Combining Intuitionistic and Classical Logic: a proof system and semantics

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Two logics or one?

Clearly these are two different logics: wars have been fought over classical vs non-classical foundations for mathematics.

Both semantics and proof theory illustrate the special nature of the intuitionistic implication (and universal quantification).

$$\mathcal{M}, u \models A \supset B \text{ if forall } u \leq v. \mathcal{M}, v \models A \text{ implies } \mathcal{M}, v \models B.$$

Enforce single-conclusion on left-introduction (Gentzen).

$$\frac{\Gamma_1 \longrightarrow A, \Delta_1 \qquad \Gamma_2, \longrightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \supset B \longrightarrow \Delta_1, \Delta_2} \supset L \quad \text{and} \quad \Delta_1 = \emptyset$$

Enforce single-conclusion on right-introduction.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B, \Delta} \supset R$$

Can we add intuitionistic implication to classical logic?

Naive schemes result with the collapse of intuitionistic implication into the classical one.

What is the cost of mixing these logics? Can one have perspicuous semantics and/or proof systems?

Previous work

$C \longrightarrow I$ via double negation translations.

$I \longrightarrow C$ via the addition of a modal operator.

Linear logic can encode $A \supset B$ as either $!A \multimap B$ (intuitionistic) or as $!A \multimap ?B$ (classical).

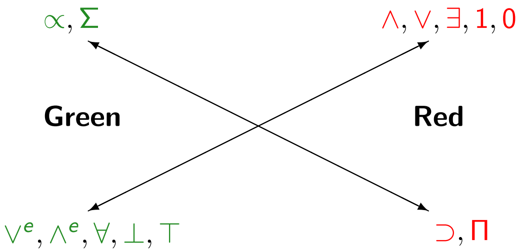
There is Girard's LU logic [Girard 1993; Vauzeilles 1993]. Maybe too ambitious and includes linear logic.

"Fibred Semantics and the Weaving of Logics", Gabbay JSL 1996.

"Combining Classical and Intuitionistic Implications," Caleiro & Ramos, FroCos 2007.

PIL: Polarized Intuitionistic Logic

Red-Polarized: $\wedge$, 1, $\vee$, 0, $\exists$, $\supset$, Π. (Syntactic variable R)
Green-Polarized: $\wedge^e$, $\top$, $\vee^e$, $\bot$, $\forall$, $\propto$, Σ. (Syntactic variable E)



Purely intuitionistic connectives: $\supset$, Π, $\propto$ and Σ
Classically-oriented connectives: $\vee$, $\wedge$, $\exists$, $\vee^e$, $\wedge^e$ and $\forall$

The formulas of PIL

Atomic formulas are (arbitrarily) classified as red. A negated $(-)^{\perp}$ atom is, thus, green.

$(B)^{\perp}$ is the negation normal form of the De Morgan dual of B.

De Morgan dualities are:

$$1/\bot,\ 0/\top,\ \supset/\propto,\ \Pi/\Sigma,\ \vee/\wedge^{e},\ \wedge/\vee^{e},\ \exists/\forall.$$

$A^{\perp\perp}$ and A are a syntactic identical for all formulas A.

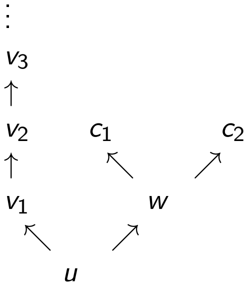
The dual of $A \supset B$ is $A \propto B^{\perp}$, and not $A^{\perp} \propto B^{\perp}$.

Classic negation $A^{\perp}$ flips between green and red.

Intuitionistic negation $A \supset 0$ is always a red formula.

A Kripke-style semantics

A terminal world in a Kripke model is a classical worlds: intuitionistic implication collapses into a classical one and the excluded middle becomes valid.



The terminal worlds c_1 and c_2 are classical: $c_1 \models p \vee \neg p$.

Worlds beyond classical worlds

We shall allow there to be worlds *beyond* classical worlds.

Such worlds will make *all* classical formulas true (one kind of inconsistency) but not all intuitionistic formulas true.

A world may validate $\bot$ (and, thus, all classical formulas) but never validate 0.

[An analogy from linear logic: for all B, $0 \vdash B$ while $\bot \vdash ?B$.]

Worlds beyond classical worlds will be called *imaginary worlds* (similar in spirit to naming $\sqrt{-1}$ as an imaginary number).

Propositional Kripke hybrid models

A *propositional Kripke hybrid model* is a tuple $\langle \mathbf{W}, \preceq, \mathbf{C}, \models \rangle$ s.t.

- $\mathbf{W}$ is a non-empty Kripke frame of possible worlds.
- $\preceq$ is a transitive and reflexive relation on $\mathbf{W}$.
- $\mathbf{C}$, the set of "classical worlds," is a subset of $\mathbf{W}$.
- $\models$ is a binary relation between elements of $\mathbf{W}$ and (red-polarized) atomic formulas.

$\triangle_{\mathbf{u}} = \{\mathbf{k} \in \mathbf{C} \mid \mathbf{u} \preceq \mathbf{k}\}$, is the set of classical worlds above $\mathbf{u}$.
A world $\mathbf{u}$ is *imaginary*, or $\perp$*-inconsistent*, if $\triangle_{\mathbf{u}}$ is empty.

The following conditions must also hold:

- $\models$ is *monotone*: for $\mathbf{u}, \mathbf{v} \in \mathbf{W}$, $\mathbf{u} \preceq \mathbf{v}$ and $\mathbf{u} \models a$ implies $\mathbf{v} \models a$.
- $\triangle_{\mathbf{k}} = \{\mathbf{k}\}$ for all $\mathbf{k} \in \mathbf{C}$, i.e., there are no classical worlds properly above other classical worlds.

Defining forcing: red connectives first

The satisfiability or *forcing* relation extends $\models$ from atoms to all propositional formulas by induction on the structure of formulas.

The key idea here is that a green formula is valid in a world $\mathbf{u}$ if it is valid in all classical worlds above $\mathbf{u}$.

First, we define the red-polarity cases using the familiar Kripke formulation. Assuming $\mathbf{u}, \mathbf{v} \in \mathbf{W}$, we have:

- $\mathbf{u} \models 1$ and $\mathbf{u} \not\models 0$
- $\mathbf{u} \models A \vee B$ iff $\mathbf{u} \models A$ or $\mathbf{u} \models B$
- $\mathbf{u} \models A \wedge B$ iff $\mathbf{u} \models A$ and $\mathbf{u} \models B$
- $\mathbf{u} \models A \supset B$ iff for all $\mathbf{v} \succeq \mathbf{u}$, $\mathbf{v} \models A$ implies $\mathbf{v} \models B$

Defining forcing: green connectives second

First define forcing of green formulas but only over classical worlds: here, $\mathbf{c} \in \mathbf{C}$ and $\mathbf{v} \in \mathbf{W}$.

- $\mathbf{c} \models a^{\perp}$ iff $\mathbf{c} \not\models a$ (a atomic).
- $\mathbf{c} \models \top$ and $\mathbf{c} \not\models \perp$
- $\mathbf{c} \models A \propto B$ iff for some $\mathbf{v} \succeq \mathbf{c}$, $\mathbf{v} \models A$ and $\mathbf{v} \not\models B^{\perp}$
- $\mathbf{c} \models A \vee^e B$ iff $\mathbf{c} \models A$ or $\mathbf{c} \models B$
- $\mathbf{c} \models A \wedge^e B$ iff $\mathbf{c} \models A$ and $\mathbf{c} \models B$

Extend $\models$ to all green formulas E in *any* $\mathbf{u} \in \mathbf{W}$:

- $\mathbf{u} \models E$ if and only if for all $\mathbf{c} \in \triangle_{\mathbf{u}}$, $\mathbf{c} \models E$.

(If $\triangle_{\mathbf{u}}$ is empty, then all green formulas are satisfied in $\mathbf{u}$.)

The $\models$ relation is well-defined: if $\mathbf{u} \in \mathbf{C}$ then the clauses above defining $\models$ for classical worlds coincide since $\triangle_{\mathbf{u}} = \{\mathbf{u}\}$.

Some simple properties about forcing

Let $\mathbf{u}, \mathbf{v} \in \mathbf{W}$, $\mathbf{c} \in \mathbf{C}$, and let A be a (propositional) formula.

a. if $\mathbf{u} \preceq \mathbf{v}$, then $\mathbf{u} \models A$ implies $\mathbf{v} \models A$ (monotonicity)

b. $\mathbf{c} \models A$ iff $\mathbf{c} \not\models A^\perp$ (excluded middle)

c. $\mathbf{u} \models A$ and $\mathbf{u} \models A^\perp$ for some A iff $\triangle_\mathbf{u}$ is empty ($\mathbf{u}$ is imaginary).

d. $\mathbf{u} \not\models E$ for some green formula E iff $\triangle_\mathbf{u}$ is non-empty.

While 0 and $\perp$ are clearly distinct, 1 and $\top$ are equivalent: they are simply red and green-polarized versions of the same truth value. Red and green formulas can be equivalent:

$$E \equiv E^\perp \supset \perp \quad \text{and} \quad (R \supset \perp) \supset \perp \equiv R \vee^e \perp.$$

A model $\mathcal{M}$ *satisfies* A, or $\mathcal{M} \models A$, if $\mathbf{u} \models A$ for every $\mathbf{u} \in \mathbf{W}$. A formula is *valid* if it is satisfied in all models.

A countermodel

The excluded middle, in the form $a \vee^e a^\perp$, is valid.

The formula $\sim a \vee^e \sim\sim a$ is not valid.

$$\begin{array}{ccc} s_1 : \{a, a^\perp\} & & s_2 : \{a^\perp\} \\ & \nwarrow \quad \nearrow & \\ & k : \{a^\perp\} & \end{array}$$

The same model shows that $a \vee^e \sim a$ is also not valid (s_2 is not needed here).

Another countermodel

The formula $(p \wedge^e q) \supset p$ is not valid. A countermodel is:

$$\begin{array}{c} k : \{p, q\} \\ \uparrow \\ s : \{\} \end{array}$$

Although every classical world *above* s satisfies p and q, s does not satisfy p.

The same model shows that several other formulas, including $(p \vee^e q) \supset (p \vee q)$, are not valid.

More generally, $E \supset p$ is never valid for green formulas E.

Heyting Algebra presentation

Every Kripke frame $\langle \mathbf{W}, \preceq, \mathbf{C}, \models \rangle$ corresponds to a Heyting algebra

$$\mathcal{H} = \langle \mathcal{U}(\mathbf{W}), \sqsubseteq, \sqcup, \sqcap, \rightarrow, \mathbf{0} \rangle, \quad \text{where}$$

- $\mathcal{U}(\mathbf{W})$ is the set of all upwardly closed subsets of $\mathbf{W}$
- The relation $\sqsubseteq$ of $\mathcal{H}$ is set inclusion; join $\sqcup$, meet $\sqcap$, and $\mathbf{0}$ are union, intersection, and the empty set, resp.
- The relative pseudo-complement $A \rightarrow B$ is the largest $x \in \mathcal{U}(\mathbf{W})$ such that $(A \sqcap x) \sqsubseteq B$ (here, it is the largest upwardly closed subset (*interior*) of $(\mathbf{W} - A) \cup B$).

The formula A is interpreted by a mapping h defined as

$$h(A) = \{\mathbf{u} \in \mathbf{W} : \ \mathbf{u} \models A\}$$

Each $h(A)$ is upwardly closed. A formula A is valid if $h(A) = \mathbf{W}$.

Imaginary worlds

The set of imaginary worlds $\perp\!\!\!\perp$ in $\mathbf{W}$ is the upwardly closed set

$$\perp\!\!\!\perp = \{\mathbf{u} \in \mathbf{W} : \ \triangle_{\mathbf{u}} \text{ is empty}\} = \{\mathbf{u} \in \mathbf{W} : \ \mathbf{u} \models \perp\} = h(\perp)$$

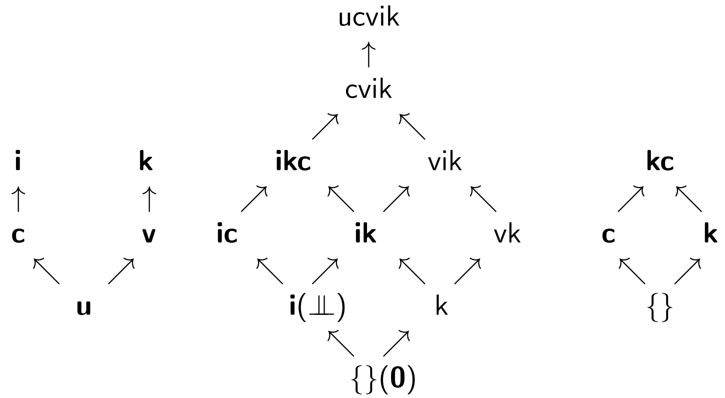
If there are no imaginary worlds in $\mathbf{W}$ then, indeed, $\perp\!\!\!\perp = \mathbf{0}$.

The following hold for h:

- $h(A \vee B) = h(A) \sqcup h(B)$, $h(A \wedge B) = h(A) \sqcap h(B)$, and $h(A \supset B) = h(A) \to h(B)$.
- $h(R^{\perp}) \ = \ h(R) \to \perp\!\!\!\perp$ for all green formulas $R^{\perp}$.

From the semantic perspective the most important addition to intuitionistic logic found in PIL is $\perp$.

A example of a Kripke frame and a Heyting algebra



Kripke Frame with $\mathbf{C} = \{\mathbf{c}, \mathbf{k}\}$ | Heyting Algebra with ⫫ | Boolean Algebra $2^{\mathbf{C}}$

For any subset K of classical worlds, $⫫ \cup K$ is an upwardly closed.

The LP Sequent Calculus

We use two-sided sequents although the use of colors makes a one-sided sequent calculus possible.

We use the symbols $\vdash_{\circ}$ and $\vdash_{\bullet}$ to represent two modes of proof.

In all rules, Γ and Θ are multisets of formulas, E is a *green* formula, R is a *red* formula, and a is any atom.

The sequent $\Gamma \vdash_{\circ} A$ is interpreted as $\bigwedge\Gamma \supset A$.
The sequent $\Gamma \vdash_{\bullet} \Theta$ is interpreted as $\bigwedge\Gamma \supset \bigvee^{e}\Theta$.
(If Δ is empty, then $\bigwedge\Delta$ is 1 and $\bigvee^{e}\Delta$ is $\perp$.)

Proofs end with sequents of the form $\Gamma \vdash_{\circ} A$ (A is any color).

A is a theorem of PIL if $\vdash_{\circ} A$ is provable.

The LP Sequent Calculus: proof rules

Structural Rules and Identity

$$\frac{\Gamma \vdash_{\bullet} E}{\Gamma \vdash_{\circ} E}\ \mathit{Signal} \qquad \frac{A^{\perp}, \Gamma \vdash_{\bullet} \Theta}{\Gamma \vdash_{\bullet} A, \Theta}\ \mathit{Store} \qquad \frac{A^{\perp}, \Gamma \vdash_{\circ} A}{A^{\perp}, \Gamma \vdash_{\bullet}}\ \mathit{Load} \qquad \frac{}{a, \Gamma \vdash_{\circ} a}\ I$$

Right-Red Introduction Rules

$$\frac{\Gamma \vdash_{\circ} A \quad \Gamma \vdash_{\circ} B}{\Gamma \vdash_{\circ} A \wedge B}\ \wedge R \qquad \frac{\Gamma \vdash_{\circ} A_i}{\Gamma \vdash_{\circ} A_1 \vee A_2}\ \vee R \qquad \frac{A, \Gamma \vdash_{\circ} B}{\Gamma \vdash_{\circ} A \supset B}\ \supset R$$

Left-Red Introduction Rules

$$\frac{A, B, \Gamma \vdash_{\circ} R}{A \wedge B, \Gamma \vdash_{\circ} R}\ \wedge L \qquad \frac{A, \Gamma \vdash_{\circ} R \quad B, \Gamma \vdash_{\circ} R}{A \vee B, \Gamma \vdash_{\circ} R}\ \vee L \qquad \frac{A \supset B, \Gamma \vdash_{\circ} A \quad B, \Gamma \vdash_{\circ} R}{A \supset B, \Gamma \vdash_{\circ} R}\ \supset L$$

Right-Green Introduction Rules

$$\frac{\Gamma \vdash_{\bullet} A \quad \Gamma \vdash_{\bullet} B}{\Gamma \vdash_{\bullet} A \wedge^{e} B}\ \wedge^{e} R \qquad \frac{\Gamma \vdash_{\bullet} A, B}{\Gamma \vdash_{\bullet} A \vee^{e} B}\ \vee^{e} R \qquad \frac{\Gamma \vdash_{\circ} A \quad \Gamma \vdash_{\bullet} B}{\Gamma \vdash_{\bullet} A \propto B}\ \propto R$$

Rules for Constants

$$\frac{}{\Gamma \vdash_{\circ} 1}\ 1R \qquad \frac{\Gamma \vdash_{\circ} R}{1, \Gamma \vdash_{\circ} R}\ 1L \qquad \frac{}{0, \Gamma \vdash_{\circ} R}\ 0L \qquad \frac{\Gamma \vdash_{\bullet}}{\Gamma \vdash_{\bullet} \perp}\ \perp R \qquad \frac{}{\Gamma \vdash_{\bullet} \top}\ \top R$$

A version of the double negation shift

Define $\neg B$ as $B \supset \bot$.

$$\dfrac{A \vdash_{\circ} A}{A, A^{\bot} \vdash_{\bullet}}\ \text{Load}$$

$$\dfrac{A, A^{\bot} \vdash_{\bullet}}{A, A^{\bot} \vdash_{\circ} \bot}\ \text{Signal},\ \bot I$$

$$\dfrac{A, A^{\bot} \vdash_{\circ} \bot}{A^{\bot} \vdash_{\circ} \neg A}\ \supset R$$

$$\dfrac{}{\vdash_{\bullet} 1}\ \text{Store}, \text{Load}, 1R$$

$$\dfrac{A^{\bot} \vdash_{\circ} \neg A \qquad \vdash_{\bullet} 1}{A^{\bot} \vdash_{\circ} \neg A \propto 1}\ \propto R$$

$$\dfrac{A^{\bot} \vdash_{\circ} \neg A \propto 1}{A^{\bot} \vdash_{\circ} \exists x.(\neg A \propto 1)}\ \exists R$$

$$\dfrac{A^{\bot} \vdash_{\circ} \exists x.(\neg A \propto 1)}{A^{\bot}, \forall x.\neg\neg A \vdash_{\bullet}}\ \text{Load}$$

$$\dfrac{A^{\bot}, \forall x.\neg\neg A \vdash_{\bullet}}{\forall x.\neg\neg A \vdash_{\bullet} A}\ \text{Store}$$

$$\dfrac{\forall x.\neg\neg A \vdash_{\bullet} A}{\forall x.\neg\neg A \vdash_{\bullet} \forall x.A}\ \forall R$$

$$\dfrac{\forall x.\neg\neg A \vdash_{\bullet} \forall x.A}{\forall x.\neg\neg A \vdash_{\circ} \forall x.A}\ \text{Signal}$$

$$\dfrac{\forall x.\neg\neg A \vdash_{\circ} \forall x.A \qquad \vdash_{\bullet} 1}{\forall x.\neg\neg A \vdash_{\circ} \forall x.A \propto 1}\ \propto R$$

$$\dfrac{\forall x.\neg\neg A \vdash_{\circ} \forall x.A \propto 1}{\forall x.\neg\neg A, \neg\forall x.A \vdash_{\bullet}}\ \text{Load}$$

$$\dfrac{\forall x.\neg\neg A, \neg\forall x.A \vdash_{\bullet}}{\forall x.\neg\neg A, \neg\forall x.A \vdash_{\circ} \bot}\ \text{Signal},\ \bot R$$

$$\dfrac{\forall x.\neg\neg A, \neg\forall x.A \vdash_{\circ} \bot}{\forall x.\neg\neg A \vdash_{\circ} \neg\neg\forall x.A}\ \supset R$$

$$\dfrac{\forall x.\neg\neg A \vdash_{\circ} \neg\neg\forall x.A}{\vdash_{\circ} \forall x.\neg\neg A \supset \neg\neg\forall x.A}\ \supset R$$

The intuitionistic fragment

If the formula A contains only red connectives and positive atoms, then the only LP proofs of $\vdash_{\circ} A$ are essentially the cut-free LJ proofs of Gentzen.

Overview of the LC proof system: polarities

The classical fragment of the LP is essentially Girard's LC proof system for classical logic [APAL 1993].

In LC, every formula is polarized as either *positive* or *negative*.

Atoms are positive. De Morgan duals flip polarities.

Compound (propositional) formulas are given their polarities as follows:

A	B	$A \wedge B$	$A \vee B$	$A \supset B$
+	+	+	+	-
-	+	+	-	+
+	-	+	-	-
-	-	-	-	-

Overview of the LC proof system: sequents

Sequents of LC are one sided sequents $\vdash \Gamma; \Delta$ where Γ and Δ are multisets of formulas and Δ is either empty or a singleton.

When Δ is the singleton S, then S is the *stoup* of $\vdash \Gamma; \Delta$.

Weakening and contraction are available in the Γ context. Here, P and Q are positive and N is negative.

$$\frac{}{\vdash \neg P; P}\ \textit{initial} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma, P;}\ \textit{dereliction}$$

$$\frac{\vdash \Gamma; P \quad \vdash \Delta, N;}{\vdash \Gamma, \Delta; P \wedge N} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma; P \vee Q} \qquad \frac{\vdash \Gamma; Q}{\vdash \Gamma; P \vee Q}$$

$$\frac{\vdash \Gamma, A, B; \Delta}{\vdash \Gamma, A \vee B; \Delta}\ \text{where } A \vee B \text{ is negative}$$

The classical fragment of LP is LC

Drop the intuitionistic connectives $\supset$, $\propto$, Π and Σ.
There are two copies of conjunction and disjunction: $\vee$, $\wedge$, $\vee^e$, $\wedge^e$.

Positive formulas are red-polarized and negative ones are green-polarized.

The polarity of an LC formula is also dependent on the polarity of its subformulas. When A and B are both positive, $A \vee B$ in LC corresponds to $A \vee B$ in PIL; otherwise, it is $A \vee^e B$.

LC sequents with a stoup correspond to the $\vdash_\circ$ while a sequent without a stoup correspond to $\vdash_\bullet$.

LC introduction rules on the stoup formula correspond to right-red introduction rules in LP; the introduction rules for “negative” connectives *in the presence of a stoup* correspond to left-red rules while those without a stoup correspond to right-green rules.

Here, P is positive and N is negative.

$$\frac{\vdash \Gamma, N, P; S}{\vdash \Gamma, N \vee P; S} \quad \longmapsto \quad \frac{\Gamma, P, N \vdash_{\circ} S}{\Gamma, P \wedge N \vdash_{\circ} S}\ \wedge L$$

$$\frac{\vdash \Gamma, N, P;}{\vdash \Gamma, N \vee P;} \quad \longmapsto \quad \frac{\Gamma \vdash_{\bullet} N, P}{\Gamma \vdash_{\bullet} N \vee^{e} P}\ \vee^{e} R$$

$$\frac{\vdash \Gamma_1; P \quad \vdash \Gamma_2, N;}{\vdash \Gamma_1, \Gamma_2; P \wedge N} \quad \longmapsto \quad \frac{\Gamma_1, \Gamma_2 \vdash_{\circ} P \quad \dfrac{\Gamma_1, \Gamma_2 \vdash_{\bullet} N}{\Gamma_1, \Gamma_2 \vdash_{\circ} N}\ \mathit{Signal}}{\Gamma_1, \Gamma_2 \vdash_{\circ} P \wedge N}\ \wedge R$$

Excluded middle

$$p \vee (p \supset 0) \quad \text{versus} \quad p \vee^{e} p^{\perp}$$

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Peirce's formula is provable in the form

$$((p \supset q) \supset p) \supset p,$$

where $\supset$ is classical implication, defined as $A \supset B = A^{\perp} \vee^{e} B$.

An approach to intermediate logics

Excluded middle

$$p \vee (p \supset 0) \quad \text{versus} \quad p \vee^e p^\perp$$

Peirce's formula is provable in the form

$$((p \supset q) \supset p) \supset p,$$

where $\supset$ is classical implication, defined as $A \supset B = A^\perp \vee^e B$.

Markov's principle

$$(\forall n(P(n) \vee \neg P(n))) \supset (\neg \forall n \neg . P(n)) \supset \exists n.P(n)$$

$$[(\Pi x. \sim P(x) \vee \sim (P(x)^\perp)) \supset (\sim \sim \exists x.P(x))] \supset \Sigma x.P(x)$$

Conclusion

Other results for PIL

- Semantic proof of cut elimination.
- Tableau style proof system. Multiple conclusion proof system.
- Decision procedure.
- Kripke hybrid model semantics for first-order quantification.

Future work

- Extend PIL to arithmetic
- Systematic investigation of various intermediate logics.
- Curry-Howard interpretation, delimited control operators (see LICS 2013).
- Mechanization of proof search (focusing proof systems).