Combining Intuitionistic and Classical Logic: a proof system and semantics

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Prelude

Given the roles of classical and intuitionistic logics in computational logic (not to mention formalized mathematics, philosophy, etc), it seems that merging these two logics should be the *holy grail*.

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Why the delays in addressing this merger?

A partial answer: In many computational settings, weak fragments of logic are used and these are often same when using classical, intuitionistic, and (sometimes) linear logics.

Prelude: Examples of weak logical settings

Databases, reachability problems, logic programming generally exploit *Horn clauses*

$$\forall \bar{x}.[A_1 \wedge \cdots \wedge A_n \supset A_0] \quad (A_i' s \text{ atomic})$$

$$H_1, \ldots, H_m \vdash_C H_0 \quad \text{if and only if} \quad H_1, \ldots, H_m \vdash_I H_0$$

This is also true for linear logic when Horn clause are encoded as

$$! \, \forall \bar{x}.[A_1 \otimes \cdots \otimes A_n \multimap A_0]$$

Prelude: Examples of weak logical settings

Weak arithmetic theories are also used in computational logic.

- Horn clauses can be used to define least fixed points.
- Model checking, bisimulation checking, winning strategies, and property-based testing use least and greatest fixed points.

Most of these tasks can be formulated using fixed points embedded into logic such that

$$\begin{array}{lcl} \vdash B \text{ in Peano arithmetic} & \longrightarrow & \vdash B \text{ in Heyting arithmetic} \\ & \longrightarrow & \vdash B \text{ in } \mu\text{MALL}. \end{array}$$

Thus, many computational tasks are not affected by ecumenical concerns since they do not promote one dogma over another.

LJ is a classical-linear hybrid

Gentzen defined sequents in LJ as sequents in LK with *0 or 1 formulas on the rhs.*

If we redefine Gentzen's $\neg B$ as $B \supset f$ (where f is additive false), then Gentzen's restriction can be changed to *exactly one formula on the rhs.*

Thus the left-hand side can use contraction and weakening (a classical context) but the right-hand side can use neither (a linear context).

Another observation from linear logic

A common minimal (intuitionistic) logic

$$\top, \&, \forall, \Rightarrow$$

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Adding $\bot$ yields all of (classical) linear logic

$$\top, \&, \forall, \Rightarrow, \multimap, \bot$$

$$B \parr C \equiv (B \multimap \bot) \multimap C$$
$$?\,B \equiv (B \multimap \bot) \Rightarrow \bot$$
$$!\,B \equiv (B \Rightarrow \bot) \multimap \bot$$

Two logics or one?

Clearly these are two different logics: wars have been fought over classical vs non-classical foundations for mathematics.

Both semantics and proof theory illustrate the special nature of the intuitionistic implication (and universal quantification).

$$\mathcal{M}, u \models A \supset B \text{ if forall } u \leq v. \mathcal{M}, v \models A \text{ implies } \mathcal{M}, v \models B.$$

Enforce single-conclusion on left-introduction (Gentzen).

$$\frac{\Gamma_1 \longrightarrow A, \Delta_1 \qquad \Gamma_2, \longrightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \supset B \longrightarrow \Delta_1, \Delta_2} \supset L \quad \text{and} \quad \Delta_1 = \emptyset$$

Enforce single-conclusion on right-introduction.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B, \Delta} \supset R$$

Previous work (as of 2013)

$C \longrightarrow I$ via double negation translations.

$I \longrightarrow C$ via the addition of a modal operator.

Linear logic can encode $A \supset B$ as either $!A \multimap B$ (intuitionistic) or as $!A \multimap ?B$ (classical).

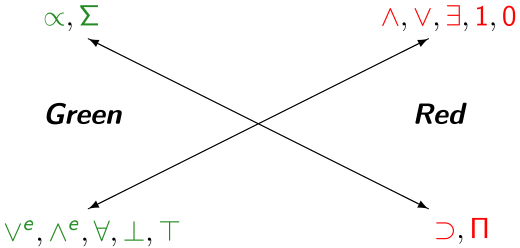
Girard's LU logic [Girard 1993; Vauzeilles 1993] includes linear logic. Maybe too ambitious.

"Fibred Semantics and the Weaving of Logics", Gabbay JSL 1996.

"Combining Classical and Intuitionistic Implications," Caleiro & Ramos, FroCos 2007.

PIL: Polarized Intuitionistic Logic

Red-Polarized: $\wedge$, 1, $\vee$, 0, $\exists$, $\supset$, Π. (Syntactic variable R)
Green-Polarized: $\wedge^e$, $\top$, $\vee^e$, $\bot$, $\forall$, $\propto$, Σ. (Syntactic variable E)



Purely intuitionistic connectives: $\supset$, Π, $\propto$ and Σ
Classically-oriented connectives: $\vee$, $\wedge$, $\exists$, $\vee^e$, $\wedge^e$ and $\forall$

The formulas of PIL

Atomic formulas are (arbitrarily) classified as red. A negated $(-)^{\perp}$ atom is, thus, green.

$(B)^{\perp}$ is the negation normal form of the De Morgan dual of B.

De Morgan dualities are:

$$1/\bot,\ 0/\top,\ \supset/\propto,\ \Pi/\Sigma,\ \vee/\wedge^{e},\ \wedge/\vee^{e},\ \exists/\forall.$$

$A^{\perp\perp}$ and A are a syntactic identical for all formulas A.

The dual of $A \supset B$ is $A \propto B^{\perp}$, and not $A^{\perp} \propto B^{\perp}$.

Classic negation $A^{\perp}$ flips between green and red.

Intuitionistic negation $\sim A := A \supset 0$ is always a red formula.

The LP Sequent Calculus

We use two-sided sequents although the use of colors makes a one-sided sequent calculus possible.

We use the symbols $\vdash_\circ$ and $\vdash_\bullet$ to represent two modes of proof.

In all rules, Γ and Θ are multisets of formulas, E is a *green* formula, R is a *red* formula, and a is any atom.

The sequent $\Gamma \vdash_\circ A$ is interpreted as $\bigwedge\Gamma \supset A$.
The sequent $\Gamma \vdash_\bullet \Theta$ is interpreted as $\bigwedge\Gamma \supset \bigvee^e\Theta$.
(If Δ is empty, then $\bigwedge\Delta$ is 1 and $\bigvee^e\Delta$ is $\bot$.)

Proofs end with sequents of the form $\Gamma \vdash_\circ A$ (A is any color).

A is a theorem of PIL if $\vdash_\circ A$ is provable.

The LP Sequent Calculus: proof rules

Red Introduction Rules

$$\frac{\Gamma \vdash_\circ A \quad \Gamma \vdash_\circ B}{\Gamma \vdash_\circ A \wedge B}\ \wedge R \qquad \frac{\Gamma \vdash_\circ A_i}{\Gamma \vdash_\circ A_1 \vee A_2}\ \vee R \qquad \frac{A, \Gamma \vdash_\circ B}{\Gamma \vdash_\circ A \supset B}\ \supset R$$

$$\frac{A, B, \Gamma \vdash_\circ R}{A \wedge B, \Gamma \vdash_\circ R}\ \wedge L \qquad \frac{A, \Gamma \vdash_\circ R \quad B, \Gamma \vdash_\circ R}{A \vee B, \Gamma \vdash_\circ R}\ \vee L \qquad \frac{A \supset B, \Gamma \vdash_\circ A \quad B, \Gamma \vdash_\circ R}{A \supset B, \Gamma \vdash_\circ R}\ \supset L$$

$$\frac{}{\Gamma \vdash_\circ 1}\ 1R \qquad \frac{\Gamma \vdash_\circ R}{1, \Gamma \vdash_\circ R}\ 1L \qquad \frac{}{0, \Gamma \vdash_\circ R}\ 0L$$

Green Introduction Rules (these are right-rules only)

$$\frac{\Gamma \vdash_\bullet A \quad \Gamma \vdash_\bullet B}{\Gamma \vdash_\bullet A \wedge^e B}\ \wedge^e R \qquad \frac{\Gamma \vdash_\bullet A, B}{\Gamma \vdash_\bullet A \vee^e B}\ \vee^e R \qquad \frac{\Gamma \vdash_\circ A \quad \Gamma \vdash_\bullet B}{\Gamma \vdash_\bullet A \propto B}\ \propto R$$

$$\frac{\Gamma \vdash_\bullet}{\Gamma \vdash_\bullet \bot}\ \bot R \qquad \frac{}{\Gamma \vdash_\bullet \top}\ \top R$$

Structural Rules and Identity

$$\frac{\Gamma \vdash_\bullet E}{\Gamma \vdash_\circ E}\ \mathit{Signal} \qquad \frac{A^\perp, \Gamma \vdash_\bullet \Theta}{\Gamma \vdash_\bullet A, \Theta}\ \mathit{Store} \qquad \frac{A^\perp, \Gamma \vdash_\circ A}{A^\perp, \Gamma \vdash_\bullet}\ \mathit{Load} \qquad \frac{}{a, \Gamma \vdash_\circ a}\ \mathit{Init}$$

A version of the double negation shift

Define $\neg B$ as $B \supset \perp$.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{
              \dfrac{
                \dfrac{
                  \dfrac{
                    \dfrac{
                      \dfrac{A \vdash_\circ A}{A, A^\perp \vdash_\bullet}\;\textit{Load}
                    }{A, A^\perp \vdash_\circ \perp}\;\textit{Signal},\ \perp I
                  }{A^\perp \vdash_\circ \neg A}\;\supset R
                  \qquad
                  \dfrac{}{\vdash_\bullet 1}\;\textit{Store}, \textit{Load}, 1R
                }{A^\perp \vdash_\circ \neg A \propto 1}\;\propto R
              }{A^\perp \vdash_\circ \exists x.(\neg A \propto 1)}\;\exists R
            }{A^\perp, \forall x.\neg\neg A \vdash_\bullet}\;\textit{Load}
          }{\forall x.\neg\neg A \vdash_\bullet A}\;\textit{Store} \;\leftarrow\; \text{LJ gets stuck here}
        }{\forall x.\neg\neg A \vdash_\bullet \forall x.A}\;\forall R
      }{\forall x.\neg\neg A \vdash_\circ \forall x.A}\;\textit{Signal}
      \qquad \vdash_\bullet 1
    }{\forall x.\neg\neg A \vdash_\circ \forall x.A \propto 1}\;\propto R
  }{\forall x.\neg\neg A, \neg\forall x.A \vdash_\bullet}\;\textit{Load}
}{
  \dfrac{
    \dfrac{\forall x.\neg\neg A, \neg\forall x.A \vdash_\circ \perp}{\forall x.\neg\neg A \vdash_\circ \neg\neg\forall x.A}\;\supset R
  }{\vdash_\circ \forall x.\neg\neg A \supset \neg\neg\forall x.A}\;\supset R
}\;\textit{Signal},\ \perp R
$$

The intuitionistic fragment

If the formula A contains only red connectives and positive atoms, then the only LP proofs of $\vdash_{\circ} A$ are essentially the cut-free LJ proofs of Gentzen.

Overview of the LC proof system: polarities

The classical fragment of the LP is essentially Girard's LC proof system for classical logic [APAL 1993].

In LC, every formula is polarized as either *positive* or *negative*.

Atoms are positive. De Morgan duals flip polarities.

Compound (propositional) formulas are given their polarities as follows:

A	B	$A \wedge B$	$A \vee B$	$A \supset B$
+	+	+	+	-
-	+	+	-	+
+	-	+	-	-
-	-	-	-	-

Overview of the LC proof system: sequents

Sequents of LC are one sided sequents $\vdash \Gamma; \Delta$ where Γ and Δ are multisets of formulas and Δ is either empty or a singleton.

When Δ is the singleton S, then S is the *stoup* of $\vdash \Gamma; \Delta$.

Weakening and contraction are available in the Γ context. Here, P and Q are positive and N is negative.

$$\frac{}{\vdash \neg P; P}\ \textit{initial} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma, P;}\ \textit{dereliction}$$

$$\frac{\vdash \Gamma; P \quad \vdash \Delta, N;}{\vdash \Gamma, \Delta; P \wedge N} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma; P \vee Q} \qquad \frac{\vdash \Gamma; Q}{\vdash \Gamma; P \vee Q}$$

$$\frac{\vdash \Gamma, A, B; \Delta}{\vdash \Gamma, A \vee B; \Delta} \text{ where } A \vee B \text{ is negative}$$

The classical fragment of LP is LC

Drop the intuitionistic connectives $\supset$, $\propto$, Π and Σ.
There are two copies of conjunction and disjunction: $\vee$, $\wedge$, $\vee^e$, $\wedge^e$.

Positive formulas are red-polarized and negative ones are green-polarized.

The polarity of an LC formula is also dependent on the polarity of its subformulas. When A and B are both positive, $A \vee B$ in LC corresponds to $A \vee B$ in PIL; otherwise, it is $A \vee^e B$.

LC sequents with a stoup correspond to the $\vdash_\circ$ while a sequent without a stoup correspond to $\vdash_\bullet$.

LC introduction rules on the stoup formula correspond to right-red introduction rules in LP; the introduction rules for "negative" connectives *in the presence of a stoup* correspond to left-red rules while those without a stoup correspond to right-green rules.

Here, P is positive and N is negative.

$$\frac{\vdash \Gamma, N, P; S}{\vdash \Gamma, N \vee P; S} \quad \longmapsto \quad \frac{\Gamma, P, N \vdash_{\circ} S}{\Gamma, P \wedge N \vdash_{\circ} S}\ \wedge L$$

$$\frac{\vdash \Gamma, N, P;}{\vdash \Gamma, N \vee P;} \quad \longmapsto \quad \frac{\Gamma \vdash_{\bullet} N, P}{\Gamma \vdash_{\bullet} N \vee^{e} P}\ \vee^{e} R$$

$$\frac{\vdash \Gamma_1; P \quad \vdash \Gamma_2, N;}{\vdash \Gamma_1, \Gamma_2; P \wedge N} \quad \longmapsto \quad \frac{\Gamma_1, \Gamma_2 \vdash_{\circ} P \qquad \dfrac{\Gamma_1, \Gamma_2 \vdash_{\bullet} N}{\Gamma_1, \Gamma_2 \vdash_{\circ} N}\ \mathit{Signal}}{\Gamma_1, \Gamma_2 \vdash_{\circ} P \wedge N}\ \wedge R$$

Excluded middle

$$p \vee (p \supset 0) \quad \text{versus} \quad p \vee^{e} p^{\perp}$$

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Peirce's formula is provable in the form

$$((p \supset q) \supset p) \supset p,$$

where $\supset$ is classical implication, defined as $A \supset B = A^{\perp} \vee^{e} B$.

An approach to intermediate logics

Excluded middle

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Markov's principle

$$(\forall n(P(n) \vee \neg P(n))) \supset (\neg \forall n \neg . P(n)) \supset \exists n.P(n)$$

$$[(\Pi x. \sim P(x) \vee \sim (P(x)^\perp)) \supset (\sim \sim \exists x.P(x))] \supset \Sigma x.P(x)$$

A Kripke-style semantics

A terminal world in a Kripke model is a classical worlds: intuitionistic implication collapses into a classical one and the excluded middle becomes valid.

$$
\begin{array}{ccccc}
\vdots & & & & \\
v_3 & & & & \\
\uparrow & & & & \\
v_2 & & c_1 & & c_2 \\
\uparrow & & \nwarrow & & \nearrow \\
v_1 & & & w & \\
& \nwarrow & & \nearrow & \\
& & u & &
\end{array}
$$

The terminal worlds c_1 and c_2 are classical: $c_1 \models p \vee \neg p$.

Worlds beyond classical worlds

We shall allow there to be worlds *beyond* classical worlds.

Such worlds will make *all* classical formulas true (one kind of inconsistency) but not all intuitionistic formulas true.

A world may validate $\bot$ (and, thus, all classical formulas) but never validate 0.

[An analogy from linear logic: for all B, $0 \vdash B$ while $\bot \vdash ?B$.]

Worlds beyond classical worlds will be called *imaginary worlds* (similar in spirit to naming $\sqrt{-1}$ as an imaginary number).

Propositional Kripke hybrid models

A *propositional Kripke hybrid model* is a tuple $\langle W, \preceq, C, \models \rangle$ s.t.

- W is a non-empty Kripke frame of possible worlds.
- $\preceq$ is a transitive and reflexive relation on W.
- C, the set of "classical worlds," is a subset of W.
- $\models$ is a binary relation between elements of W and (red-polarized) atomic formulas.

$\triangle_u = \{k \in C \mid u \preceq k\}$, is the set of classical worlds above u. A world u is *imaginary*, or $\perp$*-inconsistent*, if $\triangle_u$ is empty.

The following conditions must also hold:

- $\models$ is *monotone*: for $u, v \in W$, $u \preceq v$ and $u \models a$ implies $v \models a$.
- $\triangle_k = \{k\}$ for all $k \in C$, i.e., there are no classical worlds properly above other classical worlds.

Defining forcing: red connectives first

The satisfiability or *forcing* relation extends $\models$ from atoms to all propositional formulas by induction on the structure of formulas.

The key idea here is that a green formula is valid in a world u if it is valid in all classical worlds above u.

First, we define the red-polarity cases using the familiar Kripke formulation. Assuming $\mathsf{u}, \mathsf{v} \in \mathsf{W}$, we have:

- $\mathsf{u} \models 1$ and $\mathsf{u} \not\models 0$
- $\mathsf{u} \models A \vee B$ iff $\mathsf{u} \models A$ or $\mathsf{u} \models B$
- $\mathsf{u} \models A \wedge B$ iff $\mathsf{u} \models A$ and $\mathsf{u} \models B$
- $\mathsf{u} \models A \supset B$ iff for all $\mathsf{v} \succeq \mathsf{u}$, $\mathsf{v} \models A$ implies $\mathsf{v} \models B$

Defining forcing: green connectives second

First define forcing of green formulas but only over classical worlds: here, $c \in C$ and $v \in W$.

- $c \models a^\perp$ iff $c \not\models a$ (a atomic).
- $c \models \top$ and $c \not\models \bot$
- $c \models A \vee^e B$ iff $c \models A$ or $c \models B$
- $c \models A \wedge^e B$ iff $c \models A$ and $c \models B$
- $c \models A \propto B$ iff for some $v \succeq c$, $v \models A$ and $v \not\models B^\perp$

Extend $\models$ to all green formulas E in *any* $u \in W$:

- $u \models E$ if and only if for all $c \in \triangle_u$, $c \models E$.

(If $\triangle_u$ is empty, then all green formulas are satisfied in u.)

The $\models$ relation is well-defined: if $u \in C$ then the clauses above defining $\models$ for classical worlds coincide since $\triangle_u = \{u\}$.

Some simple properties about forcing

Let $\mathsf{u}, \mathsf{v} \in \mathsf{W}$, $\mathsf{c} \in \mathsf{C}$, and let A be a (propositional) formula.

a. if $\mathsf{u} \preceq \mathsf{v}$, then $\mathsf{u} \models A$ implies $\mathsf{v} \models A$ (monotonicity)
b. $\mathsf{c} \models A$ iff $\mathsf{c} \not\models A^{\perp}$ (excluded middle)
c. $\mathsf{u} \models A$ and $\mathsf{u} \models A^{\perp}$ for some A iff $\triangle_{\mathsf{u}}$ is empty (u is imaginary).
d. $\mathsf{u} \not\models E$ for some green formula E iff $\triangle_{\mathsf{u}}$ is non-empty.

While 0 and $\perp$ are clearly distinct, 1 and $\top$ are equivalent: they are simply red and green-polarized versions of the same truth value. Red and green formulas can be equivalent:

$$(R \supset \perp) \supset \perp \equiv R \vee^{e} \perp.$$

A model $\mathcal{M}$ *satisfies* A, or $\mathcal{M} \models A$, if $\mathsf{u} \models A$ for every $\mathsf{u} \in \mathsf{W}$. A formula is *valid* if it is satisfied in all models.

A countermodel

The excluded middle, in the form $a \vee^e a^\perp$, is valid.

The formula $\sim a \vee^e \sim\sim a$ is not valid.

$$\begin{array}{ccc} s_1 : \{a, a^\perp\} & & s_2 : \{a^\perp\} \\ & \nwarrow \quad \nearrow & \\ & k : \{a^\perp\} & \end{array}$$

The same model shows that $a \vee^e \sim a$ is also not valid (s_2 is not needed here).

Another countermodel

The formula $(p \wedge^e q) \supset p$ is not valid. A countermodel is:

$$\begin{array}{c} k : \{p, q\} \\ \uparrow \\ s : \{\} \end{array}$$

Although every classical world *above* s satisfies p and q, s does not satisfy p.

The same model shows that several other formulas, including $(p \vee^e q) \supset (p \vee q)$, are not valid.

More generally, $E \supset p$ is never valid for green formulas E.

Conclusion

Other results for PIL

- A presentation using Heyting Algebra.
- Soundness & completeness. Semantic proof of cut elimination.
- Tableau style proof system. Multiple conclusion proof system.
- Decision procedure for propositional fragment.
- Kripke hybrid model semantics for first-order quantification.

Future work

- Extend PIL to arithmetic
- Systematic investigation of various intermediate logics.
- Curry-Howard interpretation, delimited control operators (see LICS 2013).
- Mechanization of proof search (focusing proof systems).