has been used to describe canonical proof system in linear logic (Chaudhuri et al. 2008a) and
classical logic (Chaudhuri et al. 2016) and to relate sequent calculus proofs to natural
deduction proofs (Pimentel et al. 2016).

Note that the version of the $\land^+$ introduction rule in $\text{LKF}_\text{m}$ is not necessarily
invertible, while the version of that introduction rule in $\text{LKF}$ is invertible: it appears
that the true status of $\land^+$ introduction as belonging to the synchronous phase only
becomes apparent in the multifocused setting. Note also that it is immediate to prove
the completeness of $\text{LKF}_\text{m}$ given the completeness of $\text{LKF}$.

Two simple changes to the $\text{LKF}$ proof system yield a focused proof system for
multiplicative additive linear logic $\text{MALL}$ (Girard 1987). First, the set of formulas
to the right of the double arrows must be changed to multisets. Second, the fol-
lowing four inference rules must replace the corresponding inference rules in $\text{LKF}$
(Figure 3).

Here, the $\text{init}$ and $t^+$ rules do not do an implicit weakening, the $\text{decide}$ rule does not
do an implicit contraction, and the side formulas of $\land^+$ are treated multiplicatively.
The resulting proof system, called $\text{MALLF}$ in (Liang and Miller 2011), is a focused
proof system for $\text{MALL}$. Of course, the usual presentation of $\text{MALL}$ results from
replacing the logical connectives $\bot^-$, $\bot^+$, $\bot$, $\land^-$, $\land^+$, $\lor^-$, and $\lor^+$ with $\top$,
$\bot$, $\bot$, $\land$, $\lor$, $\&$, $\otimes$, $\Rightarrow$, and $\oplus$, respectively. The fact that this proof system is sound and complete
for $\text{MALL}$ immediately follows from the results about focusing in full linear logic

Another variation on focused proof systems uses a list, not a multiset, of formulas
to the left of the $\Gamma$: that is, the order by which the asynchronous inference rules
are attempted is proscribed in a fixed fashion. This variation was used by Andreoli
(1992) in his first focused proof system for linear logic.

The $\text{LKF}$ proof system was designed to support automated proof checking and
proof search (Chihani et al. 2017) as well as to provide new means for proving meta-
theoretic results for first-order classical logic (see Section 9). Other researchers,
concentrating on the Curry-Howard correspondence (proofs-as-programs) perspec-
tive, have designed other variants of focusing for classical logic. In particular, see the
$\text{LC}$ proof system (Girard 1991), the $\text{LKF}''$ (Danos et al. 1995; 1997), and the proof
system used to define the $\lambda\mu\nu$-calculus (Curien and Herbelin 2000).

11 Conclusion

We have presented the proof system $\text{LKF}$ and have proved that it is sound and
complete for $\text{LK}$ and that the cut rule and the initial rule are admissible. The proofs
of these theorems were all done directly using permutation arguments. We have
illustrated the utility of $\text{LKF}$ by applying it to some standard topics that arise in
the proof theory of classical logic. We hope that while the metatheory of $\text{LKF}$ was
established by tedious permutation arguments, many other properties of proofs in classical logic can be proved by applying LKF directly and without the need for such permutation arguments.

Acknowledgments: We thank Beniamino Accattoli, Marianna Girlando, and the anonymous reviewers for their comments on an earlier version of this paper.

References


