Discrete logarithms in small characteristic finite fields: Attacking Type 1 pairing-based cryptography

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Joint work with:

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For the general case where we don't know very specific structures on $\mathbb{H}$, this problem is believed to be hard (exponential run time in the size of N).

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  such that

  - $\hat{e}(P, P) \neq 1$ for $P \neq 0_{\mathbb{G}}$,
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- Immediate property: for any two integers k_1 and k_2,

$$\hat{e}(k_1 Q, k_2 R) = \hat{e}(Q, R)^{k_1 k_2}.$$

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  - k is the embedding degree of $\mathbb{G}$, that is the smallest positive integer k such that $r|(q^k - 1)$.
- Used pairing maps:
  - Weil pairings.
  - Tate pairings and modifications (Eta, Ate ...).

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- The $k = 6$ pairings derived from supersingular elliptic curves over $\mathbb{F}_{3^n}$:
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  - $Y^2 = X^3 - X + 1$; and
  - $Y^2 = X^3 - X - 1$.
- The $k = 12$ pairing derived from supersingular gen.-2 curves over $\mathbb{F}_{2^n}$:
  - $Y^2 + Y = X^5 + X^3$; and
  - $Y^2 + Y = X^5 + X^3 + 1$.

Example of protocols

- **Identity-based non-interactive key exchange**
  - Sakai-Oghishi-Kasahara, 2000.
- **One-round three-party** key agreement
  - Joux, 2000.
- **Identity-based encryption**
  - Boneh–Franklin, 2001.
  - Sakai–Kasahara, 2001.
- **Short digital signatures**
  - Boneh–Lynn–Shacham, 2001.
  - Zang–Safavi-Naini–Susilo, 2004.
- ...

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Reduction attack on supersingular elliptic curves:

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- For cryptographic applications on pairings over supersingular curves:
  - The embedding degree is relatively small.
  - Require the DLP in $\mathbb{G}_T$ to be hard.

Algorithm for small characteristic DLP in $\mathbb{F}_Q$

Fastest general-purpose algorithm: Coppersmith (1984) of subexponential run time $L_Q[\frac{1}{3}, 1.526]$, where $L_Q[\alpha, c]$ with $0 < \alpha < 1$ and $c > 0$ denotes

$$L_Q[\alpha, c] = e^{[c+o(1)](\log Q)^{\alpha}(\log\log Q)^{1-\alpha}} = (\log Q)^{[c+o(1)]\left(\frac{\log Q}{\log\log Q}\right)^{\alpha}}.$$

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Table: Believed security for supersingular curves till 2012

Base field ($\mathbb{F}q$)	$\mathbb{F}_{2^n}$	$\mathbb{F}_{3^n}$	$\mathbb{F}_{2^n}$
Embedding degree (k)	4	6	12
Lower security ($\approx 2^{64}$)	$n = 239$	$n = 97$	$n = 71$
Medium security ($\approx 2^{80}$)	$n = 373$	$n = 163$	$n = 127$
Higher security ($\approx 2^{128}$)	$n = 1223$	$n = 509$	$n = 367$

Joux-Lercier Algorithm for $\mathbb{F}_Q = \mathbb{F}_{q^n}$

In 2006, Joux and Lercier presented an algorithm with running time $L_Q[\frac{1}{3}, 1.442]$ when q and n are 'balanced'

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Later in 2012, Joux introduced a "pinpointing" technique that improved the Joux-Lercier algorithm to $L_Q[\frac{1}{3}, 0.961]$.

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- Apr 2013 - Göloğlu et al. solve DLP in $\mathbb{F}^*_{2^{6120}} = \mathbb{F}^*_{(2^8)^{3\cdot 255}}$ in 750 CPU hours.

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Kummer/twisted Kummer extensions: $\mathbb{F}_{q^n}$ with $n \mid q \mp 1$.

Overview on the Joux 2013 algorithm

Select polynomials $h_0, h_1 \in \mathbb{F}_{q^d}[X]$ such that

- degree of h_0 and h_1 is at most δ, a small positive integer.
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Then $\mathbb{F}_{q^{dn}} = \mathbb{F}_{q^d}(x) = \mathbb{F}_{q^d}[X]/(I_X)$ and $x^q = \frac{h_0(x)}{h_1(x)}$.

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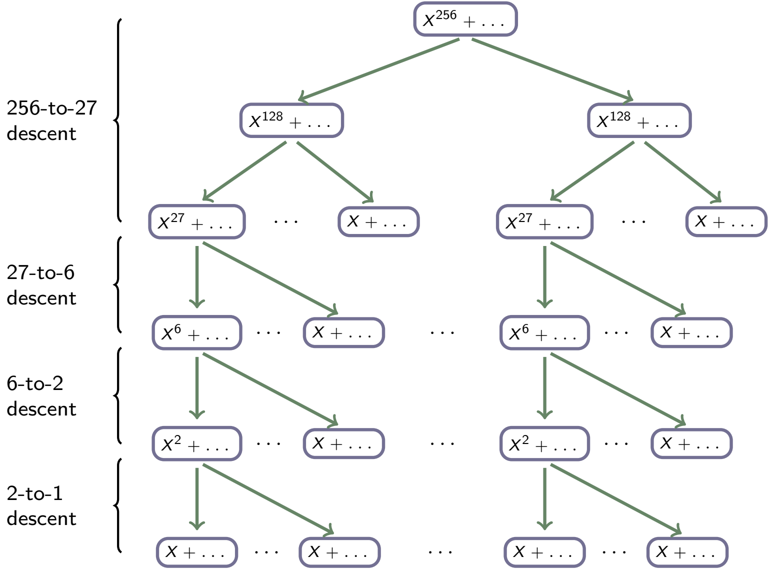
- Descent stage: $\log_g h$ is expressed as a linear combination of logs of elements in the factor base using classical methods and a new descent method (based on solving multivariate bilinear equations).

Descent Steps in $\mathbb{F}_{2^{8\cdot 3\cdot 257}}$



QPA: a much faster algorithm

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[We have another QPA by Granger-Kleinjung-Zumbrägel from April 2014.]

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A new polynomial representation

In ECC 2013, Granger presented (joint work with Zumbrägel) a modification of Joux's field representation:

- $\mathbb{F}_{q^{dn}} = \mathbb{F}_{q^d}[X]/(I_X)$ with I_X dividing $X \cdot h_1(X^q) - h_0(X^q)$.
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Resulting analysis:

DLP algorithm	Coppersmith04	Joux13-QPA13 (as analyzed by A. et al)	GZ13
Run time on $\mathbb{F}_{2^{12\cdot 367}}$	2^{128}	2^{95}	2^{76}
Run time on $\mathbb{F}_{2^{4\cdot 1223}}$	2^{128}	$\geq 2^{128}$	2^{95}

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- $h_0(X), h_1(X)$ polynomials over $\mathbb{F}_q[X]$ of small degree δ.

Very useful: allows to have $n \leq \delta \cdot q$ instead of necessarily $n \leq q + \delta$.

Resulting analysis:

DLP algorithm	Coppersmith04	Joux13-QPA13 (as analyzed by A. et al)	GZ13
Run time on $\mathbb{F}_{2^{12 \cdot 367}}$	2^{128}	2^{95}	2^{76}
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In December 2013, we used the Granger-Zumbrägel representation to show that the cryptographic DLP in $\mathbb{F}_{3^{6 \cdot 1429}}$ and $\mathbb{F}_{2^{4 \cdot 3041}}$ can be solved in time 2^{96} and 2^{129}, respectively. [Initially believed to enjoy a 2^{192} security level.]

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  - First computations of discrete logarithms in a cryptographic finite field using the new algorithms.

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Current computations

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Thanks For Your Attention!