An arithmetic site of Connes-Consani type for imaginary quadratic fields with class number 1

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Contents

1. Introduction

1.1. Brief summary of the work of A.Connes and C.Consani on the arithmetic site

The analogy between function fields (ie finite algebraic extension of $\mathbb{F}_q(T)$ for q a power of a prime number) and number fields (ie finite algebraic extension of $\mathbb{Q}$) has been and remains a fruitful principle in arithmetic geometry. As A.Weil tells in [34], thanks to this analogy, the analogous for function fields of the Riemann conjecture was proved in [33] and [19]. Since then, the hope has been to get inspiration from what happens in the function field case in order to try

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to prove the Riemann conjecture. For a long time the folklore has been to say that in order to achieve this, one should try to make "q tend to 1" and so work in "characteristic 1". Rigorously speaking it doesn't make any sense but many people since then have tried to give a reasonable meaning to the sentences "q tend to 1" and "characteristic 1" like in [30], [15], [23], [2], [26], [24], [20], [14], [22], [6], [7], [8], [28], [32], [31]. In this thesis, our main inspiration is coming from the last approach of A.Connes and C.Consani on this problem as developped in [9], [10], [12], [13].

In 1995, A.Connes ([4]) gave a spectral interpretation of the zeroes of the Riemann zeta function using the adele class space $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\star}$. In May 2014, A.Connes and C.Consani ([9], [10]) found for this space $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\star}$ an underlying structure coming from algebraic geometry by building what they have called the arithmetic site. This space is in fact a topos with a structural sheaf which has the property to be of "caracteristic 1" in the sense that it is an idempotent semiring. To introduce this structural sheaf, they drew their inspiration from what has been developped in the max-plus area by Maslov's school ([21], [25]) and by the school of the INRIA ([16], [17]).

To construct this arithmetic site, they consider the small category, denoted $\mathbb{N}^{\times}$, with only one object $\star$ and the arrows indexed by $\mathbb{N}^{\times} = \mathbb{N}\backslash\{0\}$. The composition law of arrows is given by the multiplication on $\mathbb{N}^{\times}$.

Then they consider $\widehat{\mathbb{N}^{\times}}$, later called the arithmetic site, the presheaf topos associated to this small category considered with the chaotic topology (cf [1]), in other words it is the category of contravariant functors from the category $\mathbb{N}^{\times}$ into the category of sets.

Then they show (theorem 2.1 of [10]) that the category of points (in the sense of [1]) of the topos $\widehat{\mathbb{N}^{\times}}$ is equivalent to the category of totally ordered groups isomorphic to non trivial subgroups of $(\mathbb{Q}, \mathbb{Q}_{+})$ with morphisms in the category being injective morphisms of ordered groups.

Then they show (proposition 2.5 of [10]) that the set of classes of isomorphic points of the topos $\widehat{\mathbb{N}^{\times}}$ is in natural bijection with the quotient space $\mathbb{Q}_{+}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}^{f}/\hat{\mathbb{Z}}^{\times}$.

This space is a component of the adele class space $\mathbb{Q}_{+}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}/\hat{\mathbb{Z}}^{\times}$ already used by A.Connes ([4]) to give a spectral interpretation of the zeroes of the Riemann zeta function. A.Connes and C.Consani then put on the arithmetic site as a structural sheaf the idempotent semiring $(\mathbb{Z} \cup \{-\infty\}, \max, +)$. They show then in theorem 3.8 of [10] that the points of the arithmetic site $(\widehat{\mathbb{N}^{\times}}, \mathbb{Z}_{\max})$ over $\mathbb{R}^{\max}$ is the adele class space $\mathbb{Q}_{+}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}/\hat{\mathbb{Z}}^{\times}$.

A.Connes and C.Consani end their article [10] by describing precisely the relation between the Zariski topos Spec($\mathbb{Z}$) and the arithmetic site, and by building the square of the arithmetic site. Building the square of the arithmetic site is important in the hope of adapting to the Riemann zeta function the proof given by Weil and refined by Grothendieck in [19] of the analogue of the Riemann hypothesis in the case of function fields.

1.2. Description of the main results

In this thesis, we try to generalize the constructions of A.Connes and C.Consani mentionned above to other rings of integers of number fields. We have first considered $\mathbb{Z}[\imath]$ the ring of Gaussian integers which is the simplest ring of integers to look after $\mathbb{Z}$ and it turns out that what we have done for $\mathbb{Z}[\imath]$ remains true for the 8 other rings of integers of imaginary quadratic number fields of class number 1.

In this thesis, we follow the general strategy adopted by A.Connes and C.Consani in [10] to develop the arithmetic site but the *main difficulty* in generalizing their work is that their constructions and part of their results strongly rely on the natural total order $<$ existing on $\mathbb{R}$ which is compatible with basic arithmetic operations $+$ and $\times$. Of course nothing of this sort exists in the case of $\mathbb{Z}[\imath]$ and the main part of my work has been to find the good objects to study.

The starting point of my study is, for K an imaginary quadratic field with class number 1, the small category denoted $\mathcal{O}_K$ with only one object $\star$ and arrows indexed by $\mathcal{O}_K$, the ring of integers of the number field K, the composition law of the arrows being given by the multiplication law $\times$.

In this thesis, we have shown (cf 3.2) that the category of points of the topos $\widehat{\mathcal{O}_K}$ (ie the presheaf topos on the small category $\mathcal{O}_K$ endowed with the chaotic topology) is equivalent to the category of sub-$\mathcal{O}_K$-modules of K.

We have shown (cf4.1), in the same way as A.Connes and C.Consani, that we have an adelic interpretation of the set of classes of isomorphic points of the topos $\widehat{\mathcal{O}_K}$. This set is in bijection with $\frac{\mathbb{A}_K^f}{(K^\star(\prod \mathcal{O}_\mathfrak{p}^\star))}$, which generalizes the proposition 2.5 of [10] of A.Connes and C.Consani.

Another great difficulty is to find a structural sheaf for the topos $\widehat{\mathcal{O}_K}$. It needs to be an idempotent semiring somehow linked to $\mathcal{O}_K$. In this work, we propose the set $\mathcal{C}_{\mathcal{O}_K}$ of convex polygons of the plane whose interior is non empty, invariants by the action by direct similitudes of the units $\mathcal{U}_K$ of $\mathcal{O}_K$ and whose vertices are in $\mathcal{O}_K$ to which we also add the sets $\{0\}$ and $\emptyset$ (some restrictions have to be made when K is not equal to $\mathbb{Q}(\imath)$ or $\mathbb{Q}(\imath\sqrt{3})$). We endow it with the operations $\mathrm{Conv}(\bullet \cup \bullet)$ and $+$ (the Minkowski sum). These laws turn $\mathcal{C}_{\mathcal{O}_K}$ into an idempotent semiring which we define to be the structural sheaf on $\widehat{\mathcal{O}_K}$.

Then we define $\mathcal{C}_{K,\mathbb{C}}$ as the set of convex polygons of the plane whose interior is non empty, invariants by the action by direct similitudes of the units $\mathcal{U}_K$ of $\mathcal{O}_K$ and whose vertices are in $\mathbb{C}$ to which we also add the sets $\{0\}$ and $\emptyset$ (some restrictions have to be made when K is not equal to $\mathbb{Q}(\imath)$ or $\mathbb{Q}(\imath\sqrt{3})$) and we endow it with the operations $\mathrm{Conv}(\bullet \cup \bullet)$ and $+$ (the Minkowski sum). These laws turn $\mathcal{C}_{K,\mathbb{C}}$ into an idempotent semiring. We can already remark that $\emptyset$, the neutral element of the law $\mathrm{Conv}(\bullet \cup \bullet)$, is an absorbant element for the law $+$. We then prove that $\mathrm{Aut}^+_\mathbb{B}(\mathcal{C}_{K,\mathbb{C}})$, the set of $\mathbb{B}$-automorphisms of $\mathcal{C}_{K,\mathbb{C}}$ which we will call direct, is equal to $\mathbb{C}^\star/\mathcal{U}_K$. The set of all $\mathbb{B}$-automorphisms of $\mathcal{C}_{K,\mathbb{C}}$ has a more complicated structure. This suggests heuristically that $\mathcal{C}_{K,\mathbb{C}}$ is of tropical dimension 2 which is different from what A.Connes and C.Consani did in [10] and already suggests that our spectral interpretation will be different from the one they obtained.

We prove then (cf 5.3) that the set of points of the arithmetic site $(\widehat{\mathcal{O}_K}, \mathcal{C}_{\mathcal{O}_K})$ over $\mathcal{C}_{K,\mathbb{C}}$ is in natural bijection with $\frac{\mathbb{A}_K}{(K^\star(\prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^\star \times \mathcal{U}_K))}$. This generalizes the theorem 3.8 of [10] of A.Connes and C.Consani.

Let us now denote by $\mathcal{H}$ the Hilbert space associated by A.Connes to $\frac{\mathbb{A}_K^f \times \mathbb{C}}{K^\star}$ in [4] to build the spectral realization of Hecke L functions of K. Let us denote $G = \frac{K^\star \times (\prod_{\mathfrak{p}\ \mathrm{prime}} \mathcal{O}_\mathfrak{p}^\star \times \mathcal{U}_K)}{K^\star}$. The Hilbert space associated to our space $\frac{\mathbb{A}_K^f \times \mathbb{C}}{(K^\star(\prod \mathcal{O}_\mathfrak{p}^\star \times \mathcal{U}_K))}$ is $\mathcal{H}^G$ (cf 6.2).

Let us denote $C_{K,1}$ the group of adele classes of norm 1. The Hilbert space associated in [4] to ζ_K, the Dedekind zeta function of K, is $\mathcal{H}^{C_{K,1}}$. But in our case, we can notice that $\frac{C_{K,1}}{G} = \frac{\mathbb{S}^1}{\mathcal{U}_K}$. We can therefore prove (cf theorem 6.2) that $\mathcal{H}^G = \bigoplus_{\chi \in \widehat{\mathbb{S}^1/\mathcal{U}_K}} \mathcal{H}^G_\chi$ and that the spectral interpretation tells us that the eigenvalues of the infinitesimal generator of the action of $1 \times \mathbb{R}_+^\star$ on $\mathcal{H}^G_\chi$ are exactly the $z - \frac{1}{2}$ such that $L(\chi, z) = 0$. In particular when χ is trivial we get a spectral interpretation of the zeroes of the Dedekind zeta function of K. The slight difference here with what A.Connes and C.Consani did in [10] is that the spectral interpretation gives us not only the zeta function (here of Dedekind and not of Riemann) but also some Hecke L functions. The reason for this is that heuristically $\mathcal{C}_{K,\mathbb{C}}$ is of tropical dimension 2. Our work is thus giving a family of examples where the associated topos encodes some non trivial L functions, it may give a hint on how to take into account more Hecke L functions in the future.

Then we extend to the case of K the theorem 5.3 of [10] of A.Connes and C.Consani which establish a link between $\mathrm{Spec}(\mathbb{Z})$ and the topos $\left(\hat{\mathbb{N}}, \mathbb{Z}_{\max}\right)$. More precisely (cf theorems 7.1 and 7.2), we build a geometric morphism $T : \mathrm{Spec}(\mathcal{O}_K) \to \hat{\mathcal{O}_K}$ and show that for $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$, the fiber $T^\star(\mathcal{C}_{\mathcal{O}_K})_{\mathfrak{p}}$ is the semiring $\mathcal{C}_{H_{\mathfrak{p}}}$. Moreover at the generic point, the fiber of $T^\star(\mathcal{C}_{\mathcal{O}_K})$ is $\mathbb{B}$.

Lastly, in section 10, we assume that $K = \mathbb{Q}(\imath)$. We begin (cf proposition 8.2) with giving a functional description $\mathcal{F}_{\mathbb{Z}[\imath]}$ of the structural sheaf $\mathcal{C}_{\mathbb{Z}[\imath]}$ of $\widehat{\mathcal{O}_K}$. This allows us (cf definition 8.2) to define the $\mathbb{B}$-module $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}$ and to show (cf propositions 8.3 and 8.4)that it can be naturally endowed with a structure of semiring on which $\mathbb{Z}[\imath] \times \mathbb{Z}[\imath]$ acts. It allows us then (cf definition 8.2) to define the non reduced square $\left(\mathbb{Z}[\widehat{\imath] \times \mathbb{Z}}[\imath], \mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}\right)$. It seems that this semiring is not multiplicatively cancellative. Therefore we associate to it (cf definition 8.4) its canonical multiplicatively cancellative semiring $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes}_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}$, which allows us to define (cf 8.5) the reduced square $\left(\mathbb{Z}[\widehat{\imath] \times \mathbb{Z}}[\imath], \mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes}_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}\right)$.

1.3. Future projects

In the construction of the square of the arithmetic site $(\widehat{\mathcal{O}_{\mathbb{Z}[\imath]}}, \mathcal{C}_{\mathcal{O}_{\mathbb{Z}[\imath]}})$ I have already switched viewpoints from the set $\mathcal{C}_{\mathbb{Z}[\imath]}$ of convex polygons with some special hypothesis to the set $\mathcal{F}_{\mathbb{Z}[\imath]}$ of some special convex affine by parts functions on $[1, \imath]/(1 \sim \imath)$ seen as a tropical curve. In my thesis, I have defined abstractly the tensor product over $\mathbb{B}$: $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}$. As shown in [3], the concrete description of $\mathbb{Z} \otimes_{\mathbb{B}} \mathbb{Z}$ has applications to discrete event dynamic systems. I am currently trying to find a concrete description of $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}$ and one could hope, as in the case of $\mathbb{Z} \otimes_{\mathbb{B}} \mathbb{Z}$, that the concrete description of $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}$ could be useful too in applied mathematics.

Another direction of research could consist first by noticing that what has been done in my thesis could be generalized after some work for a K a number field whose narrow class number is equal to 1. This will be the subject of a forthcoming publication.

For number fields K with narrow class number different than 1, it seems that a reasonable topos to study, would be the topos $\widehat{\mathcal{I}_K}$: the presheaf topos on the site defined by the small category $\mathcal{I}_K$, the category with one object $\star$ with arrows indexed by the elements of $\mathcal{I}_K$ (the monoid of integral ideals) and the law of composition of arrows given by the multiplication of ideals, and with the chaotic topology in the sense of [1]. It seems that the category of points of this topos is an interesting quotient of finite adeles, I am computing it now. The main difficulty will be to find a suitable structural sheaf. We would have to try to find a structural sheaf of tropical dimension 1, because thanks to the spectral interpretation, we know that in order to get in this spectral interpretation the Dedekind zeta function of K, we have to divide the adèle class space by the idèles classes of norm 1 (ie the kernel of the module map) and what is left is only an action by $\mathbb{R}_+^\star$.

Since $\widehat{\mathcal{I}_K}$ seems to be an interesting candidate for the arithmetic site for a general number field K, and since DR_K, the monoid of Deligne-Ribet of K, is closely linked to the monoid $\mathcal{I}_K$ and is playing a crucial role in the structure of Bost-Connes systems as shown in [37], it would be interesting and difficult to try to compute the category of points of the presheaf topos associated to the small category with only one object $\star$ and the arrows indexed by the elements of DR_K and the law of composition of arrows given by the law of the monoid DR_K, the delicate thing will be to put an adequate topology on this category and compute the points. One could hope that it could provide a link and maybe a better understanding between Bost-Connes systems and arithmetic sites.

It would be also interesting to see if it is possible to develop an analogue of the arithmetic site in the case of function fields on a finite field and see if the already existing proof of Weil and Grothendieck for the analogue of the Riemann hypothesis can be done also in this framework.

In march 2016 in [12] and in [13], A.Connes and C.Consani constructed by extensions of scalar a scaling site for $(\widehat{\mathbb{N}^{\times}}, \mathbb{Z}_{\max})$ and so showed that the adèle class space of $\mathbb{Q}$ which is so important in the spectral interpretation of the zeroes of the Riemann zêta function admits a natural structure of tropical curve. In the future, I intend to build similar scaling sites for more general number fields. Also regarding the scaling site, it would be interesting to see if the geometric morphism $\mathrm{Spec}(\mathbb{Z})$ and the arithmetic site $(\widehat{\mathbb{N}^{\times}}, \mathbb{Z}_{\max})$, could be extended from the Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ to the scaling site and see if a similar situation occurs also for more general number fields. In order to achieve this, one would have to use the formalism of $\mathbb{S}$-algebras developped by A.Connes and C.Consani in [11].

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2. Notations

The starting point of Alain Connes and Caterina Consani's construction is the topos $\widehat{\mathbb{N}^{\star}}$. Here we shall use the topos $\widehat{\mathcal{O}_K}$ where :

- K is a number field whose ring of integers $\mathcal{O}_K$ is principal
- $\mathcal{O}_K$ is a written shortcut for the little category which has only one object noted $\star$ and arrows indexed by the elements of $\mathcal{O}_K$ and the law of composition of arrows is determined by the multiplication law of $\mathcal{O}_K$ ($\mathcal{O}_K$ is a monoïd for the multiplication law)
- Let denote $\widehat{\mathcal{O}_K}$ the presheaf topos associated to the small category $\mathcal{O}_K$, ie the category of contravariant functors from the small category $\mathcal{O}_K$ to $\mathfrak{Sets}$ the category of sets.
- $\mathcal{U}_K$ is the set of the units of $\mathcal{O}_K$ the ring of integers of K
- $\mathbb{S}^1$ the circle, ie the set of complex number with modulus equal to 1

3. Geometric points of $\widehat{\mathcal{O}_K}$

As recalled by Alain Connes and Caterina Consani in [10] and proved by MacLane and Moerdijk in [27] : in topos theory, the category of geometric points of a presheaf topos $\widehat{\mathcal{C}}$, with $\mathcal{C}$ being a small category, is canonically equivalent to the category of covariant flat functors from $\mathcal{C}$ to $\mathfrak{Sets}$. Let us also recall that a covariant flat functor $F : \mathcal{C} \to \mathfrak{Sets}$ is said to be flat if and only if it is filtering which means :

1. $F(C) \neq \emptyset$ for at least one object C of $\mathcal{C}$

2. Given two objects A and B of $\mathcal{C}$ and two elements $a \in F(A)$ and $b \in F(B)$, then there exists an object Z of $\mathcal{C}$, two morphisms $u : Z \to A$, $v : Z \to B$ and an element $z \in F(Z)$ such that $F(u)z = a$ and $F(v)z = b$

3. Given two objects A and B of $\mathcal{C}$, two arrows $u, v : A \to B$ and $a \in F(A)$ with $F(u)a = F(v)a$, then there exists an object Z of $\mathcal{C}$, an arrow $w : Z \to A$ and an element $z \in F(Z)$ such that $F(w)z = a$ and $u \circ w = v \circ w \in \mathrm{Hom}_{\mathcal{C}}(Z, B)$.

Here in the case of $\widehat{\mathcal{O}_K}$, we deduce that a covariant functor $F : \mathcal{O}_K \to \mathfrak{Sets}$ is flat if and only if

1. $X := F(\star)$ is a non empty set

2. Given two elements $a, b \in X$, then there exists $u, v \in \mathcal{O}_K$ and $z \in X$ such that $F(u)z = a$ and $F(v)z = b$

3. Given two elements $u, v \in \mathcal{O}_K$ and $a \in X$ with $F(u)a = F(v)a$, then there exists $w \in \mathcal{O}_K$ and $z \in X$ such that $F(w)z = a$ and $u \times w = v \times w \in \mathcal{O}_K$.

Then we have :

Theorem 3.1. *Let $F : \mathcal{O}_K \to \mathfrak{Sets}$ be a flat covariant functor. Then $X :=_{def} F(\star)$ can be naturally endowed with the structure of an $\mathcal{O}_K$-module which is isomorphic (not in a canonical way) to an $\mathcal{O}_K$-module included in K*

Let us now prove this theorem with a long serie of lemma:

Let $F : \mathcal{O}_K \to \mathfrak{Sets}$ be a flat covariant functor.

Let us denote $X := F(\star)$, the image by F in $\mathfrak{Sets}$ of $\star$ the only object of the small category $\mathcal{O}_K$.

Lemma 3.1. *The group law $+$ of $\mathcal{O}_K$ will induce through F an intern law on X*

Proof. The group law $+$ of $\mathcal{O}_K$ will induce through F an intern law on X in the following way :

Let $x, \tilde{x} \in X$ be two elements of X.

By the property (ii) of the flatness of F,

Let $u, v \in \mathcal{O}_K$ and $z \in X$ such that $F(u)z = x$ and $F(v)z = \tilde{x}$.

Then we take as definition $x + \tilde{x} := F(u + v)z$.

We must now check that this definition is independent of the choices made for u, v and z.

Indeed let $u', v' \in \mathcal{O}_K$ and $z' \in X$ (not necessarily equal to u, v and z respectively) such that $F(u')z' = x$ and $F(v')z' = \tilde{x}$.

Then by property (ii) of the flatness of F,

Let $\alpha, \alpha' \in \mathcal{O}_K$ and $\hat{z} \in X$ such that $F(\alpha)\hat{z} = z$ and $F(\alpha')\hat{z} = z'$.

Then $F(u\alpha)\hat{z} = x = F(u'\alpha')\hat{z}$ and $F(v\alpha)\hat{z} = \tilde{x} = F(v'\alpha')\hat{z}$.

So by property (iii) of the flatness of F.

Let $\beta \in \mathcal{O}_K$ and $\gamma \in X$ such that $F(\beta)\gamma = \hat{z}$ and $u\alpha\beta = u'\alpha'\beta$

And let $\tilde{\beta} \in \mathcal{O}_K$ and $\gamma' \in X$ such that $F(\tilde{\beta})\gamma' = \hat{z}$ and $v\alpha\tilde{\beta} = u'\alpha'\tilde{\beta}$.

From here there are several possibilities :

- $\beta = 0$ and $\tilde{\beta} = 0$

  Then $F(0)\gamma = \hat{z} = F(0)\gamma'$

  And so $z = F(\alpha)\hat{z} = F(0)\gamma$ and $z' = F(\alpha')\hat{z} = F(0)\gamma'$

  So finally $F(u + v)z = x + \tilde{x} = F((u + v)0)\gamma = F(0)\gamma = \hat{z} = F(0)\gamma' = F((u' + v')0)\gamma' = F(u' + v')z'$

- $\beta = 0$ and $\tilde{\beta} \neq 0$ or $\beta \neq 0$ and $\tilde{\beta} = 0$ (as β and $\tilde{\beta}$ have symmetric roles we will just look the case $\beta = 0$ and $\tilde{\beta} \neq 0$)

  Then $F(0)\gamma = \hat{z} = F(\tilde{\beta})\gamma'$

  And so $z = F(\alpha)\hat{z} = F(0)\gamma$.

  So $x = F(u)z = F(0)\gamma$ and $\tilde{x} = F(v)z = F(0)\gamma$.

  But since $x = F(u')z'$ and $\tilde{x} = F(v')z'$, by property (ii) of the flatness of F,

  Let $\lambda, \mu \in \mathcal{O}_K$ and $z'' \in X$ such that $F(\lambda)z'' = \gamma$ and $F(\mu)z'' = z'$.

  So $F(0)z'' = x = F(\mu'\mu)z''$.

  And so by property (iii) of flatness of F, let $\nu \in \mathcal{O}_K$ and $z''' \in X$ such that $F(\nu)z''' = z''$ and $0\nu = \mu'\mu\nu$ and then

  - ⋆ either $\mu\nu = 0$
    Then $z' = F(0)z''$ and so $x + \tilde{x} = F(u+v)z = F(0)\gamma$ and $F(u'+v')z' = F((u'+v')0)z'' = F(0)z''$.
    So by property (ii) of flatness of F, let $l, m \in \mathcal{O}_K$ and $\bar{z} \in X$ such that $F(l)\bar{z} = \gamma$ and $F(m)\bar{z} = z''$
    And so finally $F(u+v)z = x+\tilde{x} = F(0)\gamma = F(0)\bar{z} = F(0m)\bar{z} = F(0)z'' = F(u'+v')z'$.
  - ⋆ either $u' = 0$
    Then $x = F(0)z'$
    So $F(u'+v')z' = F(v')z' = \tilde{x}$
    And we will still have $x + \tilde{x} = F(0)\gamma = \tilde{x}$
    So finally $x + \tilde{x} = F(u'+v')z'$.

- $\beta \neq 0$ and $\beta' \neq 0$

  Then $u\alpha = u'\alpha'$ and $v\alpha = v'\alpha'$

  So finally $x + \tilde{x} = F(u+v)z = F(u\alpha + v\alpha)\hat{z} = F(u'\alpha' + v'\alpha')\hat{z} = F(u'+v')z$

Therefore the definition of the law $+$ on X is independent of the choices made.

□

In fact, more is true:

Lemma 3.2. *The set $(X, +)$ with the intern law defined as before is an abelian group.*

Proof. Let us now prove $(X, +)$ that is an abelian group.

- let us first check the associativity of $+$

  It follows from the associativity of $+$ on $\mathcal{O}_K$ and more precisely :

  let $x, x', x'' \in X$, let us now apply the property (ii) of the flatness of F two times in a row,

  let us then take $a, a', a'' \in \mathcal{O}_K$ and $z \in X$ such that $x = F(a)z$, $x' = F(a')z$ and $x'' = F(a'')z$.

  Then $x + (x' + x'') = x + F(a' + a'')z = F(a)z + F(a' + a'')z = F(a + (a' + a''))z = F((a + a') + a'')z = (x + x') + x''$.

  So the law $+$ is indeed associative.

- let us now check the commutativity of $+$

  Here again it follows from the commutativity of $+$ on $\mathcal{O}_K$ and more precisely :

  let us then take $a, a' \in \mathcal{O}_K$ and $z \in X$ such that $x = F(a)z$ and $x' = F(a')z$.

  Then $x + x' = F(a + a')z = F(a' + a)z = x' + x$.

  So the law $+$ is indeed commutative.

- Let us now find the neutral element of $(X, +)$.

  We denote $0_X := F(0)x$ where $x \in X$ is any element of X.

  Let us first show that 0_X is well defined :

  Let $x, x' \in X$, by property (ii) of the flatness of F, let $a, a' \in \mathcal{O}_K$ and $z \in X$ such that $x = F(a)z$ and $x' = F(a')z$.

  Then $F(0)x = F(0a)z = F(0)z = F(0a')z = F(0)x'$.

  So 0_X is indeed well defined, let us now show that it is the neutral element of $(X, +)$.

  Let $y \in X$, by property (ii) of flatness of F, let $\tilde{a}, \tilde{a}' \in \mathcal{O}_K$ and $\tilde{z} \in X$ such that $x = F(\tilde{a})\tilde{z}$ and $0_X = F(\tilde{a}')\tilde{z}$.

  But then by the definition of 0_X, we have $F(\tilde{a}')\tilde{z} = 0_X = F(0)\tilde{z}$.

  So by property (iii) of flatness of F, let $w \in \mathcal{O}_K$ and $\hat{z} \in X$ such that $\tilde{z} = F(w)\hat{z}$ and $\tilde{a}'w = 0w = 0$.

  All in all we have $y = F(\tilde{a}w)\hat{z}$ and $0_X = F(0)\hat{z}$.

  And so $y + 0_X = F(\tilde{a}w + 0)\hat{z} = F(\tilde{a}w)\hat{z} = y$ and by commutativity $0_X + y = y + 0_X = y$.

  So 0_X is the neutral element of $(X, +)$.

- Let us finally show that each element of X admits a symmetric for the law $+$.

  Let $x \in X$, as above by property (ii) and (iii) of flatness of F, let $a \in \mathcal{O}_K$ and $z \in X$ such that $x = F(a)z$ and $0_X = F(0)z$.

  We denote $-x := F(-a)z$, then $x + (-x) = F(a + (-a))z = F(0)z = 0_X$ and by commutativity $(-x) + x = x + (-x) = 0_X$.

  But before concluding we must check that our definition of $-x$ is independent of the choices made for a and z.

  Let $a' \in \mathcal{O}_K$ and $z' \in X$ such that $x = F(a')z'$ and $0_X = F(0)z'$ too.

  By property (ii) of flatness of F, let $b, b' \in \mathcal{O}_K$ and $z'' \in X$ such that $z = F(b)z''$ and $z' = F(b')z''$.

  Then $F(ab)z'' = x = F(a'b')z''$, so by property (iii) of flatness of F, let $c \in \mathcal{O}_K$ and $z''' \in X$ such that $z'' = F(c)z'''$ and $abc = a'b'c$.

  So $-x = F(-a)z = F(-abc)z''' = F(-a'b'c)z''' = F(-a')z'$.

  So $-x$ is well defined and is the symmetric of x for the law $+$

Therefore $(X, +)$ is an abelian group. □

In fact we have a better result :

Lemma 3.3. *We can endow X with the structure of an $\mathcal{O}_K$-module.*

Proof. Let us now show that we can endow X with the structure of an $\mathcal{O}_K$-module. Let us define $\bullet$ the action law by $\bullet \begin{cases} \mathcal{O}_K \times X \longrightarrow & X \\ (\alpha, x) \longmapsto & \alpha \bullet x := F(\alpha u) z \end{cases}$

Where $u \in \mathcal{O}_K$ and $z \in X$ are such that $x = F(u)z$ and $0_X = F(0)z$ (by property (ii) and (iii) of flatness of F such elements always exist).

Let us first check that $\bullet$ is well defined, ie independent of the choices made to define it:

Let $(\alpha, x) \in \mathcal{O}_K \times X$, by property (ii) and (iii) of flatness of F, let $u, u' \in \mathcal{O}_K$ and $z, z' \in X$ such that $F(u)z = x = F(u')z'$ and $F(0)z = 0_X = F(0)z'$.

To show that $\bullet$ is well defined, let us show that $F(\alpha u)z = F(\alpha u')z'$.

Since $F(u)z = x = F(u')z'$, by property (ii) and (iii) of F, let $w, w' \in \mathcal{O}_K$ and $\hat{z}$, $\hat{w} \in \mathcal{O}_K$ and $\hat{\hat{z}} \in X$ such that $z = F(w)\hat{z}$, $z' = F(w')\hat{z}$, $\hat{z} = F(\hat{w})\hat{\hat{z}}$ and $uw\hat{w} = u'w'\hat{w}$.

Then $F(\alpha u)z = F(\alpha uw\hat{w})\hat{\hat{z}} = F(\alpha u'w'\hat{w})\hat{\hat{z}} = F(\alpha u')z'$.

So $\bullet$ is indeed well defined.

Let us now check the following relations :

- $\star$ $\forall \alpha \in \mathcal{O}_K, \forall (x, y) \in X^2, \alpha \bullet (x + y) = \alpha \bullet x + \alpha \bullet y$

  Indeed, let $\alpha \in \mathcal{O}_K$ and $x, y \in X$,

  by property (ii) of flatness of F, let $u, v \in \mathcal{O}_K$ and $z \in X$ such that $x = F(u)z$ and $y = F(v)z$.

  Then $\alpha \bullet (x + y) = \alpha \bullet (F(u + v)z) = F(\alpha(u + v))z = F(\alpha u + \alpha v)z = F(\alpha u)z + F(\alpha v)z = \alpha \bullet x + \alpha \bullet y$.

- $\star$ $\forall (\alpha, \beta) \in \mathcal{O}_K^2, \forall x \in X, (\alpha + \beta) \bullet x = \alpha \bullet x + \beta \bullet x$

  Indeed, let $\alpha, \beta \in \mathcal{O}_K$ and $x \in X$,

  By property (ii) and (iii) of flatness of F, let $u \in \mathcal{O}_K$ and $z \in X$ such that $x = F(u)z$ and $0_X = F(0)z$.

  Then $(\alpha + \beta) \bullet x = F((\alpha + \beta)u)z = F(\alpha u + \beta u)z = F(\alpha u)z + F(\beta u)z = \alpha \bullet x + \beta \bullet x$

- $\star$ $\forall (\alpha, \beta) \in \mathcal{O}_K^2, \forall x \in X, \alpha \bullet (\beta \bullet x) = (\alpha\beta) \bullet x$

  Indeed, let $\alpha, \beta \in \mathcal{O}_K$ and $x \in X$, by property (ii) and (iii) of flatness of F, let $u \in \mathcal{O}_K$ and $z \in X$ such that $x = F(u)z$ and 0_X.

  Then $\alpha \bullet (\beta \bullet x) = \alpha \bullet (F(\beta u)z) = F(\alpha\beta u)z = (\alpha\beta) \bullet F(u)z = (\alpha\beta) \bullet x$

- $\forall x \in X, 1 \bullet x = x$

  Indeed, let $x \in X$, by property (ii) and (iii) of flatness of F, let $u \in \mathcal{O}_K$ and $z \in X$ such that $x = F(u)z$ and 0_X.

  Then $1 \bullet x = 1 \bullet F(u)z = F(1 \times u)z = F(u)z = x$

So finally X is indeed an $\mathcal{O}_K$-module.

□

We end the proof of the theorem 3.1 thanks to the following lemma:

Lemma 3.4. *The $\mathcal{O}_K$-module X is isomorphic (in a non canonical way) to an $\mathcal{O}_K$-module included in K.*

Proof. We have two possibilities

* $X = \{0_X\}$ then obviously $X \simeq \{0_K\}$

* $\{0_X\} \subsetneq X$

  Then let us take $x \in X \backslash \{0_X\}$ and let us thus note $j_{X,x} \begin{cases} X \longrightarrow & K \\ \tilde{x} \longmapsto & \frac{\tilde{k}}{k} \end{cases}$ where we have $k, \tilde{k} \in \mathcal{O}_K$ and $z \in X$ such that $x = F(k)z$ and $\tilde{x} = F(\tilde{k})z$ (there always exists such elements by property (ii) of flatness of F).

  Let us first show that $j_{X,x}$ is well defined.

  Let $\tilde{x} \in X$, then by property (ii) of flatness of F, let $k, \tilde{k}, k', \tilde{k}' \in \mathcal{O}_K$ and $z, z' \in X$ such that $F(k)z = x = F(k')z'$ and $F(\tilde{k})z = \tilde{x} = F(\tilde{k}')z'$.

  According to what we have shown earlier on 0_X, $k \neq 0$ and $k' \neq 0$ (otherwise we would have $x = 0_X$ which is impossible).

  So now to show that $j_{X,x}$ is well defined, we only have left to show that $\frac{\tilde{k}}{k} = \frac{\tilde{k}'}{k'}$.

  By property (ii) and (iii) of flatness of F, let $w, w', \hat{w} \in \mathcal{O}_K$ and $\hat{z}, \hat{\hat{z}} \in X$ such that $z = F(w)\hat{z}, z' = F(w')\hat{z}, \hat{z} = F(\hat{w})\hat{\hat{z}}$ and $kw\hat{w} = k'w'\hat{w}$, so since $k \neq 0$, $w\hat{w} = \frac{k'}{k}w'\hat{w}$.

  So $0_X \neq x = F(kw\hat{w})\hat{\hat{z}} = F(k'w'\hat{w})\hat{\hat{z}}$.

  So $kw\hat{w} \neq 0$ and $k'w'\hat{w} \neq 0$, so $w'\hat{w} \neq 0$.

  We also have $F(\tilde{k}w\hat{w})\hat{\hat{z}} = \tilde{x} = F(\tilde{k}'w'\hat{w})\hat{\hat{z}}$.

  So by property (iii) of flatness of F, let $\hat{\hat{w}} \in \mathcal{O}_K$ and $\hat{\hat{\hat{z}}} \in X$ such that $\hat{\hat{z}} = F(\hat{\hat{w}})\hat{\hat{\hat{z}}}$ and $\tilde{k}w\hat{w}\hat{\hat{w}} = \tilde{k}'w'\hat{w}\hat{\hat{w}}$.

  So $\tilde{k}\frac{k'}{k}w'\hat{w}\hat{\hat{w}} = \tilde{k}'w'\hat{w}\hat{\hat{w}}$.

  And since $w'\hat{w} \neq 0$ and $k' \neq 0$, we thus have $\frac{\tilde{k}}{k}\hat{\hat{w}} = \frac{\tilde{k}'}{k'}\hat{\hat{w}}$.

  And $\hat{\hat{w}} \neq 0$ because otherwise we would have $\hat{\hat{z}} = F(0)\hat{\hat{\hat{z}}}$ and so $x = F(kw\hat{w}0)\hat{\hat{\hat{z}}} = 0_X$ which is impossible.

  And so we get that $\frac{\tilde{k}}{k} = \frac{\tilde{k}'}{k'}$ and so $j_{X,x}$ is well defined.

  Let us now check that $j_{X,x}$ is a linear map from X to K.

  Let $x', x'' \in X$ and $\lambda \in \mathcal{O}_K$. By property (ii) of flatness of F there exists $k, k', k'' \in \mathcal{O}_K$ and $z \in X$ such that $X = F(k)z$, $x' = F(k')z$ and $x'' = F(k'')z$ and as $x \neq 0$, $k \neq 0$.

  Then $\lambda x' + x'' = F(\lambda k' + k'')z$.

  And so $j_{X,x}(\lambda x' + x'') = \frac{\lambda k' + k''}{k} = \lambda\frac{k'}{k} + \frac{k''}{k} = \lambda j_{X,x}(x') + j_{X,x}(x'')$ and so $j_{X,x}$ is linear.

  Let us now show that $j_{X,x}$ is injective.

  Indeed, let $x' \in X$ such that $j_{X,x}(x') = 0$, then by property (ii) of flatness of F, let $k, k' \in \mathcal{O}_K$ and $z \in X$ such that $x = F(k)z$ and $x' = F(k')z$.

  Since $x \neq 0$, we have $k \neq 0$ and then $0 = j_{X,x}(x') = \frac{k'}{k}$.

  So $k' = 0$ and finally $x' = F(0)z = 0_X$.

  So finally we have $X \simeq \mathrm{Im}(j_{X,x})$ and of course the dependance in x makes it non canonical.

$\square$

Thanks to the theorem 3.1 we get that :

Theorem 3.2. *The category of (geometric) points of the topos* $\widehat{\mathcal{O}_K}$ *is canonically equivalent to the category of sub* $\mathcal{O}_K$*-modules of* K *and morphisms of* $\mathcal{O}_K$*-modules.*

Proof. By theorem VII.5.2bis p 382 of [27] the category of geometric ponts of $\widehat{\mathcal{O}_K}$ and natural transformations is equivalent to the category $\mathfrak{Flat}(\mathcal{O}_K)$ of the covariant flat functors from the small category $\mathcal{O}_K$ to $\mathfrak{Sets}$ and natural transformations.

Now we just have to prove that $\mathfrak{Flat}(\mathcal{O}_K)$ is equivalent to the category $\mathcal{O}_K - \mathfrak{Mod}_{\simeq\subset K}$ of $\mathcal{O}_K$-modules isomorphic to sub $\mathcal{O}_K$-modules of K and morphisms of $\mathcal{O}_K$-modules.

First we can define

$$\mathcal{E}\begin{cases} \mathfrak{Flat}(\mathcal{O}_K) \longrightarrow & \mathcal{O}_K - \mathfrak{Mod}_{\simeq\subset K} \\ F : \mathcal{O}_K \to \mathfrak{Sets} \longmapsto & F(\star) \\ \Phi : F \to G \longmapsto & \Phi : F(\star) \to G(\star) \end{cases}$$

Let us first check that $\mathcal{E}$ is well defined :

- on the objects $\mathcal{E}$ is well defined as shown by the last lemma (1.1)
- let Φ be a natural transformation from F to G with $F, G : \mathcal{O}_K \to \mathfrak{Sets}$ two flat covariant functors from the small category $\mathcal{O}_K$ to $\mathfrak{Sets}$, then by definition Φ can also be seen as an application from $F(\star)$ to $G(\star)$ since the small category $\mathcal{O}_K$ has only one object noted $\star$.

  Let us now show that $\Phi : F(\star) \to G(\star)$ is linear.

  Let $\lambda \in \mathcal{O}_K$ and $x, y \in F(\star)$, then by property (ii) of flatness of F, let $u, v \in \mathcal{O}_K$ and $z \in F(\star)$ such that $x = F(u)z$ and $y = F(v)z$.

  Then since Φ is more than a mere application from $F(\star)$ to $G(\star)$ but also a natural transformation from F to G, we have $\Phi(x) = G(u)\Phi(z)$ and $\Phi(y) = G(v)\Phi(z)$.

  And so $\Phi(\lambda x+y) = \Phi(\lambda F(u)z+F(v)z) = \Phi(F(\lambda u+v)z) = G(\lambda u+v)\Phi(z) = \lambda G(u)\Phi(z)+ G(v)\Phi(z) = \lambda\Phi(x) + \Phi(y)$.

  So it means that $\Phi : F(\star) \to G(\star)$ is indeed linear.

So $\mathcal{E}$ is well defined and is in fact a covariant functor, indeed for all F flat covariant functor from the small category $\mathcal{O}_K$ to $\mathfrak{Sets}$ we have almost by definition $\mathcal{E}(\mathrm{id}_F) = \mathrm{id}_{F(\star)}$ and also for any $F, G, H : \mathcal{O}_K \to \mathfrak{Sets}$ three flat covariant functors from the small category $\mathcal{O}_K$ to $\mathfrak{Sets}$ and any Φ be a natural transformation from F to G and Ψ be a natural transformation from G to H, we have then immediately $\mathcal{E}(\Psi \circ \Phi) = \mathcal{E}(\Psi) \circ \mathcal{E}(\Phi)$.

Let us now show that $\mathcal{E}$ is fully faithful. Indeed let $F, G : \mathcal{O}_K \to \mathfrak{Sets}$ two flat covariant functors from the small category $\mathcal{O}_K$ to $\mathfrak{Sets}$, as the small category $\mathcal{O}_K$ has only one object we deduce when we look closely at the definitions that it is rigorously the same to consider a natural transformation from F to G than a linear application from $F(\star)$ to $G(\star)$.

Let us now finally check that $\mathcal{E}$ is essentially surjective then $\mathcal{E}$ will induce an equivalence of categories. As for essential surjectivity we work up to isomorphism we can directly take X an $\mathcal{O}_K$-module included in K.

Let us then define F as follows

$$F\begin{cases} \mathcal{O}_K \longrightarrow & \mathfrak{Sets} \\ \star \longmapsto & X \\ \lambda \in \mathcal{O}_K \longmapsto & \mu_\lambda : X \to X; x \mapsto \lambda x \end{cases}$$

F is obviously a covariant functor. One now has to check that it is flat.
Of course $X \neq \emptyset$ so property (i) of flatness is satified by F.
Let us now check that property (ii) is satisfied:
let $x, y \in X$, as $X \subset K$, then :

- first case, $x = 0 = y$, then we have $x = 0 \times 0$ and $y = 0 \times 0$
- second case ($x = 0$ and $y \neq 0$) or ($y = 0$ and $x \neq 0$), without loss of generality, let us just study the case $x = 0$ and $y \neq 0$, then $x = 0y$ and $y = 1y$
- third case $x \neq 0$ and $y \neq 0$
  let us write x and y as irreducible fractions : $x = \frac{x_n}{x_d}$ and $y = \frac{y_n}{y_d}$.
  Let us note $< x, y >$ the $\mathcal{O}_K$-module generated by x and y and $t := x_d y_d$.
  Then $t < x, y >$ is an $\mathcal{O}_K$-module included in $\mathcal{O}_K$ ie an ideal of $\mathcal{O}_K$.
  Since $\mathcal{O}_K$ is principal, let us note $\delta \in \mathcal{O}_K$ a generator of $t < x, y >$.
  So let $u, v \in \mathcal{O}_K$ such that $tx = u\delta$ and $ty = v\delta$.
  Now let us note $z := \frac{\delta}{t}$, and so we have $x = uz$, $y = vz$ and $z \in < x, y > \subset X$

Let us finally check that property (iii) is satisfied by F : let $a \in X$ and let $u, v \in \mathcal{O}_K$ such that $ua = va$.

- First case $a = 0$, then let us take $w = 0 \in \mathcal{O}_K$ and $z = 0 = a \in X$, and so $wz = a$ and $uw = vw$.
- Second case $a \neq 0$, then let us take $w = 1 \in \prime_K$ and $z = a \in X$, and so $wz = a$ and $uw = vw$.

And so the theorem 3.2 is proved. $\square$

4. Adelic interpretation of the geometric points of $\widetilde{\mathcal{O}_K}$

Let us denote $\mathbb{A}_K^f :=_{def} \prod'_{\mathfrak{p} \text{ prime}} K_{\mathfrak{p}}$ (restricted product) the ring of finite adeles of K and $\widetilde{\mathcal{O}_K} := \prod_{\mathfrak{p}\,\text{prime}} \mathcal{O}_{K,\mathfrak{p}}$ its maximal compact subring.

Let us recall the definition of Dedekind's complementary module (or inverse different), it is the fractionnal ideal $\mathfrak{D}_{\mathcal{O}_K} := \{x \in K / \mathrm{tr}(x.\mathcal{O}_K) \in \mathbb{Z}\}$ of $\mathcal{O}_K$ denoted $\mathfrak{D}_{\mathcal{O}_K}$.

Then we have the following lemmas :

Lemma 4.1. *A closed sub-$\mathcal{O}_K$-module of $\widetilde{\mathcal{O}_K}$ is an ideal of the ring $\widetilde{\mathcal{O}_K}$.*

Proof. Let J be a closed sub-$\mathcal{O}_K$-module of $\widetilde{\mathcal{O}_K}$.

To prove that J is an ideal of the ring $\widetilde{\mathcal{O}_K}$, we only have to check that $\forall \alpha \in \widetilde{\mathcal{O}_K}, \forall j \in J, \alpha j \in J$.

Since J is a sub-$\mathcal{O}_K$-module of $\widetilde{\mathcal{O}_K}$, we already have that $\forall \alpha \in \mathcal{O}_K, \forall j \in J, \alpha j \in J$.

Now let $\alpha \in \widetilde{\mathcal{O}}_K$.

Since $\mathcal{O}_K$ is dense in $\widetilde{\mathcal{O}_K}$ (thanks to strong approximation theorem), let $(\alpha_n)_{n \in \mathbb{N}} \in (\mathcal{O}_K)^{\mathbb{N}}$ such that $\alpha_n \xrightarrow[n \to \infty]{} \alpha$.

Then, since we have $\alpha_n j \xrightarrow[n \to \infty]{} \alpha j$ and $\forall n \in \mathbb{N}, \alpha_n j \in J$ and also J closed, we get that $\alpha j \in J$.

Therefore J is an ideal of the ring $\widetilde{\mathcal{O}_K}$. $\square$

Lemma 4.2. *Let J be a closed sub-$\mathcal{O}_K$-module of $\widetilde{\mathcal{O}_K}$. For each prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ the projection $\pi_\mathfrak{p}(J) \subset \mathcal{O}_\mathfrak{p}$ coincides with the intersection $(\{0\} \times \mathcal{O}_{K,\mathfrak{p}}) \cap J$ and is a closed ideal of $J_\mathfrak{p} \subset \mathcal{O}_{K,\mathfrak{p}}$, moreover one has $x \in J \Leftrightarrow \forall \mathfrak{p} \quad prime, \pi_\mathfrak{p}(x) \in J_\mathfrak{p}$*

Proof. Let J be a closed sub-$\mathcal{O}_K$-module of $\widetilde{\mathcal{O}_K}$.

Thanks to the preceding lemma, J is an ideal of the ring $\widetilde{\mathcal{O}_K}$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ and let us note $a_\mathfrak{p}$ the finite adele which is zero everywhere except in $\mathfrak{p}$ where it is equal to 1.

Then we have that $\pi_\mathfrak{p}(J) = a_\mathfrak{p} J \subset J$ since J is an ideal.

And by definition $\pi_\mathfrak{p}(J) \subset \mathcal{O}_{K,\mathfrak{p}} \simeq \{0\} \times \mathcal{O}_{K,\mathfrak{p}}$.

So $\pi_\mathfrak{p}(J) \subset (\{0\} \times \mathcal{O}_{K,\mathfrak{p}}) \cap J$.

The converse inclusion is obvious, so all in all we have indeed that

$$\pi_\mathfrak{p}(J) = (\{0\} \times \mathcal{O}_{K,\mathfrak{p}}) \cap J$$

Now since $\mathcal{O}_{K,\mathfrak{p}} \simeq \{0\} \times \mathcal{O}_{K,\mathfrak{p}} \subset \widetilde{O}_K$ and since J is an ideal of $\widetilde{\mathcal{O}_K}$, we have that $\mathcal{O}_{K,\mathfrak{p}}.J \subset J$.

So $\mathcal{O}_{K,\mathfrak{p}}.\pi_\mathfrak{p}(J) = \mathcal{O}_{K,\mathfrak{p}}.\left((\{0\} \times \mathcal{O}_{K,\mathfrak{p}}) \cap J\right) \subset \left((\{0\} \times \mathcal{O}_{K,\mathfrak{p}}) \cap J\right) = \pi_\mathfrak{p}(J)$.

And since $\{0\} \times \mathcal{O}_{K,\mathfrak{p}}$ and J are closed subgroups of $\widetilde{\mathcal{O}_K}$, we get that $\pi_\mathfrak{p}(J)$ is a closed subgroup of $\mathcal{O}_{K,\mathfrak{p}}$, and so all in all $\pi_\mathfrak{p}(J)$ is a closed ideal of $\mathcal{O}_{K,\mathfrak{p}}$.

Finally the implication $x \in J \Rightarrow \forall \mathfrak{p}, \pi_\mathfrak{p}(x) \in \pi_\mathfrak{p}(J)$, is obvious.

Conversely let $x \in \widetilde{\mathcal{O}_K}$ such that $\forall \mathfrak{p}, \pi_\mathfrak{p}(x) \in \pi_\mathfrak{p}(J)$.

Since $\forall \mathfrak{p}, \pi_\mathfrak{p}(x) \in \pi_\mathfrak{p}(J) = (\{0\} \times \mathcal{O}_{K,\mathfrak{p}}) \cap J \subset J$.

Let us note for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, $a_\mathfrak{p} \in \widetilde{\mathcal{O}_K}$ the finite adele which is zero everywhere except in $\mathfrak{p}$ where it is equal to 1.

So $x = \sum_\mathfrak{p} a_\mathfrak{p} \pi_\mathfrak{p}(x) \in J$ since J is an ideal of $\widetilde{\mathcal{O}_K}$ and the sum is finite because $\pi_\mathfrak{p}(x) = 0$ almost everywhere because $x \in \widetilde{O}_K$.

Therefore we have proved that $x \in J \Leftrightarrow \forall \mathfrak{p}, \pi_\mathfrak{p}(x) \in \pi_\mathfrak{p}(J)$ □

Lemma 4.3. *For any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, any ideal of $\mathcal{O}_{K,\mathfrak{p}}$ is principal.*

Proof. Since $\mathcal{O}_{K,\mathfrak{p}}$ is a complete discrete valuation ring, every ideal of $\mathcal{O}_{K,\mathfrak{p}}$ is of the form $\pi^n \mathcal{O}_{K,\mathfrak{p}}$ with $n \in \mathbb{N}$ and π an element of valuation 1. □

With the lemmas 4.1, 4.2 and 4.3 and Pontrjagin duality, one can show that :

Theorem 4.1. *Any non trivial sub-$\mathcal{O}_K$-module of K is uniquely of the form $H_a := \{q \in K | aq \in \widehat{\mathfrak{D}_{\mathcal{O}_K}}\}$ where $a \in \mathbb{A}_K^f / \widetilde{\mathcal{O}_K}^\times$ and $\widehat{\mathfrak{D}_{\mathcal{O}_K}}$ denotes the profinite completion of the different.*

Proof. Let us recall that Tate's character for finite adeles is

$$\chi_{Tate} := \begin{cases} \mathbb{A}_K^f \longrightarrow & \mathbb{S}^1 \\ (x_\mathfrak{p}) \longmapsto & \displaystyle\prod_\mathfrak{p} \psi_{\mathrm{char}(K_\mathfrak{p})}(\mathrm{tr}(x_\mathfrak{p})) \end{cases}$$

where for all prime number p, ψ_p is $\psi_p := \mathbb{Q}_p \xrightarrow{\text{can}} \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\exp(2\pi\imath\bullet)} \mathbb{S}^1$.

Then the pairing $\langle k, a \rangle = \chi_{\mathrm{Tate}}(ka), \forall k \in K/\mathcal{O}_K, \forall a \in \mathfrak{D}_{\mathcal{O}_K}$ identifies $\widehat{\mathfrak{D}_{\mathcal{O}_K}}$ with the Pontrjagin dual of $K/\mathcal{O}_K$ by a direct application of proposition VIII.4.12 of [36].

Let us now prove the theorem.

Let H be a non trivial $\mathcal{O}_K$-module included in K. If H is included in $\mathcal{O}_K$, then H is an integral ideal and the result is obvious.

Let H be a non trivial $\mathcal{O}_K$-module included in K and containing $\mathcal{O}_K$, it is completely determined by its image $\mathcal{H}$ in $K/\mathcal{O}_K$.

Then the Pontrjagin duality implies that : $\mathcal{H} = (\mathcal{H}^\perp)^\perp = \{k \in K/\mathcal{O}_K / \forall x \in \mathcal{H}^\perp, \langle k, x\rangle = 1\}$ and $\mathcal{H}^\perp = \{x \in \mathfrak{D}_{\mathcal{O}_K} / \forall k \in K/\mathcal{O}_K, \langle k, x\rangle = 1\}$.

We also have that $\mathfrak{D}_{\mathcal{O}_K} = \prod_{\mathfrak{p} \ni F} \mathcal{O}_{K,\mathfrak{p}} \times \prod_{\mathfrak{p} \in F} \mathfrak{p}^{-n_\mathfrak{p}}$ where F is a finite set included in $\mathrm{Spec}(\mathcal{O}_K)$.

Since $\mathcal{H}^\perp \subset \mathfrak{D}_{\mathcal{O}_K}$, we have that $\left(\prod_{\mathfrak{p}\in F} \mathfrak{p}^{-n_\mathfrak{p}}\right).\mathcal{H}^\perp \subset \widetilde{\mathcal{O}_K}$ and $\left(\prod_{\mathfrak{p}\in F} \mathfrak{p}^{-n_\mathfrak{p}}\right).\mathcal{H}^\perp$ is also a closed sub-$\mathcal{O}_K$-module of $\widetilde{\mathcal{O}_K}$.

So thanks to the three last lemmas, we know that a closed sub-$\mathcal{O}_K$-module of $\widetilde{\mathcal{O}_K}$ can be written in the form $a.\widetilde{\mathcal{O}_K}$ with $a \in \widetilde{\mathcal{O}_K}$ and this a is unique up to multiplication by an element of $\widetilde{\mathcal{O}_K}^\star$, so we get the result. ☐

And so we get that:

Corollary 4.1. *There is a canonical bijection between the quotient space $\mathbb{A}_K^f/(K^\star(\prod \mathcal{O}_\mathfrak{p}^\star \times \{1\}))$ and the isomorphisms classes of the (geometric) points of the topos $\widehat{\mathcal{O}_K}$*

Proof. Thanks to theorem 3.2 and 4.1, any non trivial point of the topos $\widehat{\mathcal{O}_K}$ is obtained from a $\mathcal{O}_K$-module H_a of rank 1 included in K where $a \in \mathbb{A}_K^f/\widetilde{\mathcal{O}_K}^\times$. Two elements $a, b \in \mathbb{A}_K^f/\widetilde{\mathcal{O}_K}^\times$ determine isomorphic $\mathcal{O}_K$-modules H_a and H_b of rank 1 included in K. An isomorphism between $\mathcal{O}_K$-modules of rank 1 included in K is given by the multiplication by an element $k \in K^\star$ so that $H_b = k.H_a$ and then by theorem 3.1, $a = kb$ in $\mathbb{A}_K^f/\widetilde{\mathcal{O}_K}^\times$. Therefore the result is proved. ☐

5. The geometric points of the arithmetic site for an imaginary quadratic field with class number 1

In the rest of this paper we will restrict our attention to the simple case where we have only a finite group of symetries, therefore in the sequel we assume that K is an imaginary quadratic number field. Moreover we assume its class number is 1.

Let us denote $\widetilde{\mathcal{C}_{\mathcal{O}_K}}$ the set of $\emptyset$, $\{0\}$ and the convex polygons of the real plane (identified with $\mathbb{C}$) with non empty interior, with center 0, whose vertices have affix in $\mathcal{O}_K$ and who are invariant by the action of the elements $\mathcal{U}_K$.

Lemma 5.1. $\left(\widetilde{\mathcal{C}_{\mathcal{O}_K}}, Conv(\bullet \cup \bullet), +\right)$ *is an (idempotent) semiring whose neutral element for the first law is $\emptyset$ and for the second law is $\{0\}$*

Proof. A convex polygon is the convex hull of a finite number of points. Let $\mathcal{P} := \mathrm{Conv}\left(\bigcup_{i\in[|1,n|]} P_i\right)$ and $\widetilde{\mathcal{P}} := \mathrm{Conv}\left(\bigcup_{j\in[|1,m|]} \tilde{P}_j\right)$ be two polygones whose vertices have their affix in $\mathcal{O}_K$ and which are invariant by the action of the elements $\mathcal{U}_K$ (here P_i and $\tilde{P}_j$ mean both the points of the plane and their affixes).

Then $\mathrm{Conv}(\mathcal{P} \cup \widetilde{\mathcal{P}}) = \mathrm{Conv}(\{P_i, i \in [|1, n|]\})$ and $\mathcal{P} + \widetilde{\mathcal{P}} = \mathrm{Conv}(\{P_i + \tilde{P}_j, (i, j) \in [|1, n|] \times [|1, m|]\})$ are also polygons. From those formulae one sees also immediately that they have vertices wich have affix in $\mathcal{O}_K$ and who are invariant by the action of the elements $\mathcal{U}_K$ and that they have non empty interior and center 0. These formulae still work when one is either $\emptyset$ or $\{0\}$.

So $\widetilde{\mathcal{C}_{\mathcal{O}_K}}$ is a sub semiring of the well known semiring of the convex sets of the plane with the operations convex hull of the union and the Minkowski operation, so we have the result. ☐

Lemma 5.2. *$\mathcal{O}_K$ acts multiplicatively by direct complex similitudes on $\widetilde{\mathcal{C}_{\mathcal{O}_K}}$, that is to say that $\alpha \in \mathcal{O}_K \backslash \{0\}$ acts as the direct similitude $(\mathbb{C} \to \mathbb{C}, z \mapsto \alpha z)$ and $\emptyset$ is sent to $\emptyset$ and 0 sends everything to $\{0\}$ except $\emptyset$ which is sent to $\emptyset$.*

Proof. Direct similitudes preserve extremal points of convex sets and so we get the result. □

Lemma 5.3. *For $K = \mathbb{Q}(\imath)$ and $K = \mathbb{Q}(\imath\sqrt{3})$ (in other words the only cases where $\mathcal{U}_K$ is greater than $\{1, -1\}$),let us denote D_K the convex polygon (with center 0) whose vertices are the elements of $\mathcal{U}_K$. Then*

$$\widetilde{\mathcal{C}_{\mathcal{O}_K}} = \mathit{Semiring}(\{h.D_K, h \in \mathcal{O}_K\}) \cup \{\emptyset\}$$

ie the semiring generated by $\{h.D_K, h \in \mathcal{O}_K\}$ (to which we add $\emptyset$), it has also an action of $\mathcal{O}_K$ on it by direct similitudes of the complex plane.

Proof. Let us first note $\omega_K := \begin{cases} \imath & \text{if} \quad K = \mathbb{Q}(\imath) \\ \dfrac{1+\imath\sqrt{3}}{2} & \text{if} \quad K = \mathbb{Q}(\imath\sqrt{3}) \end{cases}$ and $\theta_K := \mathrm{Arg}(\omega_K) = \begin{cases} \dfrac{\pi}{2} & \text{if} \quad K = \mathbb{Q}(\imath) \\ \dfrac{\pi}{3} & \text{if} \quad K = \mathbb{Q}(\imath\sqrt{3}) \end{cases}$

and $\sigma_K = \begin{cases} 4 & \text{if} \quad K = \mathbb{Q}(\imath) \\ 6 & \text{if} \quad K = \mathbb{Q}(\imath\sqrt{3}) \end{cases}$.

Of course $\mathrm{Semiring}(\{h.D_K, h \in \mathcal{O}_K\}) \subset \widetilde{\mathcal{C}_{\mathcal{O}_K}}$

By definition of $\mathrm{Semiring}(\{h.D_K, h \in \mathcal{O}_K\})$, $\{\emptyset, \{0\}\} \subset \mathrm{Semiring}(\{h.D_K, h \in \mathcal{O}_K\})$.

Let $C \in \widetilde{\mathcal{C}_{\mathcal{O}_K}} \backslash \{\emptyset, \{0\}\}$.

Let us note $\mathcal{S}_{+,C}$ the set of vertices of C whose arguments (modulo 2π) belong to $[0, \theta_K[$ and $\mathcal{S}_C$ the set of all vertices of C.

Let us now show that $\mathcal{S}_C = \bigcup_{u \in \mathcal{U}_K} u\mathcal{S}_{+,C}$

Indeed since C is invariant under the action of $\mathcal{U}_K$ we have that $\bigcup_{u \in \mathcal{U}_K} u\mathcal{S}_{+,C} \subset \mathcal{S}_C$

Now let $s \in \mathcal{S}_C$, then let $k \in [|0, \sigma_K|]$ and $\alpha \in [0, \theta_K[$ such that $\mathrm{Arg}(s) \equiv k\theta_K + \alpha(2\pi)$.

So $\mathrm{Arg}(\omega_K^{-k} s) \equiv \alpha(2\pi)$ and $\omega_K^{-k} \in \mathcal{U}_K$

And since C is invariant by the action of $\mathcal{U}_K$, $\zeta_K^{-k} s \in \mathcal{S}_{+,C}$

So $s \in \omega_K^k \mathcal{S}_{+,C} \subset \bigcup_{u \in \mathcal{U}_K} u \in \mathcal{S}_{+,C}$

And so $\mathcal{S}_C \subset \bigcup_{u \in \mathcal{U}_K} u\mathcal{S}_{+,C}$ which ends the proof of $\mathcal{S}_C = \bigcup_{u \in \mathcal{U}_K} u\mathcal{S}_{+,C}$

And so finally we get that $C = \mathrm{Conv}\left(\bigcup_{s \in \mathcal{S}_{+,C}} sD_K\right) \in \mathrm{Semiring}(\{h.D_K, h \in \mathcal{O}_K\})$

So we conclude that $\mathrm{Semiring}(\{h.D_K, h \in \mathcal{O}_K\}) \supset \widetilde{\mathcal{C}_{\mathcal{O}_K}}$ and so $\widetilde{\mathcal{C}_{\mathcal{O}_K}} = \mathrm{Semiring}(\{h.D_K, h \in \mathcal{O}_K\})$ □

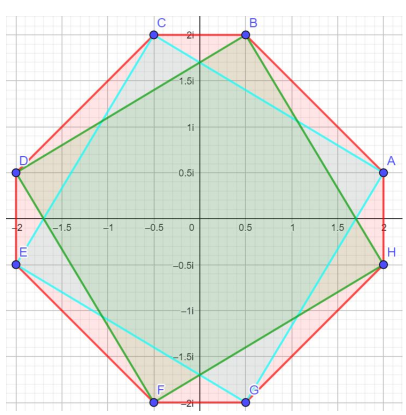

Figure 1: $ABCDEFGH = \mathrm{Conv}(z_A D_{\mathbb{Q}(\imath)} \cup z_B D_{\mathbb{Q}(\imath)}), z_A = 2 + \frac{1}{2}\imath, z_B = \frac{1}{2} + 2\imath$

Remark 5.1. *The definition of D_K in the preceding lemma does not make any sense for the seven other quadratic imaginary number fields with class number 1.*

Then for $K := \mathbb{Q}(\sqrt{d})$ with $\mathcal{U}_K = \{1, -1\}$, we adopt the following definition :

- when $d \equiv 2, 3(4)$ define $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$, define for D_K to be the convex polygone whose vertices are $1, \sqrt{d}, -1, -\sqrt{d}$
- when $d \equiv 1(4)$ define $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$, define for D_K to be the convex polygone whose vertices are $1, \frac{1+\sqrt{d}}{2}, -1, -\frac{1+\sqrt{d}}{2}$.

Definition 5.1. *Let us then denote*

$$\mathcal{C}_{\mathcal{O}_K} := \textit{Semiring}(\{h.D_K, h \in \mathcal{O}_K\}) \cup \{\emptyset\}$$

the semiring generated by $\{h.D_K, h \in \mathcal{O}_K\}$.

Lemma 5.4. *Let K be a quadratic imaginary number field with class number one. Then $\widetilde{\mathcal{C}_{\mathcal{O}_K}} = \mathcal{C}_{\mathcal{O}_K}$ if and only if $K = \mathbb{Q}(\imath)$ or $K = \mathbb{Q}(\imath\sqrt{3})$, ie if and only if $\{1, -1\} \subsetneq \mathcal{U}_K$, ie if and only if "we have enough symmetries".*

Proof. $\Leftarrow$ was shown in lemma 4.3

Let us now prove $\Rightarrow$.

Let K be a quadratic imaginary number field with class number one different from $K = \mathbb{Q}(\imath)$ and $K = \mathbb{Q}(\imath\sqrt{3})$.

So K is one of these number fields $\mathbb{Q}(\imath\sqrt{2})$, $\mathbb{Q}(\imath\sqrt{7})$, $\mathbb{Q}(\imath\sqrt{11})$, $\mathbb{Q}(\imath\sqrt{19})$, $\mathbb{Q}(\imath\sqrt{43})$, $\mathbb{Q}(\imath\sqrt{67})$ and $\mathbb{Q}(\imath\sqrt{163})$ and for all of these the group of units is reduced to $\{\pm 1\}$

- for $K = \mathbb{Q}(\imath\sqrt{2})$, D_K is the polygon with vertices $1, \imath\sqrt{2}, -1, -\imath\sqrt{2}$.

  Let us then note P the polygon with vertices $3, \imath\sqrt{2}, -3, -\imath\sqrt{2}$, we have immediately $P \in \widetilde{\mathcal{C}_{\mathcal{O}_K}}$.

  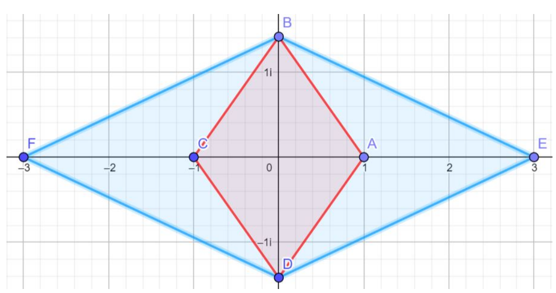

  Figure 2: $ABCD = D_{\mathbb{Q}(\imath\sqrt{2})}$, $P = EBFD$

  But $P \notin \mathcal{C}_{\mathcal{O}_K}$, indeed the only polygones in $\mathcal{C}_{\mathcal{O}_K}$ which have 3 as a vertex are $3D_K$, $D_K + D_K + D_K$, $D_K + \imath\sqrt{2}D_K$ but none of them have $\imath\sqrt{2}$ as a vertex so none of them is equal to P

- for $K = \mathbb{Q}(\imath\sqrt{7})$, D_K is the polygon with vertices $1, \frac{1+\imath\sqrt{7}}{2}, -1, \frac{-1-\imath\sqrt{7}}{2}$.

  Let us then note P the polygon with vertices $2, \frac{1+\imath\sqrt{7}}{2}, -2, \frac{-1-\imath\sqrt{7}}{2}$, we have immediately $P \in \widetilde{\mathcal{C}_{\mathcal{O}_K}}$.

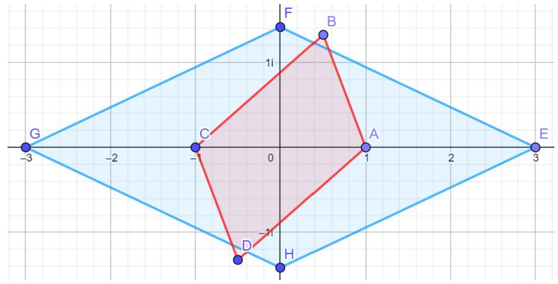

Figure 3: $ABCD = D_{\mathbb{Q}(\imath\sqrt{7})}$, $P = EFGH$

But $P \notin \mathcal{C}_{\mathcal{O}_K}$, indeed the only polygones in $\mathcal{C}_{\mathcal{O}_K}$ which have 2 as a vertex are $2D_K$, $D_K + D_K$, $\frac{-1-\imath\sqrt{7}}{2}D_K$ but none of them have $\frac{1+\imath\sqrt{7}}{2}$ as a vertex so none of them is equal to P

- for the other five cases let us write $K = \mathbb{Q}(\imath\sqrt{d})$ with $d \in \{11, 19, 43, 67, 163\}$, then D_K is the polygon with vertices $1, \frac{1+\imath\sqrt{d}}{2}, -1, \frac{-1-\imath\sqrt{d}}{2}$.

  Let us then note P the polygon with vertices $2, \frac{1+\imath\sqrt{d}}{2}, -2, \frac{-1-\imath\sqrt{d}}{2}$, we have immediately $P \in \widetilde{\mathcal{C}_{\mathcal{O}_K}}$.

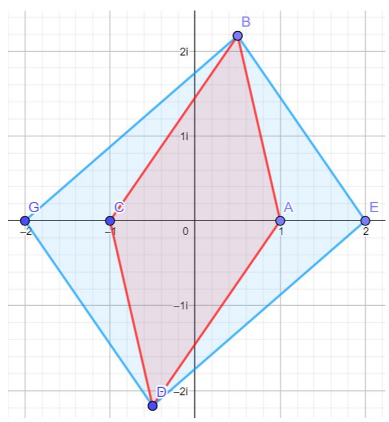

Figure 4: $ABCD = D_{\mathbb{Q}(\imath\sqrt{19})}$, $P = EFGH$

But $P \notin \mathcal{C}_{\mathcal{O}_K}$, indeed the only polygones in $\mathcal{C}_{\mathcal{O}_K}$ which have 2 as a vertex are $2D_K$ and $D_K + D_K$ but none of them have $\frac{1+\imath\sqrt{d}}{2}$ as a vertex so none of them is equal to P.

$\square$

Remark 5.2. *Why this choice of structural sheaf?*

It is because following the strategy of [10], we would like now to put a structural sheaf on the topos $\widehat{\mathcal{O}_K}$ which is an idempotent semiring, and that the points of this semiringed topos with values in something to be isomorphic to $\mathbb{A}_K^f \times \mathbb{C}/(K^\star(\prod \mathcal{O}_\mathfrak{p}^\star \times \{1\}))$.

However the set $\mathcal{U}_K$ acts trivially $\mathbb{A}_K^f/(K^\star(\prod \mathcal{O}_\mathfrak{p}^\star \times \{1\}))$ but not on $\mathbb{C}$.

This has the following consequence : let $(H, \lambda) \in ((\mathbb{A}_K^f/(\prod \mathcal{O}_\mathfrak{p}^\star) \times \{1\}) \times \mathbb{C})/K^\star$, we have $(H, \lambda) \simeq (r.H, r\lambda)$ for $r \in K^\star$, so for $r_0 \in \mathcal{U}_K$ and so $(H, \lambda) = (H, r_0\lambda)$, so λ and $r_0\lambda$ induce *the same embedding of the fiber of the structural sheaf in H into something. So 1 and r_0 should have the same action the fiber of the structural sheaf in H and more generally, 1 and r_0 should have the same action on the structural sheaf but since the action of 1 is the identity.*

We come to the conclusion that the set $\mathcal{U}_K$ of units of $\mathcal{O}_K$ should be seen as the set of symmetries *of the structural sheaf.*

Remark 5.3. *Why this choice of D_K for $K = \mathbb{Q}(\sqrt{d})$? An heuristic explanation could be that since $\mathcal{O}_K$ is a lattice in the plane, one should view it as a tiling puzzle and one of the smallest tile is the triangle $0, 1, \sqrt{d}$ or $\frac{1+\sqrt{d}}{2}$ depending on d (the last two elements form a base of $\mathcal{O}_K$ viewed as a $\mathbb{Z}$ module). Then we let the elements of $\mathcal{U}_K$ act on this tile, and the union of all tiles. We get this way D_K when $K = \mathbb{Q}(\imath)$ or $K = \mathbb{Q}(\imath\sqrt{3})$ (there were enough symmetries), in the other cases in order to get D_K we have to get the convex enveloppe of the union of all the tiles.*

Definition 5.2. *Let K be a quadratic imaginary number field with class number 1.The arithmetic site for K is the datum $\left(\widehat{\mathcal{O}_K}, \mathcal{C}_{\mathcal{O}_K}\right)$ where the topos $\widehat{\mathcal{O}_K}$ is endowed with the structure sheaf $\mathcal{C}_{\mathcal{O}_K}$ viewed as a semiring in the topos using the action of $\mathcal{O}_K$ by similitudes.*

Theorem 5.1. *The stalk of the structure sheaf $\mathcal{C}_{\mathcal{O}_K}$ at the point of the topos $\widehat{\mathcal{O}_K}$ associated with the $\mathcal{O}_K$ module H is canonically isomorphic to $\mathcal{C}_H :=$ Semiring$(\{h.D_K, h \in H\})$ the semiring generated by the polygons $h \times D_K$ with $h \in H$ (to which we add $\emptyset$) viewed here in this context as a semiring.*

Proof. By theorem 3.2 to the point of $\widehat{\mathcal{O}_K}$ associated to the $\mathcal{O}_K$ module $H \subset K$ corresponds to the flat functor $F_H : \mathcal{O}_K$ which associates to the only object $\star$ of the small category $\mathcal{O}_K$ the $\mathcal{O}_K$ module H and the endomorphism indexed by k the multiplication $F(k)$ by k in $H \subset K$.

As said in [10] and shown in [27], the inverse image functor associated to this point is the functor which associates to any $\mathcal{O}_K$ equivariant set its geometric realization

$$|-|_{F_H} \begin{cases} \mathcal{O}_K - \text{equivariant sets} \longrightarrow & \mathfrak{Sets} \\ C \longmapsto & |C|_{F_H} := (C \times_{\mathcal{O}_K} H)/\sim \end{cases}$$

where $\sim$ is the equivalence relation stating the equivalence of the couples $(C, F(k)h) \sim (kC, h)$.

Let us recall as in [10] that thanks to the property (ii) of flatness of F_H, we have

$$(C, h) \sim (C', h') \Leftrightarrow \exists \hat{h} \in H, \exists k, k' \in \mathcal{O}_K, \quad \text{such that} \quad k\hat{h} = h \quad \text{and} \quad k'\hat{h} = h' \quad \text{and} \quad kC = k'C'$$

But we would like to have a better understanding and description of the fiber.

The natural candidate we imagine to be the fiber is $\mathcal{C}_H :=$ Semiring$\{hD_K, h \in H\}$. Let us now show that this intuition is true.

Let us consider the map $\beta : \begin{cases} \mathcal{C}_{\mathcal{O}_K} \times H \to & \mathcal{C}_H \\ (C, h) \mapsto & hC \end{cases}$

Let us show that β is compatible with the equivalence relation $\sim$.

Let $(C, h), (C', h') \in \mathcal{C}_{\mathcal{O}_K} \times H$ such that $(C, h) \sim (C', h')$.

So let $\hat{h} \in H$, $k, k' \in \mathcal{O}_K$ such that $k\hat{h} = h$ and $k'\hat{h} = h'$ and $kC = k'C'$.

Then $\beta(C, h) = hC = k\hat{h}C = \hat{h}kC = \hat{h}k'C' = k'\hat{h}C' = h'C' = \beta(C', h')$.

So β is compatible with the equivalence relation.

And so β induces another application again noted β from $|\mathcal{C}_{\mathcal{O}_K}|_{F_H}$ to $\mathcal{C}_H$.

Let us now show that β is surjective.

Let $C \in \mathcal{C}_H$, let us note $\mathcal{S}_C$ the set of vertices of C. We have $C = \text{Conv}\left(\bigcup_{s \in \mathcal{S}_C} s.D_K\right)$

As $\mathcal{S}_C$ is a finite set, let $q \in \mathcal{O}_K \backslash \{0\}$ such that $\forall s \in \mathcal{S}_C, qs \in \mathcal{O}_K$.

Then $I = q < \mathcal{S}_C > \subset \mathcal{O}_K$ is an ideal where $< \mathcal{S}_C >$ means the sub-$\mathcal{O}_K$-module included in K generated by the elements of $\mathcal{S}_C$.

Since $\mathcal{O}_K$ is principal, let $d \in \mathcal{O}_K$ such that $I =< d >$.

So $\exists\, (a_s)_{s \in \mathcal{S}_C} \in (\mathcal{O}_K)^{\mathcal{S}_C}$, $d = \sum_{s \in \mathcal{S}_C} a_s q s$, so $\frac{d}{q} \in < \mathcal{S}_C > \subset H$.

And so $\beta(\text{Conv}\left(\bigcup_{s \in \mathcal{S}_C} a_s D_K, \frac{d}{q}\right) = C$, so β is surjective.

Let us now show that β is injective.

Let $(C, h), (C', h') \in |\mathcal{C}_{\mathcal{O}_K}|_{F_H}$ such that $\beta(C, h) = \beta(C', h')$, ie such that $hC = h'C'$.

By property (ii) of flatness of F_H, there exists $\hat{h} \in H$, $k, k' \in \mathcal{O}_K$ such that $h = k\hat{h}$ and $h' = k'\hat{h}$.

So we have immediately that $k\hat{h}C = k'\hat{h}C'$.

First case $\hat{h} \neq 0$ and so $kC = k'C'$ and so by definition of $\sim$, $(C, h) \sim (C', h')$.

Second case $\hat{h} = 0$, then $h = 0 = h'$ and we have also $0.C = 0.C'$ and so by definition of $\sim$, we have $(C, 0) \sim (C', 0)$.

Finally β is injective and so bijective, so $|\mathcal{C}_{\mathcal{O}_K}|_{F_H} \simeq \mathcal{C}_H$ so we have a good understanding of the fiber now.

Now we would like to know what is the semiring structure on the fiber induced by the semiring structure on $\mathcal{C}_{\mathcal{O}_K}$.

Here we follow once again [10]. The two operations $\text{Conv}(\bullet \cup \bullet)$ and $+$ of $\mathcal{C}_{\mathcal{O}_K}$ determine canonically maps of $\mathcal{O}_K$-spaces $\mathcal{C}_{\mathcal{O}_K} \times \mathcal{C}_{\mathcal{O}_K} \to \mathcal{C}_{\mathcal{O}_K}$.

Applying the geometric realization functor, we get that the induced operations on the fiber correspond to the induced maps $|\mathcal{C}_{\mathcal{O}_K} \times \mathcal{C}_{\mathcal{O}_K}|_{F_H} \to |\mathcal{C}_{\mathcal{O}_K}|_{F_H}$.

But since the geometric realization functor commutes with finite limits, we get the following identification (one could also prove it by hand) :

$$\begin{cases} |\mathcal{C}_{\mathcal{O}_K} \times \mathcal{C}_{\mathcal{O}_K}|_{F_H} = & |\mathcal{C}_{\mathcal{O}_K}|_{F_H} \times |\mathcal{C}_{\mathcal{O}_K}|_{F_H} \\ \quad (C, C', h) \mapsto & (C, h) \times (C', h) \end{cases}$$

And thanks to this identification one just needs to study the induced maps

$$|\mathcal{C}_{\mathcal{O}_K}|_{F_H} \times |\mathcal{C}_{\mathcal{O}_K}|_{F_H} \to |\mathcal{C}_{\mathcal{O}_K}|_{F_H}$$

However we already have :

$\forall C, C' \in \mathcal{C}_{\mathcal{O}_K}, \forall h \in H, h\text{Conv}(C \cup C') = \text{Conv}(hC \cup hC')$ and $h(C + C') = hC + hC'$.

So we conclude that the semiring laws on the fiber induced by the semiring laws $\text{Conv}(\bullet \cup \bullet)$ and $+$ of $\mathcal{C}_{\mathcal{O}_K}$ are the laws $\text{Conv}(\bullet \cup \bullet)$ and $+$ on $\mathcal{C}_H$.

□

Remark 5.4. *With the same notations as in the last theorem, when $K = \mathbb{Q}(\imath)$ or $K = \mathbb{Q}(\imath\sqrt{3})$, thank to the symmetries, $\mathcal{C}_H$ is the semiring of convex compact polygons with non empty interior and center zero and vertices with affixes in H and symmetric under the action of $\mathcal{U}_K$ (and with $\emptyset$ and $\{0\}$).*

Proposition 5.1. *The set of global sections $\Gamma(\widehat{\mathcal{O}_K}, \mathcal{C}_{\mathcal{O}_K})$ of the structure sheaf are given by the semiring $\{\emptyset, \{0\}\} \simeq \mathbb{B}$.*

Proof. As in [10] we recall that for a Grothendieck topos $\mathcal{T}$ the global section functor $\Gamma : \mathcal{T} \to \mathfrak{Sets}$ is given by $\Gamma(E) := \text{Hom}_{\mathcal{T}}(1, E)$ where E is an object in the topos and 1 the final object in the topos. In the special case of a topos of the form $\widehat{\mathcal{C}}$ where $\mathcal{C}$ is a small category, a global section of a contravariant functor $P : \mathcal{C} \to \mathfrak{Sets}$ is a function which assigns to each object C of $\mathcal{C}$ an element $\gamma_C \in P(C)$ in such a way that for any morphism $f : D \to C \in \mathrm{Hom}_{\mathcal{C}}(D, C)$ and D any another object of $\mathcal{C}$ one has $P(f)\gamma_C = \gamma_D$ (as explained in [27] Chap I.6.(9)).

When we apply this definition to our special case which is the small category $\mathcal{O}_K$, the global sections of sheaf $\mathcal{C}_{\mathcal{O}_K}$ are the elements of $\mathcal{C}_{\mathcal{O}_K}$ which are invariant by the action of $\mathcal{O}_K$, so the global sections of $\mathcal{C}_{\mathcal{O}_K}$ are only $\emptyset$ and $\{0\}$ and $\{\emptyset, \{0\}\} \simeq \mathbb{B}$. So the set of global sections is isomorphic to $\mathbb{B}$. □

Let us now denote for K an imaginary quadratic number field of class number 1 :

Definition 5.3. *$\mathcal{C}_{K,\mathbb{C}} :=$ Semiring$(\{h.D_K, h \in \mathbb{C}\}) \cup \{\emptyset\}$ the semiring generated by $\{h.D_K, h \in \mathbb{C}\}$ (to which we add $\emptyset$ the neutral element for Conv$(\bullet \cup \bullet)$ which is absorbant for $+$).*

Remark 5.5. *When $K = \mathbb{Q}(\imath)$ or $K = \mathbb{Q}(\imath\sqrt{3})$, due to the symmetries, $\mathcal{C}_{K,\mathbb{C}}$ is the semiring of convex compact polygons with non empty interior and center zero and symmetric under the action of $\mathcal{U}_K$ (and with $\emptyset$ and $\{0\}$).*

We have the following interesting result on $\mathcal{C}_{K,\mathbb{C}}$:

Definition 5.4. *Let us denote $Aut_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$ the group of $\mathbb{B}$-automorphisms of $\mathcal{C}_{K,\mathbb{C}}$.*

Let us give the definition of direct $\mathbb{B}$-automorphisms of $\mathcal{C}_{K,\mathbb{C}}$: an element f of $Aut_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$ is an application from $\mathcal{C}_{K,\mathbb{C}}$ to $\mathcal{C}_{K,\mathbb{C}}$ such that :

- *f is bijective*
- *$\forall C, D \in \mathcal{C}_{K,\mathbb{C}}, f(Conv(C \cup D) = Conv(f(C) \cup f(D))$*
- *$\forall C, D \in \mathcal{C}_{K,\mathbb{C}}, f(C + D) = f(C) + f(D)$*

.

Before showing the interesting result on $\mathrm{Aut}_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$, let us prove the following lemma :

Lemma 5.5. *We have that for any $f \in Aut_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$:*

- *$f(\emptyset) = \emptyset$*
- *$f(\{0\}) = \{0\}$*
- *$\forall C \in \mathcal{C}_{K,\mathbb{C}}, \forall \lambda \in \mathbb{R}^+, f(\lambda C) = \lambda f(C)$*

  In other words, the elements of $Aut_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$ commute with real homotheties.

Proof. Let $f \in \mathrm{Aut}_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$.

We can first deduce, from the definition of f, that $\forall C, D \in \mathcal{C}_{K,\mathbb{C}}, C \subset D \Rightarrow f(C) \subset f(D)$.

- Since f is bijective, let $E \in \mathcal{C}_{K,\mathbb{C}}$ be the only element of $\mathcal{C}_{K,\mathbb{C}}$ such that $f(E) = \emptyset$.

  Since $\emptyset \subset E$, we have that $f(\emptyset) \subset f(E) = \emptyset$. So $f(\emptyset) = \emptyset$.

  Therefore we have in fact that $E = \emptyset$, so in other otherwords the only element of $\mathcal{C}_{K,\mathbb{C}}$ whose image by f is $\emptyset$ is $\emptyset$.

- Since f is bijective and since $f(\emptyset)=\emptyset$, let $Z \in \mathcal{C}_{K,\mathbb{C}}\backslash\{\emptyset\}$ be the only element of $\mathcal{C}_{K,\mathbb{C}}$ such that $f(Z)=\{0\}$.

  But $\{0\} \subset Z$ so $f(\{0\}) \subset \{0\}$. But $f(\{0\}) \neq \emptyset$ since f is bijective and $f(\emptyset)=\emptyset$ so $f(\{0\})=\{0\}$.

  Therefore $Z=\{0\}$ and so $\{0\}$ is the only element of $\mathcal{C}_{K,\mathbb{C}}$ whose image by f is $\{0\}$.

- Let $f \in \mathrm{Aut}_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$.

  Then using the definition of $\mathrm{Aut}_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$, by induction, we can show that

  $$\forall C \in \mathcal{C}_{K,\mathbb{C}}, \forall \lambda \in \mathbb{N}, f(\lambda C)=\lambda f(C)$$

  Since $\{\pm 1\} \subset \mathcal{U}_K$ and since the elements of $\mathcal{C}_{K,\mathbb{C}}$ are symmetric by the action of the elements of $\mathcal{U}_K$, we get that

  $$\forall C \in \mathcal{C}_{K,\mathbb{C}}, \forall \lambda \in \mathbb{Z}, f(\lambda C)=\lambda f(C)$$

  Then classically if we take $C \in \mathcal{C}_{K,\mathbb{C}}$ and $\lambda \in \mathbb{Q}^\star$, we take $p,q \in \mathbb{Z}$ prime to each other such that $\lambda=\frac{p}{q}$,
  Then $q.f(\lambda.C)=f(q\lambda.C)=f(p.C)=p.f(C)$, so $f(\lambda.C)=\lambda.C$ so finally we have that

  $$\forall C \in \mathcal{C}_{K,\mathbb{C}}, \forall \lambda \in \mathbb{Q}, f(\lambda C)=\lambda f(C)$$

  Let $C \in \mathcal{C}$ and $\lambda \in \mathbb{R}^+$.

  Let us denote $\mathcal{D}_C^+ := \{\mu.C, \mu \in \mathbb{R}^+\}$. Equipped with the inclusion relation $\subset$, $\mathcal{D}_C^+$ is a totally ordered set.

  Since $\mathbb{R}^+$ has the least upper bound property and since $\forall \mu,\mu' \in \mathbb{R}^+, \mu.C \subset \mu'.C \Leftrightarrow \mu \leq \mu'$, $\mathcal{D}_C^+$ has the least upper bound property too, for an non empty strict subset $\mathcal{A}$ of $\mathcal{D}_C^+$, we will denote its least upper bound $\sup^{\subset}(\mathcal{A})$.

  But by its definition, f is a non decreasing map for the order $\subset$. Moreover f is bijective, so for every non empty strict subset $\mathcal{A}$ of $\mathcal{D}_C^+$, $f(\sup^{\subset}(\mathcal{A}))=\sup^{\subset}(f(\mathcal{A})$.

  But we know that $\lambda=\sup^{\leq}\{r \in \mathbb{Q}/r<\lambda\}$, so we have $f(\sup^{\subset}\{r.C/r \in \mathbb{Q} \wedge r<\lambda\})=\sup^{\subset}\{r.f(C)/r \in \mathbb{Q} \wedge r<\lambda\}$.

  But we easily have that $f(\sup^{\subset}\{r.C/r \in \mathbb{Q} \wedge r<\lambda\})=f(\lambda.C)$ and that $\sup^{\subset}\{r.f(C)/r \in \mathbb{Q} \wedge r<\lambda\}=\sup^{\leq}\{r \in \mathbb{Q}/r<\lambda\}.C=\lambda.C$.

  So we have that $f(\lambda.C)=\lambda.C$.

  And so we have proved that

  $$\forall C \in \mathcal{C}_{K,\mathbb{C}}, \forall \lambda \in \mathbb{R}^+, f(\lambda C)=\lambda f(C)$$

. □

We can now state the interesting result on $\mathrm{Aut}_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$:

Theorem 5.2. *We have that $Aut_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})=(\mathbb{C}^\star/\mathcal{U}_K) \rtimes \{id,\bar{\bullet}\}$.*

Proof. We will do this proof *only* in the case when $K = \mathbb{Q}(\imath)$, the same reasoning holds for other cases but the calculations are slighty longer.

Let $f \in \mathrm{Aut}_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$.

We can first recall, from the definition of f, that $\forall C, D \in \mathcal{C}_{K,\mathbb{C}}, C \subset D \Rightarrow f(C) \subset f(D)$.

The convex sets $\lambda.D_K$ for $\lambda \in \mathbb{C}^\star$ are caracterised abstractly by the property that they cannot be decomposed non trivially in the semiring $\mathcal{C}_{K,\mathbb{C}}$. More precisely for every $\lambda \in \mathbb{C}^\star$, if $A, B \in \mathcal{C}_{K,\mathbb{C}}$ are such that $\lambda.D_K = \mathrm{Conv}(A \cup B)$, then either A or B is equal to $\lambda.D_K$, let's say without loss of generality that it is A, and then $B \subset A$.

Indeed let $\lambda \in \mathbb{C}^\star$ and let $A, B \in \mathcal{C}_{K,\mathbb{C}}$ such that $\lambda.D_K = \mathrm{Conv}(A \cup B)$, then the set of extremal points of $\lambda.D_K$ is included in the union of the set of extremal points of A and the sets of extremal points of B. Let P be an extremal point of $\lambda.D_K$ of greater module, then P is either an extremal point of A or an extremal point of B. Without loss of generality, let's say that P is an extremal point of A. Then because the definition of $\mathcal{C}_{K,\mathbb{C}}$ and because $A \subset \lambda.D_K$, we get that $A = \lambda.D_K$. And so because $\lambda.D_K = \mathrm{Conv}(A \cup B)$, we get that $B \subset A$.

This abstract property of the $\lambda.D_K$ is preserved by any automorphism of $\mathcal{C}_{K,\mathbb{C}}$, so f sends the sets of the form $\lambda.D_K$ to other sets of the form $\lambda.D_K$. So let $\lambda \in \mathbb{C}^\star$ such that $f(D_K) = \lambda.D_K$. And since the elements of $\mathcal{C}_{K,\mathbb{C}}$ are symmetric with respect to $\mathcal{U}_K$, such a λ is unique only up to multiplication by an element of $\mathcal{U}_K$.

Let $\lambda_0 \in \mathbb{C}^\star/\mathcal{U}_K$ such that $f(D_K) = \lambda_0.D_K$, we can in fact assume that $\lambda_0 = 1$ (we would switch from f to $\frac{1}{\lambda_0} f$).

We can now define $\tau : [0; \pi/2) \to \mathbb{R}_+^\star$ and $\phi : [0; \pi/2) \to [0; \pi/2)$ such that

$$\forall \theta \in [0; \pi/2), f(e^{\imath\theta}.D_K) = \tau(\theta) e^{i\phi(\theta)}.D_K$$

Thanks to our hypothesis on λ_0, we have that $\tau(0) = 1$ and $\phi(0) = 0$.

Let us fix $\theta \in (0, \pi/2)$.

Let us denote $\lambda_{\max} \in \mathbb{R}_+^\star$ the greatest positive real number λ such that $\lambda.D_K \subset e^{\imath\theta}.D_K$.

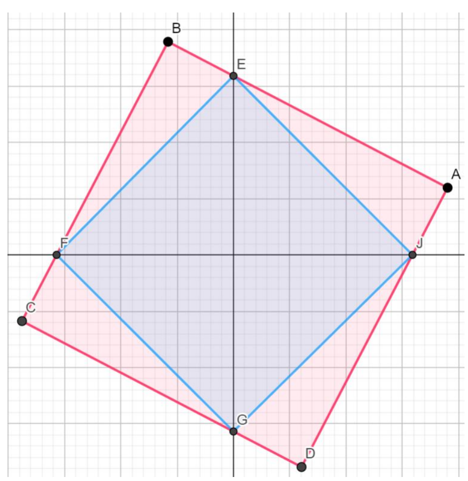

Figure 5: $ABCD = e^{\imath\theta} D_{\mathbb{Q}(\imath)}$, $EFGH = \lambda_{\max} D_{\mathbb{Q}(\imath)}$

Therefore $\imath\lambda_{\max}$ should be on the segment $[\imath e^{\imath\theta}, e^{\imath\theta}]$, so let $t \in [0, 1]$ such that

$$\imath\lambda_{\max} = t\imath e^{i\theta} + (1-t)e^{\imath\theta}$$

.

Therefore $0 = \Re(\imath\lambda_{\max}) = \Re(t\imath e^{i\theta} + (1-t)e^{\imath\theta}) = t(-\sin(\theta)) + (1-t)\cos(\theta)$.
Thus $t = \frac{\cos(\theta)}{\cos(\theta)+\sin(\theta)}$ and so $\lambda_{\max} = \Im(t\imath e^{i\theta} + (1-t)e^{\imath\theta}) = t\cos(\theta) + (1-t)\sin(\theta)$.
Finally we get that $\lambda_{\max} = \frac{1}{\cos(\theta)+\sin(\theta)}$.
But since $f \in \mathrm{Aut}_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$ and thanks to lemma 5.5, we have that

$$\forall \lambda \in \mathbb{R}^+, \lambda.D_K \subset e^{\imath\theta}.D_K \Leftrightarrow \lambda.D_K \subset \tau(\theta)e^{i\phi(\theta)}.D_K$$

Therefore we also have that $\lambda_{\max} = \frac{\tau(\theta)}{\cos(\phi(\theta))+\sin(\phi(\theta))}$ and finally

$$\tau(\theta) = \frac{\cos(\phi(\theta)) + \sin(\phi(\theta))}{\cos(\theta) + \sin(\theta)} = \frac{\sin(\phi(\theta) + \pi/4)}{\sin(\theta + \pi/4)}$$

Let us denote $\lambda_{\min} \in \mathbb{R}^+$ the lowest positive real number λ such that $e^{\imath\theta}.D_K \subset \lambda.D_K$.

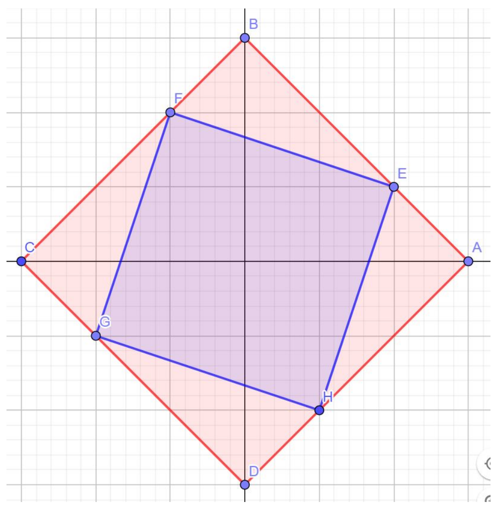

Figure 6: $EFGH = e^{\imath\theta}D_{\mathbb{Q}(\imath)}$, $ABCD = \lambda_{\min}D_{\mathbb{Q}(\imath)}$

Therefore $\imath e^{\imath\theta}$ should be on the segment $[-\lambda_{\min}, \imath\lambda_{\min}]$, so let $t \in [0,1]$ such that

$$\imath e^{\imath\theta} = t\imath\lambda_{\min} - (1-t)\lambda_{\min}$$

.

Therefore $\cos(\theta) = \Im(\imath e^{\imath\theta}) = t\lambda_{\min}$ and $\sin(\theta) = (1-t)\lambda_{\min}$.
Finally we get that $\lambda_{\min} = \cos(\theta) + \sin(\theta)$
But since $f \in \mathrm{Aut}_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})$ and thanks to lemma 5.5, we have that

$$\forall \lambda \in \mathbb{R}^+, e^{\imath\theta}.D_K \subset \lambda.D_K \Leftrightarrow \tau(\theta)e^{i\phi(\theta)}.D_K \subset \lambda.D_K$$

Therefore we also have that $\lambda_{\min} = \tau(\theta)(\cos(\phi(\theta)) + \sin(\phi(\theta)))$ and finally

$$\tau(\theta) = \frac{\cos(\theta) + \sin(\theta)}{\cos(\phi(\theta)) + \sin(\phi(\theta))} = \frac{\sin(\theta + \pi/4)}{\sin(\phi(\theta) + \pi/4)}$$

Therefore $\tau(\theta) = \frac{1}{\tau(\theta)}$ and $\tau(\theta) > 0$ and so

$$\frac{\sin(\phi(\theta) + \pi/4)}{\sin(\theta + \pi/4)} = \tau(\theta) = 1$$

.

And since $\theta, \phi(\theta) \in (0, \pi/2)$, we have :

- either $\phi(\theta)+\frac{\pi}{4}=\theta+\frac{\pi}{4}$ and so $\phi(\theta)=\theta$ which means f is the identity
- or $\phi(\theta)+\frac{\pi}{4}=\frac{\pi}{2}-\left(\theta+\frac{\pi}{4}\right)$ and so $\phi(\theta)=-\theta$ which means f is induced by $\bar{\bullet}$ the complex conjugation.

Therefore $\mathrm{Aut}^+_{\mathbb{B}}(\mathcal{C}_{K,\mathbb{C}})=(\mathbb{C}^\star/\mathcal{U}_K)\rtimes\{\mathrm{id},\bar{\bullet}\}$.

$\square$

Definition 5.5. *A point of $\left(\widehat{\mathcal{O}_K},\mathcal{C}_{\mathcal{O}_K}\right)$ over $\mathcal{C}_{K,\mathbb{C}}$ is a pair (p,f) given by a point p of the topos $\widehat{\mathcal{O}_K}$ and a direct similitude f (so it preserves the orientation and the number of vertices) from $\mathbb{C}$ to $\mathbb{C}$ which induces a morphism $f^\sharp_p:\mathcal{C}_{\mathcal{O}_K,p}\to\mathcal{C}_{K,\mathbb{C}}$ of semirings from the stalk of $\mathcal{C}_{\mathcal{O}_K}$ at the point p into $\mathcal{C}_{K,\mathbb{C}}$.*

Lemma 5.6. *We follow the notations of theorem 5.1. Let us denote $\mathcal{S}_{mod}$ the set of sub-semirings of $\mathcal{C}_{K,\mathbb{C}}$ of the form $Semiring\{hD_K,h\in H\}$ where H is a sub-$\mathcal{O}_K$ module of K.*

Let us note now Φ the map from $\left(\mathbb{A}^f_K/\widetilde{\mathcal{O}_K}^\star\right)\times(\mathbb{C}/\mathcal{U}_K)$ to $\mathcal{S}_{mod}$ defined by

$$\Phi\begin{cases}\left(K^*\backslash\mathbb{A}^f_K/\widetilde{\mathcal{O}_K}^\star\right)\times(\mathbb{C}/\mathcal{U}_K)\to & \mathcal{S}_{mod}\\ \qquad\qquad(a,\lambda)\mapsto & \Phi(a,\lambda):=Semiring\{h\lambda D_K,h\in H_a\}.\end{cases}$$

Then Φ induces a bijection between the quotient of $\left(\mathbb{A}^f_K/\widetilde{\mathcal{O}_K}^\star\right)\times(\mathbb{C}/\mathcal{U}_K)$ by the diagonal action of $K^\star$ and the set $\mathcal{S}_{mod}$.

Proof. Let us first show that the map Φ is invariant under the diagonal action of $K^\star$.

Let $k\in K^\star$ and $(a,\lambda)\in\left(\mathbb{A}^f_K/\widetilde{\mathcal{O}_K}^\star\right)\times(\mathbb{C}/\mathcal{U}_K)$

Then since $H_{ka}=k^{-1}H_a$, we have

$$\begin{aligned}\Phi(ka,k\lambda)&=\mathrm{Semiring}\{hk\lambda D_K,h\in H_{ka}\}\\&=\mathrm{Semiring}\{h\lambda D_K,h\in H_a\}\\&=\Phi(a,\lambda)\end{aligned}$$

We also have immediately thanks to theorem 3.2 that Φ is surjective.

Let us now show that Φ is injective.

Let $(a,\lambda),(b,\mu)\in\left(\mathbb{A}^f_K/\widetilde{\mathcal{O}_K}^\star\right)\times(\mathbb{C}/\mathcal{U}_K)$ such that $\Phi(a,\lambda)=\Phi(b,\mu)$.

So $\mathrm{Semiring}\{h\lambda D_K,h\in H_a\}=\mathrm{Semiring}\{\tilde{h}\mu D_K,\tilde{h}\in H_b\}$.

And since the vertices of D_K are $1,-1,s_K,-s_K$ where $s_K=\imath\sqrt{d}$ when $K=\mathbb{Q}(\sqrt{-d})$ with $-d\equiv 1,3(4)$ and $s_K=\frac{1+\imath\sqrt{d}}{2}$ when $K=\mathbb{Q}(\sqrt{-d})$ with $-d\equiv 2(4)$

We get $1.\lambda.H_a\subset 1\mu H_b+(-1).\mu.H_b+s_K\mu H_b+(-s_K).\mu.H_b$,

And $s_K.\lambda.H_a\subset 1\mu H_b+(-1).\mu.H_b+s_K\mu H_b+(-s_K).\mu.H_b$,

And $(-1).\lambda.H_a\subset 1\mu H_b+(-1).\mu.H_b+s_K\mu H_b+(-s_K).\mu.H_b$,

And $(-s_K).\lambda.H_a\subset 1\mu H_b+(-1).\mu.H_b+s_K\mu H_b+(-s_K).\mu.H_b$.

So $1\lambda H_a+(-1).\lambda.H_a+s_K\lambda H_a+(-s_K).\lambda.H_a\subset 1\mu H_b+(-1).\mu.H_b+s_K\mu H_b+(-s_K).\mu.H_b$.

And by the same process we get the other way around so we have the equality

$$\lambda(1H_a+(-1).H_a+s_K.H_a+(-s_K).H_a)=\mu(1.H_b+(-1).H_b+s_K.H_b+(-s_K).H_b)$$

But $1,-1,s_K,-s_K\in\mathcal{O}_K$ and H_a,H_b are $\mathcal{O}_K$-modules, so $\lambda H_a=\mu H_b$.

Since H_a and H_b are $\mathcal{O}_K$-modules of rank 1, there exists $k\in K^\star$ such that $\lambda=k\mu$.

So then one gets $\lambda H_a = \lambda k H_b$.

So $H_a = kH_b$, but $kH_b = H_{k^{-1}b}$.

So by theorem 3.2 we have $a = k^{-1}b$ in $\mathbb{A}_K^f/\widetilde{\mathcal{O}_K}^\star$ and then we get $\mu = k\lambda$ and $b = ka$ thus the injectivity of Φ. The lemma is proved.

$\square$

Theorem 5.3. *The set of points of the arithmetic site* $\left(\widehat{\mathcal{O}_K}, \mathcal{C}_{\mathcal{O}_K}\right)$ *over* $\mathcal{C}_{K,\mathbb{C}}$ *is naturally identified to* $\left(\mathbb{A}_K^f \times \mathbb{C}/\mathcal{U}_K\right) / \left(K^\star \times \left(\prod_{\mathfrak{p} \ prime} \mathcal{O}_\mathfrak{p}^\star \times \{1\}\right)\right)$ *which can also be identified to* $\mathbb{A}_K / \left(K^\star \left(\prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^\star \times \mathcal{U}_K\right)\right)$.

Remark 5.6. *This theorem is a generalization of the theorem of A.Connes and C.Consani in [10] on the interpretation of the points of the arithmetic site over* $\mathbb{R}_{\max}$.

Proof. Let us consider a point of the arithmetic site with values in $\mathbb{C}_{K,\mathbb{C}}$ $(p, f_p^\star)$.

By theorem 3.2, to a point of the topos $\widehat{\mathcal{O}_K}$ is associated to H an $\mathcal{O}_K$-module (of rank 1 since $\mathcal{O}_K$ is principal) included in K, and by theorem 5.1 the stalk of $\mathcal{C}_{\mathcal{O}_K}$ at the point p is $\mathcal{C}_H$.

As in [10] we consider the following two cases depending on the range of $f_p^\star$:

1. The range of $f_p^\star$ is $\mathbb{B} \simeq \{\emptyset, \{0\}\} \subset \mathcal{C}_{K,\mathbb{C}}$.

   $f_p^\star$ sends non empty sets of $\mathcal{C}_H$ to $\{0\}$ and so the pair $(p, f_p^\star)$ is uniquely determined by the point p and so by theorem 3.2 the set of those kind of points is isomorphic to $K^\star \backslash \mathbb{A}_K^f/\widetilde{\mathcal{O}_K}^\star$.

2. The range of $f_p^\star$ is not contained in $\mathbb{B}$, then by the definition of a point ($f_p^\star$ is direct similitude) the range of $f_p^\star$ is of the form Semiring$\{h\lambda D_K, h \in H\}$ and $\lambda \in \mathbb{C}/\mathcal{U}_K$ with $f_p^\star(D_K) = \lambda D_K$.

   So by lemma 5.6 the set of those points is isomorphic to $K^\star \backslash \left(\mathbb{A}_K^f/\widetilde{\mathcal{O}_K}^\star \times \mathbb{C}/\mathcal{U}_K\right)$

So all in all the set of the points of $\widehat{\mathcal{O}_K}$ with values in $\mathcal{C}_{K,\mathbb{C}}$ is isomorphic to $K^\star \backslash \left(\mathbb{A}_K^f \times \mathbb{C}/\mathcal{U}_K\right) / \left(\widetilde{\mathcal{O}_K}^\star \times \{1\}\right)$

$\square$

6. Link with the Dedekind zeta function

6.1. The spectral realization of critical zeros of L-functions by Connes

In this section, K will denote an imaginary quadratic number field with class number 1.

Let us first recall some facts ([5] and [4]) about homogeneous distributions on adeles and L-functions and then the construction of the Hilbert space $\mathcal{H}$ underlying the spectral realization of the critical zeroes.

Let k be a local field and χ a quasi-character of $k^\star$, let $s \in \mathbb{C}$, we can write χ in the following form : $\forall x \in k^\star \chi(x) = \chi_0(x)|x|^s$ with $\chi_0 : k^\star \to S^1$. Let $\mathcal{S}(k)$ denote the Schwartz Bruhat space on k.

Definition 6.1. *We say that a distribution* $D \in \mathcal{S}'(k)$ *is homogeneous of weight* χ *if one has* $\forall f \in \mathcal{S}(k), \forall a \in k^\star, \langle f^a, D\rangle = \chi(a)^{-1}\langle f, D\rangle$ *where by definition* $f^a(x) = f(ax)$.

We have the following property :

Proposition 6.1. *For* $\sigma = \Re(s) > 0$, *there exists up to normalization, only one homogeneous distribution of weight* χ *on* k *which is given by the absolutely convergent integral* $\Delta_\chi(f) = \int_{k^\star} f(x)\chi(x)d^\star x$.

Proof. Cf in [35]. □

If k is non-archimedean field, and π_k a uniformizer, let us now define and note for all $s \in \mathbb{C}$:

Definition 6.2. *The distribution $\Delta_s^{'} \in \mathcal{S}'(k)$ is defined for all $f \in \mathcal{S}(k)$ as $\Delta_s^{'} = \int_{k^\star}(f(x) - f(\pi_k x))|x|^s d^\star x$, with the multiplicative Haar measure $d^\star x$ normalized by $\langle 1_{\mathcal{O}_k^\star}, d^\star x\rangle = 1$*

The distribution Δ_s' is well defined because by the very definition of $\mathcal{S}(k)$, for $f \in \mathcal{S}(k)$, $f(x) - f(\pi_k x) = 0$ for x small enough.

This distribution has the following properties :

Proposition 6.2. *We have :*

1. $\langle 1_{\mathcal{O}_k}, \Delta_s^{'}\rangle = 1$
2. $\langle f^a, \Delta_s^{'}\rangle = |a|^{-s}\langle f, \Delta_s^{'}\rangle$
3. $\Delta_s^{'} = (1 - q^{-s})\Delta_s$ *with* $|\pi_k| = q^{-1}$

Proof. Cf §9.1 in [5]. □

Let χ be now a quasi-character from the idele class group $C_K = \mathbb{A}_K^\star/K^\star$. We can note χ as $\chi = \prod_\nu$ and $\chi(x) = \chi_0(x)|x|^s$ with $s \in \mathbb{C}$ and χ_0 a character of C_K. Let us note P the finite set of places where χ_0 is ramified. For any place $\nu \notin P$, let us denote $\Delta_\nu^{'}$ the unique homogeneous distribution of weight χ_ν normalized by $\langle \Delta_\nu^{'}, 1_{\mathcal{O}_\nu}\rangle = 1$. For any $\nu \in P$ or infinite place and for $\sigma = \Re(s) > 0$ let us denote $\Delta_\nu^{'}$ the homogeneous distribution of weight χ_ν given by proposition 5.1 (this one is unnormalized). Then the infinite tensor product $\Delta_s^{'} = \prod_\nu \Delta_\nu^{'}$ makes sense as a continuous linear form on $\mathcal{S}(\mathbb{A}_K)$ and it is homogeneous of weight χ, it is not equal to zero since $\Delta_\nu^{'} \neq 0$ for every ν and for infinite places as well and is finite by construction of the space $\mathcal{S}(\mathbb{A}_K)$ as the infinite tensor product $\bigotimes_\nu (\mathcal{S}(K_\nu), 1_{\mathcal{O}_\nu})$

Then we can see the L functions appear as a normalization factor thanks to the following property:

Proposition 6.3. *For $\sigma = \Re(s) > 1$, the following integral converges absolutely*

$$\forall f \in \mathcal{S}(\mathbb{A}_K), \int_{\mathcal{A}_K^\star} f(x)\chi_0(x)|x|^s d^\star x = \Delta_s(f)$$

and we have $\forall f \in \mathcal{S}(\mathbb{A}_K), \Delta_s(f) = L(\chi_0, s)\Delta_s^{'}(f)$.

Proof. Cf lemma 2.50 in [5] □

Let us now note $\mathcal{S}(\mathbb{A}_K)_0$ the following subspace of $\mathcal{S}(\mathbb{A}_K)$ defined by $\mathcal{S}(\mathbb{A}_K)_0 = \{f \in \mathcal{S}(\mathbb{A}_K) / f(0) = 0, \int_{\mathbb{A}_K} f(x)\mathrm{d}x = 0\}$. We can now define the operator $\mathcal{E}$:

Definition 6.3. *Let $f \in \mathcal{S}(\mathbb{A}_K)_0$ and $g \in C_K$, then we set*

$$\mathcal{E}(f)(g) = |g|^{\frac{1}{2}} \sum_{q \in K^\star} f(qg)$$

.

We have the following properties for $\mathcal{E}$:

Proposition 6.4. *We have :*

1. *for all $f \in \mathcal{S}(\mathbb{A}_K)_0$ and $g \in C_K$, the series $\mathcal{E}(f)(g)$ converges absolutely*

2. $\forall f \in \mathcal{S}(\mathbb{A}_K)_0, \forall n \in \mathbb{N}, \exists c > 0, \forall g \in C_K, |\mathcal{E}(f)(g)| \leq c \exp(-n|\log|g||)$

3. *for all $f \in \mathcal{S}(\mathbb{A}_K)_0$ and $g \in C_K$, $\mathcal{E}(\hat{f})(g) = \mathcal{E}(f)(g^{-1})$*

Proof. Cf lemma 2.51 in [5]. □

And so we can get that

Proposition 6.5. *For $\sigma = \Re(s) > 0$ and any character χ_0 of C_K, we have that*

$$\forall f \in \mathcal{S}(\mathbb{A}_K)_0, \int_{C_K} \mathcal{E}(f)(x)\chi_0(x)|x|^{s-\frac{1}{2}} d^\star x = cL(\chi_0, s)\Delta_s^{'}(f)$$

where the non zero constant c depends on the normalization of the Haar measure $d^\star x$ on C_K.

Proof. Cf lemma 2.52 in [5]. □

And we also have the following lemma :

Lemma 6.1. *There exists an approximate unit $(f_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $f_n \in \mathcal{S}(C_K)$, $\hat{f_n}$ has compact support and there exists $C > 0$ such that $||\theta_m(f_n)|| \leq C$ and that $\theta_m(f_n) \to 1$ strongly in $L^2_\delta(C_K)$ as $n \to \infty$.*

Now we are able to define the Hilbert space $\mathcal{H}$. First on $\mathcal{S}(\mathbb{A}_K)_0$ we can put the inner product corresponding to the norm $\|f\|^2_\delta = \int_{C_K} |\mathcal{E}(f)(x)|^2(1 + (\log|x|)^2)^{\frac{\delta}{2}} \mathrm{d}^\star x$.

Let us denote $L^2_\delta(\mathbb{A}_K/K^\star)_0$ the separated completion of $\mathcal{S}(\mathbb{A}_K)_0$ with respect to the inner product defined ealier. Let us also define θ_a the representation of C_K on $\mathcal{S}(\mathbb{A}_K)$ given by for $\xi \in \mathcal{S}(\mathbb{A}_K), \forall \alpha \in C_K, \forall x \in \mathbb{A}_K, (\theta_a(\alpha)\xi)(x) = \xi(\alpha^{-1}x)$.

We can also put the following Sobolev norm on $L^2_\delta(C_K)$, $\|\xi\|^2_\delta = \int_{C_K} |\xi(x)|^2(1+(\log|x|)^2)^{\frac{\delta}{2}} \mathrm{d}^\star x$.

Then by construction, the linear map $\mathcal{E} : \mathcal{S}(\mathbb{A}_K)_0 \to L^2_\delta(C_K)$ satisfies for all $f \in \mathcal{S}(\mathbb{A}_K)_0$, $\|f\|^2_\delta = \|\mathcal{E}(f)\|^2_\delta$. Thus this map extends to an isometry still denoted $\mathcal{E} : L^2_\delta(\mathbb{A}_K/K^\star)_0 \to L^2_\delta(C_K)$.

Let us also denote θ_m the regular representation of C_K on $L^2_\delta(C_K)$.

We have for any $\xi \in L^2_\delta(C_K)$:

$$\forall \alpha \in C_K, \forall x \in C_K, (\theta_a(\alpha)\xi)(x) = \xi(\alpha^{-1}x)$$

.

We then get for every $f \in L^2_\delta(\mathbb{A}_K/K^\star)_0$, $\alpha \in C_K$ and $g \in C_K$ that :

$$\begin{aligned}
\mathcal{E}(\theta_a(\alpha)f)(g) &= |g|^{\frac{1}{2}} \sum_{q\in K^\star} (\theta_a(\alpha)f)(qg) \\
&= |g|^{\frac{1}{2}} \sum_{q\in K^\star} f(\alpha^{-1}qg) \\
&= |g|^{\frac{1}{2}} \sum_{q\in K^\star} f(q\alpha^{-1}g) \\
&= |\alpha|^{\frac{1}{2}}|\alpha^{-1}g|^{\frac{1}{2}} \sum_{q\in K^\star} f(q\alpha^{-1}g) \\
&= |\alpha|^{\frac{1}{2}}(\theta_m(\alpha)\mathcal{E}(f))(g)
\end{aligned}$$

Thus we have that $\mathcal{E}\theta_a(\alpha) = |\alpha|^{\frac{1}{2}}\theta_m(\alpha)\mathcal{E}$. In other words, it shows that the natural representation θ_a of C_K on $L^2_\delta(\mathbb{A}_K/K^\star)_0$ corresponds, via the isometry $\mathcal{E}$, to the restriction of $|\alpha|^{\frac{1}{2}}\theta_m(\alpha)$ to the invariant subspace given by the range of $\mathcal{E}$.

Definition 6.4. *We denote by $\mathcal{H} = L^2_\delta(C_K)/Im(\mathcal{E})$ the cokernel of the map $\mathcal{E}$. Let us denote also $\underline{\theta}_m$ the quotient representation of C_K on $\mathcal{H}$ and finally let us denote for a character χ of $C_{K,1}$, $\mathcal{H}_\chi = \{h \in \mathcal{H}/\forall g \in C_{K,1}, \theta_m(g)h = \chi(g)h\}$*

Since N_1 is a compact group, we get that :

Proposition 6.6. *The Hilbert space $\mathcal{H}$ splits as a direct sum $\mathcal{H} = \bigoplus_{\chi \in \widehat{C_{K,1}}} \mathcal{H}_\chi$ and the representation $\underline{\theta}_m$ decomposes as a direct sum of representation $\underline{\theta}_{m,\chi} : C_K \to Aut(\mathcal{H}_\chi)$*

This situation gives rise to operators whose spectra will the critical zeroes of L functions and so the spectral interpretation of it.:

Definition 6.5. *Let us define and note D_χ the infinitesimal generator of the restriction of $\underline{\theta}_{m,\chi}$ to $1 \times \mathbb{R}^\star_+ \subset C_K$, in other words we have for every $\xi \in \mathcal{H}_\chi$, $D_\chi \xi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(\underline{\theta}_{m,\chi} - 1)\xi$*

Then the central theorem of the spectral realisation of the critical zeroes of the L functions as in [4] and in [5] is :

Theorem 6.1. *Let χ, $\delta > 1$, $\mathcal{H}_\chi$ and D_χ as above. Then D_χ has a discrete spectrum and $Sp(D_\chi) \subset \imath\mathbb{R}$ is the set of imaginary parts of zeroes of the L-function with Grössencharacter $\tilde{\chi}$ (the extension of χ to C_K) which have real part equal to $\frac{1}{2}$, ie $\rho \in Sp(D_\chi) \Leftrightarrow L\left(\tilde{\chi}, \frac{1}{2} + \rho\right) = 0$ and $\rho \in \imath\mathbb{R}$. Moreover the multiplicity of ρ in $Sp(D_\chi)$ is equal to the largest integer $n < 1 + \frac{\delta}{2}$, $n \leq$ multiplicity of $\frac{1}{2} + \rho$ as a zero of L.*

Proof. We follow the proof already of [4] and in [5] making it more precise with respect to our goal.

We first need to understand the range of $\mathcal{E}$, in order to do that, we consider its orthogonal in the dual space that is $L^2_\delta(C_K)$.

Since the subgroup $C_{K,1}$ of C_K is the group the ideles classes of norm 1, $C_{K,1}$ is a compact group and acts by the representation θ_m which is unitary when restricted to $C_{K,1}$.

Therefore we can decompose $L^2_\delta(C_K)$ and its dual $L^2_{-\delta}(C_K)$ into the direct of the following subspaces

$$L^2_{\delta,\chi_0}(C_K) = \{\xi \in L^2_\delta(C_K); \forall g \in C_K, \forall \gamma \in C_{K,1}, \xi(\gamma^{-1}g) = \chi_0(\gamma)\xi(g)\}$$

which correspond to the projections $P_{\chi_0} = \int_{C_K} \bar{\chi_0}(\gamma)\theta_m(\gamma)\mathrm{d}_1\gamma$.

And for the dual :

$$L^2_{-\delta,\chi_0}(C_K) = \{\xi \in L^2_{-\delta}(C_K); \forall g \in C_K, \forall \gamma \in C_{K,1}, \xi(\gamma g) = \chi_0(\gamma)\xi(g)\}$$

which correspond to the projections $P^t_{\chi_0} = \int_{C_K} \bar{\chi_0}(\gamma)\theta_m(\gamma)^t\mathrm{d}_1\gamma$.

Here we have used $(\theta_m(\gamma)^t\eta)(x) = \eta(\gamma x)$ which comes from the definition of the transpose $\langle\theta_m(\gamma)\xi, \eta\rangle = \langle\xi, \theta_m(\gamma)^t\eta)\rangle$ using $\int_{C_K} \xi(\gamma^{-1}x)\eta(x)d^\star x = \int_{C_K} \xi(y)\eta(\gamma y)d^\star y$.

In these formulas one only uses the character χ_0 as a character of the compact subgroup $C_{K,1}$ of C_K. One now chooses non canonically an extension $\tilde{\chi}_0$ of χ_0 as a character of C_K (ie we have $\forall\gamma \in C_{K,1}, \tilde{\chi}_0(\gamma) = \chi_0(\gamma)$). This choice is not unique and two choices of extensions only differ by a character that is principal (ie of the form $\gamma \mapsto |\gamma|^{is_0}$ with $s_0 \in \mathbb{R}$). We fix a factorization $C_K = C_{K,1} \times \mathbb{R}^\star_+$ and fix $\tilde{\chi}_0$ as being equal to 1 on $\mathbb{R}^\star_+$.

Then by definition, we can write any element η of $L^2_{-\delta,\chi_0}(C_K)$ in the form :

$$\eta : g \in C_K \mapsto \eta(g) = \tilde{\chi}_0(g)\psi(|g|)$$

where $\int_{C_K} |\psi(|g|)|^2(1+(\log|g|)^2)^{-\delta/2}d^\star g < \infty$.

A vector like this η is in the orthogonal of the range of $\mathcal{E}$ if and only if :

$$\forall f \in \mathcal{S}(\mathbb{A}_K)_0, \int_{C_K} \mathcal{E}(f)(x)\tilde{\chi}_0(x)\psi(|x|)d^\star x = 0$$

Using Mellin inversion formula $\psi(|x|) = \int_{\mathbb{R}} \hat{\psi}(t)|x|^{\imath t}dt$, we can see formally that this last equality becomes equivalent to :

$$\int_{C_K}\int_{\mathbb{R}} \mathcal{E}(f)(x)\tilde{\chi}_0(x)\hat{\psi}(t)|x|^{\imath t}dtd^\star x = \int_{\mathbb{R}} \Delta_{\frac{1}{2}+\imath t}(f)\hat{\psi}(t)dt$$

Those formal manipulations are justified by the use of the approximate units with special properties which appear in a previous lemma 6.1 and the rapid decay of $\mathcal{E}(f)$ of the proposition 6.4.

Thanks to the last formula, we are now looking for nice functions $f \in \mathcal{S}(\mathbb{A}_K)_0$ on which to test the distribution $\int_{C_K} \Delta_{\frac{1}{2}+\imath t}\hat{\psi}(t)dt$.

For the finite places, we denote by P the finite set of finite places where χ_0 ramifies, we take $f_0 := \otimes_{\nu\notin P} 1_{\mathcal{O}_\nu} \otimes f_{\chi_0}$ where f_{χ_0} is the tensor product over ramified places of the functions equal to 0 outside $\mathcal{O}_\nu^\star$ and to χ_0, ν on $\mathcal{O}_\nu^\star$.

Then by the definition of Δ'_s, for any $f \in \mathcal{S}(\mathbb{C})$ we get that $\langle \Delta'_s, f_0 \otimes f\rangle = \int_{C_K} f(x)\chi_{0;\infty}(x)|x|^s d^\star x$. Moreover if the set P of finite ramified places of χ_0 is not empty, we have $f_0(0) = 0$ and $\int_{\mathbb{A}_K^f} f_0(x)dx = 0$ so that $f_0 \otimes f \in \mathcal{S}(\mathbb{A}_K)_0$ for all $f \in \mathcal{S}(\mathbb{C})$.

We can in fact take a function f of the form $f(x) = b(x)\bar{\chi}_{0,\infty}(x)$ with $b \in \mathcal{C}_c^\infty(\mathbb{R}_+^\star)$.

So for any $s \in \mathbb{C}$ such that $\Re s > 0$, $\langle \Delta'_s, f_0 \otimes f_b\rangle = \int_{\mathbb{R}_+^\star} b(x)|x|^s d^\star x$.

So when we pair the distribution $\int_{\mathbb{R}} \Delta_{\frac{1}{2}+\imath t}\hat{\psi}(t)dt$ again such functions, we get that :

$\langle \int_{\mathbb{R}} \Delta_{\frac{1}{2}+\imath t}\hat{\psi}(t)dt, f_0 \otimes f_b\rangle = \iint_{C_K\times\mathbb{R}} L(\chi_0, \frac{1}{2}+\imath t)\hat{\psi}(t)b(x)|x|^{\frac{1}{2}+\imath t}d^\star x dt$.

But one can see that, if $\chi_0|_{C_{K,1}} \neq 1$, $L(\chi_0, \frac{1}{2}+\imath t)$ is an analytic function of t so the product $L(\chi_0, \frac{1}{2}+\imath t)\hat{\psi}(t)$ is a tempered distrubution and so is its Fourier transform. Thanks to the last equality, we have that the Fourier transform of $L(\chi_0, \frac{1}{2}+\imath t)\hat{\psi}(t)$ paired on arbitrary functions which are smooth with compact support equals 0 and so the Fourier transform $L(\chi_0, \frac{1}{2}+\imath t)\hat{\psi}(t)$ is equal to 0.

If $\chi_0|_{C_{K,1}} = 1$, we need to impose the condition $\int_{\mathbb{A}_K} f dx = 0$ ie $\int_{\mathbb{R}_+^\star} b(x)|x|d^\star x = 0$ but we can see that the space of functions $b(x)|x|^{\frac{1}{2}} \in \mathcal{C}_c^\infty(\mathbb{R}_+^\star)$ with the condition $\int_{\mathbb{R}_+^\star} b(x)|x|d^\star x = 0$ is dense in $\mathcal{S}(\mathbb{R}_+^\star)$.

Let us now recall that for the equation $\phi(t)\alpha(t) = 0$ with α a distribution on $\mathbb{S}^1$ and $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$ which has finitely many zeroes denoted x_i of order n_i with $i \in I$ with I a finite set , the distributions $\delta_{x_i}, \delta'_{x_i}, \ldots, \delta_{x_i}^{n_i-1}, i \in I$ form a basis of the space of solutions in α (of the equation $\phi(t)\alpha(t) = 0$).

Now we can come back to our main study. Thanks to what we have shown before, we now know that for η orthogonal to the range of $\mathcal{E}$ and such that $\theta_m^t(h)(\eta) = \eta$, we have that $\hat{\psi}(t)$ is a distribution with compact support satisfying the equation $L(\chi_0, \frac{1}{2}+\imath t)\hat{\psi}(t) = 0$.

Therefore thanks to what we have recalled, we get that $\hat{\psi}$ is a finite linear combination of the distributions $\delta_t^{(k)}$ with t such that $L(\chi_0, \frac{1}{2}+\imath t) = 0$ and k striclty less than the order of the zero of this L function (necessary and sufficient to get the vanishing on the range of $\mathcal{E}$) and also $k < \frac{\delta-1}{2}$ (necessary and sufficient to ensure that ψ belongs to $L^2_{-\delta}(\mathbb{R}_+^\star)$, ie $\int_{\mathbb{R}_+^\star}(\log|x|)^{2k}(1 + |\log|x||^2)^{-\delta/2}d^\star x < \infty$).

Conversely, let s be a zero of $L(\chi_0, s)$ of order $k > 0$. Then by the proposition 6.3 and the finiteness and the analyticity of Δ'_s for $\Re s > 0$, we get, for $a \in [|0, k-1|]$ and $f \in \mathcal{S}(\mathbb{A}_K)_0$, that : $\left(\frac{\partial}{\partial s}\right)^a \Delta'_s(f) = 0$.

We also have that $\left(\frac{\partial}{\partial s}\right)^a \Delta'_s(f) = \int_{C_K} \mathcal{E}(f)(x)\chi_0(x)|x|^{s-\frac{1}{2}}(\log|x|)^a d^\star x$.

Thus η belongs to the orthogonal of the range of $\mathcal{E}$ and such that $\theta^t_m(h)\eta = \eta$ if and only if it is a finite linear combination of functions of the form

$$\eta_{t,a}(x) = \chi_0(x)|x|^{\imath t}(\log|x|)^a$$

where $L\left(\chi_0, \frac{1}{2} + \imath t\right) = 0$ and $a <$ order of the zero t and $a < \frac{\delta-1}{2}$.

Therefore the restriction to the subgroup $\mathbb{R}^\star_+$ of C_K of the transpose of θ_m is given in the above basis $\eta_{t,a}$ by

$$\theta_m(\lambda)^t \eta_{t,a} = \sum_{b=0}^{a} C^b_a \lambda^{\imath t}(\log(\lambda))^b \eta_{t,b-a}$$

Therefore if $L(\chi_0, \frac{1}{2}+\imath s) \neq 0$ then $\imath s$ does not belong to the spectrum of $D^t_{\chi_0}$. This determine the spectrum of the operator $D^t_{\chi_0}$ and so the spectrum of D_{χ_0}. Therefore the theorem is proved. □

6.2. The link between the points of the arithmetic site and the Dedekind zeta function

Since the class number of K of 1, we observe that $C_{K,1}$ the ideles classes of norm 1 is given by :

$$C_{K,1} = (K^\star(\prod_{\mathfrak{p}} \mathcal{O}^\star_{\mathfrak{p}} \times \mathbb{S}^1))/K^\star$$

We still denote $\mathcal{H}$ the Hilbert space associated by Connes in [4] to $\left(\mathbb{A}^f_K \times \mathbb{C}\right)/K^\star$ and whose definition was recalled in the last section.

In the last section we have seen that one can decompose $\mathcal{H}$ in the following way : $\mathcal{H} = \bigoplus_{\chi \in \widehat{C_{K,1}}} \mathcal{H}_\chi$ with $\mathcal{H}_\chi = \{h \in \mathcal{H} / \forall g \in C_{K,1}, \theta_m(g)h = \chi(g)h\}$.

Let us note $G = \left(K^\star \times \left(\prod_{\mathfrak{p} \text{ prime}} \mathcal{O}^\star_{\mathfrak{p}} \times \mathcal{U}_K\right)\right)/K^\star$.

We can observe that $C_{K,1}/G \simeq \mathbb{S}^1/\mathcal{U}_K$.

The main idea here is that we would like to have a spectral interpretation of ζ_K linked to the space of points of the arithmetic site $\left(\widehat{\mathcal{O}_K}, \mathcal{C}_{\mathcal{O}_K}\right)$ over $\mathcal{C}_{K,\mathbb{C}}$ which is by theorem 5.3

$$\left(\mathbb{A}^f_K \times \mathbb{C}\right) / \left(K^\star \times \left(\prod_{\mathfrak{p} \text{ prime}}{}' \mathcal{O}^\star_{\mathfrak{p}} \times /\mathcal{U}_K\right)\right)$$

In Connes' formalism ([4], [5]) and as recalled in the last section, $\mathcal{H}$ is an Hilbert space associated to the adele class space $\left(\mathbb{A}^f_K \times \mathbb{C}\right)/K^\star$ linked with the spectral interpretation of L functions. More precisely if we denote $\chi_{\text{trivial}} \in \widehat{C_{K,1}}$ the trivial character of $\widehat{C_{K,1}}$, then the results of [4] and [5] show that $\mathcal{H}_{\chi_{\text{trivial}}}$ is associated to the spectral interpretation of ζ_K.

Theorem 6.2. *We have $\mathcal{H}^G = \bigoplus_{\chi \in \widehat{S^1/\mathcal{U}_K}} \mathcal{H}^G_\chi$. Then as in [4] the space $\mathcal{H}^G_\chi$ corresponds to $L(\chi, \bullet)$, so in particular when χ is trivial, $\mathcal{H}^G_\chi$ corresponds to ζ_K the Dedekind zeta function of K.*

Proof. We adapt here the same strategy as in the proof of theorem 6.1:

In our case, let us consider $L^2_\delta(C_K)^G$ (stable under the action of G) and $L^2_{-\delta}(C_K)^G$ (stable under the action of G).

Since the subgroup $C_{K,1}G \simeq \mathbb{S}^1/\mathcal{U}_K$ thanks to θ_m (as recalled earlier θ_m denotes the regular representation of C_K on $L^2_\delta(C_K)$).

Therefore we can decompose $L^2_\delta(C_K)^G$ and its dual $L^2_{-\delta}(C_K)^G$ into the direct of the following subspaces

$$L^2_{\delta,\chi_0}(C_K) = \{\xi \in L^2_\delta(C_K); \forall g \in C_K, \forall \gamma \in C_{K,1}/G, \xi(\gamma^{-1}g) = \chi_0(\gamma)\xi(g)\}$$

And for the dual :

$$L^2_{-\delta,\chi_0}(C_K) = \{\xi \in L^2_{-\delta}(C_K); \forall g \in C_K, \forall \gamma \in C_{K,1}/G, \xi(\gamma g) = \chi_0(\gamma)\xi(g)\}$$

Let us also recall that to a character $\chi_0 \in \widehat{C_{K,1}/G} \simeq \widehat{\mathbb{S}^1/\mathcal{U}_K}$, one can uniquely associate a Größencharakter $\widetilde{\chi_0}$ (the conditions being a Größencharakter and K being class number 1 give that the non archimedean part of $\widetilde{\chi_0}$ is completely determined by the archimedean part which is χ_0).

From now on we can follow exactly the same strategy as the one used in the proof of 6.1 :

Then by definition, we can write any element η of $L^2_{-\delta,\chi_0}(C_K)^G$ in the form :

$$\eta : g \in C_K \mapsto \eta(g) = \tilde{\chi}_0(g)\psi(|g|)$$

where $\int_{C_K} |\psi(|g|)|^2(1 + (\log|g|)^2)^{-\delta/2} d^\star g < \infty$.

A vector like this η is in the orthogonal of the range of $\mathcal{E}$ if and only if :

$$\forall f \in \mathcal{S}(\mathbb{A}_K)_0, \int_{C_K} \mathcal{E}(f)(x)\tilde{\chi}_0(x)\psi(|x|)d^\star x = 0$$

Using Mellin inversion formula $\psi(|x|) = \int_\mathbb{R} \hat{\psi}(t)|x|^{\imath t}dt$, we can see formally that this last equality becomes equivalent to :

$$\int_{C_K}\int_\mathbb{R} \mathcal{E}(f)(x)\tilde{\chi}_0(x)\hat{\psi}(t)|x|^{\imath t}dtd^\star x = \int_\mathbb{R} \Delta_{\frac{1}{2}+\imath t}(f)\hat{\psi}(t)dt$$

Those formal manipulations are justified by the use of the approximate units with special properties which appear in a previous lemma 6.1 and the rapid decay of $\mathcal{E}(f)$ of the proposition 6.4.

Thanks to the last formula, we are now looking for nice functions $f \in \mathcal{S}(\mathbb{A}_K)_0$ on which to test the distribution $\int_{C_K} \Delta_{\frac{1}{2}+\imath t}\hat{\psi}(t)dt$.

For the finite places, we denote by P the finite set of finite places where $\widetilde{\chi_0}$ ramifies, we take $f_0 := \otimes_{\nu\notin P} 1_{\mathcal{O}_\nu} \otimes f_{\widetilde{\chi_0}}$ where $f_{\widetilde{\chi_0}}$ is the tensor product over ramified places of the functions equal to 0 outside $\mathcal{O}^\star_\nu$ and to $\widetilde{\chi_0}_\nu$ on $\mathcal{O}^\star_\nu$.

Then by the definition of Δ'_s, for any $f \in \mathcal{S}(\mathbb{C})$ we get that $\langle \Delta'_s, f_0 \otimes f\rangle = \int_{C_K} f(x)\chi_0(x)|x|^s d^\star x$. Moreover if the set P of finite ramified places of $\widetilde{\chi_0}$ is not empty, we have $f_0(0) = 0$ and $\int_{\mathbb{A}^f_K} f_0(x)dx = 0$ so that $f_0 \otimes f \in \mathcal{S}(\mathbb{A}_K)_0$ for all $f \in \mathcal{S}(\mathbb{C})$.

We can in fact take a function f of the form $f(x) = b(x)\chi_0(x)$ with $b \in \mathcal{C}^\infty_c(\mathbb{R}^\star_+)$.

So for any $s \in \mathbb{C}$ such that $\Re s > 0$, $\langle \Delta'_s, f_0 \otimes f_b\rangle = \int_{\mathbb{R}^\star_+} b(x)|x|^s d^\star x$.

So when we pair the distribution $\int_\mathbb{R} \Delta_{\frac{1}{2}+\imath t}\hat{\psi}(t)dt$ again such functions, we get that :

$\langle \int_{\mathbb{R}} \Delta_{\frac{1}{2}+\imath t}\hat{\psi}(t)dt, f_0 \otimes f_b\rangle = \iint_{C_K\times\mathbb{R}} L(\widetilde{\chi_0}, \frac{1}{2}+\imath t)\hat{\psi}(t)b(x)|x|^{\frac{1}{2}+\imath t}d^\star x dt.$

But one can see that, if $\widetilde{\chi_0}$ is non trivial, $L(\widetilde{\chi_0}, \frac{1}{2}+\imath t)$ is an analytic function of t so the product $L(\widetilde{\chi_0}, \frac{1}{2}+\imath t)\hat{\psi}(t)$ is a tempered distrubution and so is its Fourier transform. Thanks to the last equality, we have that the Fourier transform of $L(\widetilde{\chi_0}, \frac{1}{2}+\imath t)\hat{\psi}(t)$ paired on arbitrary functions which are smooth with compact support equals 0 and so the Fourier transform $L(\widetilde{\chi_0}, \frac{1}{2}+\imath t)\hat{\psi}(t)$ is equal to 0.

If $\widetilde{\chi_0}$ is trivial, we need to impose the condition $\int_{\mathbb{A}_K} f dx = 0$ ie $\int_{\mathbb{R}_+^\star} b(x)|x|d^\star x = 0$ but we can see that the space of functions $b(x)|x|^{\frac{1}{2}} \in \mathcal{C}_c^\infty(\mathbb{R}_+^\star)$ with the condition $\int_{\mathbb{R}_+^\star} b(x)|x|d^\star x = 0$ is dense in $\mathcal{S}(\mathbb{R}_+^\star)$.

Let us now recall that for the equation $\phi(t)\alpha(t) = 0$ with α a distribution on $\mathbb{S}^1$ and $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$ which has finitely many zeroes denoted x_i of order n_i with $i \in I$ with I a finite set , the distributions $\delta_{x_i}, \delta'_{x_i}, \ldots, \delta_{x_i}^{n_i-1}, i \in I$ form a basis of the space of solutions in α (of the equation $\phi(t)\alpha(t) = 0$).

Now we can come back to our main study. Thanks to what we have shown before, we now know that for η orthogonal to the range of $\mathcal{E}$ and such that $\theta_m^t(h)(\eta) = \eta$, we have that $\hat{\psi}(t)$ is a distribution with compact support satisfying the equation $L(\widetilde{\chi_0}, \frac{1}{2}+\imath t)\hat{\psi}(t) = 0$.

Therefore thanks to what we have recalled, we get that $\hat{\psi}$ is a finite linear combination of the distributions $\delta_t^{(k)}$ with t such that $L(\widetilde{\chi_0}, \frac{1}{2}+\imath t) = 0$ and k striclty less than the order of the zero of this L function (necessary and sufficient to get the vanishing on the range of $\mathcal{E}$) and also $k < \frac{\delta-1}{2}$ (necessary and sufficient to ensure that ψ belongs to $L^2_{-\delta}(\mathbb{R}_+^\star)$, ie $\int_{\mathbb{R}_+^\star}(\log|x|)^{2k}(1 + |\log|x||^2)^{-\delta/2}d^\star x < \infty$).

Conversely, let s be a zero of $L(\widetilde{\chi_0}, s)$ of order $k > 0$. Then by the proposition 6.3 and the finiteness and the analyticity of Δ'_s for $\Re s > 0$, we get, for $a \in [|0, k-1|]$ and $f \in \mathcal{S}(\mathbb{A}_K)_0$, that : $\left(\frac{\partial}{\partial s}\right)^a \Delta'_s(f) = 0$.

We also have that $\left(\frac{\partial}{\partial s}\right)^a \Delta'_s(f) = \int_{C_K} \mathcal{E}(f)(x)\widetilde{\chi_0}(x)|x|^{s-\frac{1}{2}}(\log|x|)^a d^\star x$.

Thus η belongs to the orthogonal of the range of $\mathcal{E}$ and such that $\theta_m^t(h)\eta = \eta$ if and only if it is a finite linear combination of functions of the form

$$\eta_{t,a}(x) = \widetilde{\chi_0}(x)|x|^{\imath t}(\log|x|)^a$$

where $L\left(\widetilde{\chi_0}, \frac{1}{2}+\imath t\right) = 0$ and $a <$ order of the zero t and $a < \frac{\delta-1}{2}$.

Therefore the restriction to the subgroup $\mathbb{R}_+^\star$ of C_K of the transpose of θ_m is given in the above basis $\eta_{t,a}$ by

$$\theta_m(\lambda)^t\eta_{t,a} = \sum_{b=0}^{a} C_a^b \lambda^{\imath t}(\log(\lambda))^b \eta_{t,b-a}$$

Therefore if $L(\widetilde{\chi_0}, \frac{1}{2}+\imath s) \neq 0$ then $\imath s$ does not belong to the spectrum of $D^t_{\widetilde{\chi_0}}$. This determine the spectrum of the operator $D^t_{\widetilde{\chi_0}}$ and so the spectrum of $D_{\widetilde{\chi_0}}$. Therefore the theorem is proved. Let us remark that in the rest of the thesis and in the theorem we will make an abuse of notation and write $L(\chi_0, \bullet)$ instead of $L(\widetilde{\chi_0}, \bullet)$. □

7. Link between Spec $(\mathcal{O}_K)$ and the arithmetic site

In this section, K will still denote an imaginary quadratic number field with class number 1. We consider the Zariski topos Spec $(\mathcal{O}_K)$.

Let us denote for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, $H(\mathfrak{p}) := \{h \in K/\alpha_{\mathfrak{p}} h \in \widetilde{\mathcal{O}_K}\}$ where $\alpha_{\mathfrak{p}} \in \mathbb{A}_K^f$ is the finite adele whose components are all equal to 1 except at $\mathfrak{p}$ where the component vanishes.

Definition 7.1. *Let us denote $\mathcal{S}_K$ the sheaf of sets on Spec $(\mathcal{O}_K)$ which assigns for each Zariski open set $U \subset$ Spec $(\mathcal{O}_K)$ the set $\Gamma(U, \mathcal{S}_K) := \{U \ni \mathfrak{p} \mapsto \xi_\mathfrak{p} \in H(\mathfrak{p}) / \xi_\mathfrak{p} \neq 0$ for finitely many prime ideals $\mathfrak{p} \in U\}$. The action of $\mathcal{O}_K$ on the sections is done pointwise.*

Theorem 7.1. *The functor $T : \mathcal{O}_K \to \mathcal{S}h\,(Spec\,(\mathcal{O}_K))$ which associates to the only object $\star$ of the small category $\mathcal{O}_K$ (the one already considered earlier where $\mathcal{O}_K$ is used as a monoïd with respect to the multiplication law), the sheaf $\mathcal{S}_K$ and to endomorphisms of $\star$ the action of $\mathcal{O}_K$ on $\mathcal{S}_K$ is filtering and so defines a geometric morphism $\Theta : Spec\,(\mathcal{O}_K) \to \widehat{\mathcal{O}_K}$. The image of a point $\mathfrak{p}$ of Spec $(\mathcal{O}_K)$ associated to the prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ is the point of $\widehat{\mathcal{O}_K}$ associated to the $\mathcal{O}_K$ module $H(\mathfrak{p}) \subset K$.*

Proof. To check that the functor T is filtering, we adapt in the very same way as in [10] the definition VII 8.1 of [27] of the three filtering conditions for a functor and the lemma VII 8.4 of [27] where those conditions are reformulated and apply it to our very special case where $\mathcal{O}_K$ is the small category which has only a single object $\star$ and $\mathcal{O}_K$ (the ring of integers of K) as endomorphism and where its image under T is the object $T(\star) = \mathcal{S}$ of Spec($\mathcal{O}_K$) to get that T is filtering if and only if it respects the three following conditions:

1. For any open set U of Spec($\mathcal{O}_K$) there exists a covering $\{U_j\}$ of U and sections $\xi_j \in \Gamma(U_j, \mathcal{S})$
2. For any open set U of Spec($\mathcal{O}_K$) and sections $c, d \in \Gamma(U, \mathcal{S})$, there exists a covering $\{U_j\}$ of U and for each j arrows $u_j, v_j : \star \to \star$ in $\mathcal{O}_K$ and a section $b_j \in \Gamma(U_j, \mathcal{S})$ such that $c|_{U_j} = T(u_j)b_j$ and $d|_{U_j} = T(v_j)b_j$
3. Given two arrows $u, v : \star \to \star$ in $\mathcal{O}_K$ and a section $c \in \Gamma(U, \mathcal{S})$ with $T(u)c = T(v)c$, there exists a covering $\{U_j\}$ of U and for each j an arrow $w_j : \star \to \star$ and a section $z_j \in \Gamma(U_j, \mathcal{S})$ such that for each j, $T(w_j)z_j = c|_{U_j}$ and $u \circ w_j = v \circ w_j \in \mathrm{Hom}_{\mathcal{O}_K}(\star, \star)$

So to check that T is filtering, all we have to do now is to check the three filtering conditions.

- Let us check (i). Let U be a non empty open set of Spec($\mathcal{O}_K$), then the 0 section, ie the section hose value at each prime ideal is 0, is an element of $\Gamma(U, \mathcal{S})$ and so by considering U itself as a cover of U we have shown (i).
- Let us check (ii)

  Let U be a non empty open set of Spec($\mathcal{O}_K$).

  Let $c, d \in \Gamma(U, \mathcal{S})$ two sections of $\mathcal{S}$ over U.

  Then there exists a finite set $E \subset U$ of prime ideals of $\mathcal{O}_K$ such that both c and d vanish in the complement $V := U \backslash E$ of E.

  V is a non empty set of Spec($\mathcal{O}_K$) and let us note for each $\mathfrak{p} \in E$, $U_\mathfrak{p} := V \cup \{\mathfrak{p}\} \subset U$.

  By construction the collection $\{U_\mathfrak{p}\}_{\mathfrak{p} \in E}$ form an open covering of U.

  Then the restriction of the section c and d to $U_\mathfrak{p}$ are only determined by their value at $\mathfrak{p}$ since they vanish at every other point of $U_\mathfrak{p}$. Moreover given an element $b \in \mathcal{S}_\mathfrak{p}$, one can extend it uniquely to a section of $\mathcal{S}$ on on $U_\mathfrak{p}$ which vanishes on the complement of $\mathfrak{p}$.

  So finally for each $\mathfrak{p}$, thanks to the property (ii) of flatness of the functor associated to the stalk $\mathcal{S}_\mathfrak{p} = H_\mathfrak{p}$, as required we get that there exists arrows $u_\mathfrak{p}, v_\mathfrak{p} \in \mathcal{O}_K$ and a section $b_\mathfrak{p} \in \Gamma(U_\mathfrak{p}, \mathcal{S})$ such that $c|_{U_\mathfrak{p}} = T(u_\mathfrak{p})b_\mathfrak{p}$ and $d|_{U_\mathfrak{p}} = T(v_\mathfrak{p})b_\mathfrak{p}$. Since $\{U_\mathfrak{p}\}_{\mathfrak{p} \in E}$ is an open cover of U, we finally get (ii).

- Let us now check (iii)

  Let U be an open set of $\operatorname{Spec}(\mathcal{O}_K)$, let $c \in \Gamma(U, \mathcal{S})$ and $u, v \in \mathcal{O}_K$ such that $T(u)c = T(v)c$.

  - ⋆ Let us assume first that there is a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ such that $c_{\mathfrak{p}} \neq 0$.

    Then by property (iii) of flatness of the functor associated $\mathcal{S}_{\mathfrak{p}} = H_{\mathfrak{p}}$, let $\tilde{w} \in \mathcal{O}_K$ and $\tilde{z}_{\mathfrak{p}} \in H_{\mathfrak{p}}$ such that $T(w)\tilde{z}_{\mathfrak{p}} = c_{\mathfrak{p}}$ and $u\tilde{w} = v\tilde{w}$.

    We cannot have $\tilde{w} = 0$ because then we could have $c_{\mathfrak{p}} = 0$ which is impossible.

    So we have that $\tilde{w} =\neq 0$ and so that $u = v$.

    And then we take U itself as its own cover and with the notations of (iii) we take $z = c$ and $w = 1 \in \mathcal{O}_K$ and so (iii) is checked in this case.

  - ⋆ Otherwise c is the zero section.

    In this case we take U itself as its own cover and with the notation of (iii) we take $z = 0$ and $w = 0$.

We have thus shown that $T : \mathcal{O}_K \to \operatorname{Spec}(\mathcal{O}_K)$ respects the conditions (i), (ii) and (iii), so T is filtering.

Then by theorem VII 9.1 of [27] we get that T is flat.

Therefore by theorem VII 7.2 of [27], we get that T defines a geometric morphism $\Theta : \operatorname{Spec}(\mathcal{O}_K) \to \widehat{\mathcal{O}_K}$.

Similarly to [10], the image of a point $\mathfrak{p}$ of $\operatorname{Spec}(\mathcal{O}_K)$ is the point of $\widehat{\mathcal{O}_K}$ whose associated flat functor $F : \mathcal{O}_K \to \mathfrak{Sets}$ is the composition of the functor $T : \mathcal{O}_K \to \operatorname{Spec}(\mathcal{O}_K)$ with the stalk functor at $\mathfrak{p}$. This last funcotr associates to any sheaf on $\operatorname{Spec}(\mathcal{O}_K)$ its stlak at the point $\mathfrak{p}$ viewed as a set, so we get that F is the flat functor from $\mathcal{O}_K$ to $\mathfrak{Sets}$ associated to the stalk $\mathcal{S}_{\mathfrak{p}} = H_{\mathfrak{p}}$.

All is proven. □

Theorem 7.2. *Let us note $\Theta^\star(\mathcal{C}_{\mathcal{O}_K})$ the pullback of the structure sheaf of $\left(\widehat{\mathcal{O}_K}, \mathcal{C}_{\mathcal{O}_K}\right)$. Then:*

1. *The stalk of $\Theta^\star(\mathcal{C}_{\mathcal{O}_K})$ at the prime $\mathfrak{p}$ is the semiring $\mathcal{C}_{H_{\mathfrak{p}}}$ and at the generic point it is $\mathbb{B}$.*
2. *The sections ξ of $\Theta^\star(\mathcal{C}_{\mathcal{O}_K})$ on an open set U of Spec$(\mathcal{O}_K)$ are the maps $U \ni \mathfrak{p} \mapsto \xi_{\mathfrak{p}} \in \mathcal{C}_{H_{\mathfrak{p}}}$ which are either equal to $\{0\}$ outside a finite set or everywhere equal to the constant section $\xi_{\mathfrak{p}} = \emptyset \in \mathcal{C}_{H_{\mathfrak{p}}}, \forall \mathfrak{p} \in U$*

Proof.

1. The result follows from the fact that the stalk of $\Theta^\star(\mathcal{C}_{\mathcal{O}_K})$ at the prime $\mathfrak{p}$ is the same as the stalk of the sheaf $\mathcal{C}_{\mathcal{O}_K}$ at the $\Theta(\mathfrak{p})$ (and so associated to $H_{\mathfrak{p}}$) of $\widehat{\mathcal{O}_K}$.

   For $\{0\}$ we consider the stalk of $\mathcal{C}_{\mathcal{O}_K}$ at the point of $\widehat{\mathcal{O}_K}$ associated to the $\mathcal{O}_K$-module $\{0\}$ which is so $\mathcal{C}_{\{0\}} = \{\emptyset, \{0\}\} \simeq \mathbb{B}$.

2. It follows from theorem 6.1 and the definition of pullback.

□

8. The square of the arithmetic site for $\mathbb{Z}[\imath]$

In this section we will only treat the case of $\mathbb{Z}[\imath]$, the case for $\mathbb{Z}[\jmath]$ being similar replace $[1, \imath]$ by the segment $[1, \jmath]$.

That being said, before beginning investigating tensor products in the case of $\mathbb{Z}[i]$, we must change our point of view for an equivalent one which is functionnal, ie we will switch from convex sets to some restriction of the opposite of their support function. Although we only have an abstract description for now, I think it will be useful for the future and in other cases (for example $\mathbb{Z}[\sqrt{2}]$) to switch to the functional point of view.

Definition 8.1. *Let us note $\mathcal{F}_{\mathbb{Z}[\imath]}$ the set of all piecewise affine convex functions of the form*

$$\begin{cases} [1,\imath] = \{1-t+\imath t, t\in[0,1]\} \longrightarrow & \mathbb{R}^+_{max} \\ x+\imath y = 1-t+\imath t \longmapsto & \max\limits_{(a,b)\in\Sigma} \langle (x,y),(a,b)\rangle = ax+by = a+(b-a)t \end{cases}$$

where Σ is the set of vertices (in fact thanks to the symmetries by $\imath$ and -1 we can only take the vertices in the upper right quarter of the complex plane) of an element of $\mathcal{C}_{\mathbb{Z}[\imath]}$ (when Σ is empty, the function associated is constant equal to $-\infty$).

The easy proof of the following proposition is left to the reader.

Proposition 8.1. *Endowed with the operations max (punctual maximum) and + (punctual addition), $(\mathcal{F}_{\mathbb{Z}[\imath]}, max, +)$ is an idempotent semiring.*

We can now show that the viewpoints of the convex geometry and of those special functions are equivalent.

Proposition 8.2. $\left(\mathcal{C}_{\mathbb{Z}[\imath]}, Conv(\bullet\cup\bullet), +\right)$ *and* $(\mathcal{F}_{\mathbb{Z}[\imath]}, max, +)$ *are isomorphic semirings through the isomorphism*

$$\Phi : \begin{cases} \mathcal{C}_{\mathbb{Z}[\imath]} \longrightarrow & \mathcal{F}_{\mathbb{Z}[\imath]} \\ C \longmapsto & \begin{cases} \longrightarrow & \mathbb{R}^+_{max} \\ x+\imath y \longmapsto & \max\limits_{(a,b)\in\Sigma_C} \langle (x,y),(a,b)\rangle \end{cases} \end{cases}$$

where Σ_C stands for the set of vertices of C

Proof. This map Φ is immediately a surjective morphism between $\left(\mathcal{C}_{\mathbb{Z}[\imath]}, \mathrm{Conv}(\bullet\cup\bullet), +\right)$ and $(\mathcal{F}_{\mathbb{Z}[\imath]}, max, +)$.

Let us now show that Φ is injective.

Let $C, C' \in \mathcal{C}_{\mathbb{Z}[\imath]}\backslash\{\emptyset,\{0\}\}$ with $C \neq C'$.

Let $c' \in C'$ such that $c' \notin C$.

We identify $\mathbb{C}$ and $\mathbb{R}^2$, then thanks to Hahn-Banach theorem, there exists $\phi \in (\mathbb{R}^2)^\star$ such that $\forall c\in C, \phi(c) < \phi(c')$.

But thanks to the canonical euclidian scalar product, we can identify $(\mathbb{R}^2)^\star$ with $\mathbb{R}^2$, so let $\vec{u}\in\mathbb{R}^2$ such that $\phi = \langle \vec{u}, \bullet\rangle$, so we have that $\forall c\in C, \langle \vec{u}, c\rangle < \langle \vec{u}, c'\rangle$.

But since C is compact, let $\gamma\in C$ such that $\langle \vec{u},\gamma\rangle < \langle \vec{u}, c'\rangle = \sup_{c\in C}\langle \vec{u}, c\rangle < \langle \vec{u}, c'\rangle$

Thanks to the symmetry of C, C' by $\mathcal{U}_K$ and the identification of $\mathbb{C}$ and $\mathbb{R}^2$, we can assume that $\vec{u}, \gamma, c' \in \mathbb{C}/\mathcal{U}_K$.

Then finally we have that $(\Phi(C))(\vec{u}) \leq \langle \vec{u},\gamma\rangle < \langle \vec{u}, c'\rangle \leq (\Phi(C'))(\vec{u})$, so we have the injectivity in this case.

For the other cases, let us remark that $\Phi(\emptyset) \equiv -\infty$ the constant function equal to $'\infty$ by convention, $\Phi(\{0\}) = 0$ the constant function equal to zero by direct calculation, and that for $C \in \mathcal{C}_{\mathcal{O}_K}\backslash\{\emptyset,\{0\}\}$, if we take c a vertex of C of maximal module among the vertices of C, we immediately get that $(\Phi(C))(c) = |c|^2 > 0$ and so the injectivity is proved.

All in all, we indeed have that $\left(\mathcal{C}_{\mathbb{Z}[\imath]}, \mathrm{Conv}(\bullet\cup\bullet), +\right)$ and $(\mathcal{F}_{\mathbb{Z}[\imath]}, max, +)$ are isomorphic semirings. □

Let us now determine $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}$.

Viewing $(\mathcal{F}_{\mathbb{Z}[\imath]}, \max)$ as a $\mathbb{B}$-module, we can define $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}$ in the following way (see also [29] and [10]):

Definition 8.2. *$(\mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}, \oplus)$ is the $\mathbb{B}$-module constructed as the quotient of $\mathbb{B}$-module of finite formal sums $\sum e_i \otimes f_i$ (we can remark that no coefficients are needed since $\mathcal{F}_{\mathbb{Z}[\imath]}$ is idempotent) by the equivalence relation*

$$\sum e_i \otimes f_j \sim \sum e'_j \otimes f'_j \Leftrightarrow \forall \Psi, \sum \Psi(e_i, f_i) = \sum \Psi(e'_j, f'_j)$$

where Ψ is any bilinear map from $\mathcal{F}_{\mathbb{Z}[\imath]} \times \mathcal{F}_{\mathbb{Z}[\imath]}$ to any arbitrary $\mathbb{B}$-module and where $\oplus$ is just the formal sum.

Since $(\mathcal{F}_{\mathbb{Z}[\imath]}, \max, +)$ is moreover an idempotent semiring, we can see that the law $+$ of $\mathcal{F}_{\mathbb{Z}[\imath]}$ induces a new law again noted $+$ on $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}$ in the following way :

Proposition 8.3. *Let $a \otimes b$ and $a' \otimes b' \in \mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}$, we can define $+$ such that $(a \otimes b) + (a' \otimes b') = (a + a') \otimes (b + b')$. In this way $+$ is well defined and it turns $(\mathcal{F}_{\mathbb{Z}[\imath]} \otimes_{\mathbb{B}} \mathcal{F}_{\mathbb{Z}[\imath]}, \oplus, +)$ into an idempotent semiring.*

Proof. Let $(a, b) \in \mathcal{F}_{\mathbb{Z}[\imath]} \times \mathcal{F}_{\mathbb{Z}[\imath]}$ We define the application $\Sigma_{a,b} : \begin{cases} \mathcal{F}_{\mathbb{Z}[\imath]} \times \mathcal{F}_{\mathbb{Z}[\imath]} \rightarrow & \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]} \\ (a', b') \mapsto & (a + a') \otimes (b + b') \end{cases}$

Let $(a', b'), (a'', b') \in \mathcal{F}_{\mathbb{Z}[\imath]} \times \mathcal{F}_{\mathbb{Z}[\imath]}$,

Then we have :

$\Sigma_{a,b}(\max(a', a''), b') = (a + \max(a', a'')) \otimes (b + b') = (max(a + a', a + a'')) \otimes (b + b') = ((a + a') \otimes (b + b')) \oplus ((a + a'') \otimes (b + b')) = \Sigma_{a,b}(a', b') \oplus \Sigma_{a,b}(a'', b')$

So $\Sigma_{a,b}$ is $\mathbb{B}$-linear in the first variable. One can show in the same way that $\Sigma_{a,b}$ is $\mathbb{B}$-linear in the second variable so finally that $\Sigma_{a,b}$ is a $\mathbb{B}$-bilinear map from $\mathcal{F}_{\mathbb{Z}[\imath]} \times \mathcal{F}_{\mathbb{Z}[\imath]}$ to $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$ so it can be factorized by the universal property of tensor product by a linear map $\sigma_{a,b} : \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]} \rightarrow \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$

And consequently we denote for all $a' \otimes b' \in \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$, $(a \otimes b) + (a' \otimes b') :=_{\text{def}} \sigma_{a,b}(a' \otimes b')$.

So $+$ is well defined on elementary tensors and so after for all tensors. We deduce from this that $(\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}, \oplus, +)$ is a semiring.

□

Proposition 8.4. *$(\mathbb{Z}[\imath])^2$ acts on $(\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}, \oplus, +)$ and the action preserves the semiring structure.*

Proof. Let $(\alpha, \beta) \in (\mathbb{Z}[\imath])^2$ and let $p \in \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$.

Let I be a finite set and $f_i, g_i \in \mathcal{F}_{\mathbb{Z}[\imath]}$ for all $i \in I$ such that $p = \bigoplus_{i \in I} f_i \otimes g_i$.

Then we define the action of (α, β) on p by $(\alpha, \beta) \bullet p = \sum_{i \in I} \Phi(\alpha \bullet \Phi^{-1}(f_i)) \otimes \Phi(\beta \bullet \Phi^{-1}(g_i))$ where Φ is the isomorphism between $\mathcal{C}_{\mathbb{Z}[\imath]}$ and $\mathcal{F}_{\mathbb{Z}[\imath]}$.

With this definition, the action of $(\mathbb{Z}[\imath])^2$ on $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$ is directly compatible with the law $\oplus$ and so preserves the structure of $\mathbb{B}$-module.

Let $a \otimes b, a' \otimes b' \in \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$, then we have $a \otimes b + a' \otimes b' = (a + a') \otimes (b + b')$.

And for $(\alpha, \beta) \in (\mathbb{Z}[\imath])^2$, we have $(\alpha, \beta) \bullet ((a + a') \otimes (b + b')) = \Phi(\alpha \bullet \Phi^{-1}(a + a')) \otimes \Phi(\beta \bullet \Phi^{-1}(b + b'))$.

But $\alpha \bullet \Phi^{-1}(a + a') = \alpha \bullet \Phi^{-1}(a) + \alpha \bullet \Phi^{-1}(a')$ and $\beta \bullet \Phi^{-1}(b + b') = \beta \bullet \Phi^{-1}(b) + \beta \bullet \Phi^{-1}(b')$.

So we have $(\alpha, \beta) \bullet ((a + a') \otimes (b + b')) = (\alpha, \beta) \bullet a \otimes b + (\alpha, \beta) \bullet a' \otimes b'$.

the action of $(\mathbb{Z}[\imath])^2$ on $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$ is directly compatible with the law $+$. □

Thanks to this last proposition, we can therefore view $(\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}, \oplus, +)$ as an idempotent semiring in the topos $\widehat{(\mathbb{Z}[\imath])^2}$ (the topos of sets with an action of $(\mathbb{Z}[\imath])^2$ where the composition of arrows is the multiplication component by component). It allows us to define the unreduced square of the arithmetic site for $\mathbb{Z}[\imath]$ as follows :

Definition 8.3. *The unreduced square* $\left(\widehat{(\mathbb{Z}[\imath])^2}, \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}\right)$ *is the topos* $\widehat{(\mathbb{Z}[\imath])^2}$ *with the structure sheaf* $(\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}, \oplus, +)$ *viewed as an idempotent semiring in the topos.*

The idempotent semiring $(\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}, \oplus, +)$ is not necessarily a multiplicative cancellative semiring. In the case it is not, we can send it into a multiplicative cancellative semiring in the following way :

Let us set $\mathcal{P} := \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$.

Let us denote $\mathcal{R}$ the idempotent semiring (with laws $\oplus$ and $+$ being defined component wise) $\mathcal{R} := \mathcal{P} \times \mathcal{P}/\sim$ where $\sim$ is the equivalence relation defined as follows

$$(a, b) \sim (a', b') \Leftrightarrow \exists c \in \mathcal{P}, a + b' + c = a' + b + c$$

Proposition 8.5. *The semiring $\mathcal{R}$ is multiplicatively cancellative.*

Proof. Let $(a, b), (a', b'), (c, d) \in \mathcal{R}$ with $(c, d) \neq (-\infty, -\infty)$ such that $(c, d) + (a, b) = (c, d) + (a', b')$, so we have $(a + c, b + d) = (a' + c, b' + d)$.

So we have $a + c + b' + d = a' + c + b + d$, ie $a + b' + (c + d) = a' + b + (c + d)$.

So in $\mathcal{R}$, we have $(a, b) = (a', b')$ and so $\mathcal{R}$ is multiplicatively cancellative. □

Definition 8.4. *Let us denote $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}$ the image of $\mathcal{P} = \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$ by the application*

$$\gamma : \begin{cases} \mathcal{P} \to & \mathcal{R} \\ a \mapsto & (a, 0) \end{cases}$$

It is an idempotent multiplicatively cancellative semiring.

Proposition 8.6. *The reduced tensor product of $\mathcal{F}_{\mathbb{Z}[\imath]}$ by $\mathcal{F}_{\mathbb{Z}[\imath]}$ is given by $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}$, it satisfies the following universal property. For any multiplicative cancellative ring R and any homomorphism $\rho : \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]} \to R$ such that $\rho^{-1}(\{0\}) = \{(-\infty, -\infty)\}$, then there exists a unique homomorphism $\rho' : \mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]} \to R$ such that $\rho = \rho' \circ \gamma$.*

Proof. Let R a multiplicative cancellative ring and a homomorphism $\rho : \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]} \to R$ such that $\rho^{-1}(\{0\}) = \{(-\infty, -\infty)\}$.

Let $a, b \in \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$ such that $a = b$ in $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}$.

Then there exists $c \in \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]} \backslash \{(-\infty, -\infty)\}$ such that $a + c = b + c$.

Then $\rho(a + c) = \rho(b + c)$, so $\rho(a) \times_R \rho(c) = \rho(b) \times_R \rho(c)$.

Since $\rho^{-1}(\{0\}) = \{(-\infty, -\infty)\}$, $\rho(c) \neq 0_R$.

And so since R is multiplicatively cancellative, we have $\rho(a) = \rho(b)$, so the image of an element of $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$ by the application ρ depends only on the class of this latter element in $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}$ and so we can take $\rho' : \mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]} \ni \gamma(a) \mapsto \rho(a)$. We have shown that the application ρ' is well defined and we have $\rho = \rho' \circ \gamma$. Therefore the result is proved.

□

Proposition 8.7. *The action of $\mathbb{Z}[\imath] \times \mathbb{Z}[\imath]$ on $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$ induces an action on $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}$ which is compatible with the semiring structure.*

Proof. Let $a, b \in \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$ such that in $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}$, a is equal to b (ie $\gamma(a) = \gamma(b)$).

Then let $c \in \mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$ such that $a + c = b + c$.

Then for any $(\alpha, \beta) \in \mathbb{Z}[\imath] \times \mathbb{Z}[\imath]$, we have $(\alpha, \beta) \bullet (a + c) = (\alpha, \beta) \bullet (b + c)$ and so $(\alpha, \beta) \bullet a + (\alpha, \beta) \bullet c = (\alpha, \beta) \bullet b + (\alpha, \beta) \bullet c$ and finally $(\alpha, \beta) \bullet a$ equal to $(\alpha, \beta) \bullet b$ in $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}$ (ie $\gamma((\alpha, \beta) \bullet a) = \gamma((\alpha, \beta) \bullet b)$.

Consequently the action of $\mathbb{Z}[\imath] \times \mathbb{Z}[\imath]$ is compatible with the relation $\sim$ used to define $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}$ and since the action of $\mathbb{Z}[\imath] \times \mathbb{Z}[\imath]$ was compatible with the semiring structure of $\mathcal{F}_{\mathbb{Z}[\imath]} \otimes \mathcal{F}_{\mathbb{Z}[\imath]}$, the induced action of $\mathbb{Z}[\imath] \times \mathbb{Z}[\imath]$ is compatible with the semiring structure on $\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}$.

$\square$

Definition 8.5. *The reduced square* $\left(\widehat{(\mathbb{Z}[\imath])^2}, \mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}\right)$ *is the topos* $\widehat{(\mathbb{Z}[\imath])^2}$ *with the structure sheaf* $(\mathcal{F}_{\mathbb{Z}[\imath]} \hat{\otimes} \mathcal{F}_{\mathbb{Z}[\imath]}, \oplus, +)$ *viewed as an idempotent semiring in the topos.*

9. Appendix by Alain Connes : some remarks on the semiring $\mathcal{C}_{\mathbb{Q}(\imath),\mathbb{C}}$

Let us start with the field $K := \mathbb{Q}(\imath)$ of Gaussian numbers. Sagnier considers the semiring $R = \mathcal{C}_{K,\mathbb{C}}$ whose elements are $\emptyset$, $\{0\}$ and all closed convex bounded polygons invariant under the group of units $\mathcal{U} = \{\pm 1, \pm \imath\}$. The two operations are :

$$A \vee B := \mathrm{Conv}(A \cup B)\,,\; A + B := \{a + b | a \in A, b \in B\}$$

One lets $D_K := \mathrm{Conv}(\mathcal{U})$

Lemma 9.1. *Let* $\lambda, \lambda' \in \mathbb{C}^\star$ *and assume*

$$\lambda' \in \mathit{Conv}(\lambda \mathbb{R}_+, \imath\lambda \mathbb{R}_+)$$

One then has

$$\lambda D_K + \lambda' D_K = \mathit{Conv}((\lambda + \lambda')D_K, (\imath\lambda + \lambda')D_K) \tag{1}$$

Moreover provided that $\lambda' \in \lambda\mathbb{R}_+ \bigcup \imath\lambda\mathbb{R}_+$ *both* $(\lambda + \lambda')$ *and* $(\imath\lambda + \lambda')$ *are extreme points of* $\lambda D_K + \lambda' D_K$.

Proof. The extreme points of $S = \lambda D_K + \lambda' D_K$ are contained in the 16 elements of $E = \lambda\mathcal{U} + \lambda'\mathcal{U}$ since an extreme point of S is reaching the unique maximum for a suitable $\mathbb{R}$-linear form L while

$$\sup\{L(a + b) | a \in A, b \in B\} = \sup\{L(a) | a \in A\} + \sup\{L(b) | b \in B\}$$

and a linear form on a convex polygon reaches its maximum on an extreme point. We can assume $\lambda = 1$ and $\lambda' = z = a + \imath b$ with $a > 0$, $b > 0$. The only elements of E which can be in the first quadrant are in the list

$$E' = \{1 + z, \imath + z, z - 1, z - \imath, \imath z + 1, -\imath z + \imath\}$$

One has $\imath z + 1 = \imath(z - \imath)$, $-\imath z + \imath = -\imath(z - 1)$ and thus what is needed is to show that $z - 1$, $z - \imath$ are not extreme points. Asociated to each λD_K, one has a linear form L_λ such that

$$\lambda D_K = \{\xi | L_\lambda(u\xi) \leq \lambda\bar{\lambda}, \forall u \in \mathcal{U}\}, L_\lambda(\xi) = \Re((1 - \imath)\xi\bar{\lambda}) \tag{2}$$

There are 4 linear forms which suffice to determine S by inequalities of the form $|L(u)| \leq v$, they are $L_1, L_\imath, L_z, L_{\imath z}$. The value of $\sup L_1(w)$ for $w \in zD_K$ is reached on the extreme point

$z \in zD_K$ and is given by $a+b$. The value of $\sup L_z(w)$ for $w \in D_K$ is given again by $a+b$. Thus the relevant inequalities are

$$|L_1(w)| \le 1+a+b, |L_1(\imath w)| \le 1+a+b, |L_z(w)| \le a^2+b^2+a+b, |L_z(\imath w)| \le a^2+b^2+a+b$$

We take first $w = z-1$. One has $L_1(w) = a+b-1$ and thus $|L_1(w)| < 1+a+b$. One has $L_1(\imath w) = a-b-1$ and thus $|L_1(\imath w)| < 1+a+b$. One has $L_z(w) = z\bar z - a + b$ and thus $|L_z(w)| < a^2+b^2+a+b$. One has $L_z(w) = z\bar z - a - b$ and thus $|L_z(\imath w)| < a^2+b^2+a+b$. These strict inequalities show that $w = z-1$ is in the interior of $\lambda D_K + \lambda' D_K$. For $t = z - \imath$ one gets

$$L_1(t) = a+b-1, L_1(\imath t) = a-b+1, L_z(t) = z\bar z - a - b, L_z(\imath t) = z\bar z + a - b$$

and again the strict inequalities show that t is in the interior of $\lambda D_K + \lambda' D_K$. Thus the only possible extreme points in the first quadrant are $1+z$ and $\imath + z$ and this proves the first part of the lemma. Assume now that $a>0$ and $b>0$ and let us show that $1+z$ and $\imath+z$ are extreme points of $\lambda D_K + \lambda' D_K$. For $s = 1+z$ one has

$$L_1(s) = a+b+1, L_1(\imath s) = a-b+1, L_z(s) = z\bar z + a - b, L_z(\imath s) = z\bar z + a + b$$

thus we see that both L_1 and $L_z(\imath)$ reach their maximum on s and since they are distinct linear forms this shows that s is an extreme point. For $r = \imath + z$ one has

$$L_1(r) = a+b+1, L_1(\imath r) = a-b-1, L_z(r) = z\bar z + a + b, L_z(\imath r) = z\bar z - a + b$$

thus we see that both L_1 and L_z reach their maximum on r. This proves the second part of the lemma. $\square$

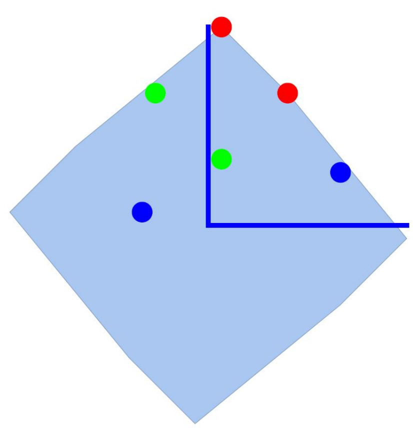

Figure 7: $\lambda D_K + \lambda' D_K$ for $\lambda = 1$, $\lambda' = z = \frac{1}{5} + 2\imath$. In red the points $1+z, \imath+z$. In green $z-1, z-\imath$.In blue $1+\imath z, -\imath z+\imath$

This suggests to define a multivalued law related to the addition in the quotient $H := \mathbb{C}/\mathcal{U}$ which is related to the addition in the hyperfield $\mathbb{C}/\mathcal{U}$. In fact it seems that one can select naturally a subset with two elements among the elements of $(\lambda\mathcal{U} + \lambda'\mathcal{U})/\mathcal{U}$. Assuming $\lambda' \in \mathrm{Conv}(\lambda\mathbb{R}_+, \imath\lambda\mathbb{R}_+)$ and that λ and λ' are not colinear, one takes

$$\lambda \oplus \lambda' := \{\lambda + \lambda', \imath\lambda + \lambda'\}$$

In fact this operation is commutative and one just selects the two elements which are obtained by adding the points which for an angle $< \pi/2$.By the lemma this means that one associates the extreme points of $\lambda D_K + \lambda' D_K$. One needs to study associativity. There is a natural subset of the hypersum $(\lambda\mathcal{U} + \lambda'\mathcal{U} + \lambda''\mathcal{U})/\mathcal{U}$ given by the extreme points of $\lambda D_K + \lambda' D_K + \lambda'' D_K$. But one should expect that the Minkowski sum of three squares in a dodecagon and this does not correspond to the iteration of the operation $\oplus$ which produces 4 elements. Thus we cannot expect associativity.

Let us now work out the condition under which $\lambda D_K \subset \lambda' D_K$. This is equivalent to $\lambda \in \lambda' D_K$ and by (2) to

$$|\Re((1 \pm \imath)\lambda\bar{\lambda}')| \leq |\lambda'|^2$$

It is by construction an order relation on $\mathbb{C}/\mathcal{U}$. This partial order, together with the scaling by $\mathbb{R}_+^\star$ determines the topology since the interior λD_K° of λD_K is the union of the $(1-\epsilon)\lambda D_K$ for $\epsilon > 0$ and its exterior is open, as all sets of the form $\lambda D_K^\circ \cap \lambda' D_K^c$. These sets form a basis of neighborhood of any point and hence a basis of the topology. Indeed the λD_K° form a basis of neighborhoods of 0 and the $(1+\epsilon)D_K^\circ \cap ((1-\epsilon)(1+\imath)D_K)^c$ form a basis of neighborhoods of 1. It follows that any order automorphism is automatically continuous since the inverse image of an open set is open as a union of intervals of the above form.

Theorem 9.1. *The group $Aut(R)$ of automorphisms of the semiring R is the direct product $(\mathbb{C}^\star/\mathcal{U}) \rtimes_\sigma \mathbb{Z}/2\mathbb{Z}$ of the quotient of $\mathbb{C}^\star$ by the units by the action of the Galois group of $\mathbb{C}$ over $\mathbb{R}$. The group is the quotient $W/\mathcal{U}$ of the Weil group by the group of units.*

Proof. The λD_K are the only elements of R which cannot be written ia non-trivial manner as $B \vee C$ where both term are $\neq \lambda D_K$. indeed in such case the action of $\mathcal{U}$ on the extreme points is transitive and conversely. Since any element of R is of the form $\vee \lambda_j D_K$, it follows that $\theta \in \mathrm{Aut}(R)$ is uniquely of the form

$$\theta(\vee \lambda_j D_K) = \vee f(\lambda_j) D_K$$

where the map $f : \mathbb{C}/\mathcal{U} \to \mathbb{C}/\mathcal{U}$ is uniquely determined by the action on the λD_K. This map is an isomorphism for the partial order (given by inclusion) and is hence continuous, it commutes with the positive real scaling. Moreover since θ preserves the Minkowski sum one has

$$\theta(\lambda D_K + \lambda' D_K) = \theta(\lambda D_K) + \theta(\lambda' D_K)$$

which implies, by 9.1,

$$f(\lambda \oplus \lambda') = f(\lambda) \oplus f(\lambda')$$

Since f commutes with positive real scaling and is additive on a ray, the behavior of $f(\lambda \oplus \lambda')$ is known on a ray. Let us assume that λ' is in the interior of $\mathrm{Conv}(\lambda\mathbb{R}_+, \imath\lambda\mathbb{R}_+)$. One gets

$$\{f(\lambda + \lambda'), f(\imath\lambda + \lambda')\} = f(\lambda) \oplus f(\lambda')$$

We can assume (using composition by a complex scaling and if needed by the complex conjugation) $f(1) = 1$ and that $f(1 + \imath\epsilon)$ is in the first quadrant for $\epsilon > 0$ small enough. Since f commutes with positive real scaling and is continuous and bijective $\mathbb{C}/\mathcal{U}$ it determines, undder the above conditions, a bijection of the first quadrant $\mathcal{Q} = \mathrm{Conv}(\mathbb{R}_+, \imath\mathbb{R}_+)$ onto itself which is the identity on the boundary. The rays form an interval I and the induced map is continuous and increasing. It follows that when the argument $\lambda' \in Q$ larger than the argument of $\lambda \in \mathcal{Q}$ the same holds after applying f and thus

$$f(\lambda) \oplus f(\lambda') = \{f(\lambda) + f(\lambda'), \imath f(\lambda) + f(\lambda')\}$$

One then obtains, using continuity, that

$$f(\lambda+\lambda')f(\lambda)+f(\lambda'), \forall \lambda, \lambda' \in \mathcal{Q}$$

But f is the identity on the boundary of $\mathcal{Q}$ and one concludes that it is the identity in $\mathcal{Q}$ by writing any element of $\mathcal{Q}$ as a sum of two elements of the boundary. Finally note that the Weil group W is the subgroup of the multiplicative group $\mathbb{H}^\star$ of the non-zero quaternions generated by $\mathbb{C}^\star$ and j (or k). Thus since $-1 \in \mathcal{U}$ the quotient $W/\mathcal{U}$ is the semi-direct product $(\mathbb{C}^\star/\mathcal{U}) \rtimes_\sigma \mathbb{Z}/2\mathbb{Z}$. $\square$

The next question is to see if one can characterize the elements of the form μD_K by the condition (1). One views the action of $\mathbb{C}^\star/\mathcal{U}$ as an extension of the Frobenius action of $\mathbb{R}_+^\star$. We adopt momentarily a multiplicative notation and rewrite (1) in the form

$$X^\lambda X^{\lambda'} = X^{\lambda\oplus\lambda'}, \forall \lambda, \lambda' \tag{3}$$

Assume that X, Y are the solutions of this equation and take $X+Y$ one then has by distributivity and since $X \to X^\lambda$ is an endomorphism (which is a complex generalization of the Frobenius)

$$(X+Y)^\lambda (X+Y)^{\lambda'} = (X^\lambda + Y^\lambda)(X^{\lambda'} + Y^{\lambda'}) = X^\lambda X^{\lambda'} + X^\lambda Y^{\lambda'} + Y^\lambda X^{\lambda'} + Y^\lambda Y^{\lambda'}$$

while with $\lambda \oplus \lambda' = \{\mu, \mu'\}$ one has

$$(X+Y)^{\lambda\oplus\lambda'} = (X+Y)^\mu + (X+Y)^{\mu'} = X^\lambda X^{\lambda'} + Y^\lambda Y^{\lambda'}$$

Thus the presence of the cross terms should prevent $X+Y$ from being a solution, and allow one to characterize the monomials by the above equation. in fact one can consider the general case of a polynomial of the form

$$P = \sum X^{\alpha_j}, X = D_K$$

One then has $P^\lambda = \sum X^{\lambda\alpha_j}$ and

$$P^\lambda P^{\lambda'} = \sum X^{\lambda\alpha_j} X^{\lambda'\alpha_k} = \sum_{\mu \in \cup(\lambda\alpha_j \oplus \lambda'\alpha_k)} X^\mu$$

while

$$P^{\lambda\oplus\lambda'} = \sum_{\mu \in \cup(\lambda\alpha_j \oplus \lambda'\alpha_j)} X^\mu$$

Thus in order to obtain a contradiction it is enough to show that for suitable choice of (λ, λ') one of the cross terms $\lambda\alpha_j \oplus \lambda'\alpha_k$ $j \neq k$ gives an extreme point. Note that an extreme point ϵ in a Minkowski sum is uniquely a sum of extreme points $\epsilon = \epsilon' + \epsilon''$ in the summands. This follows by choosing a linear form whihch is not critical for the two summands and on which the maximum is reached by ϵ.

Lemma 9.2. *1. The λD_K are the only solutions of equation* (3)*.*

2. The same statement holds assuming that (3) *for all $\lambda, \lambda' \in \mathcal{O}_K$.*

Proof. 1. Consider a polynomial $P = \sum X^{\alpha_j}$ which is not a monomial. Then there are two distinct orbits of $\mathcal{U}$, $z\mathcal{U}$ and $z'\mathcal{U}$ among the extreme points of the convex hull C of the $\alpha_j D_K$. It follows that there are two $\mathbb{R}$-linear forms L, L' from $\mathbb{C}$ to $\mathbb{R}$, whose maximum on C is reached on z and z'. Any $\mathbb{R}$-linear form is uniquely written as $L_l(u) = \Re(ul)$ and thus one can find $v \in \mathbb{C}^\star$ such that $L'(u) = L(vu)$ for all $u \in \mathbb{C}$. The product PP^v corresponds to the Minkowski sum $C + vC$. The linear form reaches its maximum on $z \in C$ and on vz' on vC since $L(vz') = L'(z')$ is larger than any other $L(vu) = L'(u)$. This shows that $\epsilon = z + vz'$ is an extreme point $C + vC$. Since the orbits $z\mathcal{U}$ and $z'\mathcal{U}$ are distinct, the extreme point ϵ comes from the contribution of a cross term in PP^v. By the uniqueness of the decomposition as a sum of extreme points $\epsilon = \epsilon' + \epsilon''$, it follows that one gets a contradiction with the validity of (3) for $X = P$ since $P^{1\oplus v}$ only involves diagonal terms, i.e. terms in $\cup(\alpha_j \oplus v\alpha_j)$. This shows that the λD_K are the only solutions equation (3).

2. We can assume that the two $\mathbb{R}$-linear forms L, L' from $\mathbb{C}$ to $\mathbb{R}$ are of the form $L_l, L_{l'}$ with $l, l' \in \mathcal{O}_K$ because they are defined by an open cone whose intersection with $\mathcal{O}_K$ is non-empty. One then gets an equation of the form $L'(lu) = L(l'u)$ for all $u \in \mathbb{C}$. the above reasoning shows that $lz + l'z'$ is an extreme point of $lC + l'C$ which corresponds to a cross term so that weaker version of (3) fails.

□

We denote by Fr_λ the action on convex polygons by multiplication by λ. We view it as an extension of the Frobenius action on multiplicatively cancellative semirings.

Theorem 9.2. *The morphisms ρ from the Sagnier semiring $\mathcal{C}_{K,\mathbb{C}}$ to the semiring $\mathcal{C}_{\mathbb{Z}[\imath]}$ of $\mathcal{U}$-invariant convex polygons in $\mathbb{C}$, which fulfill $\rho \circ Fr_\lambda = Fr_\lambda \circ \rho$ for any $\lambda \in \mathcal{O}_K$ are given by multiplication by a complex number.*

Proof. By th equation $\rho \circ \mathrm{Fr}_\lambda = \mathrm{Fr}_\lambda \circ \rho$ the morphism ρ is determined uniquely by $X = \rho(D_K)$. The compatibility with the two operations $\vee$ and Minkowski sum shows that X fulfills the weaked form of (3). Thus, by lemma 9.2, there exists $\lambda \in \mathbb{C}$ such that $X = \lambda D_K$ and it follows that ρ is given by multiplication by λ. □

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