Finding Unity in Computational Logic

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ISCL: International School on Computational Logic
Bertinoro, 11-15 April 2011

Lecture 2: Some slides about sequent calculus.

Simply typed λ-terms

The traditional approach to first-order logic is to first define *terms* and then *formulas*.

In the higher-order logic setting, formulas can appear in terms.

Thus, terms and formulas need to be defined simultaneous: types distinguish terms from formulas.

The simply typed λ-calculus [Church 1940] provides a unifying approach to terms and formulas.

The rules of α and *β-conversions* are used to describe equality and substitution of formulas and terms into formulas and terms.

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Various dependently type λ-calculus have been proposed to unify the notion of term, formula, and proof.

- We do not following this approach since we need a much more flexible and open notion of proof.

Inference rules: structural rules

There are exactly three sets of these: *exchange, contraction, weakening*.

$$\frac{\Sigma : \Gamma', B, C, \Gamma'' \vdash \Delta}{\Sigma : \Gamma', C, B, \Gamma'' \vdash \Delta}\ xL \qquad \frac{\Sigma : \Gamma \vdash \Delta', B, C, \Delta''}{\Sigma : \Gamma \vdash \Delta', C, B, \Delta''}\ xR$$

$$\frac{\Sigma : \Gamma, B, B \vdash \Delta}{\Sigma : \Gamma, B \vdash \Delta}\ cL \qquad \frac{\Sigma : \Gamma \vdash \Delta, B, B}{\Sigma : \Gamma \vdash \Delta, B}\ cR$$

$$\frac{\Sigma : \Gamma \vdash \Delta}{\Sigma : \Gamma, B \vdash \Delta}\ wL \qquad \frac{\Sigma : \Gamma \vdash \Delta}{\Sigma : \Gamma \vdash \Delta, B}\ wR$$

Inference rules: identity rules

There are exactly two: *initial, cut*.

$$\frac{}{\Sigma\colon B \vdash B}\ init \qquad \frac{\Sigma\colon \Gamma_1 \vdash \Delta_1, B \qquad \Sigma\colon B, \Gamma_2 \vdash \Delta_2}{\Sigma\colon \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

Inference rules: identity rules

There are exactly two: *initial*, *cut*.

$$\frac{}{\Sigma\colon B \vdash B}\ init \qquad \frac{\Sigma\colon \Gamma_1 \vdash \Delta_1, B \qquad \Sigma\colon B, \Gamma_2 \vdash \Delta_2}{\Sigma\colon \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

Notice that if we have weakening and exchange on the left and right, then

$$\frac{}{\Sigma\colon B, \Gamma \vdash B, \Delta}\ init$$

is *admissible*.

Inference rules: introduction rules (some examples)

$$\frac{\Sigma\colon B, \Gamma \vdash \Delta}{\Sigma\colon B \wedge C, \Gamma \vdash \Delta}\ \wedge L \qquad \frac{\Sigma\colon C, \Gamma \vdash \Delta}{\Sigma\colon B \wedge C, \Gamma \vdash \Delta}\ \wedge L$$

$$\frac{\Sigma\colon \Gamma \vdash \Delta, B \qquad \Sigma\colon \Gamma \vdash \Delta, C}{\Sigma\colon \Gamma \vdash \Delta, B \wedge C}\ \wedge R$$

$$\frac{\Sigma\colon \Gamma_1 \vdash \Delta_1, B \qquad \Sigma\colon C, \Gamma_2 \vdash \Delta_2}{\Sigma\colon B \supset C, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \supset L \quad \frac{\Sigma\colon B, \Gamma \vdash \Delta, C}{\Sigma\colon \Gamma \vdash \Delta, B \supset C}\ \supset R$$

$$\frac{\Sigma \vdash t : \tau \qquad \Sigma\colon \Gamma, B[t/x] \vdash \Delta}{\Sigma\colon \Gamma, \forall_\tau x\, B \vdash \Delta}\ \forall L \qquad \frac{\Sigma, y : \tau\ :\ \Gamma \vdash \Delta, B[y/x]}{\Sigma\colon \Gamma \vdash \Delta, \forall_\tau x\, B}\ \forall R$$

Permutations of inference rules

$$\dfrac{\dfrac{\Sigma\colon \Gamma, p, r \vdash s, \Delta \qquad \Sigma\colon \Gamma, q, r \vdash s, \Delta}{\Sigma\colon \Gamma, p \vee q, r \vdash s, \Delta}\ \vee L}{\Sigma\colon \Gamma, p \vee q \vdash r \supset s, \Delta}\ \supset R$$

$$\dfrac{\dfrac{\Sigma\colon \Gamma, p, r \vdash s, \Delta}{\Sigma\colon \Gamma, p \vdash r \supset s, \Delta}\ \supset R \qquad \dfrac{\Sigma\colon \Gamma, q, r \vdash s, \Delta}{\Sigma\colon \Gamma, q \vdash r \supset s, \Delta}\ \supset R}{\Sigma\colon \Gamma, p \vee q \vdash r \supset s, \Delta}\ \vee L$$

Permutations of inference rules (continued)

$$\cfrac{\cfrac{\Sigma\colon \Gamma_1, r \vdash \Delta_1, p \qquad \Sigma\colon \Gamma_2, q \vdash \Delta_2, s}{\Sigma\colon \Gamma_1, \Gamma_2, p \supset q, r \vdash \Delta_1, \Delta_2, s}\ \supset\!\mathrm{L}}{\Sigma\colon \Gamma_1, \Gamma_2, p \supset q \vdash \Delta_1, \Delta_2, r \supset s}\ \supset\!\mathrm{R}$$

To switch the order of these two inference rules requires introduction some weakenings and a contraction.

$$\cfrac{\cfrac{\cfrac{\cfrac{\Sigma\colon \Gamma_1, r \vdash \Delta_1, p}{\Sigma\colon \Gamma_1, r \vdash \Delta_1, p, s}\ wR}{\Sigma\colon \Gamma_1 \vdash \Delta_1, p, r \supset s}\ \supset\!\mathrm{R} \qquad \cfrac{\cfrac{\Sigma\colon \Gamma_2, q \vdash \Delta_2, s}{\Sigma\colon \Gamma_2, q, r \vdash \Delta_2, s}\ wL}{\Sigma\colon \Gamma_2, q \vdash \Delta_2, r \supset s}\ \supset\!\mathrm{R}}{\Sigma\colon \Gamma_1, \Gamma_2, p \supset q \vdash \Delta_1, \Delta_2, r \supset s, r \supset s}\ \supset\!\mathrm{L}}{\Sigma\colon \Gamma_1, \Gamma_2, p \supset q \vdash \Delta_1, \Delta_2, r \supset s}\ cR$$