Focused proof systems for Intuitionistic Logics

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Main reference: Liang & M. Focusing and polarization in linear, intuitionistic, and classical logics. TCS, 2009.

Invertible rules and the asynchronous phase

Some inference rules are *invertible*, *e.g.*,

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash B[y/x]}{\Gamma \vdash \forall x.B}$$

First focusing principle: when proving a sequent, apply invertible rules exhaustively and in any order.

This is the *asynchronous phase* of proof search: if formulas are "processes" in an "environment," then these formulas "evolve" without communications with the environment.

Vocabulary: Here, invertible = asynchronous = negative.

Non-invertible rules and the synchronous phase

Some inference rules are not generally invertible, *e.g.*,

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \qquad \frac{\Gamma \vdash B[t/x]}{\Gamma \vdash \exists x.B}$$

Some *backtracking* is generally necessary within proof search using these inference rules.

Second focusing principle: non-invertible rules are applied in a "chain-like" fashion, focusing on a formula and its synchronous subformulas.

This is the *synchronous phase* of proof search: we will not know if our use of the inference rule is successful without checking with the formula's environment.

Vocabulary: Here, non-invertible = synchronous = positive.

Extending the asyn/syn distinction to atoms

Focusing proof systems generally extend the asyn/syn distinction to atoms.

We shall assume that somehow all atoms are given a *bias*, that is, they are either positive (syn-like) or negative (asyn-like).

A *positive formula* is either a positive atom or has a top-level synchronous connective.

A *negative formula* is either a negative atom or has a top-level asynchronous connective.

Goal-directed search as focusing

In a sequent calculus presentation of logic programming, "backchaining" is described as "focused application of left-rules."

$$\frac{\Gamma \vdash G \qquad \overset{\Xi}{\Gamma, D \vdash A}}{\Gamma, G \supset D \vdash A} \supset L$$

What is the last inference rule in Ξ?

If formulas are over only $\supset$, $\forall$, and if A is atomic, the following restriction is complete: If D is atomic, then $D = A$ and Ξ is initial; otherwise, Ξ ends with an introduction rule for D.

If one selects the left-hand formula

$$\forall \bar{x}_1(G_1 \supset \forall \bar{x}_2(G_2 \supset \cdots \forall \bar{x}_n(G_n \supset A') \ldots))$$

to prove the atom A on the right, then there is a θ such that $A = A'\theta$ and $\Gamma \vdash G_i\theta$ are provable $(i = 1, \ldots, n)$.

Various focusing-like proof system

Uniform proofs [Miller, Nadathur, Scedrov] and *LJT* [Herbelin] permits backward chaining proof.

LLF: Andreoli's focusing proof system for linear logic

LKT/LKQ/LK^η*:* Focusing systems for classical logic [Danos, Joinet, Schellinx]

LJQ [Herbelin] permits forward-chaining proof. *LJQ'* [Dyckhoff, Lengrand] extends it.

λ*RCC* [Jagadeesan, Nadathur, Saraswat] allows mixing forward chaining and backward chaining (in a subset of intuitionistic logic).

LJF (following) allows forward and backward proof in all of intuitionistic logic. LJT, LJQ, λRCC, and LJ are subsystems.

LKF (derived from LJF) provides focusing for all of classical logic.

Backward and Forward Chaining

$$\frac{\Gamma \vdash a \qquad \Gamma, b \vdash G}{\Gamma, a \supset b \vdash G} \; a, b \text{ are atoms, focus on } a \supset b$$

Negative atoms: The right branch is trivial; i.e., $b = G$. Continue with $\Gamma \vdash a$ (backward chaining).
Positive atoms: The left branch is trivial; i.e., $\Gamma = \Gamma', a$. Continue with $\Gamma', a, b \vdash G$ (forward chaining).

Backward and Forward Chaining

$$\frac{\Gamma \longmapsto a \qquad \Gamma, b \longmapsto G}{\Gamma, a \supset b \longmapsto G} \quad a, b \text{ are atoms, focus on } a \supset b$$

Negative atoms: The right branch is trivial; i.e., $b = G$. Continue with $\Gamma \longmapsto a$ (backward chaining).
Positive atoms: The left branch is trivial; i.e., $\Gamma = \Gamma', a$. Continue with $\Gamma', a, b \longmapsto G$ (forward chaining).

Let G be $fib(n, f)$ and let Γ contain $fib(0, 0)$, $fib(1, 1)$, and

$$\forall n \forall f \forall f' [fib(n, f) \supset fib(n+1, f') \supset fib(n+2, f+f')].$$

The nth Fibonacci number is f iff $\Gamma \vdash G$.
If all $fib(\cdot, \cdot)$ are negative then the unique proof is *exponential* in n.
If all $fib(\cdot, \cdot)$ are positive then there are many proofs, with the shortest proof *linear* in n.

The full picture behind focusing

Andreoli (1991) was the first to give a focused proof system for a full logic (linear logic).

The proof system for MALL (multiplicative-additive linear logic) is remarkably elegant and unambiguous.

Some complexity arises from using the exponentials (!, ?): in particular, exponentials terminate focusing phases.

Next, we describe two comprehensive focused proof systems.

- LJF for *intuitionistic logic*
- LKF for *classical logic*

LJF: Annotations

Assign *bias* to all atoms: they are either negative or positive.

Annotate every conjunction $\wedge$ as either $\wedge^{+}$ or $\wedge^{-}$.

Annotations do not effect provability, although the structure of proofs can vary greatly as annotations change.

Positive formulas are among positive atoms and

$$\top,\ \bot,\ A \wedge^{+} B,\ A \vee B, \exists x A.$$

Negative formulas are among negative atoms and

$$A \wedge^{-} B,\ A \supset B,\ \forall x A.$$

LJF: The four different sequents

1. $[\Gamma], \Theta \vdash \mathcal{R}$: an *unfocused sequent,* Γ contains negative formulas and positive atoms and $\mathcal{R}$ represents either a formula R or $[R]$.

2. $[\Gamma] \vdash [R]$: all asynchronous formulas have been decomposed: focus is ready for selection.

3. $[\Gamma] \xrightarrow{B} [R]$: *left-focusing* (the focus is B). Means $\Gamma, B \vdash R$.

4. $[\Gamma] -_B\!\!\rightarrow$: *right-focusing* (the focus is B). Means $\Gamma \vdash B$.

You get a "regular" sequent if you drop the brackets and move the focused formula to either the left or right.

Structural Rules: Decision and Reaction

$$\frac{[N,\Gamma] \xrightarrow{N} [R]}{[N,\Gamma] \vdash [R]}\ Lf \qquad \frac{[\Gamma] -_P\rightarrow}{[\Gamma] \vdash [P]}\ Rf$$

$$\frac{[\Gamma] \vdash N}{[\Gamma] -_N\rightarrow}\ R_r \qquad \frac{[\Gamma], P \vdash [R]}{[\Gamma] \xrightarrow{P} [R]}\ R_l$$

$$\frac{[C,\Gamma], \Theta \vdash \mathcal{R}}{[\Gamma], \Theta, C \vdash \mathcal{R}}\ []_l \qquad \frac{[\Gamma], \Theta \vdash [D]}{[\Gamma], \Theta \vdash D}\ []_r$$

Two forms of the Initial Rule

$$\frac{}{[P,\Gamma] -_P\rightarrow}\ I_r,\ \text{atomic } P \qquad \frac{}{[\Gamma] \xrightarrow{N} [N]}\ I_l,\ \text{atomic } N$$

P is positive; N is negative; C is negative or a positive atom; and D is positive or a negative atom.

Introduction Rules

$$\frac{[\Gamma] \xrightarrow{A_i} [R]}{[\Gamma] \xrightarrow{A_1 \wedge^- A_2} [R]} \wedge^- L \qquad \frac{[\Gamma], \Theta \vdash A \quad [\Gamma], \Theta \vdash B}{[\Gamma], \Theta \vdash A \wedge^- B} \wedge^- R$$

$$\frac{[\Gamma], \Theta, A, B \vdash \mathcal{R}}{[\Gamma], \Theta, A \wedge^+ B \vdash \mathcal{R}} \wedge^+ L \qquad \frac{[\Gamma] -_A\rightarrow \quad [\Gamma] -_B\rightarrow}{[\Gamma] -_{A \wedge^+ B}\rightarrow} \wedge^+ R$$

$$\frac{[\Gamma], \Theta, A \vdash \mathcal{R} \quad [\Gamma], \Theta, B \vdash \mathcal{R}}{[\Gamma], \Theta, A \vee B \vdash \mathcal{R}} \vee L \qquad \frac{[\Gamma] -_{A_i}\rightarrow}{[\Gamma] -_{A_1 \vee A_2}\rightarrow} \vee R$$

Each connective has an asynchronous and a synchronous introduction rule.

Introduction Rules (cont.)

$$\frac{[\Gamma] \; -_{A}\!\rightarrow \qquad [\Gamma] \xrightarrow{B} [R]}{[\Gamma] \xrightarrow{A \supset B} [R]} \supset L \qquad \frac{[\Gamma], \Theta, A \vdash B}{[\Gamma], \Theta \vdash A \supset B} \supset R$$

$$\frac{[\Gamma], \Theta, A \vdash \mathcal{R}}{[\Gamma], \Theta, \exists y A \vdash \mathcal{R}} \exists L^{\dagger} \qquad \frac{[\Gamma] \; -_{A[t/x]}\!\rightarrow}{[\Gamma] \; -_{\exists x A}\!\rightarrow} \exists R$$

$$\frac{[\Gamma] \xrightarrow{A[t/x]} [R]}{[\Gamma] \xrightarrow{\forall x A} [R]} \forall L \qquad \frac{[\Gamma], \Theta \vdash A}{[\Gamma], \Theta \vdash \forall y A} \forall R^{\dagger}$$

(†) As usual, y is not free in the lower sequent.

About the structural rules in LJF

The only form of *contraction* is in the Lf rule

$$\frac{[N,\Gamma] \xrightarrow{N} [R]}{[N,\Gamma] \vdash [R]} \; Lf$$

The only occurrence of *weakening* is in the initial rule

$$\frac{}{[P,\Gamma] \; -_P\!\!\rightarrow} \; I_r, \text{ atomic } P \qquad \frac{}{[\Gamma] \xrightarrow{N} [N]} \; I_l, \text{ atomic } N$$

The context Γ in $[\Gamma], \Theta \vdash \mathcal{R}$ contains either negative formulas or positive atoms.

Thus: positive non-atomic formulas on the left and negative formulas on the right are treated *linearly!*

Soundness and Completeness of LJF

Theorem. Let B be an intuitionistic formula. Let $\hat{B}$ be an annotation of the conjunctions in B. Fix a bias assignment to atomic formulas. Then $\vdash_I B$ if and only if $[\cdot] \dashv \hat{B}$ is provable in LJF.

Proof. Soundness is easy: an LJF immediately yields an LJ proof. Completeness is more difficult. It can be proved using a standard, permutation argument. It can also be proved by mapping intuitionistic logic into linear logic using polarities to insert the exponential !: for example,

$$(P \supset B)^{+1} = P^{-1} \multimap B^{+1} \qquad (N \supset B)^{+1} = !\, N^{-1} \multimap B^{+1}$$

$$(A \supset B)^{-1} = A^{+1} \multimap B^{-1}$$

This translation is inspired by Girard's analysis behind LU.

Cut rules

The cut rule for *LJF* takes many forms:

$$\frac{[\Gamma], \Theta \vdash P \quad [\Gamma'], \Theta', P \vdash \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \vdash \mathcal{R}} \qquad \frac{[\Gamma], \Theta \vdash C \quad [C, \Gamma'], \Theta' \vdash \mathcal{R}}{[\Gamma\Gamma'], \Theta\Theta' \vdash \mathcal{R}}$$

$$\frac{[\Gamma] \xrightarrow{B} [P] \quad [\Gamma'], P \vdash [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]} \qquad \frac{[\Gamma] \vdash N \quad [N, \Gamma'] \xrightarrow{B} [R]}{[\Gamma\Gamma'] \xrightarrow{B} [R]}$$

$$\frac{[\Gamma] -_C\!\rightarrow \quad [C, \Gamma'] -_R\!\rightarrow}{[\Gamma\Gamma'] -_R\!\rightarrow}$$

As before, P is positive, N is negative, and C is negative or a positive atom.

Notice that the last three cut rules retain focus in the conclusion.

These rules are admissible.

Size of Connectives

Connectives are small. Forget the focusing result. A great deal of interleaving/parallelism of introduction rules takes place.
Connectives are big. Connectives are maximal collections of async or sync connectives.

By inserting "delays" into formulas, the "big connective" view yields the "small connective" view.

Delays: $\partial^-(B) = true \supset B$ and $\partial^+(B) = true \wedge^+ B$. Clearly, B, $\partial^-(B)$, and $\partial^+(B)$ are logically equivalent, but $\partial^-(B)$ is always negative and $\partial^+(B)$ is always positive.

For example, LJQ' is embedded into LJF by inserting some delays:
$B^l = B^r = B$ (atom B), $(A \wedge B)^l = \partial^-(A^l \wedge^+ B^l)$,
$(A \wedge B)^r = A^r \wedge^+ B^r$, $(A \vee B)^l = \partial^-(A^l \vee B^l)$, $(A \vee B)^r = A^r \vee B^r$,
$(A \supset B)^l = A^r \supset \partial^+(B^l)$, $(A \supset B)^r = \partial^+(A^l \supset B^r)$.