"Introduction to Deep Inference and Proof Nets"

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LECTURE 1

Classical logic

Content:

- recall sequent calculus
  - LK
  - GS1p
- Classical logic in the calculus of structures
  - SKSg
  - SKS
- Correspondence GS1p vs SKSg
- cut-elimination in SKS

CLASSICAL LOGIC – Sequent Calculus

System LK with Cut:

STRUCTURAL RULES

$$\text{Ax}\ \frac{}{A \vdash A} \qquad\qquad \text{cut}\ \frac{\phi \vdash A \quad A \vdash \psi}{\phi \vdash \psi}$$

$$W_L\ \frac{\phi \vdash \psi}{A, \phi \vdash \psi} \qquad\qquad W_R\ \frac{\phi \vdash \psi}{\phi \vdash \psi, A} \qquad \text{(weekening)}$$

$$C_L\ \frac{A, A, \phi \vdash \psi}{A, \phi \vdash \psi} \qquad\qquad C_R\ \frac{\phi \vdash \psi, A, A}{\phi \vdash \psi, A} \qquad \text{(contraction)}$$

LOGICAL RULES

$$\vee_L\ \frac{A, \phi \vdash \psi \quad B, \phi \vdash \psi}{A \vee B, \phi \vdash \psi} \qquad\qquad \vee_{RL}\ \frac{\phi \vdash \psi, A}{\phi \vdash \psi, A \vee B} \qquad \vee_{RR}\ \frac{\phi \vdash \psi, B}{\phi \vdash \psi, A \vee B}$$

$$\wedge_{LL}\ \frac{A, \phi \vdash \psi}{A \wedge B, \phi \vdash \psi} \qquad \wedge_{LR}\ \frac{B, \phi \vdash \psi}{A \wedge B, \phi \vdash \psi} \qquad\qquad \wedge_R\ \frac{\phi \vdash \psi, A \quad \phi \vdash \psi, B}{\phi \vdash \psi, A \wedge B}$$

$$\neg_L\ \frac{\phi \vdash \psi, A}{\neg A, \phi \vdash \psi} \qquad\qquad \neg_R\ \frac{A, \phi \vdash \psi}{\phi \vdash \psi, \neg A}$$

Example: Prove $\vdash A \vee \neg A$ in LK

$$C_R\ \dfrac{\vee_{RL}\ \dfrac{\vee_{RR}\ \dfrac{\text{Ax}\ \dfrac{}{\vdash A, \neg A}}{\vdash A, A \vee \neg A}}{\vdash A \vee \neg A, A \vee \neg A}}{\vdash A \vee \neg A}$$

CLASSICAL LOGIC - Sequent Calculus

System GS1p with Cut

$$\top\ \frac{}{\vdash \top} \qquad \mathrm{Ax}\ \frac{}{\vdash A, \bar{A}} \qquad \mathrm{cut}\ \frac{\vdash \Phi, A \quad \vdash \Psi, \bar{A}}{\vdash \Phi, \Psi}$$

$$\wedge\ \frac{\vdash \Phi, A \quad \vdash \Psi, B}{\vdash \Phi, \Psi, A \wedge B} \qquad \vee\ \frac{\vdash \Phi, A, B}{\vdash \Phi, A \vee B}$$

$$\mathrm{C}\ \frac{\vdash \Phi, A, A}{\vdash \Phi, A} \qquad \mathrm{W}\ \frac{\vdash \Phi}{\vdash \Phi, A}$$

- negation only on atoms
- units $\top, \bot$

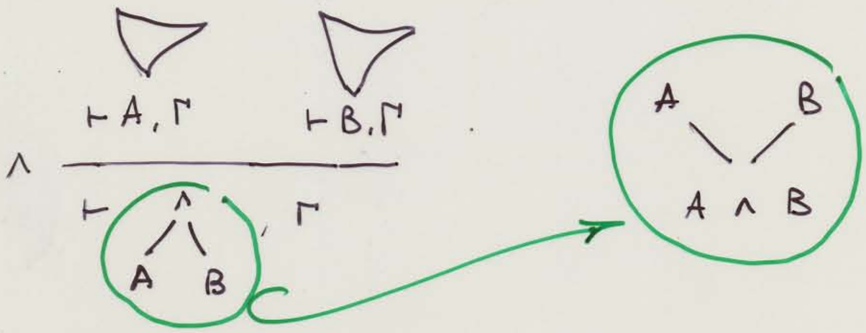
Example: Prove $\vdash A \vee \bar{A}$ in GS1p

$$\vee\ \dfrac{\mathrm{Ax}\ \dfrac{}{\vdash A, \bar{A}}}{\vdash A \vee \bar{A}}$$

Key notes on sequent calculus :

- Logical Inference rules are applied to the <u>main connective</u> of the formula

- The <u>formula tree</u> shapes the <u>proof tree</u> :



(additive conjunction)

- Lk and GS1p (and many more presentations of classical logic) are logically equivalent : they prove the same tautologies.

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- Cut-elimination in systems in proof theory is <u>one</u> of the fundamental properties desired. It says and implies
  - the cut rule can be disposed of, without affecting completeness
  - more determinism in <u>proof search</u>
  - more theoretical results ....
- Cut-Elimination holds for Lk + cut and GS1p + Cut

CLASSICAL LOGIC IN THE CALCULUS OF STRUCTURES

- **Structures (formulae)**

written in context notation

$$S ::= f \mid t \mid a \mid \underbrace{[S, \dots, S]}_{>0} \mid \underbrace{(S, \dots, S)}_{>0} \mid \bar{S}$$

$[\dots]$ disjunction

$(\dots)$ conjunction

(positive and negative atoms)

- **Equivalence on formulae**

$$\left.\begin{array}{l}[R, [T, U], V] = [R\ T\ U\ V] \\ (R\ (T\ U), V) = (R\ T\ U\ V)\end{array}\right\} \text{assoc.}$$

$$\left.\begin{array}{l}[R\ T] = [T\ R] \\ (R\ T) = (T\ R)\end{array}\right\} \text{comm.}$$

$$\left.\begin{array}{ll}(f, f) = f & [f\ R] = R \\ [t, t] = t & (t\ R) = R\end{array}\right\} \text{units}$$

$$\left.\begin{array}{ll}\bar{f} = t & \overline{[R\ T]} = (\bar{R}, \bar{T}) \\ \bar{t} = f & \overline{(R\ T)} = [\bar{R}\ \bar{T}] \\ \bar{\bar{R}} = R & \end{array}\right\} \text{negation}$$

$$\text{if } R = T \text{ then } \left.\begin{cases} S\{R\} = S\{T\} \\ \bar{R} = \bar{T}\end{cases}\right\} \text{context closure}$$

- **Inference rules applied deep in context**

with shape

$$\pi \frac{S\{T\}}{S\{R\}}$$

$S\{\_\}$ a context

Top down reading: a rewrite rule

$$T \longrightarrow R$$

where '$\longrightarrow$' is logical implication.

SYSTEM SKS q (Calculus of structures)

$$i\downarrow \frac{S\{t\}}{S[R\,\bar{R}]} \qquad i\uparrow \frac{S(R\,\bar{R})}{S\{f\}} \qquad \text{interaction}$$

$$s \frac{S\{([R\,U]\,T)\}}{S\{[(R\,T)\,U]\}} \qquad \text{switch}$$

$$w\downarrow \frac{S\{f\}}{S\{R\}} \qquad w\uparrow \frac{S\{R\}}{S\{t\}} \qquad \text{weakening}$$

$$c\downarrow \frac{S\{[R\,R]\}}{S\{R\}} \qquad c\uparrow \frac{S\{R\}}{S\{(R\,R)\}} \qquad \text{contraction}$$

- rules applied deep in context $S\{\ \}$
- up- and down- version of rules
- $\rho \frac{S\{T\}}{S\{R\}}$ means '$T \rightarrow R$' $\rightarrow$ logical implication (rewriting rule)
- Duality of up-/down-rules (contraposition)

$$\rho\downarrow \frac{S\{T\}}{S\{R\}} \qquad \rho\uparrow \frac{S\{\bar{R}\}}{S\{\bar{T}\}}$$

(s is self-dual, so no arrow)

- Derivation in (any) system $\mathcal{S}$ in CoS: finite sequence of instances of inference rules in $\mathcal{S}$

$$\begin{array}{l} \rho_{n+1} \dfrac{T}{U} \\ \rho_n \dfrac{}{\vdots} \\ \rho_2 \dfrac{}{V} \\ \rho_1 \dfrac{}{R} \end{array} \qquad \Delta \begin{array}{c} T \\ \Big\Vert \mathcal{S} \\ R \end{array}$$

T premiss

R conclusion

- Proof in system $\mathcal{S}$ in CoS: a derivation where T is t

$$\begin{array}{c} \overline{\Big\Vert} \\ R \end{array} \qquad \begin{array}{c} t \\ \Vert \\ R \end{array}$$

Examples: proofs & derivations

$$
\mathsf{c}{\downarrow}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{t}{[(a\,a)\;a\;a]}}{[(a\,a)\;a]}
\qquad\qquad
\mathsf{c}{\downarrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{t}{[(a\,a)\;a\;a]}}{[(a\,a)\;a\;a\;b]}}{[(a\,a)\;a\;b]}
\qquad
\mathsf{w}{\downarrow}\,\dfrac{[(a\,a)\;b]}{[(a\,a)\;a\;b]}
$$

- Duality of proofs / derivations:
  The dual of a derivation is obtained by turning it upside-down, and replacing each rule, each connective and each atom by its dual

  Example:

$$
\mathsf{c}{\downarrow}\,\dfrac{\mathsf{w}{\uparrow}\,\dfrac{[(a\,\bar{b})\;a]}{[a\,a]}}{a}
\qquad \text{is dual to} \qquad
\mathsf{w}{\downarrow}\,\dfrac{\mathsf{c}{\uparrow}\,\dfrac{\bar{a}}{(\bar{a}\,\bar{a})}}{([\bar{a}\;b]\;\bar{a})}
$$

- Derivations in contexts: Given Δ obtain $S\{\Delta\}$:

$$
\Delta:\;
\begin{array}{rc}
\rho_{n+1} & T \\ \hline
\rho_n & U \\ \hline
 & \vdots \\ \hline
\rho_2 & V \\ \hline
\rho_1 & R
\end{array}
\qquad\qquad
S\{\Delta\}:\;
\begin{array}{rc}
 & S\{T\} \\ \hline
\rho_{n+1} & S\{U\} \\ \hline
\rho_n & \vdots \\ \hline
\rho_2 & S\{V\} \\ \hline
\rho_1 & S\{R\}
\end{array}
$$

- Rule ρ is derivable in $\mathcal{S}$ if

  $\rho\,\dfrac{T}{R}$ and there exists a derivation $\begin{array}{c} T \\ \Vert\,\mathcal{S} \\ R \end{array}$, forall T, R.

- Rule ρ is admissible for $\mathcal{S}$ if for every proof $\begin{array}{c} \Pi\,\mathcal{S}\cup\{\rho\} \\ R \end{array}$ there is a proof $\begin{array}{c} \Pi\,\mathcal{S} \\ R \end{array}$.

- The up-fragment of SKSg is admissible ; system KSg = $\{\mathsf{i}{\downarrow}, \mathsf{s}, \mathsf{w}{\downarrow}, \mathsf{c}{\downarrow}\}$

- Admissibility of the up-fragment implies cut-elimination (elimination of the cut-rule).

- **Deduction Thm:**

$$\begin{array}{c} T \\ \Delta \; \Big\| \; \mathsf{SKSg} \\ R \end{array} \qquad \text{iff} \qquad \begin{array}{c} \Pi \; \Big\| \; \mathsf{SKSg} \\ [\bar{T}\; R] \end{array}$$

<u>proof</u>:

$$\begin{array}{c} T \\ \Delta \; \Big\| \; \mathsf{SKSg} \\ R \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \mathsf{i}{\downarrow}\,\dfrac{t}{[\bar{T}, T]} \\ [\bar{T}, \Delta] \; \Big\| \; \mathsf{SKSg} \\ [\bar{T}\; R] \end{array}$$

$$\begin{array}{c} \Pi \; \Big\| \; \mathsf{SKSg} \\ R \end{array} \quad \rightsquigarrow \quad \begin{array}{c} T \\ (T, \Pi) \; \Big\| \; \mathsf{SKSg} \\ \mathsf{i}{\uparrow}\,\dfrac{\dfrac{(T\;[\bar{T}\; R])}{[R\;(T\;\bar{T})]}}{R} \end{array}$$

Locality : Rules in atomic form – SKS

- From SKSg: restrict identity, weakening and contraction into atomic form. Restricting contraction introduces a new rule, "medial" (m).

$$\mathsf{ai}{\downarrow}\ \frac{S\{t\}}{S\{[a,\bar{a}]\}} \qquad \mathsf{ai}{\uparrow}\ \frac{S\{(a,\bar{a})\}}{S\{f\}}$$

$$\mathsf{s}\ \frac{S\{([R\,U]\ T)\}}{S\{[(R\,T)\ U]\}}$$

$$\mathsf{m}\ \frac{S\{([R\,U]\ [T\,V])\}}{S\{[(R\,T)\ (U\,V)]\}}$$

$$\mathsf{aw}{\downarrow}\ \frac{S\{f\}}{S\{a\}} \qquad \mathsf{aw}{\uparrow}\ \frac{S\{a\}}{S\{t\}}$$

$$\mathsf{ac}{\downarrow}\ \frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\ \frac{S\{a\}}{S\{(a\,a)\}}$$

- KS = SKSg \ {ai↑, aw↑, ac↑}

- i↓ is derivable for {ai↓, s}
  w↓ is derivable for {aw↓, s}
  c↓ is derivable for {ac↓, m}

  and dually,

  i↑ is derivable for {ai↑, s}
  w↑ is derivable for {aw↑, s}
  c↑ is derivable for {ac↑, m}

Example: $w\downarrow$ derivable for $\{aw\downarrow, s\}$

$$w\downarrow \frac{S\{f\}}{S\{R\}}$$

- By induction on R:
  - R atom : The instance of the general rule is also an instance of the atomic rule
  - $R = t$: Consider

$$= \frac{s \frac{= \frac{S\{f\}}{S([t\,t]\, f)}}{S[t\,(t\,f)]}}{S\{t\}}$$

  - $R = [P, Q]$: Apply the i.h.

$$w\downarrow \frac{w\downarrow \frac{= \frac{S\{f\}}{S[f\,f]}}{S[f, Q]}}{S[P, Q]}$$

  - $R = (P, Q)$: Apply the i.h.

$$w\downarrow \frac{w\downarrow \frac{= \frac{S\{f\}}{S(f\,f)}}{S(f, Q)}}{S(P, Q)}$$

- Thm: systems SKS and SKSg are strongly equivalent (every derivation can be transformed from one system into the other.)

NB. m is derivable in SKSg (for $\{c\downarrow, w\downarrow\}$)

CORRESPONDENCE : GS1p vs SKSg

$$\frac{}{\vdash A,\bar A} \qquad i\!\downarrow\frac{t}{[A,\bar A]}$$

$$\mathrm{cut}\,\frac{\vdash \Phi A \quad \vdash \Psi, A}{\vdash \Phi,\Psi} \qquad \begin{array}{rc} & ([\Phi,A]\ [\Psi\,\bar A]) \\ \cline{2-2} s & [\Phi([\Psi\,\bar A],A)] \\ \cline{2-2} s & [\Phi,\Psi,(A,\bar A)] \\ \cline{2-2} i\!\uparrow & [\Phi,\Psi,f] \\ \cline{2-2} = & [\Phi,\Psi] \end{array}$$

$$\wedge\,\frac{\vdash \Phi,A \quad \vdash \Psi,B}{\vdash \Phi,\Psi,A\wedge B} \qquad \begin{array}{rc} & ([\Phi,A]\,[\Psi,B]) \\ \cline{2-2} s & [\Phi(A[\Psi,B])] \\ \cline{2-2} s & [\Phi,\Psi,(AB)] \end{array}$$

$$c\,\frac{\vdash \Phi,AA}{\vdash \Phi,A} \qquad c\!\downarrow\frac{[\Phi,AA]}{[\Phi,A]}$$

$$w\,\frac{\vdash \Phi}{\vdash \Phi,A} \qquad \begin{array}{rc} & \Phi \\ \cline{2-2} = & [\Phi f] \\ \cline{2-2} w\!\downarrow & [\Phi,A] \end{array}$$

1. From Sequent Calculus to CoS:

a • Map formulae and sequents of GS1p into structures of SKSg

b • THM: for every derivation $\begin{array}{c}\Sigma_1\ldots\Sigma_h\\ \Delta \\ \Sigma\end{array}$ in $GS\wedge p$ + Cut there is a derivation $\begin{array}{c}(\Sigma_1\ldots\Sigma_h)\\ \Big\Vert\, SKSg\setminus\{c\!\uparrow, w\!\uparrow\} \\ \Sigma\end{array}$ with the same number of cuts.

Proof : structural induction on the given Δ.

- Base cases: $\Delta = \Sigma$: take Σ.

  $\Delta = T\,\frac{}{\vdash T}$: take t

  $\Delta = Ax\,\frac{}{\vdash A,\bar A}$: Take $i\!\downarrow\frac{t}{[A\,\bar A]}$

- Inductive cases:

$$\Delta = \wedge\,\frac{\begin{array}{c}\Sigma_1\ldots\Sigma_k\\ \Delta_1\\ \vdash \Phi,A\end{array} \quad \begin{array}{c}\Sigma_1'\ldots\Sigma_e'\\ \Delta_2\\ \vdash \Psi,B\end{array}}{\vdash \Phi,\Psi,A\wedge B}$$

By i.h. $\exists$ $\begin{array}{c}(\Sigma_1\ldots\Sigma_k)\\ \Delta_1\Big\Vert\\ [\Phi,A]\end{array}$ $\begin{array}{c}(\Sigma_1'\ldots\Sigma_e')\\ \Delta_2\Big\Vert\\ [\Psi,B]\end{array}$

Compose them as:

$$\begin{array}{c} (\Sigma_1\ldots\Sigma_k\ \Sigma_i'\ldots\Sigma_i') \\ \Vert \\ ([\Phi,A]\ \Sigma_i'\ldots\Sigma_i') \\ \Vert \\ ([\Phi,A]\ \ [\Psi,B]) \\ \hline [\Psi([\Phi A],B)] \\ \hline [\Phi,\Psi,(AB)] \end{array}$$

(other cases are analogous).

2. From calculus of structures to sequent calculus

a. Map structures into formulae

$$a \rightsquigarrow a \qquad [R,T] \rightsquigarrow R \vee T$$
$$t \rightsquigarrow \top \qquad (R\,T) \rightsquigarrow R \wedge T$$
$$f \rightsquigarrow \bot$$

b. 'Imitate' deep inference

Lemma: For every two formulas A, B and every context $C\{\}$ there is a derivation

$$\begin{array}{c} \vdash A, \bar{B} \\ \nabla \\ \vdash C\{A\}, \overline{C\{B\}} \end{array}$$

in GS1p

__Proof:__ structural induction on $C\{\}$.

- Base case : $C\{\} = \{\}$ trivial.
- $C\{\} = C_1 \wedge C_2\{\}$: Take

$$\mathsf{V}\,\dfrac{\wedge\,\dfrac{\mathsf{Ax}\,\dfrac{}{\vdash C_1, \bar{C}_1} \qquad \dfrac{\vdash A, \bar{B}}{\vdash C_2\{A\}, \overline{C_2\{B\}}}\,\Delta}{\vdash C_1 \wedge C_2\{A\}, \bar{C}_1, \overline{C_2\{B\}}}}{\vdash C_1 \wedge C_2\{A\}, \bar{C}_1 \vee \overline{C_2\{B\}}}$$

where Δ exists by i.h.

- $C\{\} = C_1 \vee C_2\{\}$: similar.

c. All rules $\rho\,\dfrac{R}{T}$ have a proof $\begin{array}{c}\Pi\\ \vdash \bar{R}, T\end{array}$ in GS1p

Example: $\mathsf{s}\,\dfrac{S([U,V]T)}{S[(UT),V]}$ switch rule. Consider:

$$2\times\vee\,\dfrac{\wedge\,\dfrac{\wedge\,\dfrac{\mathsf{Ax}\,\dfrac{}{\vdash U, \bar{U}} \qquad \mathsf{Ax}\,\dfrac{}{\vdash V, \bar{V}}}{\vdash U, \bar{U} \wedge \bar{V}, V} \qquad \mathsf{Ax}\,\dfrac{}{\vdash T, \bar{T}}}{\vdash (U \wedge T), V, (\bar{U} \wedge \bar{V}), \bar{T}}}{\vdash (U \wedge T) \vee V, (\bar{U} \wedge \bar{V}) \vee \bar{T}}$$

d • **Thm:** for every derivation $\Delta \;\overset{Q}{\underset{P}{\Big\|}}\;\mathsf{SKSg}$ there is a derivation $\overset{\vdash Q}{\underset{\vdash P}{\bigtriangledown}}$ in GSip + cut

Proof: by induction on Δ

- Base: $\Delta = P$ (P, Q coincide). Take $\vdash P$
- Inductive case: Take the topmost rule in Δ:

$$
\begin{array}{ccc}
Q & \cdots\cdots & \rho\,\dfrac{S\{T\}}{S\{R\}} \\
\Delta\,\Big\| & & \Delta'\,\Big\|\;\mathsf{SKSg} \\
P & \cdots\cdots & P
\end{array}
$$

From ρ (previous prop.):

$$
\overset{\Pi}{\bigtriangledown} \\
\vdash R, \bar{T}
$$

Consider this derivation

$$
\dfrac{\dfrac{\overset{\Pi}{\bigtriangledown}}{\vdash R, \bar{T}}\;\overset{\Delta_1}{\bigtriangledown}\;\vdash S\{R\}, \overline{S\{T\}} \qquad \vdash S\{T\}}{\vdash S\{R\}}\,\mathsf{cut} \\
\overset{\Delta_2}{\bigtriangledown} \\
\vdash P
$$

Δ_1: from previous lemma

Δ_2: by i.h.

e • **Cor:** if S has a proof in SKSg then $\vdash S$ has a proof in GSip + cut

CUT ELIMINATION

- Cut-elimination in the calculus of structures is very different than in the sequent calculus. Why?

- Because in the sequent calculus the main connective "drives" the reduction:

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, A} \qquad \dfrac{\Pi_2}{\vdash \Gamma, B}}{\vdash \Gamma, A \wedge B}\,\wedge \qquad \dfrac{\dfrac{\Pi_3}{\vdash \Delta, \bar{A}}}{\vdash \Delta, \bar{A} \vee B}\,\vee}{\vdash \Gamma, \Delta}\,\mathsf{cut}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, A} \qquad \dfrac{\Pi_3}{\vdash \Delta, \bar{A}}}{\vdash \Gamma, \Delta}\,\mathsf{cut}$$

- In CoS, if we follow this idea, what are we supposed to do?

$$\begin{array}{c} \Pi \\ \Vert \\ \mathsf{s}\,\dfrac{(a\,[d\,(b\,c\,[\bar{a}\,\bar{b}\,\bar{c}])])}{[d\,(a\,b\,c\,[\bar{a}\,\bar{b}\,\bar{c}])]} \\ = \dfrac{}{\mathsf{i}{\uparrow}\,\dfrac{S(R\,T\,[\bar{R}\,\bar{T}])}{S\{f\}}} \end{array} \qquad \begin{array}{l} R = (a.b) \\ T = c \\ S = [d\,\{\ \}] \end{array}$$

- Atomicity helps a lot!

CUT ELIMINATION

- **Lemma 1:** Each rule in SKS is derivable for identity, cut, switch and its dual rule.

Proof: Ex:

$$\rho\uparrow \frac{S\{T\}}{S\{R\}} \quad \leadsto \quad i\downarrow \frac{\dfrac{\dfrac{\dfrac{S\{T\}}{S(T\,[R\,\bar{R}])}\,s}{S[R\,(T\,\bar{R})]}\,\rho\downarrow}{S[R\,(T\,\bar{T})]}\,i\uparrow}{S\{R\}}$$

∴ We have to deal only with atomic cut $(ai\uparrow)$

- **Def:** shallow atomic cut $ai\uparrow \dfrac{[S\,(a\,\bar{a})]}{S}$

- **Lemma 2:** Atomic cut is derivable for shallow atomic cut and switch.

Proof

$$ai\uparrow \frac{\dfrac{S([R\,(a\,\bar{a})]\,T)}{S([R\,f]\,T)}}{S(R\,T)}\,= \quad \leadsto \quad s\,\frac{\dfrac{\dfrac{S([R\,(a\,a)],T)}{S[(R\,T)\,(a\,\bar{a})]}\,ai\uparrow}{S[(R\,T)\,f]}\,=}{S(R\,T)}$$

∴ We have to deal only with shallow atomic cuts.

CUT ELIMINATION

- **Lemma 3:** Each proof $\overset{\Pi\ \mathsf{KS}}{T\{a\}}$ can be transformed into a proof of $\overset{\Pi\ \mathsf{KS}}{T\{t\}}$.

proof:

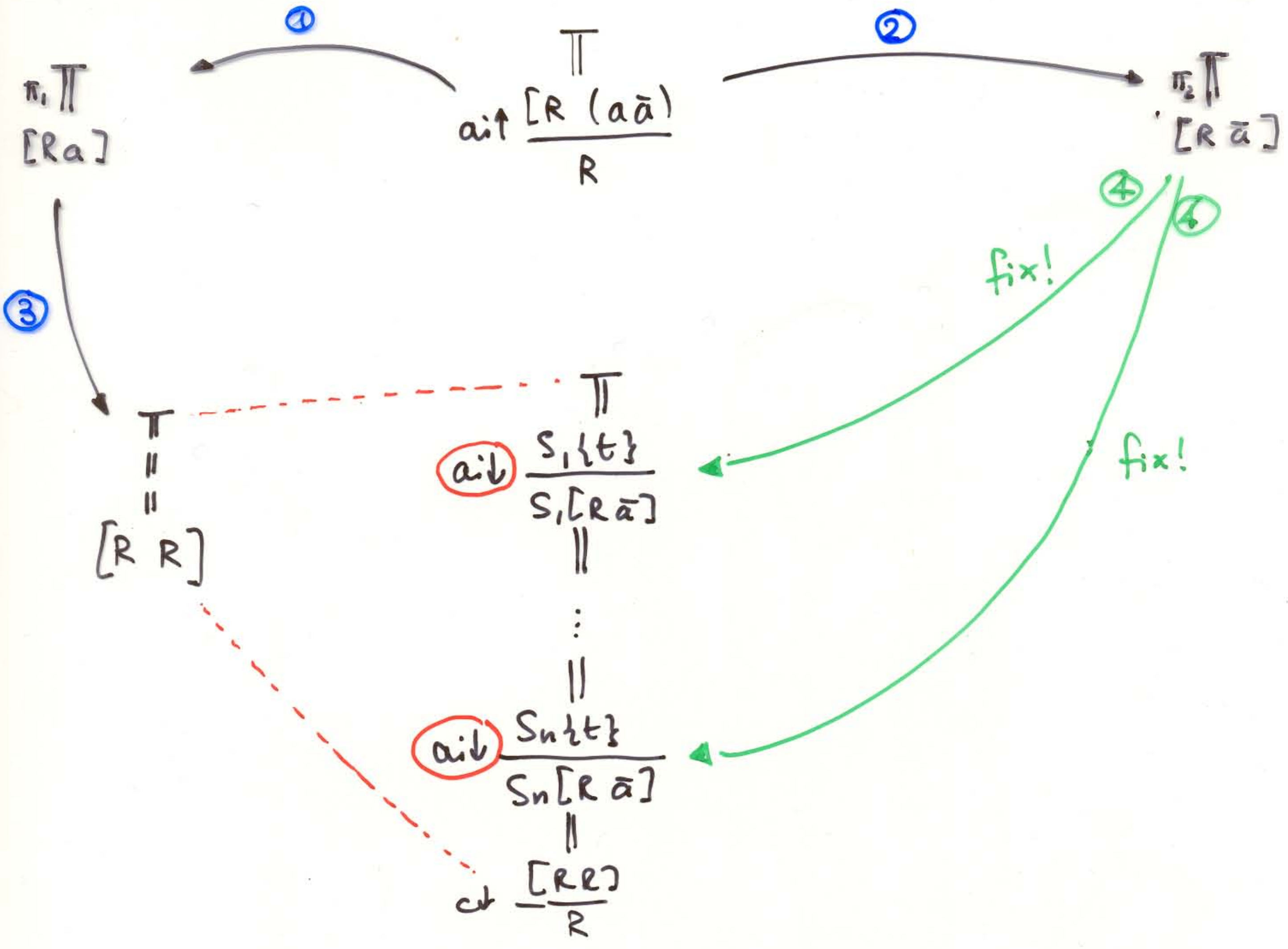
- visit the proof bottom up, replacing a with t:
  - replacing inside the context : no effect
  - replacing in m / s instances : no effect
  - in other rules : consider these derivations:

$$\mathsf{ac}\!\downarrow \frac{S[a\ a]}{S\{a\}} \quad \rightsquigarrow \quad = \frac{S[t\ t]}{S\{t\}}$$

$$\mathsf{aw}\!\downarrow \frac{S\{f\}}{S\{a\}} \quad \rightsquigarrow \quad \begin{array}{l} = \dfrac{S\{f\}}{S([t\ t]\ f)} \\ \mathsf{s}\,\dfrac{}{S[t\ (t\ f)]} \\ = \dfrac{}{S\{t\}} \end{array}$$

$$\mathsf{ai}\!\downarrow \frac{S\{t\}}{S[a\ \bar{a}]} \quad \rightsquigarrow \quad \begin{array}{l} = \dfrac{S\{t\}}{S[t\ f]} \\ \mathsf{aw}\!\downarrow \dfrac{}{S[t\ \bar{a}]} \end{array}$$

CUT ELIMINATION



①② : Lemma 3, replace $\bar{a} \leftarrow t$ $\quad a \leftarrow t$ in ① ② respectively

③ : replace $a \leftarrow R$: broken proof!

④ : • $ac\downarrow$, $aw\downarrow$ are replaced by $c\downarrow$, $w\downarrow$

fix • $ai\downarrow$ is replaced by $S\{\Pi_2\}$

$$ai\downarrow \frac{S\{t\}}{S[a\,\bar{a}]} \quad \rightsquigarrow \quad S\{\Pi_2\} \left\| \begin{array}{c} S\{t\} \\ \\ S[R\,\bar{a}] \end{array} \right.$$

add a contraction at the end.

Repeat for all instances of $ai\uparrow$.