Lecture 5: Normal forms for classical logic proofs
Classical logic is polarized as follow:

1. $B \supset C$ is replaced with $\neg B \lor C$,
2. negations are pushed to the atoms,
3. atoms are assigned bias (either $+$ or $-$), and
4. $\land$, $\lor$, $\top$, and $\bot$ are annotated with either $+$ or $-$.

LKF is a focused, one-sided sequent calculus with the sequents

$$\vdash \Theta \uparrow \downarrow \Gamma \quad \text{and} \quad \vdash \Theta \downarrow B$$

Here, $\Theta$ is a multiset of positive formulas and negative literals, $\Gamma$ is a multiset of formulas, and $B$ is a formula.
LKF: focused proof systems for classical logic

\[ \vdash \Theta \uparrow \Gamma, t^\rightarrow \quad \vdash \Theta \uparrow \Gamma, A \quad \vdash \Theta \uparrow \Gamma, B \]

\[ \vdash \Theta \uparrow \Gamma, f^\rightarrow \quad \vdash \Theta \uparrow \Gamma, A \quad \vdash \Theta \uparrow \Gamma, B \]

\[ \vdash \Theta \uparrow \Gamma, A \land B \quad \vdash \Theta \uparrow \Gamma, A \lor B \]

\[ \vdash \Theta \uparrow \Gamma, A[y/x] \quad \vdash \Theta \uparrow \Gamma, \forall x A \]
LKF: focused proof systems for classical logic

\[
\begin{align*}
\Gamma, A \vdash \Theta & \quad \Gamma, B \vdash \Theta \\
\Gamma, t^- \vdash \Theta & \quad \Gamma, A \wedge^\neg B \vdash \Theta \\
\Gamma \vdash \Theta & \quad \Gamma, A, B \vdash \Theta \\
\Gamma, f^- \vdash \Theta & \quad \Gamma, A \neg \vdash B \vdash \Theta \\
\Gamma, A[y/x] \vdash \Theta & \quad \Gamma, \forall x A \vdash \Theta \\
A \vdash \Theta & \quad B \vdash \Theta \\
\Gamma, A \wedge^+ B \vdash \Theta & \quad \Gamma, A_1 \lor^+ A_2 \vdash \Theta \\
\Gamma, \exists x A \vdash \Theta & \quad \Gamma, P \vdash \Theta \\
\Gamma, P, A \vdash \Theta & \quad \Gamma, C \vdash \Theta \\
\end{align*}
\]
LKF: focused proof systems for classical logic

\[ \vdash \Theta \uparrow \Gamma, t^- \quad \vdash \Theta \uparrow \Gamma, A \quad \vdash \Theta \uparrow \Gamma, B \]

\[ \vdash \Theta \uparrow \Gamma, \neg A \quad \vdash \Theta \uparrow \Gamma, A \land \neg B \quad \vdash \Theta \uparrow \Gamma, \neg B \]

\[ \vdash \Theta \uparrow \Gamma, f^- \quad \vdash \Theta \uparrow \Gamma, A \land B \quad \vdash \Theta \uparrow \Gamma, A \lor B \]

\[ \vdash \Theta \uparrow \Gamma, \forall x A \quad \vdash \Theta \uparrow \Gamma, f^+ \quad \vdash \Theta \uparrow \Gamma, A \lor B \quad \vdash \Theta \uparrow \Gamma, A \land B \]

\[ \vdash \Theta \uparrow \Gamma, \forall A \quad \vdash \Theta \uparrow \Gamma, A \quad \vdash \Theta \uparrow \Gamma, B \]

\[ \vdash \Theta \downarrow A \quad \vdash \Theta \downarrow B \quad \vdash \Theta \downarrow A_i \quad \vdash \Theta \downarrow A[t/x] \]

\[ \vdash \Theta \downarrow t^+ \quad \vdash \Theta \downarrow A \land B \quad \vdash \Theta \downarrow A_1 \lor A_2 \quad \vdash \Theta \downarrow \exists x A \]

**Init**

\[ \vdash \neg P_a, \Theta \downarrow P_a \quad \vdash \Theta, C \uparrow \Gamma \quad \vdash \Theta \uparrow \Gamma, C \quad \vdash \Theta \downarrow N \quad \vdash P, \Theta \uparrow P \]

**Store**

\[ \vdash \Theta \uparrow \Gamma, C \quad \vdash \Theta \downarrow N \quad \vdash \Theta \downarrow N \quad \vdash \Theta \uparrow P \quad \vdash P, \Theta \uparrow P \]

**Release**

**Decide**

- \( P \) positive; \( P_a \) positive literal; \( N \) negative; \( C \) positive formula or negative literal.
The only form of \textit{contraction} is in the \textit{Decide} rule

\[
\frac{\vdash P, \Theta \downarrow P}{\vdash P, \Theta \uparrow}.
\]

The only occurrence of \textit{weakening} is in the \textit{Init} rule.

\[
\frac{\vdash \neg P_a, \Theta \downarrow P_a}{\vdash \neg P_a, \Theta \uparrow}.
\]

Thus: negative non-atomic formulas are treated \textit{linearly}!

Only positive formulas are contracted.
Let $B$ be a first-order logic formula and let $\hat{B}$ result from $B$ by placing $+$ or $-$ on $t$, $f$, $\land$, and $\lor$ (there are exponentially many such placements).

**Theorem.** $B$ is a first-order theorem if and only if $\hat{B}$ has an LKF proof. [Liang & M, TCS 2009]

Thus the different polarizations do not change *provability* but can radically change the *proofs*.
Assume that $\Theta$ contains the formula $a \land^+ b \land^+ \neg c$ and that we have a derivation that decides on this formula.

This derivation is possible iff $\Theta$ is of the form $\neg a, \neg b, \Theta'$. Thus, the "macro-rule" is:

$$\vdash \neg a, \neg b, \neg c, \Theta' \uparrow.$$
Two certificates for propositional logic: negative

Use $\land^-$ and $\lor^-$. Their introduction rules are invertible. The initial “macro-rule” is huge, having all the clauses in the conjunctive normal form of $B$ as premises.

$$
\vdash L_1, \ldots, L_n \downarrow L_i \quad \text{Init}
$$

$$
\vdash L_1, \ldots, L_n \uparrow \quad \text{Decide}
$$

$$
\vdash \cdot \uparrow B
$$

The proof certificate can specify the complementary literals for each premise or it can ask the checker to search for them.

Proof certificates can be tiny but require exponential time for checking.
Two certificates for propositional logic: positive

Use $\land^+$ and $\lor^+$. Sequents are of the form $\vdash B, \mathcal{L} \uparrow \cdot$ and $\vdash B, \mathcal{L} \downarrow P$, where $B$ is the original formula to prove, $P$ is positive, and $\mathcal{L}$ is a set of negative literals.

Macro rules are in one-to-one correspondence with $\phi \in DNF(B)$. Divide $\phi$ into $\phi^-$ (negative literals) and $\phi^+$ (positive literals).

$$\{ \vdash B, \mathcal{L}, N \uparrow \cdot \mid N \in \phi^- \} \quad \vdash B, \mathcal{L} \downarrow B \quad \vdash B, \mathcal{L} \uparrow \cdot \quad \text{provided } \neg \phi^+ \in \mathcal{L}$$

Proof certificates are sequences of members of $DNF(B)$. Size and processing time can be reduced (in response to “cleverness”).
Let $B$ be a quantifier-free first-order formula. $\exists \bar{x}. B$ is valid if and only if there is an $n \geq 1$ and substitutions $\theta_1, \ldots, \theta_n$ such that $B\theta_1 \lor \cdots \lor B\theta_n$ is tautologous.

It is well known that Herbrand’s theory can be proved by a permutation argument based on the completeness of cut-free proofs. Given LKF, this proof is transparent.
Other Possible Applications

*Oracles* as proofs: when there is no choice in searching for a proof, just continue; when there is a choice, the oracle provides information to resolve the choice. Oracles can be small but fragile certificates. Focusing should help to develop a more declarative and robust version of oracles.

*Tables* of lemma (M & Nigam, CSL07): polarities can be used to enforce *re-use* instead of *re-prove*.

There are close links between *games semantics* and logic provided by focused proofs. See Delande, M, & Saurin, Annals of Pure and Applied Logic, 2010.

Mixing polarities might relate to *mixing evaluation strategies* (call-by-name, call-by-value) in functional programming languages.