Focusing Gentzen's sequent calculus

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Outline for three hours of lectures

What's new with the sequent calculus?

LKF: A focused version of LK

Some applications of LKF

LJF: A focused version of LJ

Applications of LJF

Outline

What's new with the sequent calculus?

LKF: A focused version of LK

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Applications of LJF

Gentzen 1935: "Investigations into Logical Deduction"

Gentzen was interested in proving the consistency of arithmetic and first-order logic in both classical and intuitionistic logics.

His sequent calculi LJ and LK (for intuitionistic and classical logics, respectively) were central to his success with that project. He also developed some decision procedures.

Ketonen [1944, 2022] pushed further, particularly with LK (classical logic) and established some algorithms for normalizing formulas (CNF), sharpened Hauptsatz, and some independence results.

Early application of the sequent calculus were: consistency results, independence of results from axioms, proof systems for novel logics, harmony, etc.

Recent demands on proof theory

Several demands on proof theory have arisen from computer science since the 1980s.

- Functional programming and the Curry-Howard Correspondence, especially for classical logic.
- Type inference for rich λ-calculi.
- Logic programming and goal-directed search.
- Automated deduction. Contraction-free sequent calculus, cycle detection,
- Term representation, substitution, sharing.

Innovations since Gentzen's 1935 paper

The following advances in proof theory will *not* be touched in these lectures.

- semantics (algebraic / model-theoretic)
- new proof structures (hypersequents, deep inference, proof nets, etc)
- Constructive reasoning, program extraction
- Proof mining, reverse mathematics
- etc

Innovations since Gentzen's 1935 paper

The following advances are the topic of these lectures.

- Lessons learned from linear logic [Girard, 1987]
  - importance of *weakening* and *contraction*
  - distinction between *additive* and *multiplicative* inference rules
  - introduction of the *exponentials* !, ?
  - *polarization*
  - *focused proof systems*
- Two focused sequent calculus proof systems
  - LKF - a focused version of LK
  - LJF - a focused version of LJ
- The completeness of LKF and LJF entails various proof-theoretic results.

How to read an inference rule

An inference rule can be understood in two senses:

1. It takes complete proofs of its premises and builds a complete proof of its conclusion.
2. It describes a way to reduce the attempt to prove its conclusion to attempts to prove its premises.

Both readings of inference rules are, of course, valid. While the former reading is more historical, I will often use the latter.

Invertibility of inference rules

A key observation about an inference rule is whether or not it is invertible: i.e., if the conclusion has a proof then all of its premises must have a proof.

The notion of invertibility did not occur to Gentzen [von Plato, 2009], but does appear in [Ketonen, 1944], where cut elimination is used to prove invertibility of some rules.

e.g., if $\Gamma \vdash A \wedge B, \Delta$ has proof Ξ, it has a proof that introduces $\wedge$.

$$
\dfrac{
  \dfrac{\begin{array}{c}\Xi\\ \Gamma \vdash A \wedge B, \Delta\end{array} \quad \dfrac{\dfrac{}{A \vdash A}\,\mathit{init}}{A \wedge B \vdash A}\,\wedge L}{\Gamma \vdash A, \Delta}\,\mathit{cut}
  \quad
  \dfrac{\begin{array}{c}\Xi\\ \Gamma \vdash A \wedge B, \Delta\end{array} \quad \dfrac{\dfrac{}{B \vdash B}\,\mathit{init}}{A \wedge B \vdash B}\,\wedge L}{\Gamma \vdash B, \Delta}\,\mathit{cut}
}{\Gamma \vdash A \wedge B, \Delta}\,\wedge R
$$

Polarity

One of the lessons learned from linear logic is that invertible is more than a proof-search heuristic. In Linear Logic, we have:

the right introduction of a connective is not invertible
if and only if
the right introduction of the dual connective is invertible!

Terminology: Since duality is involved, a positive/negative distinction seems appropriate.

- positive = not invertible ($\Downarrow$)
- negative = invertible ($\Uparrow$)

Do not confuse with positive or negative subformula occurrences!

Gentzen's inference rules for two-sided sequents

Identity rules

$$\frac{}{B \vdash B}\ init \qquad \frac{\Gamma \vdash \Delta, B \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

Introduction rules

$$\frac{\Gamma, B_i \vdash \Delta}{\Gamma, B_1 \wedge B_2 \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, B \quad \Gamma \vdash \Delta, C}{\Gamma \vdash \Delta, B \wedge C} \qquad \frac{}{\Gamma \vdash \Delta, \boldsymbol{t}}$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, B \vee C \vdash \Delta} \qquad \frac{}{\Gamma, \boldsymbol{f} \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, B_i}{\Gamma \vdash \Delta, B_1 \vee B_2}$$

$$\frac{\Gamma \vdash \Delta, B \quad \Gamma', C \vdash \Delta'}{\Gamma, \Gamma', B \supset C \vdash \Delta, \Delta'} \qquad \frac{\Gamma, B \vdash \Delta, C}{\Gamma \vdash \Delta, B \supset C}$$

$$\frac{\Gamma, Bs \vdash \Delta}{\Gamma, \forall x.Bx \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, By}{\Gamma \vdash \Delta, \forall x.Bx} \qquad \frac{\Gamma, By \vdash \Delta}{\Gamma, \exists x.Bx \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, Bs}{\Gamma \vdash \Delta, \exists x.Bx}$$

Structural rules, zones, LJ vs LK

Structural rules

$$\frac{\Gamma, B, B \vdash \Delta}{\Gamma, B \vdash \Delta}\ cL \quad \frac{\Gamma \vdash \Delta, B, B}{\Gamma \vdash \Delta, B}\ cR \quad \frac{\Gamma \vdash \Delta}{\Gamma, B \vdash \Delta}\ wL \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, B}\ wR$$

In LK: admits cL, wL, cR, wR. That is, the LHS (left-hand side) and RHS (right-hand side) are treated classically.

In LJ: admits only cL, wL. That is, the LHS is treated classically and the RHS is treated linearly.

As a result, every sequent in an LJ proof of $\vdash B$ is a single-conclusion proof: hence, every sequent has exactly one formula on the right.

Observations about these proof rules

- The usual eigenvariable restriction holds for $\forall R$ and $\exists L$.
- First-order quantification is over a first-order terms.
- The structural rule of exchange is built into this presentation. The LHS and RHS are multisets.

- Gentzen's $\neg B$ is replaced with $B \supset \boldsymbol{f}$, allowing us to change "at most one formula on the right" to "exactly one formula on the right."

- Intuitionistic logic is a hybridization of linear and classical logics. The two zones (LHS and RHS) are distinct.
- In classical logic, the distinction between these two zones can be reduced to just one zone (via a one-sided sequent calculus).

Additive versus multiplicative inference rules

An identity or introduction rule is classified as follows:

additive
: every side formula in the conclusion appears in *every* premise.

multiplicative
: every side formula in the conclusion appears in *exactly one* premise.

It is possible for an inference rule to be neither or both (e.g., if there is only one premise).

In Gentzen's LK and LJ the introduction rules for conjunction and disjunction are additive while cut and initial rules and the left implication introduction are multiplicative.

The cost of checking an additive vs a multiplicative rule varies greatly between reading them premise-to-conclusion or vice versa.

Four shortcomings of the sequent calculus

1. The collision of cut and the structural rules
2. Permutations of inference rules
3. Chose either the additive or multiplicative versions of binary inference rules, but not both
4. No provision for synthetic inference rules

1: The collision of cut and the structural rules

Consider the following instance of the cut rule.

$$\frac{\Gamma \vdash C \qquad \Gamma', C \vdash B}{\Gamma, \Gamma' \vdash B}\ cut$$

1: The collision of cut and the structural rules

Consider the following instance of the cut rule.

$$\dfrac{\Gamma \vdash C \qquad \dfrac{\Gamma', C, C \vdash B}{\Gamma', C \vdash B}}{\Gamma, \Gamma' \vdash B}\ cut$$

If the right premise is proved by a left-contraction rule from the sequent $\Gamma', C, C \vdash B$, then permute the *cut* rule to the right:

$$\dfrac{\dfrac{\Gamma \vdash C \qquad \dfrac{\Gamma \vdash C \qquad \Gamma', C, C \vdash B}{\Gamma, \Gamma', C \vdash B}\ cut}{\Gamma, \Gamma, \Gamma' \vdash B}\ cut}{\Gamma, \Gamma' \vdash B}\ cL.$$

1: The collision of cut and the structural rules

Consider the following instance of the cut rule.

$$\dfrac{\dfrac{\Gamma \vdash C, C}{\Gamma \vdash C} \qquad \Gamma', C \vdash B}{\Gamma, \Gamma' \vdash B}\ cut$$

If the left premise is proved by a right-contraction rule from the sequent $\Gamma \vdash C, C$, then permute the *cut* rule to the left:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash C, C \quad \Gamma', C \vdash B}{\Gamma, \Gamma' \vdash C, B}\ cut \qquad \Gamma', C \vdash B}{\Gamma, \Gamma', \Gamma' \vdash B, B}\ cut}{\Gamma, \Gamma' \vdash B}\ cL,\ cR$$

1: The collision of cut and the structural rules

Consider the following instance of the cut rule.

$$\dfrac{\dfrac{\Gamma \vdash C, C}{\Gamma \vdash C} \qquad \dfrac{\Gamma', C, C \vdash B}{\Gamma', C \vdash B}}{\Gamma, \Gamma' \vdash B}\ cut$$

What if both premises are contractions? Cut can *non-deterministically* move to either premises.

In intuitionistic logic, this non-determinism is avoided since contraction on the right is simply forbidden.

1: The collision of cut and the structural rules (continued)

Such nondeterminism in cut-elimination is even more pronounced when we consider the collision of the cut rule with weakening.

$$\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash B}}{\vdash C, B}\ wR \qquad \dfrac{\dfrac{\Xi_2}{\vdash B}}{C \vdash B}\ wL}{\dfrac{\vdash B, B}{\vdash B}\ cR}\ cut$$

Cut-elimination can yield either Ξ_1 or Ξ_2.

This kind of example does not occur in the intuitionistic (single-sided) version of the sequent calculus.

These are often called *Lafont's examples* [Girard et al., 1989].

Polarization will allow us to say something more general.

2. Permutations of inference rules

The following two deviations differ by permuting an inference rule.

$$\dfrac{\dfrac{\Gamma, B_i, C_j \vdash \Delta}{\Gamma, B_i, C_1 \wedge C_2 \vdash \Delta}}{\Gamma, B_1 \wedge B_2, C_1 \wedge C_2 \vdash \Delta} \qquad \dfrac{\dfrac{\Gamma, B_i, C_j \vdash \Delta}{\Gamma, B_1 \wedge B_2, C_j \vdash \Delta}}{\Gamma, B_1 \wedge B_2, C_1 \wedge C_2 \vdash \Delta}$$

These two derivations are different are often considered equal.

Permutation of inference rules is a huge issue in trying to see structure in the sequent calculus.

The existence of such permutations is probably the main reason for the revolt again sequent calculus, giving rise to natural deduction/typed λ-calculi, expansion trees, proof nets, etc.

3. Choose only one among additive or multiplicative rules

Gentzen used the additive versions of conjunction and disjunction.

People in classical logic theorem proving usually use the invertible rules for conjunction and disjunction (which is multiplicative).

Things can then be arranged so that the only non-invertible rule in classical logic is the $\exists$R rule.

Why not allow *both* the additive and multiplicative versions of these rules?

4. No provision for synthetic inference rules

Inference rules in LK are too small. Consider the axiom stating that the predicate *path* is transitive.

$$\forall x \forall y \forall z\, (path\ x\ y \supset path\ y\ z \supset path\ x\ z).$$

Using this axiom involves at least five LK introduction rules. It is more natural to view that formula as yielding an inference rule.

$$\frac{\Gamma \vdash \Delta,\ path\ x\ y \quad \Gamma \vdash \Delta,\ path\ y\ z}{\Gamma \vdash \Delta, path\ x\ z}$$

$$\frac{path\ x\ y,\ path\ y\ z,\ path\ x\ z,\ \Gamma \vdash \Delta}{path\ x\ y,\ path\ y\ z, \Gamma \vdash \Delta}$$

One of these *synthetic rules* might be a more appropriate way to invoke the transitivity axiom.

How can we build such synthetic rules? Can we guarantee cut-elimination holds when we add them?

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Unpolarized formulas

The logical connectives already seen, namely,

$$\boldsymbol{t}, \quad \wedge, \quad \boldsymbol{f}, \quad \vee, \quad \supset, \quad \forall, \quad , \exists$$

are used to build *unpolarized formulas* for both classical and intuitionistic logics.

In two-sided presentations of classical logic and intuitionistic logics, the negation of B is written as $B \supset \boldsymbol{f}$.

In one-sided presentations of classical logic, we restrict negations to have atomic scope: we write $\neg A$ as a primitive connective (where A is atomic).

LKF: polarized formulas

Positive connectives are $\boldsymbol{f}^+$, $\vee^+$, $\boldsymbol{t}^+$, $\wedge^+$, and $\exists$.
Negative connectives are $\boldsymbol{t}^-$, $\wedge^-$, $\boldsymbol{f}^-$, $\vee^-$, and $\forall$.
Literals are atomic formulas and negated atomic formulas.

An *atomic bias assignment* is a function $\delta(\cdot)$ that maps atomic formulas to the set $\{+, -\}$.

Extend $\delta(\cdot)$ to literals: $\delta(\neg A)$ is the opposite polarity of $\delta(A)$.

A polarized formula is *positive* if its top-level connective is positive or it is a literal L and $\delta(L) = +$.

A polarized formula is *negative* if its top-level connective is negative or its a literal L and $\delta(L) = -$.

We require that $\delta(\cdot)$ is *stable* under substitution: $\delta(\theta A) = \delta(A)$.
Thus, $\delta(A)$ is determined by the predicate symbol of A.

LKF: polarized formulas (continued)

Linear logic has other names for the polarized connectives.

	conjunction	true	disjunction	false
multiplicative	$\wedge^+$, $\otimes$	$\boldsymbol{t}^+$, 1	$\vee^-$, $\parr$	$\boldsymbol{f}^-$, $\bot$
additive	$\wedge^-$, $\&$	$\boldsymbol{t}^-$, $\top$	$\vee^+$, $\oplus$	$\boldsymbol{f}^+$, 0

Logical connectives have *four attributes*:

- arity: 0, 1, 2, ...
- variety: additive, multiplicative
- polarity: positive, negative
- junctiveness: conjunction, disjunction.

De Morgan duality flips the last 2 and leaves the first 2 unchanged.

Given any two of variety, polarity, junctiveness, the third is uniquely determined.

LKF: negation normal form

Polarized formulas are in *negation normal form* (nnf), meaning that there are no occurrences of implication $\supset$, and that the negation symbol $\neg$ has only atomic scope.

The negation symbol $\neg$ is extended also to non-atomic polarized formulas.

- $\neg\neg A = A$ for atomic formula A
- $\neg(A \wedge^{+} B) = \neg A \vee^{-} \neg B, \quad \neg(A \vee^{-} B) = \neg A \wedge^{+} \neg B$
- $\neg(A \vee^{+} B) = \neg A \wedge^{-} \neg B, \quad \neg(A \wedge^{-} B) = \neg A \vee^{+} \neg B$
- $\neg\exists x.A = \forall x.\neg A, \quad \neg\forall x.A = \exists x.\neg A$

Delays and polarization

For certain technical reasons, it is useful to have *delays*:

- $\partial_+(B)$ is always positive and equivalent to B.
  - $B \vee^+ \boldsymbol{f}^+$ or $B \wedge^+ \boldsymbol{t}^+$ or $\exists x.B$ where x is bound vacuously, or
  - as a 1-ary version of $\vee^+$ or $\wedge^+$.
- $\partial_-(B)$ is always negative and equivalent to B.
  - $B \vee^- \boldsymbol{f}^-$ or $B \wedge^- \boldsymbol{t}^-$ or $\forall x.B$ where x is bound vacuously, or
  - as a 1-ary version of $\vee^-$ or $\wedge^-$.

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  - as a 1-ary version of $\vee^-$ or $\wedge^-$.

Let B be an unpolarized formula and let $\hat{B}$ be the result of

- annotating occurrences of $\boldsymbol{t}$, $\wedge$, $\boldsymbol{f}$, $\vee$ in B with a $+$ or $-$, and
- insert any number of delays.

If $\delta(\cdot)$ is an atomic bias assignment, then the pair $\langle \delta(\cdot), \hat{B} \rangle$ is a *polarization* of B.

Generally, there is (at least) an exponential number of polarizations of an unpolarized formula.

LKF: sequent

LKF uses the following one-sided sequents with two *zones*:

$$\vdash \Gamma \Uparrow \Theta \quad \text{and} \quad \vdash A \Downarrow \Theta$$

The zones Γ and Θ are multisets of polarized formulas.
A is a polarized formula.

Introductions take place in the zone between $\vdash$ and the $\Uparrow$ or $\Downarrow$.

The zone Θ is called *storage* and has *classical maintenance*, i.e., they admit contraction and weakening.

Those structural rules are implicit by adopting the convention:

> A classical zone is treated *additively* in *multiplicative* rules.

The zone Γ is called the *staging area* and has *linear maintenance*.

LKF: proof rules (without cut)

Negative introduction rules

$$\frac{}{\vdash \boldsymbol{t}^-, \Gamma \Uparrow \Theta} \qquad \frac{\vdash A, \Gamma \Uparrow \Theta \qquad \vdash B, \Gamma \Uparrow \Theta}{\vdash A \wedge^- B, \Gamma \Uparrow \Theta}$$

$$\frac{\vdash \Gamma \Uparrow \Theta}{\vdash \boldsymbol{f}^-, \Gamma \Uparrow \Theta} \qquad \frac{\vdash A, B, \Gamma \Uparrow \Theta}{\vdash A \vee^- B, \Gamma \Uparrow \Theta} \qquad \frac{\vdash [y/x]B, \Gamma \Uparrow \Theta}{\vdash \forall x.B, \Gamma \Uparrow \Theta}$$

Positive introduction rules

$$\frac{}{\vdash \boldsymbol{t}^+ \Downarrow \Theta} \qquad \frac{\vdash A \Downarrow \Theta \qquad \vdash B \Downarrow \Theta}{\vdash A \wedge^+ B \Downarrow \Theta} \qquad \frac{\vdash B_i \Downarrow \Theta}{\vdash B_1 \vee^+ B_2 \Downarrow \Theta} \qquad \frac{\vdash [s/x]B \Downarrow \Theta}{\vdash \exists x.B \Downarrow \Theta}$$

Non-introduction rules

$$\frac{}{\vdash p \Downarrow \neg p, \Theta}\ \mathit{init} \qquad \frac{\vdash N \Uparrow \Theta}{\vdash N \Downarrow \Theta}\ \mathit{release} \qquad \frac{\vdash \Gamma \Uparrow Q, \Theta}{\vdash Q, \Gamma \Uparrow \Theta}\ \mathit{store}$$

$$\frac{\vdash P \Downarrow P, \Theta}{\vdash \cdot \Uparrow P, \Theta}\ \mathit{decide}$$

Here: P is positive, N is negative, Q is positive or a literal, and p is a positive literal.

Observations about LKF proof rules

The polarized formula B has an LKF proof if the sequent $\vdash B \Uparrow \cdot$ has an LKF proof

Storage (the Θ context) is non-decreasing as we move from conclusion to premise.

Key observations:

1. *Contraction* occurs only in the *decide* rule and only for *positive* formulas. A negative formula is never contracted.
2. *Weakening* occurs only at the leaves (in the *init* and $\boldsymbol{t}^+$ rules) and only on *positive formulas* and *negative literals*.

Theorem (Completeness of LKF)

Let B be an unpolarized formula that is provable in LK.
If $\hat{B}$ is any polarization of B then $\hat{B}$ has an LKF proof.

Liang & M proved this using a translation into linear logic [2009] and later with a direct proof [2024].

The central dichotomies of focused proof systems

When reading sequent calculus rules from conclusion to premises:

rule application	invertible	vs	non-invertible
oracle	no information	vs	essential information
non-determinism	don't care	vs	don't know
phase	negative $\Uparrow$	vs	positive $\Downarrow$

Andreoli [1992] used the terms asynchronous/synchronous terminology: these are used less in recent years.

The polarity of linear logic connectives is *unambiguous*. In classical and intuitionistic logic, there are some ambiguities.

The structure of (cut-free) focused proofs

A sequent of the form $\vdash \cdot \Uparrow \Theta$ is called a *border sequent*.

Such sequents can only be proved by using the *decide* rule.

A *synthetic inference rule* is defined as one occurrence each of the $\Downarrow$ and $\Uparrow$-phases, with border sequents as the conclusion and the premises.

$$
\dfrac{\dfrac{\cdots \qquad \dfrac{\cdots \qquad \dfrac{\dfrac{\cdots \quad \vdash \cdot \Uparrow \Theta' \quad \cdots}{\vdash N \Uparrow \cdots}\ \text{neg phase}}{\vdash N \Downarrow \cdots}\ \textit{release} \qquad \cdots}{\vdash \cdots \Downarrow \cdots}\ \text{pos phase} \qquad \cdots}{\vdash P \Downarrow \Theta}}{\vdash \cdot \Uparrow \Theta}\ \textit{decide}\ P \in \Theta
$$

The $\Downarrow$-phase is multiplicative. The $\Uparrow$-phase is additive.

Application of LKF: Two proof systems

The LKneg proof system is based on invertible inference rules.

$$\frac{\vdash B \mid \cdot}{\vdash B}\ \mathit{start} \qquad \frac{\vdash \Gamma \mid \Delta, L}{\vdash L, \Gamma \mid \Delta}\ \mathit{store} \qquad \frac{}{\vdash \cdot \mid \Delta, L, \neg L}\ \mathit{init}$$

$$\frac{\vdash \Gamma \mid \Delta}{\vdash \boldsymbol{f}, \Gamma \mid \Delta} \qquad \frac{\vdash B, C, \Gamma \mid \Delta}{\vdash B \vee C, \Gamma \mid \Delta} \qquad \frac{}{\vdash \boldsymbol{t}, \Gamma \mid \Delta} \qquad \frac{\vdash B, \Gamma \mid \Delta \quad \vdash C, \Gamma \mid \Delta}{\vdash B \wedge C, \Gamma \mid \Delta}$$

Here, L denotes a literal.

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The LKneg proof system is based on invertible inference rules.

$$\frac{\vdash B \mid \cdot}{\vdash B}\ \textit{start} \qquad \frac{\vdash \Gamma \mid \Delta, L}{\vdash L, \Gamma \mid \Delta}\ \textit{store} \qquad \frac{}{\vdash \cdot \mid \Delta, L, \neg L}\ \textit{init}$$

$$\frac{\vdash \Gamma \mid \Delta}{\vdash \boldsymbol{f}, \Gamma \mid \Delta} \qquad \frac{\vdash B, C, \Gamma \mid \Delta}{\vdash B \vee C, \Gamma \mid \Delta} \qquad \frac{}{\vdash \boldsymbol{t}, \Gamma \mid \Delta} \qquad \frac{\vdash B, \Gamma \mid \Delta \quad \vdash C, \Gamma \mid \Delta}{\vdash B \wedge C, \Gamma \mid \Delta}$$

Here, L denotes a literal.

The LKpos proof system is based on non-invertible rules.

$$\frac{\vdash B \mid \cdot \mid B}{\vdash B}\ \textit{start} \qquad \frac{\vdash B \mid \mathcal{N}, \neg A \mid B}{\vdash \neg A \mid \mathcal{N} \mid B}\ \textit{restart} \qquad \frac{}{\vdash A \mid \mathcal{N}, \neg A \mid B}\ \textit{init}$$

$$\frac{\vdash B_i \mid \mathcal{N} \mid B}{\vdash B_1 \vee B_2 \mid \mathcal{N} \mid B} \qquad \frac{}{\vdash \boldsymbol{t} \mid \mathcal{N} \mid B} \qquad \frac{\vdash B_1 \mid \mathcal{N} \mid B \quad \vdash B_2 \mid \mathcal{N} \mid B}{\vdash B_1 \wedge B_2 \mid \mathcal{N} \mid B}$$

Completeness of both systems follow immediately from the completeness of LKF.

Proof sizes can vary greatly: consider $(p \vee C) \vee \neg p$.

The cut rule for LKF

The *cut* rule operates on $\Uparrow$ sequents.

$$\frac{\vdash B \Uparrow \Theta \quad \vdash \neg B \Uparrow \Theta'}{\vdash \cdot \Uparrow \Theta, \Theta'}\ cut$$

During the proof of cut-elimination, the following four variants of the cut rule need to be considered and eliminated as well.

$$\frac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ cut_u \qquad \frac{\vdash A \Downarrow \Theta \quad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_f$$

$$\frac{\vdash \Gamma \Uparrow \Theta, P \quad \vdash \neg P, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u \qquad \frac{\vdash B \Downarrow \Theta, P \quad \vdash \neg P \Uparrow \Theta'}{\vdash B \Downarrow \Theta, \Theta'}\ dcut_f$$

Here, A and B are arbitrary polarized formulas and P is a positive polarized formula.

Outline of completeness proof

1. Prove that all four cuts are admissible.
2. Prove the admissibility of the general *init* rule.
3. Prove some generalized invertibility lemmas.
4. Embed Gentzen's LK into LKF by choosing an appropriate polarization.
5. Prove that all LK rules are admissible in LKF.

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Applications of LKF: Admissibility of *cut* in LK

Theorem

The cut rule for LK is admissible in the cut-free fragment of LK.

Follows immediately from the meta-theory of LKF.

Applications of LKF: Lafont's examples disappear

In all occurrences of the *cut* rule in LKF,

$$\frac{\vdash B \Uparrow \Theta \quad \vdash \neg B \Uparrow \Theta'}{\vdash \cdot \Uparrow \Theta, \Theta'}\ cut$$

exactly one of B and $\neg B$ is negative and one is positive. Hence, contraction is available only for one of these (the positive one) and not both.

Application of LKF: Synthetic inference rules

Let Θ contain the negated and polarized transitivity axiom:

$$\exists x \exists y \exists z.(path\ x\ y \wedge^{+} path\ y\ z \wedge^{+} \neg path\ x\ z)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\Xi_1}{\vdash path\ r\ s \Downarrow \Theta} \qquad \cfrac{\Xi_2}{\vdash path\ s\ t \Downarrow \Theta} \qquad \cfrac{\Xi_3}{\vdash \neg path\ r\ t \Downarrow \Theta}}{\vdash path\ r\ s \wedge^{+} path\ s\ t \wedge^{+} \neg path\ r\ t \Downarrow \Theta}\ \wedge^{+} \times 2}{\vdash \exists x \exists y \exists z.(path\ x\ y \wedge^{+} path\ y\ z \wedge^{+} \neg path\ x\ z) \Downarrow \Theta}\ \exists \times 3}{\vdash \cdot \Uparrow \Theta}\ decide$$

The shape of Ξ_1, Ξ_2, and Ξ_3 depends on the polarity of the *path* predicate.

Application of LKF: Synthetic inference rules (continued)

If path-atoms are negative, then Ξ_1 and Ξ_2 end with the *release* and *store* rules while the proof Ξ_3 is trivial. This synthetic rule is

$$\frac{\vdash \cdot \Uparrow \mathit{path}\ r\ s, \Theta \qquad \vdash \cdot \Uparrow \mathit{path}\ s\ t, \Theta}{\vdash \cdot \Uparrow \mathit{path}\ r\ t, \Theta}$$

If path atoms are positive, then Ξ_3 end with the *release* and *store* rules while the proof Ξ_1 and Ξ_2 are trivial. This synthetic rule is

$$\frac{\vdash \cdot \Uparrow \neg \mathit{path}\ r\ s, \neg \mathit{path}\ s\ t, \neg \mathit{path}\ r\ t, \Theta}{\vdash \cdot \Uparrow \neg \mathit{path}\ r\ s, \neg \mathit{path}\ s\ t, \Theta}$$

These synthetic inference rules are the one-sided version of the *back-chaining* and *forward-chaining* rules displayed earlier (see [Chaudhuri et al., 2008b]).

Cut-elimination holds when synthetic inference rules are added [Marin et al., 2022].

Application of LKF: Herbrand's theorem

The formula $\exists\bar{x}.B$ is provable if and only if there are substitutions $\theta_1, \ldots, \theta_m$ ($m \geq 1$) such that $B\theta_1 \vee \cdots \vee B\theta_m$ is provable.

Let $\hat{B}$ be a polarized version of B in which all propositional connectives in B are polarized negatively. Since $\exists\bar{x}.B$ is provable, the sequent $\vdash \exists\bar{x}.\hat{B} \Uparrow \cdot$ and $\vdash \cdot \Uparrow \exists\bar{x}.\hat{B}$ must have LKF proofs.

Let C be $\hat{B}\theta_1 \vee^+ \ldots \vee^+ \hat{B}\theta_m$ where θ_i is the i^{th} instantiate of $\exists\bar{x}.B$ in that LKF proof. (C may contain $\vee^-$ and $\vee^+$.)

$$\dfrac{\dfrac{\vdash \hat{B}\theta_i \Uparrow \exists\bar{x}.\hat{B}, \mathcal{L}}{\vdash \hat{B}\theta_i \Downarrow \exists\bar{x}.\hat{B}, \mathcal{L}}\ \textit{release}}{\vdash \exists\bar{x}.\hat{B} \Downarrow \exists\bar{x}.\hat{B}, \mathcal{L}}\ \exists \times n \qquad \Longrightarrow \qquad \dfrac{\dfrac{\vdash \hat{B}\theta_i \Uparrow C, \mathcal{L}}{\vdash \hat{B}\theta_i \Downarrow C, \mathcal{L}}\ \textit{release}}{\vdash \hat{B}\theta_1 \vee^+ \ldots \vee^+ \hat{B}\theta_m \Downarrow C, \mathcal{L}}\ \vee^+$$

Except for the details *inside* the $\Downarrow$-phase, these proofs are *identical*.

Application of LKF: Hosting other proof systems

The LKQ and LKT proof systems of [Danos et al., 1995] can be seen as LKF proofs in which the following polarization functions are used. Here, A ranges over atomic formulas.

LKT	LKQ
Atoms are negative	Atoms are positive
$(A)^l = \neg A$	$(A)^l = \neg A$
$(A)^r = A$	$(A)^r = A$
$(B \supset C)^l = (B)^r \wedge^+ (C)^l$	$(B \supset C)^l = (B)^r \wedge^+ \partial_-((C)^l)$
$(B \supset C)^r = (B)^l \vee^- \partial_+((C)^r)$	$(B \supset C)^r = \partial_+((B)^l \vee^- (C)^r)$

Cut-free proofs in LKT (resp, LKQ) of B correspond to LKF proofs of $(B)^r$ using the LKT (resp, LKQ) definition.

Gentzen's LK proof system can also be hosted inside LKF by using lots of delays.

Variants of focusing in classical logic: multifocusing

Positive introduction rules

$$\frac{}{\vdash \boldsymbol{t}^+ \Downarrow \Gamma} \qquad \frac{\vdash B_1, \Theta_1 \Downarrow \Gamma \qquad \vdash B_2, \Theta_2 \Downarrow \Gamma}{\vdash B_1 \wedge^+ B_2, \Theta_1, \Theta_2 \Downarrow \Gamma} \qquad \frac{\vdash B_i, \Theta \Downarrow \Gamma}{\vdash B_1 \vee^+ B_2, \Theta \Downarrow \Gamma}\ i \in \{1, 2\}$$

Release and decide rules

$$\frac{\vdash \Delta \Uparrow \Gamma}{\vdash \Delta \Downarrow \Gamma}\ \mathit{release}^\dagger \qquad \frac{\vdash \Delta \Downarrow \Delta, \Gamma}{\vdash \cdot \Uparrow \Delta, \Gamma}\ \mathit{decide}^\ddagger$$

Proviso $\dagger$: Δ consists of only negative formulas.
Proviso $\ddagger$: Δ is a non-empty multiset of positive formulas.

We have argued that *maximal multifocused* (MMF) proofs provide a *canonical* proof representation.

[Chaudhuri et al., 2008a] proof nets and MMF proofs in MALLF.
[Chaudhuri et al., 2016] expansion trees and MMF proofs in LKF.

Variants of focusing in classical logic: MALLF

Simple changes to four rules of LKF yields MALLF, a focused proof system for the MALL fragment of linear logic, first proposed in [Andreoli, 1992].

Change the storage from classical to linear maintenance.

This change in maintenance forces the following rules to change as well.

$$\frac{}{\vdash \boldsymbol{t}^+ \Downarrow \cdot} \qquad \frac{\vdash A \Downarrow \Theta_1 \quad \vdash B \Downarrow \Theta_2}{\vdash A \wedge^+ B \Downarrow \Theta_1, \Theta_2}$$

$$\frac{}{\vdash p \Downarrow \neg p}\ init \qquad \frac{\vdash P \Downarrow \Theta}{\vdash \cdot \Uparrow P, \Theta}\ decide$$

Outline

What's new with the sequent calculus?

LKF: A focused version of LK

Some applications of LKF

LJF: A focused version of LJ

Applications of LJF

LJF: two-sided sequents

$$\Gamma \Uparrow \Theta \vdash \Delta \Uparrow \Delta' \qquad \Gamma \Downarrow \Theta \vdash \Delta \Downarrow \Delta'$$

All four *zones* Γ, Θ, Δ, and Δ' are multisets of polarized formulas. The multiset union $\Delta \cup \Delta'$ is always a singleton.

Γ and Δ' are called the left and right *storage zones*.
Θ and Δ are called the left and right *staging zones*.

$\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta$ are called *border sequents*: these sequents form the conclusion and premises of synthetic inference rules.
Notation conventions

- drop $\cdot \Downarrow$ and $\cdot \Uparrow$ when they appear on the right,
- drop $\Downarrow \cdot$ and $\Uparrow \cdot$ when they appear on the left.
- Thus, $\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E$ can be written as $\Gamma \vdash E$. Border sequents in LJF resemble sequents in LJ.

LJF: the logical connectives

Positive connectives: $\wedge^+$, $\boldsymbol{t}^+$, $\vee^+$, $\boldsymbol{f}^+$, $\exists$.
Negative connectives: $\wedge^-$, $\boldsymbol{t}^-$, $\supset$, $\forall$.

The only ambiguous logical connectives in intuitionistic logic is conjunction and its unit: $\wedge^-$, $\boldsymbol{t}^-$, $\wedge^+$, $\boldsymbol{t}^+$.

There is only one disjunction and its unit: $\vee^+$, $\boldsymbol{f}^+$.

Atomic formulas are also ambiguous. We employ atomic bias assignments, $\delta(\cdot)$ with LJF as well.

Notation: The RHS can be simplified as follows: write $B \Downarrow \cdot$ as simply $B \Downarrow$ and write $B \Uparrow \cdot$ as simply $B \Uparrow$.

LJF: the invertible introduction rules

Right introduction rules

$$\frac{\Gamma \Uparrow \Theta \vdash B_1 \Uparrow \qquad \Gamma \Uparrow \Theta \vdash B_2 \Uparrow}{\Gamma \Uparrow \Theta \vdash B_1 \wedge^- B_2 \Uparrow} \qquad \frac{}{\Gamma \Uparrow \Theta \vdash \boldsymbol{t}^- \Uparrow}$$

$$\frac{\Gamma \Uparrow B_1, \Theta \vdash B_2 \Uparrow}{\Gamma \Uparrow \Theta \vdash B_1 \supset B_2 \Uparrow} \qquad \frac{\Gamma \Uparrow \Theta \vdash [y/x]B \Uparrow}{\Gamma \Uparrow \Theta \vdash \forall x.B \Uparrow}$$

Left introduction rules

$$\frac{\Gamma \Uparrow B_1, B_2, \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow B_1 \wedge^+ B_2, \Theta \vdash \Delta_1 \Uparrow \Delta_2} \qquad \frac{\Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow \boldsymbol{t}^+, \Theta \vdash \Delta_1 \Uparrow \Delta_2}$$

$$\frac{\Gamma \Uparrow B_1, \Theta \vdash \Delta_1 \Uparrow \Delta_2 \qquad \Gamma \Uparrow B_2, \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow B_1 \vee^+ B_2, \Theta \vdash \Delta_1 \Uparrow \Delta_2} \qquad \frac{}{\Gamma \Uparrow \boldsymbol{f}^+, \Theta \vdash \Delta_1 \Uparrow \Delta_2}$$

$$\frac{\Gamma \Uparrow [y/x]B, \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow \exists x.B, \Theta \vdash \Delta_1 \Uparrow \Delta_2}$$

Here, B ranges over arbitrary polarized formulas.

LJF: the non-invertible introduction rules

Right introduction rules

$$\frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \vdash B_2 \Downarrow}{\Gamma \vdash B_1 \wedge^+ B_2 \Downarrow} \qquad \frac{}{\Gamma \vdash \boldsymbol{t}^+ \Downarrow}$$

$$\frac{\Gamma \vdash B_i \Downarrow}{\Gamma \vdash B_1 \vee^+ B_2 \Downarrow} \qquad \frac{\Gamma \vdash [t/x]B \Downarrow}{\Gamma \vdash \exists x.B \Downarrow}$$

Left introduction rules

$$\frac{\Gamma \Downarrow B_i \vdash D}{\Gamma \Downarrow B_1 \wedge^- B_2 \vdash D} \qquad \frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \Downarrow B_2 \vdash D}{\Gamma \Downarrow B_1 \supset B_2 \vdash D} \qquad \frac{\Gamma \Downarrow [t/x]B \vdash D}{\Gamma \Downarrow \forall x.B \vdash D}$$

In the introduction rules for $\vee_r^+$ and $\wedge_l^-$, i is either 1 or 2.

LJF: Release, Store, Decide, and Initial rules

$$\frac{C, \Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma \Uparrow C, \Theta \vdash \Delta_1 \Uparrow \Delta_2}\ \mathit{storeL} \qquad \frac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow D}{\Gamma \Uparrow \cdot \vdash D \Uparrow \cdot}\ \mathit{storeR}$$

$$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow D}{\Gamma \Downarrow P \vdash D}\ \mathit{releaseL} \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \cdot}{\Gamma \vdash N \Downarrow}\ \mathit{releaseR}$$

$$\frac{\Gamma, N \Downarrow N \vdash D}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow D}\ \mathit{decideL} \qquad \frac{\Gamma \vdash P \Downarrow}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P}\ \mathit{decideR}$$

$$\frac{}{\Gamma \Downarrow N_a \vdash N_a}\ \mathit{initL} \qquad \frac{}{\Gamma, P_a \vdash P_a \Downarrow}\ \mathit{initR}$$

P is positive, P_a is a positive atom.
N is negative, N_a is a negative atom.
C is a negative formula or positive atom.
D is a positive formula or negative atom.

Observations about LJF

We say that the polarized formula B has an LJF proof if the sequent $\cdot \Uparrow \cdot \vdash B \Uparrow \cdot$ has an LJF proof.

Storage (the Γ context) is non-decreasing as we move from conclusion to premise.

Key observations:

1. *Contraction* occurs only in the *decideL* rule and only for *negative* formulas.
2. *Weakening* occurs only at the leaves (in the *init* and $\boldsymbol{t}^+$ rules) and only on *negative formulas* and *positive literals*.

Theorem (Completeness of LJF)

Let B be an unpolarized formula that is provable in LJ.
If $\hat{B}$ is any polarization of B then $\hat{B}$ has an LJF proof.

Liang & M [2009] prove this using a translations into linear logic.

Outline

What's new with the sequent calculus?

LKF: A focused version of LK

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LJF: A focused version of LJ

Applications of LJF

Additional applications of LJF

Some applications to the proof-theory of intuitionistic logic.

- Synthetic connectives
- Forward and backward chaining
- Term representations
- Proof of Harrop's theorem
- Completeness of G3i and G4ip

Some applications to relating classical and intuitionistic logic.

- Barr's theorem

Synthetic inference rules

A *left synthetic inference rule* for B is an inference rule of the form

$$\frac{\Gamma_1 \Uparrow \vdash \Uparrow A_1 \quad \dots \quad \Gamma_n \Uparrow \vdash \Uparrow A_n}{\Gamma \Uparrow \vdash \Uparrow A}\ B$$

justified by a derivation (in LJF) of the form

$$\cfrac{\cfrac{\begin{array}{c}\Gamma_1 \Uparrow \cdot \vdash \cdot \Uparrow A_1 \dots \Gamma_n \Uparrow \cdot \vdash \cdot \Uparrow A_n \\ \vdots \Uparrow \\ \vdots \Downarrow \end{array}}{\Gamma \Downarrow B \vdash A}}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow A}\ \textit{decideL},\ B \in \Gamma$$

There may be multiple synthetic rule for a given formula B.

Two definitions

We name two specific *atomic bias assignments*:

- $\delta^{-}(A) = -$ for all atomic A.
- $\delta^{+}(A) = +$ for all atomic A.

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- $\delta^-(A) = -$ for all atomic A.
- $\delta^+(A) = +$ for all atomic A.

The *order of a formula* is defined as follows:

- $ord(B) = 0$ if B is atomic or $\boldsymbol{t}$ or $\boldsymbol{f}$
- $ord(B \supset C) = \max(ord(B) + 1, ord(C))$
- $ord(B \wedge C) = ord(B \vee C) = \max(ord(B), ord(C))$
- $ord(\forall x.B) = ord(\exists x.B) = ord(B)$

For example, $ord(a \supset (b \supset c)) = 1$ and $ord((a \supset b) \supset c) = 2$.

Horn clauses have order 0 or 1.

The order of polarized formulas is defined analogously.

Axioms as rules

Let $\mathcal{T}$ be a finite set of polarized formulas of order 1 or 2. Let δ be an atomic bias assignment.

$\mathrm{LJ}\lfloor\delta, \mathcal{T}\rfloor$ extends LJ with the left synthetic inference rules for $\mathcal{T}$: if $B \in \mathcal{T}$ then the synthetic rule

$$\frac{B, \Gamma_1 \Uparrow \cdot \vdash \cdot \Uparrow A_1 \quad \cdots \quad B, \Gamma_n \Uparrow \cdot \vdash \cdot \Uparrow A_n}{B, \Gamma \Uparrow \cdot \vdash \cdot \Uparrow A}\ B$$

justifies the rule

$$\frac{\Gamma_1 \vdash A_1 \quad \cdots \quad \Gamma_n \vdash A_n}{\Gamma \vdash A}\ B$$

which is added to $\mathrm{LJ}\lfloor\delta, \mathcal{T}\rfloor$.

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$$\frac{B, \Gamma_1 \Uparrow \cdot \vdash \cdot \Uparrow A_1 \quad \cdots \quad B, \Gamma_n \Uparrow \cdot \vdash \cdot \Uparrow A_n}{B, \Gamma \Uparrow \cdot \vdash \cdot \Uparrow A}\ B$$

justifies the rule

$$\frac{\Gamma_1 \vdash A_1 \quad \cdots \quad \Gamma_n \vdash A_n}{\Gamma \vdash A}\ B$$

which is added to LJ$\lfloor \delta, \mathcal{T} \rfloor$.

Theorem

$\mathcal{T}, \Gamma \Uparrow \cdot \vdash \cdot \Uparrow A$ is provable in LJ if and only if $\Gamma \vdash A$ is provable in LJ$\lfloor \delta, \mathcal{T} \rfloor$.

For related work, see Negri and von Plato [1998].

Cut-elimination for LJ$\lfloor \delta, \mathcal{T} \rfloor$

The following theorem states that cut is admissible for the extensions of LJ with polarized theories based on synthetic inference rules.

Theorem (Cut admissibility for LJ$\lfloor \delta, \mathcal{T} \rfloor$)

Let $\mathcal{T}$ be a finite polarized theory of order 2 or less. Then the cut rule is admissible for the proof system LJ$\lfloor \delta, \mathcal{T} \rfloor$.

The proof is given by Marin, M, Pimentel, & Volpe in [2022] for both LJF and LKF.

Back-chaining and Forward-chaining

Let $\mathcal{T}$ be the collection of formulas

$$D_1 = a_0 \supset a_1, \; D_2 = a_0 \supset a_1 \supset a_2, \; \cdots, \; D_n = a_0 \supset \cdots \supset a_n, \; \cdots$$

where a_i are atomic.

Back-chaining and Forward-chaining

Let $\mathcal{T}$ be the collection of formulas

$$D_1 = a_0 \supset a_1,\ D_2 = a_0 \supset a_1 \supset a_2,\ \cdots,\ D_n = a_0 \supset \cdots \supset a_n,\ \cdots$$

where a_i are atomic.

Back-chaining: The inference rules in $\mathrm{LJ}\lfloor \delta^-, \mathcal{T} \rfloor$ include

$$\frac{\Gamma \vdash a_0 \quad \cdots \quad \Gamma \vdash a_{n-1}}{\Gamma \vdash a_n}$$

Back-chaining and Forward-chaining

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$$D_1 = a_0 \supset a_1,\ D_2 = a_0 \supset a_1 \supset a_2,\ \cdots,\ D_n = a_0 \supset \cdots \supset a_n,\ \cdots$$

where a_i are atomic.

Back-chaining: The inference rules in $\mathrm{LJ}\lfloor \delta^-, \mathcal{T} \rfloor$ include

$$\frac{\Gamma \vdash a_0 \ \cdots \ \Gamma \vdash a_{n-1}}{\Gamma \vdash a_n}$$

Forwardchaining: The inference rules in $\mathrm{LJ}\lfloor \delta^+, \mathcal{T} \rfloor$ include

$$\frac{\Gamma, a_0, \cdots, a_{n-1}, a_n \vdash A}{\Gamma, a_0, \cdots, a_{n-1} \vdash A}$$

Back-chaining and Forward-chaining

What are the proofs of $a_0 \vdash a_n$ using synthetic rules?

Back-chaining and Forward-chaining

What are the proofs of $a_0 \vdash a_n$ using synthetic rules?

When a_i are all given the negative bias, we have:

$$\frac{\Gamma \vdash a_0}{\Gamma \vdash a_1} \quad \frac{\Gamma \vdash a_0 \quad \Gamma \vdash a_1}{\Gamma \vdash a_2} \quad \cdots \quad \frac{\Gamma \vdash a_0 \quad \cdots \quad \Gamma \vdash a_{n-1}}{\Gamma \vdash a_n} \quad \cdots$$

The *unique* proof of $a_0 \vdash a_n$ has **exponential** size.

Back-chaining and Forward-chaining

What are the proofs of $a_0 \vdash a_n$ using synthetic rules?

When a_i are all given the negative bias, we have:

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The *unique* proof of $a_0 \vdash a_n$ has **exponential** size.

When a_i are all given the positive bias, we have:

$$\frac{\Gamma, a_0, a_1 \vdash A}{\Gamma, a_0 \vdash A} \qquad \frac{\Gamma, a_0, a_1, a_2 \vdash A}{\Gamma, a_0, a_1 \vdash A} \qquad \cdots \qquad \frac{\Gamma, a_0, \ldots, a_{n-1}, a_n \vdash A}{\Gamma, a_0, \ldots, a_{n-1} \vdash A}$$

There are an infinite number of proofs.
The *smallest* proof of $a_0 \vdash a_n$ has **linear** size.

Term representation: Annotating rules and proofs

Now we annotate the inference rules in the previous example.

$$\frac{\Gamma \vdash a_0}{\Gamma \vdash a_1} \qquad \frac{\Gamma \vdash a_0 \quad \Gamma \vdash a_1}{\Gamma \vdash a_2} \qquad \cdots$$

$$\frac{\Gamma \vdash a_0 \quad \ldots \quad \Gamma \vdash a_{n-1}}{\Gamma \vdash a_n}$$

Term representation: Annotating rules and proofs

Now we annotate the inference rules in the previous example.

$$\frac{\Gamma \vdash t_0 : a_0}{\Gamma \vdash E_1\, t_0 : a_1} \qquad \frac{\Gamma \vdash t_0 : a_0 \quad \Gamma \vdash t_1 : a_1}{\Gamma \vdash E_2\, t_0\, t_1 : a_2} \qquad \cdots$$

$$\frac{\Gamma \vdash t_0 : a_0 \quad \ldots \quad \Gamma \vdash t_{n-1} : a_{n-1}}{\Gamma \vdash E_n\, t_0 \cdots t_{n-1} : a_n}$$

Term representation: Annotating rules and proofs

Now we annotate the inference rules in the previous example.

$$\frac{\Gamma \vdash t_0 : a_0}{\Gamma \vdash E_1 t_0 : a_1} \qquad \frac{\Gamma \vdash t_0 : a_0 \quad \Gamma \vdash t_1 : a_1}{\Gamma \vdash E_2 t_0 t_1 : a_2} \qquad \cdots$$

$$\frac{\Gamma \vdash t_0 : a_0 \quad \ldots \quad \Gamma \vdash t_{n-1} : a_{n-1}}{\Gamma \vdash E_n t_0 \cdots t_{n-1} : a_n}$$

Consider the proofs of $d : a_0 \vdash t : a_4$.

Term representation: Annotating rules and proofs

Now we annotate the inference rules in the previous example.

$$\frac{\Gamma \vdash t_0 : a_0}{\Gamma \vdash E_1 t_0 : a_1} \qquad \frac{\Gamma \vdash t_0 : a_0 \quad \Gamma \vdash t_1 : a_1}{\Gamma \vdash E_2 t_0 t_1 : a_2} \qquad \cdots$$

$$\frac{\Gamma \vdash t_0 : a_0 \quad \ldots \quad \Gamma \vdash t_{n-1} : a_{n-1}}{\Gamma \vdash E_n t_0 \cdots t_{n-1} : a_n}$$

Consider the proofs of $d : a_0 \vdash t : a_4$. The term t is

$$\begin{array}{l}(E_4\ (E_3\ (E_2\ (E_1\ d)\ (E_1\ d))\\ \qquad\qquad (E_2\ (E_1\ d)\ (E_1\ d)))\\ \qquad (E_3\ (E_2\ (E_1\ d)\ (E_1\ d))\\ \qquad\qquad (E_2\ (E_1\ d)\ (E_1\ d))))\end{array}$$

Sharing of subterms is not supported.

Term representation: Annotating rules and proofs

Now we annotate the inference rules in the previous example.

$$\frac{\Gamma, a_0, a_1 \vdash A}{\Gamma, a_0 \vdash A} \quad \frac{\Gamma, a_0, a_1, a_2 \vdash A}{\Gamma, a_0, a_1 \vdash A} \quad \dots$$

$$\frac{\Gamma, a_0, \cdots, a_{n-1}, a_n \vdash A}{\Gamma, a_0, \cdots, a_{n-1} \vdash A}$$

Consider the proofs of $a_0 \vdash a_4$.

Term representation: Annotating rules and proofs

Now we annotate the inference rules in the previous example.

$$\frac{\Gamma, x_0 : a_0, x_1 : a_1 \vdash t : A}{\Gamma, x_0 : a_0 \vdash F_1 x_0(\lambda x_1.t) : A} \quad \frac{\Gamma, x_0 : a_0, x_1 : a_1, x_2 : a_2 \vdash t : A}{\Gamma, x_0 : a_0, x_1 : a_1 \vdash F_2 x_0 x_1(\lambda x_2.t) : A} \quad \ldots$$

$$\frac{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1}, x_n : a_n \vdash t : A}{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1} \vdash F_n x_0 \cdots x_{n-1}(\lambda x_n.t) : A}$$

Consider the proofs of $d : a_0 \vdash t : a_4$.

Term representation: Annotating rules and proofs

Now we annotate the inference rules in the previous example.

$$\frac{\Gamma, x_0 : a_0, x_1 : a_1 \vdash t : A}{\Gamma, x_0 : a_0 \vdash F_1 x_0 (\lambda x_1 . t) : A} \quad \frac{\Gamma, x_0 : a_0, x_1 : a_1, x_2 : a_2 \vdash t : A}{\Gamma, x_0 : a_0, x_1 : a_1 \vdash F_2 x_0 x_1 (\lambda x_2 . t) : A} \quad \ldots$$

$$\frac{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1}, x_n : a_n \vdash t : A}{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1} \vdash F_n x_0 \cdots x_{n-1} (\lambda x_n . t) : A}$$

Consider the proofs of $d : a_0 \vdash t : a_4$.

The term t annotating the shortest proof is

$$\begin{array}{ll}
(F_1\ d & (\lambda x_1. \\
(F_2\ d\ x_1 & (\lambda x_2. \\
(F_3\ d\ x_1\ x_2 & (\lambda x_3. \\
(F_4\ d\ x_1\ x_2\ x_3 & (\lambda x_4.\ x_4))))))))
\end{array}$$

Term representation: Annotating rules and proofs

Now we annotate the inference rules in the previous example.

$$\frac{\Gamma, x_0 : a_0, x_1 : a_1 \vdash t : A}{\Gamma, x_0 : a_0 \vdash F_1 x_0(\lambda x_1.t) : A} \quad \frac{\Gamma, x_0 : a_0, x_1 : a_1, x_2 : a_2 \vdash t : A}{\Gamma, x_0 : a_0, x_1 : a_1 \vdash F_2 x_0 x_1(\lambda x_2.t) : A} \quad \ldots$$

$$\frac{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1}, x_n : a_n \vdash t : A}{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1} \vdash F_n x_0 \cdots x_{n-1}(\lambda x_n.t) : A}$$

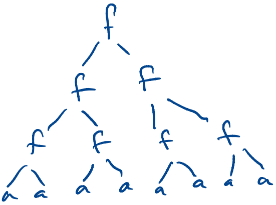
Consider the proofs of $d : a_0 \vdash t : a_4$.

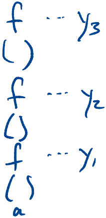
A better syntax might be

$$\begin{array}{ll}
name\ x_1 = (F_1\ d) & in \\
name\ x_2 = (F_2\ d\ x_1) & in \\
name\ x_3 = (F_3\ d\ x_1\ x_2) & in \\
name\ x_4 = (F_4\ d\ x_1\ x_2\ x_3) & in\ \ x_4
\end{array}$$

Sharing of subterms is explicitly supported. See M & Wu [2023].

Another example $f : i \to i \to i$ $a : i$



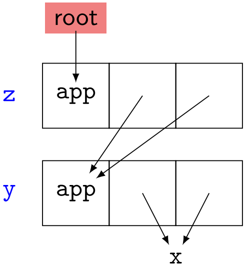


name $y_1 = f a a$ in
name $y_2 = f y_1 y_1$ in
name $y_3 = f y_2 y_2$ in
y_3

name $x_1 = f a a$ in
name $x_2 = f a a$ in
name $x_3 = f x_1 x_2$ in x_2

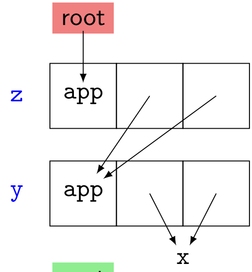
Showing can be <u>redundant</u> and <u>vacuous</u>.

An example

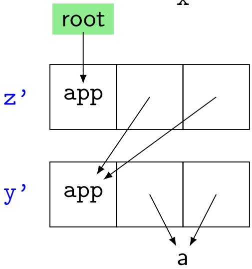



```
name y = app x x in
name z = app y y in
z
```

An example

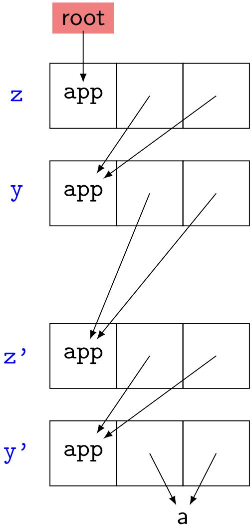



```
name y = app x x in
name z = app y y in
z
```




```
name y' = app a a in
name z' = app y' y' in
z'
```

An example




```
name y = app x x in
name z = app y y in
z
```

```
name y' = app a a in
name z' = app y' y' in
name y = app z' z' in
name z = app y y in z
```

```
name y' = app a a in
name z' = app y' y' in
z'
```

Equality on terms

We have two different formats for untyped λ-terms.

When should two such expressions be considered the same?

- Unfold sharing in two terms, then check as regular terms. Exponential.

Bisimulation on such graphs can be checked in linear time: see A. Condoluci, B. Accattoli, & C. Sacerdoti Coen, *Sharing equality is linear*, *PPDP 2019*.

Harrop's theorem

Harrop formulas are defined as: (A is atomic, B is arbitrary).

$$H := A \mid B \supset H \mid \forall x\, H \mid H_1 \wedge H_2.$$

Polarize atoms and $\wedge$ negatively.

A simple induction proves that if C is a polarized positive formula, then the sequent $\Gamma \Downarrow \hat{H} \vdash C$ is not provable. Let $\mathcal{P}$ be a set of H-formulas.

Theorem

$\mathcal{P} \vdash B_1 \vee B_2$ has LJ proof $\;\Rightarrow\;$ $\mathcal{P} \vdash B_i$ has LJ proof for $i = 1, 2$.
$\mathcal{P} \vdash \exists x.B$ has LJ proof $\;\Rightarrow\;$ $\mathcal{P} \vdash B[t/x]$ has LJ proof for some t.

Proof: By completeness, there is an LJF proof of $\mathcal{P} \Uparrow \cdot \vdash \cdot \Uparrow B_1 \vee B_2$. Since the last inference rule of that proof cannot be *decideL*, it must be *decideR*. Similarly for $\exists$. QED.

G3i proof system of Troelstra and Schwichtenberg [2000]

$$\frac{}{P, \Gamma \vdash P}\ P \text{ atomic} \qquad \frac{}{\boldsymbol{f}, \Gamma \vdash A}$$

$$\frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \vee A_1}\ (i = 0, 1)$$

$$\frac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C} \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B}$$

$$\frac{\forall x.A, A[x/t], \Gamma \vdash C}{\forall x.A, \Gamma \vdash C} \qquad \frac{\Gamma \vdash A[x/y]}{\Gamma \vdash \forall x.A}$$

$$\frac{A[x/y], \Gamma \vdash B}{\exists x.A, \Gamma \vdash B} \qquad \frac{\Gamma \vdash A[x/t]}{\Gamma \vdash \exists x.A}$$

Contraction is built into $\forall L$ and into the left premise of $\supset L$.
If we polarize $\wedge^+$ and atoms positively, then this is almost LJF.

Completeness of G3i

The binary relation $B \preceq C$ is defined on formulas via:

$$\frac{}{C \preceq C} \qquad \frac{C \preceq C_2}{C \preceq C_1 \supset C_2}$$

Intuitively, $B \preceq C$ means that B is the *better choice* to use with *decideL* since a focus on C leads to a focus on B but with other subgoals required.

The admissibility and *invertibility* in LJF of the following *strengthening* rule is easy to prove.

$$\frac{\Gamma, B \Uparrow \cdot \vdash \cdot \Uparrow D}{\Gamma, B, C \Uparrow \cdot \vdash \cdot \Uparrow D}\ B \preceq C$$

An LJF proof with this strengthening rule applied to the right premise of every $\supset L$ rule yields a G3i proof.

Completeness of G4ip

Replace $\supset L$ in G3i with four rules to get G4ip.

$$\frac{B, P, \Gamma \vdash E}{P \supset B, P, \Gamma \vdash E}\ L0\supset$$

$$\frac{C \supset (D \supset B), \Gamma \vdash E}{(C \wedge D) \supset B, \Gamma \vdash E}\ L\wedge\supset$$

$$\frac{C \supset B, D \supset B, \Gamma \vdash E}{(C \vee D) \supset B, \Gamma \vdash E}\ L\vee\supset$$

$$\frac{D \supset B, C, \Gamma \vdash D \quad B, \Gamma \vdash E}{(C \supset D) \supset B, \Gamma \vdash E}\ L\supset\supset$$

Completeness of G4ip: Polarize atoms positive and use $\wedge^{+}$.

1. First rule follows since atomic formulas are positive polarity.
2. Focusing on $(C \wedge^{+} D) \supset B$ or $C \supset (D \supset B)$ are indistinguishable. Same for third rule.
3. When the left context contains C then *decideL* on $(C \supset D) \supset B$ is the same as on $D \supset B$.

Relating classical and intuitionistic logics

Propositional geometric formulas C have the form

$$(p_1 \wedge \cdots \wedge p_n) \supset (q_1 \vee \cdots \vee q_m),$$

where $n, m \geq 0$ and $p_1, \ldots, p_n, q_1, \ldots, q_m$ are atomic.

Theorem

The sequent $C_1, \ldots, C_r \vdash C_0$ is provable in classical logic if and only if it is provable in intuitionistic logic.

Assume that $C_1, \ldots, C_r \vdash C_0$ is provable in LK.
Polarization using $\wedge^+$, $\vee^+$, and atomic formulas positive.
By completeness, $\hat{C}_1, \ldots, \hat{C}_r \vdash \hat{C}_0$ is provable in LKF.
The border sequents in such a proof have the form

$$C_1, \ldots, C_r, p_1, \ldots, p_n \Uparrow \cdot \vdash \cdot \Uparrow q_1 \vee^+ \cdots \vee^+ q_m$$

These are proved using *decideR* on $q_1 \vee^+ \cdots \vee^+ q_m$ or *decideL* on C_i. Thus, we have just two kinds of synthetic rules in this proof.

Apply *decideL* on $(p_1 \wedge \cdots \wedge p_n) \supset (q_1 \vee \cdots \vee q_m)$ yields the synthetic rule

$$\frac{p_1, \ldots, p_n, \Gamma, q_1 \Uparrow \cdot \vdash \cdot \Uparrow \Delta \quad \cdots \quad p_1, \ldots, p_n, \Gamma, q_m \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{p_1, \ldots, p_n, \Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta}$$

Apply *decideR* on $q_1 \vee \cdots \vee q_m$ yields the synthetic rule

$$\frac{}{q_i, \Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta}$$

A simple induction proves that if we start with one formula on the right (i.e., C_0) then all border sequents have exactly one formula on the right. This proof is, thus, an LJF-proof.

The same argument works when clauses are generalized to

$$\forall \bar{x}.[(p_1 \wedge \cdots \wedge p_n) \supset \exists \bar{y}.(q_1 \vee \cdots \vee q_m)]$$

Negation translations vs Polarization

$$\begin{tikzcd}
LK \arrow[r, bend left, "\neg\neg"] \arrow[d, "\text{polarization}"'] & LJ \arrow[d, "\text{fixed polarization}"] \\
LKF \arrow[r, leftrightarrow, "\text{same structure}"'] & LJF
\end{tikzcd}$$

Classical polarizations yield double-negation translations
Chihani, Ilic, M, 2016. Cut-free proofs only.

Conclusion

- Many lessons from linear logic can be applied to classical and intuitionistic logic.
- We have factored some of these lessons into the design of focused proof systems for LJ and LK.
  - flexible polarizations
  - control on contraction and weakening
  - large scale inference rules
- The completeness of LJF and LKF can yield various well known proof-theoretic results.

References

Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *J. of Logic and Computation*, 2(3): 297–347, 1992. doi: 10.1093/logcom/2.3.297.

Kaustuv Chaudhuri, Dale Miller, and Alexis Saurin. Canonical sequent proofs via multi-focusing. In G. Ausiello, J. Karhumäki, G. Mauri, and L. Ong, editors, *Fifth International Conference on Theoretical Computer Science*, volume 273 of *IFIP*, pages 383–396. Springer, September 2008a. doi: 10.1007/978-0-387-09680-3_26.

Kaustuv Chaudhuri, Frank Pfenning, and Greg Price. A logical characterization of forward and backward chaining in the inverse method. *J. of Automated Reasoning*, 40(2-3):133–177, 2008b. doi: 10.1007/s10817-007-9091-0.

Kaustuv Chaudhuri, Stefan Hetzl, and Dale Miller. A multi-focused proof system isomorphic to expansion proofs. *J. of Logic and Computation*, 26(2):577–603, 2016. doi: 10.1093/logcom/exu030.

V. Danos, J.-B. Joinet, and H. Schellinx. LKT and LKQ: sequent calculi for second order logic based upon dual linear decompositions of classical implication. In J.-Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, number 222 in London Mathematical Society Lecture Note Series, pages 211–224. Cambridge University Press, 1995. doi: 10.1017/CBO9780511629150.

Gerhard Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland, Amsterdam, 1935. doi: 10.1007/BF01201353. Translation of articles that appeared in 1934-35. Collected papers appeared in 1969.

Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987. doi: 10.1016/0304-3975(87)90045-4.

Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and Types*. Cambridge University Press, 1989.

Oiva Ketonen. Untersuchungen zum prädikatenkalkul. *Annales Academiae Scientiarum Fennicae, series A, I. Mathematica-physica 23*, 1944.

Oiva Ketonen. *Investigations into the Predicate Calculus*. College Publications, 2022. Ed. by S. Negri and J. von Plato.

Chuck Liang and Dale Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, 2009. doi: 10.1016/j.tcs.2009.07.041.

Chuck Liang and Dale Miller. Focusing Gentzen's LK proof system. In Thomas Piecha and Kai Wehmeier, editors, *Peter Schroeder-Heister on Proof-Theoretic Semantics*, Outstanding Contributions to Logic. Springer, March 2024. URL `https://hal.archives-ouvertes.fr/hal-03457379`.

Sonia Marin, Dale Miller, Elaine Pimentel, and Marco Volpe. From axioms to synthetic inference rules via focusing. *Annals of Pure and Applied Logic*, 173(5):1–32, 2022. doi: 10.1016/j.apal.2022.103091.

Dale Miller and Jui-Hsuan Wu. A positive perspective on term representations. In Bartek Klin and Elaine Pimentel, editors, *31st EACSL Annual Conference on Computer Science Logic (CSL 2023)*, volume 252 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 3:1–3:21, Dagstuhl, Germany, 2023. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi: 10.4230/LIPIcs.CSL.2023.3.

Sara Negri and Jan von Plato. Cut elimination in the presence of axioms. *Bulletin of Symbolic Logic*, 4(4):418–435, 1998. doi: 10.2307/420956.

Jan von Plato. Proof theory of classical and intuitionistic logic. In Leila Haaparanta, editor, *The Development of Modern Logic*. Oxford University Press, 2009.

A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 2nd edition, 2000.