Proving the completeness of proof systems using LKF: some examples

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Abstract

We consider how focusing system LKF for classical logic can be used to prove the completeness of various proof systems. We consider, in particular, Herbrand expansions, DPLL, etc.

1 Introduction

Focusing proof systems are sequent style proof systems that allow one to organize the "micro-rules" of inference, given by the introduction rules, into "macro-rules" of inference. The LKF proof system [3] is such a proof system for first-order classical logic. Other such proof systems exist for intuitionistic logic (LJF, also found in [3]) and for linear logic [1].

Recently, Nigam and Miller [4] have used linear logic and its focused proof system to faithfully encode and to proof the (relative) completeness of a range of proof systems: in particular, sequent calculus (with and without the cut rule), natural deduction (normal and non-normal), natural deduction with generalized elimination, free deduction, and tableaux. Notice that all of these proof systems can be seen as various kinds of tree structures based on inference rules.

Also recently, Delande and Miller [2] have used the focused proof system for multiplicative, additive linear logic (MALL) to show how games can be used to provide for both proofs and refutations.

In this note, we explore how well focusing proof systems can be used to analyze proof systems that are not structured as trees of inference rules (such as games, themselves).

2 LKF

Liang and Miller presented the LKF proof system in [3], given here in Figure 1, as a focusing proof system for classical logic. Here, P is positive, N is negative, C is

$$\frac{\vdash [\Theta, C], \Gamma}{\vdash [\Theta], \Gamma, C}\ [] \qquad \frac{\mapsto [P, \Theta], P}{\vdash [P, \Theta]}\ Focus \qquad \frac{\vdash [\Theta], N}{\mapsto [\Theta], N}\ Release$$

$$\frac{}{\mapsto [\neg P, \Theta], P}\ ID^{+},\ \text{atom } P \qquad \frac{}{\mapsto [N, \Theta], \neg N}\ ID^{-},\ \text{atom } N$$

$$\frac{}{\mapsto [\Theta], \mathcal{T}} \qquad \frac{}{\vdash [\Theta], \Gamma, \neg\mathcal{F}} \qquad \frac{\vdash [\Theta], \Gamma}{\vdash [\Theta], \Gamma, \neg\mathcal{T}}$$

$$\frac{\vdash [\Theta], \Gamma, A \quad \vdash [\Theta], \Gamma, B}{\vdash [\Theta], \Gamma, A \wedge^{-} B} \qquad \frac{\vdash [\Theta], \Gamma, A, B}{\vdash [\Theta], \Gamma, A \vee^{-} B}$$

$$\frac{\vdash [\Theta], \Gamma, A}{\vdash [\Theta], \Gamma, \forall x A} \qquad \frac{\mapsto [\Theta], A \quad \mapsto [\Theta], B}{\mapsto [\Theta], A \wedge^{+} B}$$

$$\frac{\mapsto [\Theta], A_i}{\mapsto [\Theta], A_1 \vee^{+} A_2} \qquad \frac{\mapsto [\Theta], A[t/x]}{\mapsto [\Theta], \exists x A}$$

Figure 1: The Focused Proof System LKF

a positive formula or a negative literal, Θ consists of positive formulas and negative literals, and x is not free in Θ, Γ. Focused and unfocused sequents have the form $\mapsto [\Theta], A$ and $\vdash [\Theta], \Gamma$, respectively.

The additive and multiplicative versions of conjunction and disjunction are available in LKF. In particular, the top-level logical symbol determines this reading: $\wedge^{-}$ and $\vee^{+}$ are additive while $\wedge^{+}$ and $\vee^{-}$ are multiplicative. The difference between the two conjunctions and two disjunctions lies in the focused proofs that they admit: they are, however, provability equivalent.

The main result about the LKF proof system is that it is sound and complete for classical, first order logic. Clearly, if a sequent proof has a focused proof, it has a LK proof (in the sense of [**?**]): one simply deletes the additional, focusing-oriented syntax in the sequents and inference rules. The more surprising thing is the following completeness proof (proved in [3]).

Theorem 1 (Completeness for LKF)**.** *If a sequent is provable in LK (equivalently, if it is classically valid) and if we polarize the atomic formulas, disjunctions, and conjunction in that sequent arbitrarily, then the resulting sequent must have an LKF proof.*

3 Herbrand's theorem

To illustrate the utility of the LKF proof system, we shall that Herbrand's Theorem is an immediate consequence of the completeness of LKF.

Theorem 2 (Herbrand's Theorem)**.** *Let B be a quantifier-free formula all of whose free variables are in the list of variables $\bar{x} = x_1, \ldots, x_n$ ($n \geq 0$). If $\exists \bar{x}.B$ is valid then there exists substitutions $\theta_1, \ldots, \theta_m$ on the domain $\bar{x}$ such that $B\theta_1 \wedge \cdots \wedge B\theta_m$ is a tautology.*

$$\frac{\Omega;\Delta,L^{\perp}\vdash\Phi,L\wedge C\quad \Omega;\Delta,L\vdash\Phi,L\wedge C}{L,\Omega;\Delta\vdash\Phi,L\wedge C}\ Split$$

$$\frac{\Omega;\Delta,L^{\perp}\vdash\Phi}{L^{\perp},\Omega;\Delta\vdash\Phi,L}\ Assert$$

$$\frac{\Omega;\Delta,L^{\perp}\vdash\Phi}{\Omega;\Delta,L^{\perp}\vdash\Phi,L\wedge C}\ Subsume$$

$$\frac{}{\Omega;\Delta,L\vdash\Phi,L}\ Empty$$

$$\frac{\Omega;\Delta,L\vdash\Phi,C}{\Omega;\Delta,L\vdash\Phi,L\wedge C}\ Resolve$$

Figure 2: The DPLL rules. Ω is just an alias to express literals present in the sequent but not in Δ.

Proof. Before using any focused proof system, such as LKF, we must polarize the connectives of B: for this application, we choose the asynchronous propositional connectives $\wedge^-$ and $\vee^-$ (the polarity of literals must also be picked but that choice is not important in this example). Assume that $\exists\bar{x}.B$ is valid. By the completeness of LKF (Theorem 1), there is an LKF proof Ξ of $\exists\bar{x}.B$.

Every occurrence of the *Focus* rule in Ξ has a conclusion of the form $\vdash[\exists\bar{x}.B,\mathcal{L}]$, for where $\mathcal{L}$ is some multiset of literals. The premise of such a rule has one of the following two forms.

1. The premise is a sequent such as $\mapsto[\exists\bar{x}.B,\mathcal{L}]L$, where L is a literal whose complement occurs in $\mathcal{L}$. In that case, this premise is immediately proved using an ID rule.

2. The premise is of the form $\mapsto[\exists\bar{x}.B,\mathcal{L}]\exists\bar{x}.B$. This sequent is the conclusion of a synchronous phase that starts with a sequent of the form $\mapsto[\exists\bar{x}.B,\mathcal{L}]B\theta$. This later sequent is itself the conclusion of an asynchronous decomposition phase, all of whose premises are of the form $\vdash[\exists\bar{x}.B,\mathcal{L},\mathcal{L}']$, for some set of literal $\mathcal{L}'$.

Let $\theta_1,\ldots,\theta_m$ be the set of substitutions that are attached to such synchronous phases. It is now immediate to transform Ξ to a proof Ξ' of $\vdash B\theta_1\vee^+\cdots\vee^+B\theta_m$ in such a way that whenever the formula $\exists\bar{x}.B$ is selected and reduced to $B\theta_i$ within Ξ, the corresponding synchronous phase in Ξ' focuses on $B\theta_1\vee^+\cdots\vee^+B\theta_m$ and similarly reduces that formula to $B\theta_i$ (via the introduction rules for $\vee^+$). All other aspects of the proof Ξ are left unchanged in Ξ'. $\square$

4 DPLL and focusing

4.1 The DPLL algorithme

Theorem 3 (soundness)**.** *If* $\Omega;\Delta\vdash\Phi$ *then* $\Delta\supset\Phi$ *is true in classical logic.*

4.2 LKF with polarization rules

Soundness

Proposition 4.1 (soundness of LKF^p). *LKF^p is sound.*

Proof. The D, R, $\wedge^+$ and $Init$ rules are the one of LKF. The cut rule is admissible since it's just a restriction of the original cut rule. The pol rule is like a decide rule so it's sound. The $elim$ rule is sound, indeed assume we allow focusing (the only rule appliable elsewhere) on the formula instead of the $elim$ rule .

$$\dfrac{\dfrac{\dfrac{\dfrac{(loop)}{O; N, L \vdash \Gamma, L \Uparrow}}{O; N, L \vdash \Gamma, L \Downarrow L}\ R \qquad \ldots}{O; P; N, L \vdash \Gamma, L \Downarrow L \wedge^+ C}\ \wedge^+}{O; N, L \vdash \Gamma, L, L \wedge^+ C \Uparrow}\ D$$

We get a loop, so there no way to use the formula : it can be deleted. So this system is sound. □

Completeness

Proposition 4.2. *In LKF, all macro rules (decomposition of one formula) commute to each other: any restriction on the D rule which fix an order is admissible.*

Proof. In LK all rules are invertible ($init$ and $\wedge$). So we can enforce any order while all formula may be focused on it. Focusing just compell to decompose a whole formula in one phase but let the order between formulas free. □

Proposition 4.3. *LKF^p ensure that there are no rule that cannot be focus on (it fixs an order without excluding any formula).*

Proof. Here the order is fixed by the apparition of polarity on all atoms of the formula. We can focus on a formula completely polarized with D, on one with just one literal unassigned (with pol or $elim$ according to the fact that all literal are positive there are at least on negative). And if any of these cases occur the cut rule increase strictly the number of polarized atoms such that before failure all formula have been focusable. □

4.3 The LKF emulation

Tranlation of process step into sequent We can't give a one to one coresspondance between a DPLL step and a sequent of LKF. We say that an sequent correspond to a DPLL step if the sequent (not in a focused phase) is

$$O; N \vdash (\bigwedge p_1, \ldots, p_k, o_1, \ldots, o_j), \ldots, (\bigwedge p'_1, \ldots, p'_k, o'_1, \ldots, o'_j), (\bigwedge p_1, \ldots, p'_k, o_1, \ldots, o'_j, n_1, \ldots, n_i) \Uparrow$$

and the DPLL step is

$$O; N \vdash (\overset{+}{\bigwedge} o_1, \ldots, o_j), \ldots, (\overset{+}{\bigwedge} o'_1, \ldots, o'_j)$$

$$\dfrac{l, O; N \vdash \Gamma, l \Uparrow \quad l, O; N \vdash \Gamma, l^{\perp} \Uparrow}{l, O; N \vdash \Gamma \Uparrow}\ \mathit{cut}$$

$$\dfrac{O; N \vdash \Gamma \Downarrow C}{O; N \vdash \Gamma, C \Uparrow}\ D$$

$$\dfrac{O; N, l \vdash \Gamma, l \Uparrow}{O; N, l \vdash \Gamma \Downarrow l}\ R$$

$$\dfrac{O; N \vdash \Gamma \Downarrow C \quad O; N \vdash \Gamma \Downarrow C}{O; N \vdash \Gamma \Downarrow C \wedge^{+} C'}\ \wedge^{+}$$

$$\dfrac{}{O; N, l^{\perp} \vdash \Gamma, l^{\perp} \Downarrow l}\ \mathit{Init}$$

$$\dfrac{O; N, l_1^{\perp}, \ldots, l_n^{\perp}, l \vdash \Gamma \Downarrow Cl_i \wedge^{+} l}{O, l; N, l_1^{\perp}, \ldots, l_n^{\perp} \vdash \Gamma, Cl_i \wedge^{+} l \Uparrow}\ \mathit{pol}$$

$$\dfrac{O; N, l \vdash l, \Phi}{O; N, l \vdash \Phi, l, C \wedge^{+} l}\ \mathit{elim}$$

Figure 3: LKF^{p} rules. Cut rule is allowed only if all formulas have at least two unassigned litterals; D rule only if all literals are positive.

Assert When we apply the $assert$ rule in DPLL, we have the equivalent rules in LKF:

$$\dfrac{\dfrac{\dfrac{\dfrac{O; N, C_1^{\perp}, C_2^{\perp}, l \vdash \Gamma, c_i, l \Uparrow}{O; N, C_1^{\perp}, C_2^{\perp}, l \vdash \Gamma, c_i \Downarrow l}\ R \quad \dfrac{}{O; N, C_1^{\perp}, C_2^{\perp}, l \vdash \Gamma, c_i \Downarrow C_2}\ \mathit{Init}}{O; N, C_1^{\perp}, C_2^{\perp}, l \vdash \Gamma, c_i \Downarrow C_2 \wedge^{+} l}\ \wedge^{+} \quad \dfrac{}{O; N, C_1^{\perp}, C_2^{\perp}, l \vdash \Gamma, c_i \Downarrow C_1}\ \mathit{Init}}{O; N, C_1^{\perp}, C_2^{\perp}, l \vdash \Gamma, c_i \Downarrow (C_2 \wedge^{+} l) \wedge^{+} C_1}\ \wedge^{+}}{O, l; C_1^{\perp}, C_2^{\perp}, N \vdash \Gamma, c_i, (C_1 \wedge^{+} l) \wedge^{+} C_2 \Uparrow}\ \mathit{pol}$$

Split When we apply the $Split$ rule in DPLL, we have the equivalent rules in LKF:

$$\dfrac{\dfrac{\dfrac{O; N, l \vdash \Gamma, l \Uparrow}{O; N, l \vdash \Gamma, \Downarrow l}\ R}{O, l; N \vdash \Gamma, l \Uparrow}\ \mathit{pol} \quad \dfrac{\dfrac{O; N, l^{\perp} \vdash \Gamma, l \Uparrow}{O; N, l^{\perp} \vdash \Gamma \Downarrow l}\ R}{O, l^{\perp}; N \vdash \Gamma, l^{\perp} \Uparrow}\ \mathit{pol}}{O, l; N \vdash \Gamma \Uparrow}\ \mathit{cut}$$

Subsume Subsume is emulated by the $elim$ rule.

$$\dfrac{O; N, L \vdash \Gamma, L \Uparrow}{O; N, L \vdash \Gamma, L, L \wedge^{+} C \Uparrow}\ \mathit{elim}$$

Empty When we apply the $empty$ rule in DPLL, we have the equivalent rules in LKF:

$$
\cfrac{\cfrac{\cfrac{}{O; N, C_1^\perp, C_2^\perp \vdash \Gamma, c_i \Downarrow C_2}\ Init \quad \cfrac{}{O; N, C_1^\perp, C_2^\perp \vdash \Gamma, c_i \Downarrow C_1}\ Init}{O; N, C_1^\perp, C_2^\perp \vdash \Gamma, c_i \Downarrow C_2 \wedge^+ C_1}\ \wedge^+}{O; C_1^\perp, C_2^\perp, N \vdash \Gamma, c_i, C_1 \wedge^+ C_2 \Uparrow}\ D
$$

Resolve They are no rules for $resolve$: this rule is captured by setting polarity.

Conclusion Each step of DPLL may be emulated by LKF. We need, now, to be sure there are no behavior from LKF that doesn't correspond to DPLL.

4.4 Strict behavior

We need to ensure that there are no consecutive unfocused sequents that does not correspond to LKF. There several big step possible in this system :

The cut rule If we applie the cut rule, we have the assumption that all formulas have at least two unassigned atoms. So after the cut rule, it's not possible to use one of the $elim$,pol or D rule except on the new atomic formulas. And because of the new formulas, the cut rule cannot be applied again. The only choice is to apply a pol rule on each atomic formula. And then realease focus.

The pol rule We have the propertie: if a litteral is negtive, it's present inside the context. So once a pol rule is applied, there just the Assert scheme available (just one branch left open and the other closed).

The D rule If the D allows a successfull focus phase, it means all literals have polarity and positive one have they complement present inside the context (and in our case it's always the case). Because we enforce the D rule to be applied only on completely positive formulas, the proof always finish as in the $Empty$ scheme.

The $elim$ rule Trivial.

Completeness of DPLL LKF^p is complete and emulate all rules of DPLL and just these rules, so DPLL is complete.

5 Rewriting proofs

Functions are relations. Is it possible to use a proof system for relations to actually achieve familiar rewriting proofs that are based, instead, on function and terms?

If functions are taken as primitive (as in denotational semantics), then relations / non-determinism is treated as set-valued functions. If relations are primitive, then functions are seen as just a kind of relation. The restriction on relations that yields functions (ie, functionality) also has an immediate, focusing-related counterpart....

Translation of rewriting term into logic formula We can define a rewriting rule with this grammar :

$$rw := X \to Y$$
$$X := f(X, \dots)|cst|var$$
$$Y := f(X, \dots)|cst|var from X$$

So , in certain way, a rewriting rule is a transformation of syntactic tree. We will use linear logic to translate each node of the tree into a literal. A rewriting rule is translated this way :

$$\mathcal{T}(X \to Y) = \forall r \Pi_2(\mathcal{F}(X, r)) \, (\Pi_1(\mathcal{F}(X, r)) \multimap \exists \Pi_2(\mathcal{F}(X), r) \Pi_1 \mathcal{F}(X, r))$$
$$\mathcal{F}(f(X, \dots), k) = (\Pi_1 \mathcal{F}(X, k_1) \otimes \dots \otimes f(k_1, \dots, r); \Pi_2 \, (\mathcal{F}(X, k_1)) \dots, k_1, \dots, r)$$

Proposition 5.1. *If we have*

$$X \to^* Y$$

then we have

$$!(\mathcal{T}(rw_1), \mathcal{T}(rw_2), \dots) \otimes \Pi_1(X, r) \vdash \exists \Pi_2(Y, r) \Pi_1(Y, r)$$

Proposition 5.2. *All neutral sequents got in the proof of*

$$!(\mathcal{T}(rw_1), \mathcal{T}(rw_2), \dots) \otimes \Pi_1(X, r) \vdash \exists \Pi_2(Y, r) \Pi_1(Y, r)$$

Have the shape :

$$!(\mathcal{T}(rw_1), \mathcal{T}(rw_2), \dots) \otimes \Pi_1(Z, r) \vdash \exists \Pi_2(Y, r) \Pi_1(Y, r)$$

Where Z is such that $X \to Z$.

6 More...

References

[1] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *J. of Logic and Computation*, 2(3):297–347, 1992.

[2] Olivier Delande and Dale Miller. A neutral approach to proof and refutation in MALL. In F. Pfenning, editor, *23th Symp. on Logic in Computer Science*, pages 498–508. IEEE Computer Society Press, 2008.

[3] Chuck Liang and Dale Miller. Focusing and polarization in intuitionistic logic. In J. Duparc and T. A. Henzinger, editors, *CSL 2007: Computer Science Logic*, volume 4646 of *LNCS*, pages 451–465. Springer, 2007. Extended version to appear in TCS.

[4] Vivek Nigam and Dale Miller. Focusing in linear meta-logic. In *Proceedings of IJCAR: International Joint Conference on Automated Reasoning*, volume 5195 of *LNAI*, pages 507–522. Springer, 2008.