Faster Individual Discrete logarithm in non-prime finite fields GF(p^n) with the Number and Function Field Sieve Algorithms

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Outline

Motivation: Pairing-based cryptography

The Number Field Sieve algorithm

Individual Discrete Log

Asymetric cryptography

Factorization (RSA cryptosystem)

Discrete logarithm problem (Diffie–Hellman, etc)

Given a finite cyclic group $(\mathbf{G}, \cdot)$, a generator g and $y \in \mathbf{G}$, compute x s.t. $y = g^x$.

Common choice of $\mathbf{G}$:
prime finite field $\mathbb{F}_p$ (since 1976), characteristic 2 finite field $\mathbb{F}_{2^n}$, elliptic curve $E(\mathbb{F}_p)$ (since 1985)

Elliptic curves in cryptography

$$E : y^2 = x^3 + ax + b, \; a, b \in \mathbb{F}_p$$

- proposed in 1985 by Koblitz, Miller
- $E(\mathbb{F}_p)$ has an efficient group law (chord an tangent rule) $\rightarrow$ **G**
- $\#E(\mathbb{F}_p) = p + 1 - t$, trace t: $|t| \leqslant 2\sqrt{p}$

Need a prime-order (or with tiny cofactor) elliptic curve:

$$h \cdot \ell = \#E(\mathbb{F}_p), \quad \ell \text{ is prime}, \quad h \text{ tiny, e.g. } h = 1, 2$$

- compute t
- slow to compute in 1985: can use *supersingular curves* whose trace is known.

Supersingular elliptic curves

Example over $\mathbb{F}_p$, $p \geq 5$

$$E : y^2 = x^3 + x \ / \ \mathbb{F}_p, \quad p = 3 \bmod 4$$

s.t. $t = 0$, $\#E(\mathbb{F}_p) = p + 1$.
take p s.t. $p + 1 = 4 \cdot \ell$ where ℓ is prime.

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1993: Menezes-Okamoto-Vanstone and Frey-Rück attacks
There exists a pairing e that embeds the group $E(\mathbb{F}_p)$ into $\mathbb{F}_{p^2}$
where **DLP is much easier**.
Do not use supersingular curves.

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Do not use supersingular curves.
But computing a pairing is **very slow**:
[Harasawa Shikata Suzuki Imai 99]: 161467s (112 days) on a 163-bit supersingular curve, where $\mathbf{G}_T \subset \mathbb{F}_{p^2}$ of 326 bits.

Pairing-based cryptography

1999: Frey–Muller–Rück: actually, Miller Algorithm can be **much faster**.

2000: *[Joux ANTS]* Computing a pairing can be done efficiently (1s on a supersingular 528-bit curve, $\mathbf{G}_T \subset \mathbb{F}_{p^2}$ of 1055 bits).

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \ : \ E(\mathbb{F}_{q^n})[\ell] \times E(\mathbb{F}_{q^n})[\ell] \longrightarrow \mathbb{F}_{q^n}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

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Attacks

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- inversion of e : hard problem (exponential)

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- discrete logarithm computation in $\mathbb{F}_{q^n}^*$: **easier, subexponential** $\rightarrow$ take a large enough field

Common target groups $\mathbb{F}_{q^n}$

- $\mathbb{F}_{2^{4n}}$, $\mathbb{F}_{3^{6n}}$ where $E/\mathbb{F}_{2^n}$, $E/\mathbb{F}_{3^n}$ is supersingular
- $\mathbb{F}_{p^2}$ where $E/\mathbb{F}_p$ is a supersingular curve
- $\mathbb{F}_{p^3}, \mathbb{F}_{p^4}, \mathbb{F}_{p^6}$ where $E/\mathbb{F}_p$ is an ordinary MNT curve *[Miyaji Nakabayashi Takano 01]*
- $\mathbb{F}_{p^{12}}$ where $E/\mathbb{F}_p$ is a BN curve *[Barreto-Naehrig 05]*

DLP hardness for a 3072-bit finite field:

- **hard** in $\mathbb{F}_p$ where p is a 3072-bit prime
- **easy** in $\mathbb{F}_{2^{4n}}$, $\mathbb{F}_{3^{6n}}$ *[Barbulescu, Gaudry, Joux, Thomé 14, Granger et al. 14]*
- what about $\mathbb{F}_{p^n}$ where $2 \leq n \leq 12$ and p^n is a 3072-bit?

Number Field Sieve algorithm for DL in GF(p^n)

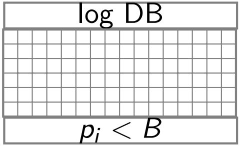
How to generate relations ?
Use *two* distinct rings $R_f = \mathbb{Z}[x]/(f(x))$, $R_g = \mathbb{Z}[x]/(g(x))$ and two maps ρ_f, ρ_g that map $x \in R_f$, resp. $x \in R_g$ to *the same* element $z \in \mathbb{F}_{p^n}$:

$$\begin{cases} \rho_f : x \in R_f \mapsto z, \\ \rho_g : x \in R_g \mapsto z \end{cases}$$

$$\begin{tikzcd}
 & \mathbb{Z}[x] \arrow[dl] \arrow[dr] & \\
R_f = \mathbb{Z}[x]/(f(x)) \arrow[dr, "R_f \ni a - bx \mapsto a - bz"'] & & R_g = \mathbb{Z}[x]/(g(x)) \arrow[dl, "R_g \ni a - bx \mapsto a - bz"] \\
 & \mathbb{F}_{p^n} = \mathbb{F}_p[z]/(\varphi(z)) &
\end{tikzcd}$$

Number Field Sieve algorithm for DL in GF(p^n)

1. Polynomial selection
2. Relation collection
3. Linear algebra



   - We know the log of *small* elements in $\mathbb{Z}[x]/(f(x))$ and $\mathbb{Z}[x]/(g(x))$
   - *small* elements are of the form $a_i - b_i x = \mathfrak{p}_i \in \mathbb{Z}[x]/(f(x))$, s.t. $|\,\mathrm{Norm}(\mathfrak{p}_i)| = p_i < B$

4. Individual discrete logarithm

Initial Splitting in $\mathbb{F}_p$

An integer T is B-smooth if $N = \prod_i p_i^{e_i}$ and $p_i \leq B$

Algorithm 1: Generic Initial Splitting

Input: Target $T_0 \in \mathbb{F}_p$, generator g, subgroup order ℓ, bound B
Output: $t \in \mathbb{Z}/\ell\mathbb{Z}$, $\mathbf{T} \in \mathbb{Z}$ a preimage of $T = g^t T_0$, such that $\mathbf{T}$ is B-smooth

repeat
  take t at random in $\{1, \ldots, \ell - 1\}$
  $T \leftarrow g^t T_0$
  $\mathbf{T} \leftarrow u/v \equiv T \bmod p$ a rational reconstruction of $T \bmod p$
until $\mathbf{T}$ *is B-smooth, i.e. u and v are B-smooth*
return $\mathbf{T} = u/v, t$ // $\log_g T_0 = \log_g \rho(\mathbf{T}) - t$

Initial Splitting in $\mathbb{F}_p$, $\mathbb{F}_{p^n}$, $\mathbb{F}_{2^n}$, $\mathbb{F}_{3^n}$

- $\mathbb{F}_p$: Rational Reconstruction. $T \in \mathbb{Z}/p\mathbb{Z}$, $\mathbf{T}$ is an integer $< p$. Rational Reconstruction gives $\mathbf{T} = u/v \bmod p$ with $u, v < \sqrt{p}$
- *[Blake Fuji-Hara Mullin Vanstone 84]* Waterloo algorithm in $\mathbb{F}_{2^n}$: $\mathbb{F}_2[x] \ni \mathbf{T} \equiv U/V = \frac{u_0 + \ldots + u_{\lfloor n/2 \rfloor} x^{\lfloor n/2 \rfloor}}{v_0 + \ldots + v_{\lfloor n/2 \rfloor} x^{\lfloor n/2 \rfloor}}$ **reduce degree**
- *[Joux Lercier Smart Vercauteren 06]* in $\mathbb{F}_{p^n}$: $\mathbf{T} \equiv U/V = \frac{u_0 + \ldots + u_d x^d}{v_0 + \ldots + v_d x^d}$, $d = \deg f \geq n$, $|u_i|, |v_i| \sim p^{n/(2 \deg f)}$ **reduce coefficient size**

Individual Discrete Log of target $T_0 \in \mathbb{F}_{p^n}^*$

Given g and a DL database s.t. for all $p_i < B_0 \sim 2^{27}$, $\log p_i$ is known,

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   DO
   1.1 take t at random in $\{1, \ldots, \ell - 1\}$ and set $T = g^t T_0$ (hence $\log_g(T_0) = \log_g(T) - t$)
   1.2 factorize
   $$\mathrm{Norm}(\mathbf{T}) = \underbrace{q_1 \cdots q_i}_{\text{too large: } 2^{27} < q_i \leq 2^{90}} \times (\text{elements in DL database}),$$
   UNTIL $q_i \leq B_1 \sim 2^{90}$

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2. Descent strategy: set $\mathcal{S} = \{q_i : \ B_0 < q_i \leq B_1\}$
   while $\mathcal{S} \neq \emptyset$ **do**
   - set $B_j < B_i$
   - find a relation $q_i = \prod_{B_0 < q_j < B_j} q_j \times$ (elements in DL database)
   - $\mathcal{S} \leftarrow \mathcal{S} \setminus \{q_i\} \cup \{q_j\}_{j \in J}$

   end while

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3. log combination to find the individual target DL

508-bit $\mathbb{F}_{p^3}$ Polynomial Selection

$$
\begin{aligned}
p &= 908761003790427908077548955758380356675829026531247 \\
&\quad \text{of 170 bits} \\
A &= 28y^2 + 16y - 109 \\
\varphi &= x^3 - yx^2 - (y+3)x - 1, \quad \sigma(x) \mapsto -x - 1/x \\
f &= \mathrm{Res}_y(A, \varphi) \\
&= 28x^6 + 16x^5 - 261x^4 - 322x^3 + 79x^2 + 152x + 28 \\
&\quad \|f\|_\infty = 8.33 \text{ bits} \\
\alpha(f) &= -2.9
\end{aligned}
$$

$$
\begin{aligned}
g &= 24757815186639197370442122x^3 + 40806897040253680471775183x^2 \\
&\quad -33466548519663911639551183x - 24757815186639197370442122 \\
&\quad \|g\|_\infty = 85.01 \text{ bits} \\
\alpha(g) &= -4.1 \\
&\quad \text{Murphy's E value:} \\
\mathbb{E}(f,g) &= 1.31 \cdot 10^{-12}
\end{aligned}
$$

508-bit $\mathbb{F}_{p^3}$ individual discrete logarithm

Target:

$$T_0 = 0x11a2f1f13fa9b08703a033ee3c4321539156f865ee9 + 0x1098c3b7280ef2cf8b091d08197de0a9ba935ff79c6\ z + 0x221205020e7729cb46166a9edfd5acb3bf59dd0a7d4\ z^2 \in \mathbb{F}_p[z]/(\varphi(z))$$

Preimage: $\mathbf{T}_0 = t_0 + t_1 x + t_2 x^2 \in \mathbb{Z}[x]$

$$\mathrm{Norm}_f(\mathbf{T}) = Res(f, \mathbf{T}) \leq A ||\mathbf{T}||_\infty^{\deg f} ||f||_\infty^{\deg \mathbf{T}}$$

$\mathrm{Norm}_f(\mathbf{T}_0) = \mathrm{Res}(f, \mathbf{T}_0)$ of 1032 bits $\approx p^6 = Q^2$
$\mathrm{Norm}_g(\mathbf{T}_0) = \mathrm{Res}(g, \mathbf{T}_0)$ of 670 bits $\approx p^4 = Q^{4/3}$
Joux–Lercier:
$\mathrm{Norm}_f(JL_f(\mathbf{T}_0)) \approx p^3 = Q$
$\mathrm{Norm}_g(JL_g(\mathbf{T}_0)) \approx p^4 = Q^{4/3}$

Preimage improvement *[G. 15]*

Lemma

Let $T \in \mathbb{F}_{p^n}$.
$\log(T) = \log(u \cdot T) \bmod \ell$ *for any* u *in a proper subfield of* $\mathbb{F}_{p^n}$.

Preimage improvement *[G. 15]*

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- $\mathbb{F}_p$ is a proper subfield of $\mathbb{F}_{p^n}$
- target $T = t_0 + t_1 x + \ldots + t_d x^d$
- we divide the target by its leading term:

$$\log(T) = \log(T/t_d) \bmod \ell$$

We can assume that the target is monic.

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Similar technique in pairing computation: Miller loop denominator elimination *[Boneh Kim Lynn Scott 02]*

Subfield Simplification + LLL

We want to reduce $||\mathbf{T}||_\infty$. Example with $\mathbb{F}_{p^3}$:

- $\varphi = x^3 - yx^2 - (y+3)x - 1,\ y \in \mathbb{Z}$
- $\mathbf{T} = t_0 + t_1 x + x^2$
- define $L = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ t_0 & t_1 & 1 & 0 & 0 & 0 \\ \varphi_0 & \varphi_1 & \varphi_2 & 1 & 0 & 0 \\ 0 & \varphi_0 & \varphi_1 & \varphi_2 & 1 & 0 \\ 0 & 0 & \varphi_0 & \varphi_1 & \varphi_2 & 1 \end{bmatrix}$
- LLL(L) outputs a short vector r, linear combination of L's rows. $r = \lambda_0 p + \lambda_1 px + \lambda_2 T + \lambda_3 \varphi + \lambda_4 x\varphi + \lambda_5 x^2 \varphi$. $r = r_0 + \ldots + r_5 x^5,\ ||r_i||_\infty \leq C \det(L)^{1/6} = O(p^{1/3})$

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- $\mathrm{Norm}_f(r) = O(p^2)$ of ≈ 340 bits instead of $O(p^3)$ of 508 bits

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- LLL(L) outputs a short vector r, linear combination of L's rows. $r = \lambda_0 p + \lambda_1 px + \lambda_2 T + \lambda_3 \varphi + \lambda_4 x\varphi + \lambda_5 x^2 \varphi$.
  $r = r_0 + \ldots + r_5 x^5$, $||r_i||_\infty \le C \det(L)^{1/6} = O(p^{1/3})$
- $\mathrm{Norm}_f(r) = O(p^2)$ of ≈ 340 bits instead of $O(p^3)$ of 508 bits
- $\log \rho(r) = \log(T) \bmod \ell$ because $\rho(r) = \lambda_2 T$ with $\lambda_2 \in \mathbb{F}_p$

Initial Splitting step complexity

Given a target $T_0 \in \mathbb{F}_{p^n}^*$, and g a generator of $\mathbb{F}_{p^n}^*$
repeat

1. take t at random in $\{1, \ldots, \ell - 1\}$ and set $T = g^t T_0$
2. factorize $\mathrm{Norm}(\mathbf{T})$

until it is B_1-smooth: $\mathrm{Norm}(\mathbf{T}) = \prod_{q_i \leq B_1} q_i \prod_{p_i \leq B_0} p_i$

L-notation: $\mathbf{c} > 0$,

$$Q = p^n, \quad L_Q[1/3, \mathbf{c}] = \mathrm{e}^{(\mathbf{c}+o(1))(\log Q)^{1/3} \ (\log \log Q)^{2/3}} .$$

Norm factorization done with ECM method, in time $L_{B_1}[1/2, \sqrt{2}]$

Lemma (Initial Splitting step running-time)

If $\mathrm{Norm}(\mathbf{T}) \leq Q^e$, *take* $B_1 = L_Q[2/3, (e^2/3)^{1/3}]$, *then the running-time is* $L_Q[1/3, (3e)^{1/3}]$ *(and this is optimal).*

Subfield Simplification + LLL

$$\mathrm{Norm}_f(\mathbf{T}) = Res(f, \mathbf{T}) \leq A||\mathbf{T}||_\infty^{\deg f}||f||_\infty^{\deg \mathbf{T}}$$

- $\mathrm{Norm}_f(r) \leq ||r||_\infty^6 ||f||_\infty^5 = O(p^2) = O(Q^{2/3}) < O(Q)$

MNT example: $\log Q = 508$ bits

	$\mathrm{Norm}_f(\mathbf{T})$		$\mathrm{Norm}_g(\mathbf{T})$		$L_Q[1/3, c]$		$q_i \leq B_1 =$
	Q^e	bits	Q^e	bits	c	time	$L_Q[\frac{2}{3}, c]$
Nothing	Q^2	1010	$Q^{4/3}$	667	1.58	2^{53}	2^{109}
[JLSV06]	Q	508	$Q^{5/3}$	847	1.44	2^{48}	2^{90}
Subfield	$\mathbf{Q^{2/3}}$	**340**	$\mathbf{Q}$	**508**	**1.26**	$\mathbf{2^{42}}$	$\mathbf{2^{69}}$

Combined with Pomerance Early Abort Strategy, we obtained a 54-bit smooth initial splitting for $g^{35313}T_0$ and a 59-bit smooth initial splitting for g^{52154} in 32 core-hours.
The descent took 13.4 and 10.7 core hours.

With more subfields: e.g. $\mathbb{F}_{p^6}$

JLSV1 polynomial selection: $\|f\|_\infty = \|g\|_\infty = \sqrt{p}$,
$\deg f = \deg g = 6$
$\mathrm{Norm}_f(\mathbf{T}_0) = \|f\|_\infty^{\deg \mathbf{T}} \|\mathbf{T}\|_\infty^6$
Let $\{1, U, U^2\}$ be a polynomial basis of $\mathbb{F}_{p^3} \subset \mathbb{F}_{p^6}$, e.g. $U = g^{1+p^3}$

$$E = \begin{bmatrix} * & * & * & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & * & 1 \end{bmatrix} = \mathrm{RowEchelonedForm}\left(\begin{bmatrix} T \\ UT \\ U^2 T \end{bmatrix} \right)$$

- E obtained with $\mathbb{F}_p$-linear combinations of $\{T, UT, U^2T\}$
- for each row $\leftrightarrow r_i \in \mathbb{Z}[x]$,
  $r_i = \lambda_0 T + \lambda_1 UT + \lambda_2 U^2 T = \underbrace{(\lambda_0 + \lambda_1 U + \lambda_2 U^2)}_{=u_i \in \mathbb{F}_{p^3}} T$

  $\log_g \rho(r_i) = \log_g(T) \bmod \ell$

Subfield Simplification + LLL in $\mathbb{F}_{p^6}$

We want to reduce $||\mathbf{T}||_\infty$.

- $\mathbf{T} = t_0 + t_1 x + x^2$
- define $L = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & * & 1 \end{bmatrix}$
- $\mathrm{LLL}(L) \to r = \lambda_0 p + \lambda_1 px + \lambda_2 px^2 + \lambda_3 u_0 T + \lambda_4 u_1 T + \lambda_5 u_2 T$.
  $r = r_0 + \ldots + r_5 x^5,\ ||r_i||_\infty \leq C \det(L)^{1/6} = O(p^{1/2})$

Subfield Simplification + LLL in $\mathbb{F}_{p^6}$

We want to reduce $||\mathbf{T}||_\infty$.

- $\mathbf{T} = t_0 + t_1 x + x^2$
- define $L = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & * & 1 \end{bmatrix}$
- $\mathrm{LLL}(L) \rightarrow r = \lambda_0 p + \lambda_1 px + \lambda_2 px^2 + \lambda_3 u_0 T + \lambda_4 u_1 T + \lambda_5 u_2 T$.
  $r = r_0 + \ldots + r_5 x^5,\ ||r_i||_\infty \le C \det(L)^{1/6} = O(p^{1/2})$
- $\mathrm{Norm}_f(r) = O(p^{3+\frac{5}{2}}) = O(Q^{11/12})$ of ≈ 470 bits instead of $O(p^3)$ of 508 bits

Subfield Simplification + LLL in $\mathbb{F}_{p^6}$

We want to reduce $||\mathbf{T}||_\infty$.

- $\mathbf{T} = t_0 + t_1 x + x^2$
- define $L = \begin{array}{cl} \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ * & * & * & * & * & 1 \end{bmatrix} & \begin{array}{l} \rho(p) = 0 \in \mathbb{F}_{p^n} \\ \\ \\ u_0 T \\ u_1 T \\ u_2 T \end{array} \end{array}$
- $\mathrm{LLL}(L) \to r = \lambda_0 p + \lambda_1 px + \lambda_2 px^2 + \lambda_3 u_0 T + \lambda_4 u_1 T + \lambda_5 u_2 T$.
  $r = r_0 + \ldots + r_5 x^5,\ ||r_i||_\infty \leq C \det(L)^{1/6} = O(p^{1/2})$
- $\mathrm{Norm}_f(r) = O(p^{3+\frac{5}{2}}) = O(Q^{11/12})$ of ≈ 470 bits instead of $O(p^3)$ of 508 bits
- $\log \rho(r) = \log(T) \bmod \ell$ because
  $\rho(r) = (\lambda_3 u_0 + \lambda_4 u_1 + \lambda_5 u_2) T$ with $\lambda_{i+3} u_i \in \mathbb{F}_{p^3}$

Theorem

Let $T \in \mathbb{F}_{p^n}^*$ an element which is not in a proper subfield of $\mathbb{F}_{p^n}$. We want to compute its discrete logarithm modulo a (large) prime ℓ, where $\ell \mid \Phi_n(p)$. Let f, R_f given by a polynomial selection method. Let d be the largest divisor of n, $d < n$ and $d = 1$ if n is prime.

Then there exists a preimage $\mathbf{T}$ in $\mathbb{Z}[x]/(f(x))$ of $T \in \mathbb{F}_{p^n}^*$, such that $\log \rho(\mathbf{T}) \equiv \log T \bmod \ell$ and whose norm in R_f is bounded by $O(q^e)$, where $q = p^n$ and q^e equals

1. *$q^{1-d/n}$ for the GJL, Conjugation, Joux-Pierrot, Sarkar-Singh and TNFS-like methods (and for all the possible methods where $\|f\|_\infty = o(p)$);*
2. *$q^{\frac{3}{2}-\frac{d}{n}-\frac{1}{2n}}$ for the JLSV1 method;*
3. *$q^{2-\frac{d}{n}-\frac{2}{D+1}}$ for the JLSV2 method, where D is the degree of g.*

Small characteristic $\mathbb{F}_{2^{4m}}$ and $\mathbb{F}_{3^{6m}}$

Same idea as for $\mathbb{F}_{p^6}$ but without LLL:
The largest subfield of $\mathbb{F}_{2^{4m}}$ is $\mathbb{F}_{2^{2m}}$, let $d = 2m$: Compute two $\mathbb{F}_2$-linear Gaussian eliminations

$$A = \begin{bmatrix} T \\ UT \\ \vdots \\ U^{d-1}T \end{bmatrix} \to \begin{bmatrix} * & * & * & 0 & \dots & 0 \\ 0 & * & * & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & \dots & 0 & * & * & * \end{bmatrix}$$

Each row r_i corresponds to $u_i T_i = u_i X^i \cdot T_i'$, where T_i' is of degree $n/2 = 2m$ and $\log_g(X^i T_i') = \log_g T \bmod \ell$

- Need one poly of degree $n/2$ to be B-smooth instead of two polys
- cost of Gaussian elimination shared over $n/2$ tests
- Magma implementation for $\mathbb{F}_{3^{6 \cdot 509}}$ and $\mathbb{F}_{2^{12 \cdot 367}}$ available

Asymptotic complexity

L-notation: $\mathbf{c} > 0$,

$$Q = p^n, \quad L_Q[1/3, \mathbf{c}] = \mathrm{e}^{(\mathbf{c}+o(1))(\log Q)^{1/3}\ (\log\log Q)^{2/3}}$$

Set $B = \log L_{2^n}[2/3, \gamma]$

- Blake–Fuji-Hara–Mullin–Vanstone Waterloo alg.: $L_{2^n}\left[\frac{1}{3}, \frac{1}{3\gamma}\right]$
- Subfield alg.: $L_{2^n}\left[\frac{1}{3}, \frac{d-1}{d}\frac{1}{3\gamma}\right]$ where d is the largest proper divisor of n (best case: $d = n/2$, $L_{2^n}\left[\frac{1}{3}, \frac{1}{2}\frac{1}{3\gamma}\right]$)

For $\mathbb{F}_{3^{6\cdot 509}} = \mathbb{F}_{3^6}[x]/(I(x))$: T_0 is a degree 508 polynomial over $\mathbb{F}_{3^6}$.

- We found a 30-smooth polynomial over $\mathbb{F}_{3^6}$
- much less elements to "descent"
- improve also the width and depth of "descent" tree

Thank you!

Pre-print available soon.