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Coursework for "Introduction to Deep Inference and Proof Nets"

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Solution for Exercise 1 (7 points)

$$
\mathsf{c_R}\dfrac{\vee_{\mathsf{RL}}\dfrac{\vee_{\mathsf{RR}}\dfrac{\mathsf{c_R}\dfrac{\wedge_{\mathsf{R}}\dfrac{\mathsf{c_R}\dfrac{\vee_{\mathsf{RL}}\dfrac{\vee_{\mathsf{RR}}\dfrac{\wedge_{\mathsf{R}}\dfrac{\vee_{\mathsf{RL}}\dfrac{\neg_{\mathsf{R}}\dfrac{\mathsf{ax}\dfrac{}{a\vdash a}}{\vdash \neg a, a}}{\vdash \neg a, a\vee b}\quad \vee_{\mathsf{RL}}\dfrac{\neg_{\mathsf{R}}\dfrac{\mathsf{ax}\dfrac{}{c\vdash c}}{\vdash \neg c, c}}{\vdash \neg c, c\vee d}}{\vdash \neg a, \neg c, [a\vee b]\wedge[c\vee d]}}{\vdash \neg a, \neg a\vee\neg c, [a\vee b]\wedge[c\vee d]}}{\vdash \neg a\vee\neg c, \neg a\vee\neg c, [a\vee b]\wedge[c\vee d]}}{\vdash \neg a\vee\neg c, [a\vee b]\wedge[c\vee d]}\quad \mathsf{c_R}\dfrac{\vee_{\mathsf{RL}}\dfrac{\vee_{\mathsf{RR}}\dfrac{\wedge_{\mathsf{R}}\dfrac{\vee_{\mathsf{RR}}\dfrac{\neg_{\mathsf{R}}\dfrac{\mathsf{ax}\dfrac{}{b\vdash b}}{\vdash \neg b, b}}{\vdash \neg b, a\vee b}\quad \vee_{\mathsf{RR}}\dfrac{\neg_{\mathsf{R}}\dfrac{\mathsf{ax}\dfrac{}{d\vdash d}}{\vdash \neg d, d}}{\vdash \neg d, c\vee d}}{\vdash \neg b, \neg d, [a\vee b]\wedge[c\vee d]}}{\vdash \neg b, \neg b\vee\neg d, [a\vee b]\wedge[c\vee d]}}{\vdash \neg b\vee\neg d, \neg b\vee\neg d, [a\vee b]\wedge[c\vee d]}}{\vdash \neg b\vee\neg d, [a\vee b]\wedge[c\vee d]}}{\vdash [\neg a\vee\neg c]\wedge[\neg b\vee\neg d], [a\vee b]\wedge[c\vee d], [a\vee b]\wedge[c\vee d]}}{\vdash [\neg a\vee\neg c]\wedge[\neg b\vee\neg d], [a\vee b]\wedge[c\vee d]}}{\vdash [\neg a\vee\neg c]\wedge[\neg b\vee\neg d], ([\neg a\vee\neg c]\wedge[\neg b\vee\neg d])\vee([a\vee b]\wedge[c\vee d])}}{\vdash ([\neg a\vee\neg c]\wedge[\neg b\vee\neg d])\vee([a\vee b]\wedge[c\vee d]), ([\neg a\vee\neg c]\wedge[\neg b\vee\neg d])\vee([a\vee b]\wedge[c\vee d])}}{\vdash ([\neg a\vee\neg c]\wedge[\neg b\vee\neg d])\vee([a\vee b]\wedge[c\vee d])}\quad .
$$

Solution for Exercise 2 (10 points) In system KS:

$$
\mathsf{m}\dfrac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{\mathsf{ai}\!\downarrow\dfrac{\mathsf{ai}\!\downarrow\dfrac{\mathsf{s}\dfrac{\mathsf{ai}\!\downarrow\dfrac{\mathsf{ai}\!\downarrow\dfrac{\mathsf{t}}{\bar{d}\vee d}}{([\bar{c}\vee c]\wedge\bar{d})\vee d}}{(\bar{c}\wedge\bar{d})\vee[c\vee d]}}{(\bar{c}\wedge\bar{d})\vee([b\vee\bar{b}]\wedge[c\vee d])}}{(\bar{c}\wedge\bar{d})\vee([b\vee([a\vee\bar{a}]\wedge\bar{b})]\wedge[c\vee d])}}{(\bar{c}\wedge\bar{d})\vee([a\vee b\vee(\bar{a}\wedge\bar{b})]\wedge[c\vee d])}}{(\bar{a}\wedge\bar{b})\vee(\bar{c}\wedge\bar{d})\vee([a\vee b]\wedge[c\vee d])}}{([\bar{a}\vee\bar{c}]\wedge[\bar{b}\vee\bar{d}])\vee([a\vee b]\wedge[c\vee d])}\quad .
$$

In system KSg:

$$
\begin{array}{rc}
 & \mathsf{t} \\ \cline{2-2}
\mathsf{i}{\downarrow} & (\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge \bar{d}) \vee ([a \vee b] \wedge [c \vee d]) \\ \cline{2-2}
\mathsf{w}{\downarrow} & (\bar{a} \wedge \bar{b}) \vee (\bar{c} \wedge [\bar{b} \vee \bar{d}]) \vee ([a \vee b] \wedge [c \vee d]) \\ \cline{2-2}
\mathsf{w}{\downarrow} & (\bar{a} \wedge \bar{b}) \vee ([\bar{a} \vee \bar{c}] \wedge [\bar{b} \vee \bar{d}]) \vee ([a \vee b] \wedge [c \vee d]) \\ \cline{2-2}
\mathsf{w}{\downarrow} & (\bar{a} \wedge [\bar{b} \vee \bar{d}]) \vee ([\bar{a} \vee \bar{c}] \wedge [\bar{b} \vee \bar{d}]) \vee ([a \vee b] \wedge [c \vee d]) \\ \cline{2-2}
\mathsf{w}{\downarrow} & ([\bar{a} \vee \bar{c}] \wedge [\bar{b} \vee \bar{d}]) \vee ([\bar{a} \vee \bar{c}] \wedge [\bar{b} \vee \bar{d}]) \vee ([a \vee b] \wedge [c \vee d]) \\ \cline{2-2}
\mathsf{c}{\downarrow} & ([\bar{a} \vee \bar{c}] \wedge [\bar{b} \vee \bar{d}]) \vee ([a \vee b] \wedge [c \vee d])
\end{array}
\quad .
$$

Solution for Exercise 3 (8 points) The rule m is derivable in KSg:

$$
\mathsf{m}\,\frac{(R \wedge U) \vee (T \wedge V)}{[R \vee T] \wedge [U \vee V]}
\qquad \text{is obtained as} \qquad
\begin{array}{rc}
 & (R \wedge U) \vee (T \wedge V) \\ \cline{2-2}
\mathsf{w}{\downarrow} & (R \wedge U) \vee (T \wedge [U \vee V]) \\ \cline{2-2}
\mathsf{w}{\downarrow} & (R \wedge U) \vee ([R \vee T] \wedge [U \vee V]) \\ \cline{2-2}
\mathsf{w}{\downarrow} & (R \wedge [U \vee V]) \vee ([R \vee T] \wedge [U \vee V]) \\ \cline{2-2}
\mathsf{w}{\downarrow} & ([R \vee T] \wedge [U \vee V]) \vee ([R \vee T] \wedge [U \vee V]) \\ \cline{2-2}
\mathsf{c}{\downarrow} & [R \vee T] \wedge [U \vee V]
\end{array}
\quad .
$$

This means that there is a (scheme of) derivation Δ_{m}, in KSg, such that

$$
\begin{array}{c}
(R \wedge U) \vee (T \wedge V) \\
\Delta_{\mathsf{m}} \Big\Vert\, \mathsf{KSg} \\
[R \vee T] \wedge [U \vee V]
\end{array}
\quad ,
$$

for any formulae R, T, U and V. Hence, we have the following derivation in KSg:

$$
\begin{array}{rc}
 & (R \wedge P) \vee (P \wedge Q) \vee (S \wedge Q) \\ \cline{2-2}
\mathsf{i}{\downarrow} & (R \wedge P) \vee ([a \vee \bar{a}] \wedge Q) \vee P) \vee (S \wedge Q) \\ \cline{2-2}
\mathsf{s} & (R \wedge P) \vee ([a \vee (\bar{a} \wedge Q)] \wedge P) \vee (S \wedge Q) \\ \cline{2-2}
\mathsf{s} & (a \wedge P) \vee (R \wedge P) \vee (\bar{a} \wedge Q) \vee (S \wedge Q) \\
 & \Delta_{\mathsf{m}} \Big\Vert\, \mathsf{KSg} \\
 & (a \wedge P) \vee (R \wedge P) \vee ([\bar{a} \vee S] \wedge [Q \vee Q]) \\
 & \Delta_{\mathsf{m}} \Big\Vert\, \mathsf{KSg} \\
 & ([a \vee R] \wedge [P \vee P]) \vee ([\bar{a} \vee S] \wedge [Q \vee Q]) \\ \cline{2-2}
\mathsf{c}{\downarrow} & ([a \vee R] \wedge [P \vee P]) \vee ([\bar{a} \vee S] \wedge Q) \\ \cline{2-2}
\mathsf{c}{\downarrow} & ([a \vee R] \wedge P) \vee ([\bar{a} \vee S] \wedge Q)
\end{array}
\quad .
$$

Solution for Exercise 4 (7 points) Proceed by way of contradiction and assume that there is a formula R such that both R and $\bar{R}$ are provable in S, i.e., there exist proofs

$$
\begin{array}{c}
\mathsf{t} \\
\Pi_1 \Big\Vert \\
R
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\mathsf{t} \\
\Pi_2 \Big\Vert \\
\bar{R}
\end{array}
\quad .
$$

From Π_2 we obtain the derivation Δ_2 by 'flipping' it (exploiting the fact that exchanging premise and conclusion of each rule instance, and taking their negations, you still get a valid derivation):

$$\Delta_2 \left\| \begin{array}{c} R \\ \\ \mathsf{f} \end{array} \right. .$$

Now composing Π_1 and Δ_2

$$\begin{array}{rc} & \mathsf{t} \\ \Pi_1 & \| \\ & R \\ \Delta_2 & \| \\ & \mathsf{f} \end{array}$$

we would derive f from t, contradicting the hypothesis.

Solution for Exercise 5 (5 points) A possible solution is the following proof:

$$\mathsf{ai}\uparrow \cfrac{\mathsf{ai}\uparrow \cfrac{\mathsf{s} \cfrac{\mathsf{s} \cfrac{\mathsf{ai}\uparrow \cfrac{\mathsf{s} \cfrac{\mathsf{s} \cfrac{d \vee ([a \vee (b \wedge \bar{b})] \wedge [b \vee \bar{b}] \wedge \bar{a})}{d \vee ([b \vee \bar{b}] \wedge [(a \wedge \bar{a}) \vee (b \wedge \bar{b})])}}{d \vee (a \wedge \bar{a}) \vee ([b \vee \bar{b}] \wedge b \wedge \bar{b})}}{d \vee ([b \vee \bar{b}] \wedge b \wedge \bar{b})}}{d \vee ([b \vee (b \wedge \bar{b})] \wedge \bar{b})}}{d \vee (b \wedge \bar{b}) \vee (b \wedge \bar{b})}}{d \vee (b \wedge \bar{b})}}{d} .$$

An alternative solution can be obtained as in Exercise 9.

Solution for Exercise 6 (6 points) Consider the implication formula expressing the cut rule: $(A \wedge \neg A) \to \mathsf{f}$. This implication is valid, i.e., $\models (A \wedge \neg A) \to \mathsf{f}$. But in KSg there is no way of building a derivation Δ

$$\Delta \left\| \begin{array}{c} (A \wedge \neg A) \\ \\ \mathsf{f} \end{array} \right. \mathsf{KSg}$$

provided that A contains at least one atom, because no rule in KSg can, while going up in a derivation, introduce an atom that is not present in the conlusion. (The only rules in SKSg that can do that are $\mathsf{i}\uparrow$ and $\mathsf{w}\uparrow$, but they are not present in KSg.) So, KSg is not implicationally complete.

Solution for Exercise 7 (7 points) In the lecture, we have shown the translation of a Frege system with 17 axioms into SKSg, and we have argued that SKSg polynomially simulates that Frege system. Furthermore, the Robustness theorem says that all Frege system are p-equivalent. In particular also the Frege system in the exercise and the Frege system used in the lecture p-simulate each other. Consequently, SKSg also p-simulates the Frege system in the exercise.

Solution for Exercise 8 (5 points)

$$\otimes\,\dfrac{\begin{array}{c}\Pi_1\\ \vdash b,\Gamma\end{array} \qquad \bot\,\dfrac{\begin{array}{c}\Pi_2\\ \vdash a,\Delta\end{array}}{\vdash \bot, a,\Delta}}{\vdash b\otimes\bot,\Gamma,a,\Delta}$$

Solution for Exercise 9 (5 points) This exercise is the same as Exercise 5. Hence the solution 5 is also good for this one (if you replace $\otimes$ with $\wedge$ and $\parr$ with $\vee$). Here is an alternative one

$$\mathsf{ai}\!\uparrow\dfrac{\mathsf{ai}\!\uparrow\dfrac{\mathsf{s}\,\dfrac{\mathsf{ai}\!\uparrow\dfrac{\mathsf{s}\,\dfrac{[d \parr ([a \parr (b \otimes b^\bot)] \otimes [b \parr b^\bot] \otimes a^\bot)]}{[d \parr ([(a \otimes a^\bot) \parr (b \otimes b^\bot)] \otimes [b \parr b^\bot])]}}{[d \parr (b \otimes b^\bot \otimes [b \parr b^\bot])]}}{[d \parr (b \otimes [(b^\bot \otimes b) \parr b^\bot])]}}{[d \parr (b \otimes b^\bot)]}}{d}$$

Of course, this solution is also good for Exercise 5.

Solution for Exercise 10 (5 points) We proceed by structural induction on A. If $A = a$ for some atom, we have immediately $a^{\bot\bot} = a$ by definition. If $A = a^\bot$, we have $(a^\bot)^{\bot\bot} = (a^{\bot\bot})^\bot = a^\bot$. If $A = 1$, we have $1^{\bot\bot} = \bot^\bot = 1$, and similarly for $A = \bot$. If $A = (B \otimes C)$, then

$$(B \otimes C)^{\bot\bot} = [B^\bot \parr C^\bot]^\bot = (B^{\bot\bot} \otimes C^{\bot\bot}) = (B \otimes C)$$

where the last equation holds by induction hypothesis. For $A = [B \parr C]$, we proceed similarly.

Solution for Exercise 11 (9 points) Applying splitting to Π' gives us

$$\begin{array}{c}[Q_1 \parr Q_2]\\ \mathsf{MLS}\,\Big\Vert\,\Pi_1\\ K_4\end{array} \qquad\text{and}\qquad \begin{array}{c}\mathsf{MLS}\,\Big\Vert\,\Pi_2\\ [a \parr K_1 \parr K_3 \parr Q_1]\end{array} \qquad\text{and}\qquad \begin{array}{c}\mathsf{MLS}\,\Big\Vert\,\Pi_3\\ [K_2 \parr Q_2]\end{array}$$

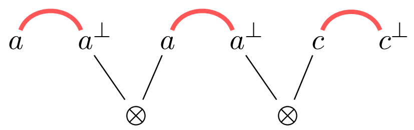
where $\mathsf{size}(\Pi_2) + \mathsf{size}(\Pi_3) < \mathsf{size}(\Pi')$. In particular, we have $\mathsf{size}(\Pi_2) < \mathsf{size}(\Pi')$. Hence we can apply the induction hypothesis to Π_2. From this we get

$$\begin{array}{c}a^\bot\\ \mathsf{MLS}\,\Big\Vert\,\Pi_4\\ [K_1 \parr K_3 \parr Q_1]\end{array}$$

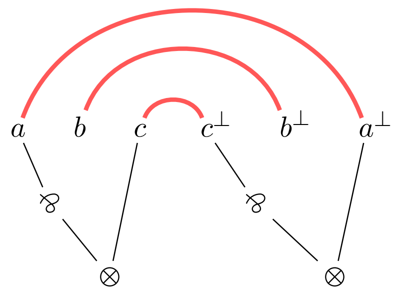
We can build Π_a as follows:

$$
\begin{array}{c}
a^\perp \\
\mathsf{MLS}\,\Big\Vert\,\Pi_4 \\
= \dfrac{[K_1 \parr K_3 \parr Q_1]}{[(K_1 \otimes 1) \parr K_3 \parr Q_1]} \\
\mathsf{MLS}\,\Big\Vert\,\Pi_3 \\
\mathsf{s}\,\dfrac{[(K_1 \otimes [K_2 \parr Q_2]) \parr K_3 \parr Q_1]}{[(K_1 \otimes K_2) \parr K_3 \parr Q_1 \parr Q_2]} \\
\mathsf{MLS}\,\Big\Vert\,\Pi_1 \\
[(K_1 \otimes K_2) \parr K_3 \parr K_4]
\end{array}
$$

Solution for Exercise 12 (4 points)



Solution for Exercise 13 (12 points) (a) There is a disconnected (and cyclic) DR-switching:



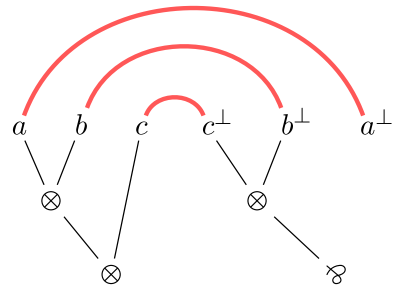
(b)

$$
\begin{array}{c}
\mathsf{ai}{\downarrow}\,\dfrac{}{[a \parr a^\perp]} \\
\mathsf{ai}{\downarrow}\,\dfrac{}{[(a \otimes [b \parr b^\perp]) \parr a^\perp]} \\
\mathsf{s}\,\dfrac{}{[(a \otimes b) \parr b^\perp \parr a^\perp]} \\
\mathsf{ai}{\downarrow}\,\dfrac{}{[(a \otimes b) \parr ([c \parr c^\perp] \otimes b^\perp) \parr a^\perp]} \\
\mathsf{s}\,\dfrac{}{[(a \otimes b) \parr c \parr (c^\perp \otimes b^\perp) \parr a^\perp]}
\end{array}
$$

(c)

$$\mathsf{s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{}{[a \parr a^\perp]}}{[(a\otimes[b\parr b^\perp])\parr a^\perp]}}{[(a\otimes b)\parr b^\perp\parr a^\perp]}}{[(a\otimes b\otimes[c\parr c^\perp])\parr b^\perp\parr a^\perp]}}{[(a\otimes b\otimes c)\parr c^\perp\parr b^\perp\parr a^\perp]}$$

(d) There is a cyclic switching:



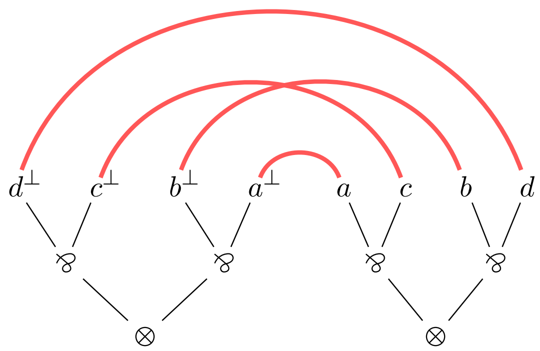
Solution for Exercise 14 (10 points) We proceed by way of contradiction. If the rule was derivable, there would be a derivation

$$\begin{array}{c}[(a\otimes b)\parr(c\otimes d)]\\ \mathsf{SMLS}^- \,\Big\Vert \\ ([a\parr c]\otimes[b\parr d])\end{array}$$

which is equivalent to having

$$\begin{array}{c}\mathsf{MLS}^- \,\Big\Vert\, \Pi \\ [([d^\perp\parr c^\perp]\otimes[b^\perp\parr a^\perp])\parr([a\parr c]\otimes[b\parr d])]\end{array}$$

Then, the only way to get proof net corresponding to Π is the following:



But this has a cyclic (and disconnected) linking and is therefore not correct: