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Coursework for "Introduction to Deep Inference and Proof Nets"

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Guidelines and Conditions

- This coursework constitute the only examination for the course "Introduction to Deep Inference and Proof Nets", and it includes 14 exercises, each of them is assigned a score, clearly indicated at the beginning of the text. The total number of points is 100.
- The text of this coursework is available for download from the course web page, starting from Monday, December 17, 2007. A paper version is made available during the first lecture in the lecture hall for the attendees.
- Deadline for returning the worked solutions is Friday, December 21, 2007 at 12:00 (noon). This is a strict deadline. Worked solutions can be handed in to any of the lecturers.
- Students are allowed to work in groups, but each student has to return her/his own paperwork, for administrative reasons.
- We remind students to clearly write their names on all their submitted sheets, and to bind them, for assessment.
- A sample solution will be made available on the web at the end of the course and this is the only form of public feedback provided.

Exercise 1 (7 points) Prove the sequent $\vdash ([\neg a \vee \neg c] \wedge [\neg b \vee \neg d]) \vee ([a \vee b] \wedge [c \vee d])$ in system LK.

Exercise 2 (10 points) Prove the formula $([\bar{a} \vee \bar{c}] \wedge [\bar{b} \vee \bar{d}]) \vee ([a \vee b] \wedge [c \vee d])$ in system KS and in system KSg.

Exercise 3 (8 points) Consider the following deep inference rule ρ, where a is an atom, and R, P, Q and S are formulae:

$$\rho\,\frac{(R \wedge P) \vee (S \wedge Q) \vee (P \wedge Q)}{([a \vee R] \wedge P) \vee ([\bar{a} \vee S] \wedge Q)} \quad .$$

Show that ρ is derivable in KSg.

Exercise 4 (7 points) For this exercise, we use the following definition: A system (presenting classical logic) is *consistent* if it does not allow to prove both R and $\neg R$, for some formula R.

Now consider a system S in deep inference, in which for every rule in S also its dual rule is in S, and prove the following statement: If S does not allow to derive f from t then the system S is consistent. (Note: You cannot use the rule $\mathsf{i}{\uparrow}$, since it might not be in S.)

Exercise 5 (5 points) Replace the following derivation in the calculus of structures with one that uses $\mathsf{ai}{\uparrow}$ instead of $\mathsf{i}{\uparrow}$:

$$\mathsf{i}{\uparrow}\,\frac{d \vee ([a \vee (b \wedge \bar{b})] \wedge [b \vee \bar{b}] \wedge \bar{a})}{d} \quad .$$

Exercise 6 (6 points) Show that KSg is not implicationally complete.

Exercise 7 (7 points) Consider the Frege system with the following axiom schemata:

$$\begin{aligned}
F_1 &\equiv A \to (B \to A) \\
F_2 &\equiv (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\
F_3 &\equiv \neg A \to (A \to B) \\
F_4 &\equiv (A \to B) \to ((\neg A \to B) \to B)
\end{aligned}$$

and with modus ponens as inference rule. Show that it admits a polynomial simulation in SKSg. (Hint: You can use all theorems that have been mentioned in the lecture.)

Exercise 8 (5 points) What is the result of eliminating the cut from the following proof in $\mathsf{MLL} + \mathsf{cut}$:

$$\otimes\,\dfrac{\dfrac{\Pi_1}{\vdash b, \Gamma} \qquad \mathsf{cut}\,\dfrac{\bot\,\dfrac{⅋\,\dfrac{\bot\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash a^\bot, a}}{\vdash a^\bot, \bot, a}}{\vdash a^\bot ⅋ \bot, a}}{\vdash \bot, a^\bot ⅋ \bot, a} \qquad \otimes\,\dfrac{\dfrac{\Pi_2}{\vdash a, \Delta} \qquad 1\,\dfrac{}{\vdash 1}}{\vdash a \otimes 1, \Delta}}{\vdash \bot, a, \Delta}}{\vdash b \otimes \bot, \Gamma, a, \Delta}$$

You can assume that Π_1 and Π_2 are cut free.

Exercise 9 (5 points) Replace the following derivation in the calculus of structures with one that uses $\mathsf{ai}\!\uparrow$ instead of $\mathsf{i}\!\uparrow$.

$$\mathsf{i}\!\uparrow \frac{[d \parr ([a \parr (b \otimes b^\perp)] \otimes [b \parr b^\perp] \otimes a^\perp)]}{d}$$

Exercise 10 (5 points) Show that the De-Morgan-laws for defining negation (for MLL) imply that $A^{\perp\perp} = A$ for all formulas A.

Exercise 11 (9 points) In this exercise you will do one case of the proof of the "atomic splitting" lemma, which says that

$$\text{If} \quad \begin{array}{c} \mathsf{MLS} \Big\Vert \Pi \\ [a \parr K] \end{array} \quad \text{then} \quad \begin{array}{c} a^\perp \\ \mathsf{MLS} \Big\Vert \Pi_a \\ K \end{array} \quad .$$

The proof is by induction on $\mathsf{size}(\Pi)$. Consider only the case where Π is

$$\begin{array}{c} \mathsf{MLS} \Big\Vert \Pi' \\ \mathsf{s}\,\dfrac{[([a \parr K_1 \parr K_3] \otimes K_2) \parr K_4]}{[a \parr (K_1 \otimes K_2) \parr K_3 \parr K_4]} \end{array}$$

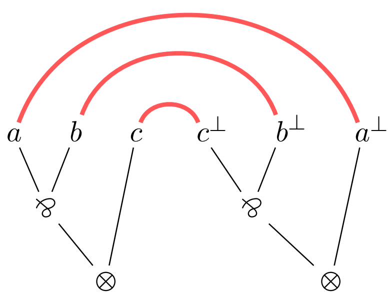
Apply the general splitting lemma and the induction hypothesis to build Π_a.

Exercise 12 (4 points) Draw the proof net for the following derivation:

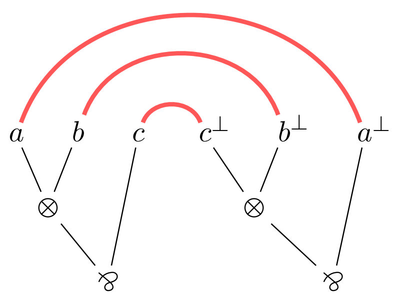
$$\mathsf{s}\,\dfrac{\mathsf{ai}\!\downarrow \dfrac{\mathsf{s}\,\dfrac{\mathsf{ai}\!\downarrow \dfrac{\mathsf{ai}\!\downarrow \dfrac{}{[a \parr a^\perp]}}{[a \parr (a^\perp \otimes [c \parr c^\perp])]}}{[a \parr (a^\perp \otimes c) \parr c^\perp]}}{[([a \parr a^\perp] \otimes a) \parr (a^\perp \otimes c) \parr c^\perp]}}{[a \parr (a^\perp \otimes a) \parr (a^\perp \otimes c) \parr c^\perp]}$$

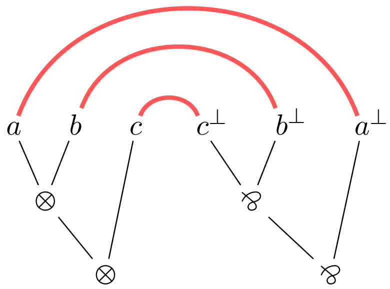
Exercise 13 (12 points) For each of the following four (pre-)proof nets do the following: Either give an MLS^- derivation that corresponds to the net, or explain why such a derivation does not exist.

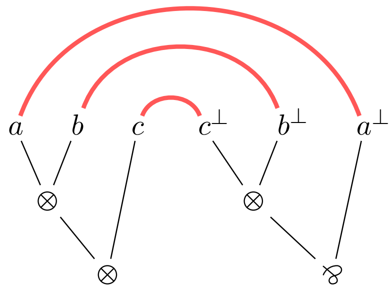
(a)



(b)



(*c*) 

(*d*) 

Exercise 14 (10 points) Consider the following inference rule:

$$\mathsf{m}\,\frac{S\{(A \otimes B) \parr (C \otimes D)\}}{S\{[A \parr C] \otimes [B \parr D]\}}$$

Show that the rule m is not derivable in SMLS^-. (Hint: use proof nets.)