MELL in the Calculus of Structures

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Abstract

The *calculus of structures* is a new proof theoretical formalism, like natural deduction, the sequent calculus and proof nets, for specifying logical systems syntactically. In a rule in the calculus of structures, the premise as well as the conclusion are *structures*, which are expressions that share properties of formulae and sequents. In this paper, I study a system for MELL, the multiplicative exponential fragment of linear logic, in the calculus of structures. It has the following features: a local promotion rule, no non-deterministic splitting of the context in the times rule and a modular proof for the cut elimination theorem. Further, derivations have a new property, called decomposition, that cannot be observed in any other known proof theoretical formalism.

Keywords: Calculus of structures, linear logic, proof theory, cut elimination.

Contents

1 Introduction

Sequent calculus [10, 11], natural deduction [10, 24] and proof nets [12] are proof theoretical formalisms that are used to define logical systems syntactically and to prove properties of those systems. Such syntactic tools are particularly important if semantics is missing, incomplete or under development, as it is often the case in computer science.

Proof theory plays an increasing role in theoretical computer science, mainly via the two paradigms of proof reduction and proof construction [3]. Proof reduction, also known as proof normalization, is via the Curry-Howard isomorphism [19], which identifies formulas and types, tightly connected to the functional programming paradigm. Correct proofs correspond to well-typed programs, and the normalization of the proof corresponds to the computation of the program. Proof construction, or proof search, is connected to the logic programming paradigm via the notion of uniform proof [22]. Intuitively, formulae correspond to instructions, and (possibly incomplete) proofs correspond to states. In other words, the search for the proof corresponds to the computation.

The *calculus of structures*, which is a new proof theoretical formalism, is a generalization of the one-sided sequent calculus. It has been introduced by Guglielmi in [14] for specifying a non-commutative logic. It has then been shown that the calculus of structures is also suitable for classical logic [7, 5] and linear logic [16, 28]. Preliminary research shows that also modal logic [27] and intuitionistic logic [6] can benefit from the presentation in the calculus of structures. The basic principles of the calculus of structures are that the notions of formulae and sequents are merged into a single kind of expression, called *structure*, and that inference rules can be applied anywhere deep inside structures. Since the calculus of structures allows for cut elimination and a subformula property, it can have impact on the proof reduction paradigm as well as the proof construction paradigm.

In this paper, I will study the multiplicative exponential fragment of linear logic (MELL) [12] within this new formalism. The main results have been presented in a very brief form in [16]. The starting point for this research are the following (well-known) observations on the sequent calculus system for MELL.

- Almost all rules in the sequent calculus system for MELL have the following property: if a rule has to be applied during a proof search, only the main connective of one formula has to be investigated. For instance, for the application of the *par* rule

$$\parr \frac{\vdash A, B, \Phi}{\vdash A \parr B, \Phi} \quad ,$$

only the main connective $\parr$ of the formula $A \parr B$ has to be considered. From the point of view of proof search this is a very good property, because the computational resources (time and space) for applying a rule are bounded. This is particularly important if the proof search is done by a distributed system.

However, there is one exception in MELL: For applying the *promotion* rule

$$
! \frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n} \quad ,
$$

it is necessary to check for each formula in the context of $!A$, whether it has the form $?B$. Up to now there is no known system for MELL without this problem, which also occurs in proof nets associated to boxes.

- There is another disturbing fact connected to the promotion rule: The formula $A \parr ?B_1 \parr \cdots \parr ?B_n$, which corresponds to the sequent in the premise, does not linearly imply the formula $!A \parr ?B_1 \parr \cdots \parr ?B_n$, which corresponds to the sequent in the conclusion, whereas for all other rules in MELL we have a proper implication between premise and conclusion. The reason why the promotion rule is correct is that if the formula $A \parr ?B_1 \parr \cdots \parr ?B_n$ is provable, then $!A \parr ?B_1 \parr \cdots \parr ?B_n$ is also provable. It might be interesting to note here that the sequent calculus rules for the quantifiers do have the same problem [7].

- Consider the *times* rule

$$
\otimes \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \quad .
$$

From the point of view of proof search, this rule presents a serious problem: One has to decide how to split the context of the formula $A \otimes B$ at the moment the rule is applied. For n formulas in Φ, Ψ, there are 2^n possibilities. Although there are methods, like lazy evaluation, that can circumvent this problem inside an implementation [18], there still remains the question whether it can be solved inside the logical system.

- In the sequent calculus system for linear logic, the general identity axiom

$$
\mathsf{id} \frac{}{\vdash A, A^\perp} \quad ,
$$

where A is any formula, can be reduced to its atomic version

$$
\mathsf{id} \frac{}{\vdash a, a^\perp} \quad ,
$$

where a is an atom. This is done via an inductive argument on the size of the formula A. For example, if $A = B \otimes C$ we can replace

$$
\mathsf{id} \frac{}{\vdash B \otimes C, B^\perp \parr C^\perp} \qquad \text{by} \qquad \parr \frac{\otimes \dfrac{\mathsf{id} \dfrac{}{\vdash B, B^\perp} \quad \mathsf{id} \dfrac{}{\vdash C, C^\perp}}{\vdash B \otimes C, B^\perp, C^\perp}}{\vdash B \otimes C, B^\perp \parr C^\perp} \quad .
$$

However, for the general cut rule

$$
\mathsf{cut} \frac{\vdash A, \Phi \quad \vdash A^\perp, \Psi}{\vdash \Phi, \Psi} \quad ,
$$

such an argument is impossible. The cut cannot be reduced to its atomic version.

An interesting question is whether these facts are inherently connected to the logic of MELL or not: In the former case one has to use a different logic in order to avoid the problems mentioned above, and in the latter case one has to find a different presentation for MELL.

One of the contributions of this paper is to show that it is *not* MELL itself which is responsible. As already observed in [14, 16, 15], the reason is due to the following two properties of the sequent calculus: First, a proof in the sequent calculus is a tree where branching occurs when inference rules with more than one premise are used, and there is a proof of the conclusion if there are proofs of each premise. Second, the main connective plays a central role in the application of an inference rule. A rule gives a meaning to the main connective in the conclusion by saying that the conclusion is provable if certain subformulae obtained by removing that connective are provable. These two properties together have remarkable success in making the study of systems independent of their semantics, but they also make the sequent calculus unnecessarily rigid. The calculus of structures allows to relax the two properties of the branching of derivation trees and the decomposition of formulae around the main connective without losing the good properties like cut elimination.

In the calculus of structures, inference rules have the shape $\rho\,\dfrac{S\{T\}}{S\{R\}}$, i.e. all rules have only one premise. Premise and conclusion are structures. The structure $S\{R\}$ consists of the structural context $S\{\ \}$ and the structure R, which fills the hole of $S\{\ \}$. The rule ρ above simply says that if (during the proof search) a structure matches the conclusion $S\{R\}$, then it can be rewritten as $S\{T\}$, where the context $S\{\ \}$ does not change (or vice versa if one reasons top-down). The rule ρ corresponds to the implication $T \Rightarrow R$, where $\Rightarrow$ stands for the implication that is modelled in the system. In the case of MELL it is linear implication $\multimap$. For instance, the implication $!(A \mathbin{\invamp} B) \multimap\, !A \mathbin{\invamp} ?B$ gives us a local promotion rule:

$$\mathsf{p}{\downarrow}\,\frac{S\{!(A \mathbin{\invamp} B)\}}{S\{!A \mathbin{\invamp} ?B\}} \quad .$$

Observe that this rule is sound. The non-deterministic splitting of the context in the times rule of linear logic is avoided by using the linear implication $A \otimes (B \mathbin{\invamp} C) \multimap (A \otimes B) \mathbin{\invamp} C$ in a rule:

$$\mathsf{s}\,\frac{S\{A \otimes (B \mathbin{\invamp} C)\}}{S\{(A \otimes B) \mathbin{\invamp} C\}} \quad .$$

This rule, called *switch* [14], is also the key to the reduction of the general cut rule to its atomic version.

Observe that there is a danger here, because any axiom $T \Rightarrow R$ of a Hilbert system could be used in a rule, with the consequence that there would be no structural relation between T and R. And so, all good proof theoretical properties, like cut elimination, would be lost. Therefore, the challenge is to design inference rules that, on the one hand, are liberal enough to overcome the strictness of the sequent calculus and, on the other hand, are conservative enough to allow a proof of cut elimination and a subformula property.

Since, in the calculus of structures, derivations are chains of instances of inference rules (and not trees as in the sequent calculus), they show a top-down symmetry, which is not present in the sequent calculus. An important consequence of this new symmetry is that the cut rule

$$\mathsf{i}{\uparrow}\,\frac{S\{A \otimes A^{\perp}\}}{S\{\perp\}}$$

becomes top-down symmetric to the identity rule

$$\mathsf{i}{\downarrow}\,\frac{S\{1\}}{S\{A \parr A^{\perp}\}} \quad .$$

With this, it is possible to reduce the general cut rule to its atomic version

$$\mathsf{ai}{\uparrow}\,\frac{S\{a \otimes a^{\perp}\}}{S\{\perp\}}$$

in the same way as this can be done for the identity. Furthermore, new manipulations of derivations become possible. For instance, we can negate a derivation and flip it upside down, and it remains a valid derivation.

Because of the new top-down symmetry, the calculus of structures allows for a *modular* cut elimination proof. This is another reason to study known logics, like MELL, within this new formalism (in [13], p. 15, Girard deems the lack of modularity in the sequent calculus as one of the main technical limitations of proof theory).

The top-down symmetry of the calculus of structures does also allow to formulate new properties of derivations, that are not observable in other proof theoretical formalisms. An important such property is decomposition, which basically says the following: every derivation can be transformed into a derivation consisting of three phases:

- a creation phase, which contains only rules that increase the size of the structure,
- a merging phase, which contains only rules that do not change the size of the structure (like the rules $\mathsf{p}{\downarrow}$ and s shown above), and
- a destruction phase, which contains only rules that decrease the size of the structure.

Such decomposition theorems have been also considered for other systems in the calculus of structures: for a non-commutative logic in [14, 17] and for classical logic in [7, 5].

Let me now sketch the outline of this paper. In the next section, I will give a short introduction to MELL and its sequent calculus presentation. In Section 3, I will introduce the language of structures and some basic notions of the calculus of structures.

Then, in Section 4, I will present two systems, called system SELS (*Symmetric or Self-dual multiplicative Exponential Linear logic in the calculus of Structures*) and system ELS (*multiplicative Exponential Linear logic in the calculus of Structures*). The first system corresponds to MELL with cut. It is self-dual because for every rule in the system, there is a dual (i.e. contrapositive) rule in the system. It is also called symmetric because it demonstrates the top-down symmetry of the calculus of structures. The second system corresponds to MELL without cut. In Section 5, I will show the correspondence between these two systems in the calculus of structures and the system for MELL in the sequent calculus. As a consequence, we obtain a cut elimination result for system ELS, which follows (easily) from the cut elimination proof using the sequent calculus presentation for MELL.

In Section 6, I will study the permutation of rules. This is the basis for the decomposition of derivations in system SELS and the cut elimination proof within the calculus of structures. Sections 7 and 8 are devoted to the proof of the decomposition theorem for system SELS.

In Section 9, I will give a cut elimination proof for system ELS which will completely be carried out inside the calculus of structures, without the detour of using the sequent calculus. It will be very different from all known cut elimination proofs for MELL because it uses the result of the decomposition theorem and because it will be modular. For a more detailed explanation of cut elimination in the calculus of structures let me refer the reader to the introductory part of that section.

2 The Multiplicative Exponential Fragment of Linear Logic

The calculus of structures, being a proof theoretical formalism, is not tied to any particular logic. It can be used to define many different logical systems, in the same way as the sequent calculus has been used for various systems, for instance classical and intuitionistic logic [10], the Lambek calculus [20] or linear logic [12]. In this paper, I will restrict myself to the multiplicative exponential fragment of linear logic.

2.1 Definition The multiplicative exponential fragment of linear logic (MELL) is defined as follows:

- *Formulae*, denoted with A, B and C, are built over atoms according to the following syntax:

$$A ::= a \mid \mathsf{1} \mid \bot \mid A \parr A \mid A \otimes A \mid !A \mid ?A \mid A^{\bot} \quad ,$$

  where a stands for any atom, $\mathsf{1}$ and $\bot$ are constants, called *one* and *bottom*, respectively, the binary connectives $\parr$ and $\otimes$ are called *par* and *times*, respectively, the unary connectives $!$ and $?$ are called *of-course* and *why-not*, respectively, and $A^{\bot}$ is the *negation* of A. When necessary, parentheses are used to disambiguate

$$
\mathsf{id}\ \frac{}{\vdash A, A^{\perp}} \qquad \mathsf{cut}\ \frac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}
$$

$$
\invamp\ \frac{\vdash A, B, \Phi}{\vdash A \invamp B, \Phi} \qquad \otimes\ \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \qquad \perp\ \frac{\vdash \Phi}{\vdash \perp, \Phi} \qquad \mathsf{1}\ \frac{}{\vdash \mathsf{1}}
$$

$$
\mathsf{dr}\ \frac{\vdash A, \Phi}{\vdash ?A, \Phi} \qquad \mathsf{ct}\ \frac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi} \qquad \mathsf{wk}\ \frac{\vdash \Phi}{\vdash ?A, \Phi} \qquad !\ \frac{\vdash A, ?B_1, \dots, ?B_n}{\vdash !A, ?B_1, \dots, ?B_n}\ (\text{for } n \geqslant 0)
$$

Figure 1: System MELL in the sequent calculus

expressions. Negation obeys the De Morgan laws:

$$
\begin{array}{rcl}
(A \invamp B)^{\perp} & = & A^{\perp} \otimes B^{\perp} \quad , \\
(A \otimes B)^{\perp} & = & A^{\perp} \invamp B^{\perp} \quad , \\
(!A)^{\perp} & = & ?A^{\perp} \quad , \\
(?A)^{\perp} & = & !A^{\perp} \quad , \\
\mathsf{1}^{\perp} & = & \perp \quad , \\
\perp^{\perp} & = & \mathsf{1} \quad , \\
A^{\perp\perp} & = & A \quad .
\end{array}
$$

Formulae are considered equivalent modulo the smallest congruence satisfying the equations above.

- *Sequents*, denoted with Σ, are expressions of the kind

  $$\vdash A_1, \dots, A_h \quad ,$$

  where $h \geqslant 0$ and the comma between the formulae $A_1, \dots, A_h$ stands for multiset union. Multisets of formulae are denoted with Φ and Ψ.

- *Derivations*, denoted with Δ, are trees where the nodes are sequents to which a finite number (possibly zero) of instances of the inference rules shown in Figure 1 are applied. The sequents in the leaves are called *premises*, and the sequent in the root is the *conclusion*. A derivation with no premises is a *proof*, denoted with Π.

2.2 Example The following derivation shows an example for a proof in MELL:

$$
\mathsf{ct}\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash ?a, !a^{\perp}} \qquad \parr\,\dfrac{!\,\dfrac{\mathsf{dr}\,\dfrac{\otimes\,\dfrac{\mathsf{dr}\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash a, a^{\perp}}}{\vdash ?a, a^{\perp}} \qquad \mathsf{id}\,\dfrac{}{\vdash b, b^{\perp}}}{\vdash ?a, a^{\perp} \otimes b, b^{\perp}}}{\vdash ?a, ?(a^{\perp} \otimes b), b^{\perp}}}{\vdash ?a, ?(a^{\perp} \otimes b), !b^{\perp}}}{\vdash ?a, ?(a^{\perp} \otimes b) \parr !b^{\perp}}}{\vdash ?a, ?a, (?(a^{\perp} \otimes b) \parr !b^{\perp}) \otimes !a^{\perp}}}{\vdash ?a, (?(a^{\perp} \otimes b) \parr !b^{\perp}) \otimes !a^{\perp}} \quad .
$$

3 The Language of Structures

In the sequent calculus, rules apply to sequents, which in turn are built from formulae. In the calculus of structures, rules apply to structures, which are a kind of intermediate expressions between formulae and sequents.

In order to present a system in the sequent calculus, we need first to define a language of formulae and sequents, as I did in the previous section. For presenting a system in the calculus of structures we have to do the same, i.e. defining a language of structures first. In this section, I will define the language ELS of structures for the systems that are discussed in this paper.

3.1 Definition There are countably many *atoms*, which are denoted with a, b, c, ... The *structures* of the language ELS are denoted with P, Q, R, S, ..., and are generated by

$$
R ::= a \mid \perp \mid 1 \mid [\underbrace{R, \dots, R}_{>0}] \mid (\underbrace{R, \dots, R}_{>0}) \mid !R \mid ?R \mid \bar{R} \quad ,
$$

where a stands for any atom, 1 and $\perp$ are constants, called *one* and *bottom*. A structure $[R_1, \dots, R_h]$ is called a *par structure*, $(R_1, \dots, R_h)$ is called a *times structure*, $!R$ is called an *of-course structure*, and $?R$ is called a *why-not structure*; $\bar{R}$ is the *negation* of the structure R. Structures are considered to be equivalent modulo the relation $=$, which is the smallest congruence relation induced by the equations shown in Figure 2, where $\vec{R}$ and $\vec{T}$ stand for finite, non-empty sequences of structures. Then by definition we have for all structures $R, R', R_1, R'_1, \dots, R_h, R'_h$ and $h > 0$,

- if $R = R'$, then $!R = !R'$ and $?R = ?R'$ and $\bar{R} = \bar{R}'$;
- if $R_i = R'_i$ for $i = 1, \dots, h$, then $[R_1, \dots, R_h] = [R'_1, \dots, R'_h]$ and $(R_1, \dots, R_h) = (R'_1, \dots, R'_h)$.

Associativity

$$\begin{array}{rcl}[\vec{R},[\vec{T}]] & = & [\vec{R},\vec{T}] \\ (\vec{R},(\vec{T})) & = & (\vec{R},\vec{T})\end{array}$$

Commutativity

$$\begin{array}{rcl}[\vec{R},\vec{T}] & = & [\vec{T},\vec{R}] \\ (\vec{R},\vec{T}) & = & (\vec{T},\vec{R})\end{array}$$

Units

$$\begin{array}{rcl}[\bot,\vec{R}] & = & [\vec{R}] \\ (1,\vec{R}) & = & (\vec{R})\end{array}$$

Singleton

$$[R] = R = (R)$$

Exponentials

$$\begin{array}{rcl}?\bot & = & \bot \\ !1 & = & 1 \\ ??R & = & ?R \\ !!R & = & !R\end{array}$$

Negation

$$\begin{array}{rcl}\overline{\bot} & = & 1 \\ \bar{1} & = & \bot \\ \overline{[R_1,\dots,R_h]} & = & (\bar{R}_1,\dots,\bar{R}_h) \\ \overline{(R_1,\dots,R_h)} & = & [\bar{R}_1,\dots,\bar{R}_h] \\ \overline{?R} & = & !\bar{R} \\ \overline{!R} & = & ?\bar{R} \\ \bar{\bar{R}} & = & R\end{array}$$

Figure 2: Basic equations for the syntactic congruence =

3.2 Definition In the same setting, we can define *structure contexts*, which are structures with a hole. Formally, they are generated by

$$S ::= \{\ \} \mid [\underbrace{R,\dots,R}_{\geqslant 0}, S, \underbrace{R,\dots,R}_{\geqslant 0}] \mid (\underbrace{R,\dots,R}_{\geqslant 0}, S, \underbrace{R,\dots,R}_{\geqslant 0}) \mid !S \mid ?S \quad .$$

Because of the De Morgan laws there is no need to include the negation into the definition of the context, which means that the structure that is plugged into the hole of a context will always be positive. Structure contexts will be denoted with $R\{\ \}$, $S\{\ \}$, $T\{\ \}$, ... Then, $S\{R\}$ denotes the structure that is obtained by replacing the hole $\{\ \}$ in the context $S\{\ \}$ by the structure R. The structure R is a *substructure* of $S\{R\}$ and $S\{\ \}$ is its *context*. For a better readability, I will omit the context braces if no ambiguity is possible, e.g. I will write $S[R,T]$ instead of $S\{[R,T]\}$.

3.3 Example Let $S\{\ \} = [(a,![\{\ \},?a],\bar{b}),b]$ and $R = c$ and $T = (\bar{b},\bar{c})$ then

$$S[R,T] = [(a,![c,(\bar{b},\bar{c}),?a],\bar{b}),b] \quad .$$

3.4 Definition In the calculus of structures, an *inference rule* is a scheme of the kind

$$\rho\frac{T}{R} \quad ,$$

where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion.* An inference rule is called an *axiom* if its premise is empty, i.e. the rule is of the shape

$$\rho \frac{}{R} \quad .$$

A typical rule has shape $\rho \dfrac{S\{T\}}{S\{R\}}$ and specifies a step of rewriting, by the implication $T \Rightarrow R$, inside a generic context $S\{\ \}$. Rules with empty contexts correspond to the case of the sequent calculus.

3.5 Definition A (*formal*) *system* $\mathscr{S}$ is a set of inference rules.

3.6 Definition A *derivation* Δ in a certain formal system is a finite sequence of instances of inference rules in the system:

$$\begin{array}{r c}
 & R \\
\rho & \overline{} \\
 & R' \\
\rho' & \overline{} \\
 & \vdots \\
\rho'' & \overline{} \\
 & R''
\end{array} \quad .$$

A derivation can consist of just one structure. The topmost structure in a derivation, if present, is called the *premise* of the derivation, and the bottommost structure is called its *conclusion.* A derivation Δ whose premise is T, whose conclusion is R, and whose inference rules are in $\mathscr{S}$ will be indicated with $\begin{array}{c}T\\ \Delta \Vert \mathscr{S}\\ R\end{array}$. A *proof* Π in the calculus of structures is a finite derivation whose topmost inference rule is an axiom. It will be denoted by $\begin{array}{c}\Pi \overline{\Vert} \mathscr{S}\\ R\end{array}$.

3.7 Definition A rule ρ is *derivable* in a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every application of $\rho \dfrac{T}{R}$ there is a derivation $\begin{array}{c}T\\ \Delta \Vert \mathscr{S}\\ R\end{array}$. A rule ρ is *admissible* for a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every proof $\begin{array}{c}\Pi \overline{\Vert} \mathscr{S}\cup\{\rho\}\\ R\end{array}$ there is a proof $\begin{array}{c}\Pi' \overline{\Vert} \mathscr{S}\\ R\end{array}$.

3.8 Definition Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if for every derivation $\begin{array}{c}T\\ \Delta \Vert \mathscr{S}\\ R\end{array}$ there is a derivation $\begin{array}{c}T\\ \Delta' \Vert \mathscr{S}'\\ R\end{array}$, and vice versa. Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *(weakly) equivalent* if for every proof $\begin{array}{c}\Pi \overline{\Vert} \mathscr{S}\\ R\end{array}$ there is a proof $\begin{array}{c}\Pi' \overline{\Vert} \mathscr{S}'\\ R\end{array}$, and vice versa.

3.9 Definition The function $\underline{\ \cdot\ }_{\mathsf{S}}$ defines the obvious translation from MELL formulae

$$
\mathsf{id}'\ \frac{}{[R,\bar{R}]}\qquad \mathsf{id}''\ \frac{S}{(S,[R,\bar{R}])}\qquad \mathsf{cut}'\ \frac{(S,[R,P],[\bar{R},Q])}{(S,[P,Q])}
$$

$$
\parr'\ \frac{(S,[R,T,P])}{(S,[[R,T],P])}\qquad \otimes'\ \frac{(S,[R,P],[T,Q])}{(S,[(R,T),P,Q])}\qquad \bot'\ \frac{(S,P)}{(S,[\bot,P])}\qquad 1'\ \frac{}{1}\qquad 1''\ \frac{S}{(S,1)}
$$

$$
\mathsf{dr}'\ \frac{(S,[R,P])}{(S,[?R,P])}\qquad \mathsf{ct}'\ \frac{(S,[?R,?R,P])}{(S,[?R,P])}\qquad \mathsf{wk}'\ \frac{(S,P)}{(S,[?R,P])}\qquad !'\ \frac{(S,[R,?T_1,\dots,?T_n])}{(S,[!R,?T_1,\dots,?T_n])}\ (\text{for } n \geqslant 0)
$$

Figure 3: System MELL' in the calculus of structures

into ELS structures:

$$
\begin{array}{rcl}
\underline{a}_{\mathsf{s}} & = & a \quad , \\
\underline{\bot}_{\mathsf{s}} & = & \bot \quad , \\
\underline{1}_{\mathsf{s}} & = & 1 \quad , \\
\underline{A \parr B}_{\mathsf{s}} & = & [\underline{A}_{\mathsf{s}},\underline{B}_{\mathsf{s}}] \quad , \\
\underline{A \otimes B}_{\mathsf{s}} & = & (\underline{A}_{\mathsf{s}},\underline{B}_{\mathsf{s}}) \quad , \\
\underline{?A}_{\mathsf{s}} & = & ?\underline{A}_{\mathsf{s}} \quad , \\
\underline{!A}_{\mathsf{s}} & = & !\underline{A}_{\mathsf{s}} \quad , \\
\underline{A^{\bot}}_{\mathsf{s}} & = & \overline{\underline{A}_{\mathsf{s}}} \quad .
\end{array}
$$

The domain of $\underline{\cdot}_{\mathsf{s}}$ is extended to sequents by

$$
\begin{array}{rcl}
\underline{\vdash}_{\mathsf{s}} & = & \bot \quad \text{and} \\
\underline{\vdash A_1,\dots,A_h}_{\mathsf{s}} & = & [\underline{A_1}_{\mathsf{s}},\dots,\underline{A_h}_{\mathsf{s}}] \text{ for } h \geqslant 0 \quad .
\end{array}
$$

The translation $\underline{\cdot}_{\mathsf{s}}$ induces trivially a set of rules for the calculus of structures that are able to mimic the derivations in MELL. These rules form system MELL' which is shown in Figure 3. (The rules $\parr'$, $\bot'$, and $1''$ are vacuous.) These rules are a one-to-one translation of the rules of the sequent calculus shown in Figure 1. The structures R and T (possibly indexed) in Figure 3 correspond to the formulas A and B, respectively, in Figure 1. The structures P and Q correspond to the contexts Φ and Ψ in the sequent calculus. The structure S carries the information about the sequent calculus tree, which is not directly visible in the calculus of structures. It is easy to see that for every derivation in MELL there is a corresponding derivation in the calculus of structures using system MELL', and vice versa.

3.10 Example The corresponding proof in MELL' for the proof in MELL in Example 2.2 becomes:

$$
\mathsf{ct}'\dfrac{\otimes'\dfrac{\mathsf{id}''\dfrac{!'\dfrac{\mathsf{dr}'\dfrac{\otimes'\dfrac{\mathsf{dr}'\dfrac{\mathsf{id}''\dfrac{\mathsf{id}'\dfrac{}{[b,\bar{b}]}}{([a,\bar{a}],[b,\bar{b}])}}{([?a,\bar{a}],[b,\bar{b}])}}{[?a,(\bar{a},b),\bar{b}]}}{[?a,?(\bar{a},b),\bar{b}]}}{[?a,?(\bar{a},b),!\bar{b}]}}{([?a,!\bar{a}],[?a,?(\bar{a},b),!\bar{b}])}}{[?a,?a,([?(\bar{a},b),!\bar{b}]),!\bar{a}]}}{[?a,([?(\bar{a},b),!\bar{b}]),!\bar{a}]} \quad .
$$

This shows that the calculus of structures is at least as powerful as the sequent calculus, because, by this method, any system in the sequent calculus that admits a one-sided presentation can be ported, trivially, to the calculus of structures. But this hardly justifies the use of the calculus of structures. In the next section, I will build two systems that are equivalent to MELL (one to MELL with cut and one to MELL without cut) and that will use the the new freedom and symmetry of the calculus of structures. As a consequence they will be much simpler than MELL′ shown above.

3.11 Definition The translation from ELS structures into MELL formulae is realized by the function $\underline{\cdot}_{\mathsf{L}}$:

$$
\begin{array}{rcl}
\underline{a}_{\mathsf{L}} & = & a \quad , \\
\underline{\bot}_{\mathsf{L}} & = & \bot \quad , \\
\underline{1}_{\mathsf{L}} & = & 1 \quad , \\
\underline{[R_1,\dots,R_h]}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \parr \cdots \parr \underline{R_h}_{\mathsf{L}} \quad , \\
\underline{(R_1,\dots,R_h)}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \otimes \cdots \otimes \underline{R_h}_{\mathsf{L}} \quad , \\
\underline{?R}_{\mathsf{L}} & = & ?\,\underline{R}_{\mathsf{L}} \quad , \\
\underline{!R}_{\mathsf{L}} & = & !\,\underline{R}_{\mathsf{L}} \quad , \\
\underline{\bar{R}}_{\mathsf{L}} & = & (\underline{R}_{\mathsf{L}})^{\bot} \quad .
\end{array}
$$

3.12 Remark Although ELS structures are in fact equivalence classes and MELL formulae are not, the translations $\underline{\cdot}_{\mathsf{S}}$ and $\underline{\cdot}_{\mathsf{L}}$ work because the DeMorgan laws are imposed on both and the other equations on structures are logical equivalences in MELL.

4 A Symmetric Set of Rules

In the calculus of structures, rules come in pairs, a down-version $\rho{\downarrow}\,\dfrac{S\{T\}}{S\{R\}}$ and an up-version $\rho{\uparrow}\,\dfrac{S\{\bar{R}\}}{S\{\bar{T}\}}$. This duality derives from the duality between $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$, where $\Rightarrow$ is the implication modelled in the system. In our case it is linear implication.

4.1 Definition The structural rules

$$\mathsf{s}\,\frac{S([R,T],U)}{S[(R,U),T]} \quad , \quad \mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \quad , \quad \mathsf{p}{\uparrow}\,\frac{S(?R,!T)}{S\{?(R,T)\}} \quad ,$$

$$\mathsf{w}{\downarrow}\,\frac{S\{\bot\}}{S\{?R\}} \quad , \quad \mathsf{w}{\uparrow}\,\frac{S\{!R\}}{S\{1\}} \quad , \quad \mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}} \qquad \text{and} \qquad \mathsf{b}{\uparrow}\,\frac{S\{!R\}}{S(!R,R)}$$

are called *switch* (s), *promotion* ($\mathsf{p}{\downarrow}$), *copromotion* ($\mathsf{p}{\uparrow}$), *weakening* ($\mathsf{w}{\downarrow}$), *coweakening* ($\mathsf{w}{\uparrow}$), *absorption* ($\mathsf{b}{\downarrow}$) and *coabsorption* ($\mathsf{b}{\uparrow}$), respectively.

Observe that the switch rule is self-dual, i.e. if premise and conclusion are negated and exchanged, we obtain again an instance of switch, whereas all other rules have a dual co-rule.

4.2 Definition The rules

$$\mathsf{i}{\downarrow}\,\frac{S\{1\}}{S[R,\bar{R}]} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\,\frac{S(R,\bar{R})}{S\{\bot\}}$$

are called *interaction* and *cut* (or *cointeraction*), respectively.

Observe that these rules correspond to the identity and cut rule in the sequent calculus (the exact correspondence is shown in the proof of Theorem 5.2), with the difference that the duality between identity and cut is more vivid.

4.3 Definition The rules

$$\mathsf{ai}{\downarrow}\,\frac{S\{1\}}{S[a,\bar{a}]} \qquad \text{and} \qquad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar{a})}{S\{\bot\}}$$

are called *atomic interaction* and *atomic cut* (or *atomic cointeraction*), respectively.

The rules $\mathsf{ai}{\downarrow}$ and $\mathsf{ai}{\uparrow}$ are obviously instances of the rules $\mathsf{i}{\downarrow}$ and $\mathsf{i}{\uparrow}$ above. It is well known that in many systems in the sequent calculus, the identity rule can be reduced to its atomic version. In the calculus of structures we can do the same. But furthermore, by duality, we can do the same to the cut rule. This is not possible in the sequent calculus because whenever an atomic cut is applied in the sequent calculus a branching occurs and there is no way to reunite two branches in a sequent calculus derivation.

4.4 Proposition *The rule* $\mathsf{i}{\downarrow}$ *is derivable in the system* $\{\mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{p}{\downarrow}\}$. *Dually, the rule* $\mathsf{i}{\uparrow}$ *is derivable in* $\{\mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{p}{\uparrow}\}$.

Proof: For a given application of $\mathsf{i}\!\downarrow \frac{S\{1\}}{S[R,\bar{R}]}$, by structural induction on R, we will construct an equivalent derivation that contains only $\mathsf{ai}\!\downarrow$, s and $\mathsf{p}\!\downarrow$.

- $R = \bot$ or $R = 1$: In this case $S[R,\bar{R}] = S\{1\}$.
- R is an atom: Then the given instance of $\mathsf{i}\!\downarrow$ is an instance of $\mathsf{ai}\!\downarrow$.
- $R = [P,Q]$, where $P \neq \bot \neq Q$: Apply the induction hypothesis on

$$\mathsf{s}\,\frac{\mathsf{s}\,\dfrac{\mathsf{i}\!\downarrow\dfrac{\mathsf{i}\!\downarrow\dfrac{S\{1\}}{S[Q,\bar{Q}]}}{S([P,\bar{P}],[Q,\bar{Q}])}}{S[Q,([P,\bar{P}],\bar{Q})]}}{S[P,Q,(\bar{P},\bar{Q})]} \quad .$$

- $R = (P,Q)$, where $P \neq 1 \neq Q$: Similar to the previous case.
- $R = ?P$, where $P \neq \bot$: Apply the induction hypothesis on

$$\mathsf{p}\!\downarrow\frac{\mathsf{i}\!\downarrow\dfrac{S\{1\}}{S\{![P,\bar{P}]\}}}{S[?P,!\bar{P}]} \quad .$$

  (Note that $S\{1\} = S\{!1\}$.)
- $R = !P$, where $P \neq 1$: Similar to the previous case.

The second statement is dual to the first. For the sake of convenience let me show the two interesting derivations:

$$\mathsf{i}\!\uparrow\frac{\mathsf{i}\!\uparrow\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{S(P,Q,[\bar{P},\bar{Q}])}{S(Q,[(P,\bar{P}),\bar{Q}])}}{S[(P,\bar{P}),(Q,\bar{Q})]}}{S(Q,\bar{Q})}}{S\{\bot\}} \qquad \text{and} \qquad \mathsf{i}\!\uparrow\frac{\mathsf{p}\!\uparrow\dfrac{S(!P,?\bar{P})}{S\{?(P,\bar{P})\}}}{S\{\bot\}} \quad .$$

$\square$

4.5 Definition The system $\{\mathsf{ai}\!\downarrow, \mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{p}\!\downarrow, \mathsf{p}\!\uparrow, \mathsf{w}\!\downarrow, \mathsf{w}\!\uparrow, \mathsf{b}\!\downarrow, \mathsf{b}\!\uparrow\}$, shown in Figure 4 is called *Symmetric (or Self-dual) multiplicative Exponential Linear logic in the calculus of Structures*, or system SELS. The set $\{\mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{p}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow\}$ is called the *down-fragment* and $\{\mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{p}\!\uparrow, \mathsf{w}\!\uparrow, \mathsf{b}\!\uparrow\}$ is called the *up-fragment*.

There is another strong admissibility result involved here, that has already been observed in [14]. If the rules $\mathsf{i}\!\downarrow$, $\mathsf{i}\!\uparrow$ and s are in a system, then any other rule ρ makes its *co-rule* ρ', i.e. the rule obtained from ρ by exchanging and negating premise and

$$
\left.\begin{array}{cc}
\mathsf{ai}\!\downarrow \dfrac{S\{1\}}{S[a,\bar{a}]} & \mathsf{ai}\!\uparrow \dfrac{S(a,\bar{a})}{S\{\bot\}}
\end{array}\right\}\text{Interaction}
$$

$$
\left.\begin{array}{cc}
\multicolumn{2}{c}{\mathsf{s}\,\dfrac{S([R,U],T)}{S[(R,T),U]}} \\[2ex]
\mathsf{p}\!\downarrow \dfrac{S\{![R,T]\}}{S[!R,?T]} & \mathsf{p}\!\uparrow \dfrac{S(?R,!T)}{S\{?(R,T)\}}
\end{array}\right\}\text{Core}
$$

$$
\left.\begin{array}{cc}
\mathsf{w}\!\downarrow \dfrac{S\{\bot\}}{S\{?R\}} & \mathsf{w}\!\uparrow \dfrac{S\{!R\}}{S\{1\}} \\[2ex]
\mathsf{b}\!\downarrow \dfrac{S[?R,R]}{S\{?R\}} & \mathsf{b}\!\uparrow \dfrac{S\{!R\}}{S(!R,R)}
\end{array}\right\}\text{Non-core}
$$

Figure 4: System SELS

conclusion, be derivable: Let $\rho\,\dfrac{S\{P\}}{S\{Q\}}$ be given. Then any instance of $\rho'\,\dfrac{S\{\bar{Q}\}}{S\{\bar{P}\}}$ can be replaced by the following derivation:

$$
\begin{array}{l}
\mathsf{i}\!\downarrow \dfrac{S\{\bar{Q}\}}{\mathsf{s}\,\dfrac{S(\bar{Q},[P,\bar{P}])}{\rho\,\dfrac{S[(\bar{Q},P),\bar{P}]}{\mathsf{i}\!\uparrow \dfrac{S[(\bar{Q},Q),\bar{P}]}{S\{\bar{P}\}}}}}
\end{array}
\quad .
$$

4.6 Proposition *Every rule $\rho\!\uparrow$ in* SELS *is derivable in* $\{\mathsf{i}\!\downarrow, \mathsf{i}\!\uparrow, \mathsf{s}, \rho\!\downarrow\}$.

Propositions 4.4 and 4.6 together say, that the general cut rule $\mathsf{i}\!\uparrow$ is as powerful as the whole up-fragment of the system and vice versa.

Observe that in Proposition 4.4 only the rules s, $\mathsf{p}\!\downarrow$ and $\mathsf{p}\!\uparrow$ are used to reduce the general interaction and the general cut to their atomic version, whereas the rules $\mathsf{w}\!\downarrow$, $\mathsf{w}\!\uparrow$, $\mathsf{b}\!\downarrow$ and $\mathsf{b}\!\uparrow$ are not used. This motivates the following definition.

4.7 Definition In system SELS, the rules s, $\mathsf{p}\!\downarrow$ and $\mathsf{p}\!\uparrow$ are called *core* part, whereas the rules $\mathsf{w}\!\downarrow$, $\mathsf{w}\!\uparrow$, $\mathsf{b}\!\downarrow$ and $\mathsf{b}\!\uparrow$ are *non-core*.

So far we are only able to describe derivations. In order to describe proofs, we need an axiom.

$$
\mathsf{1}\!\downarrow \frac{}{\mathsf{1}} \qquad \mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}
$$

$$
\mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \qquad \mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]}
$$

Figure 5: System ELS

4.8 Definition The following rule is called *one*:

$$
\mathsf{1}\!\downarrow \frac{}{\mathsf{1}} \quad .
$$

In the language of the sequent calculus it simply says that $\vdash \mathsf{1}$ is provable. I will put this rule to the down-fragment of system SELS and by this break the top-down symmetry of derivations and observe proofs.

4.9 Definition The system $\{\mathsf{1}\!\downarrow, \mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{p}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow\}$, shown in Figure 5, which is obtained from the down-fragment of system SELS together with the axiom, is called *multiplicative Exponential Linear logic in the calculus of Structures*, or system ELS.

Observe that in every proof in system ELS, the rule $\mathsf{1}\!\downarrow$ occurs exactly once, namely as the topmost rule of the proof.

As an immediate consequence of Propositions 4.4 and 4.6 we get the following:

4.10 Theorem *The systems* $\mathsf{ELS} \cup \{\mathsf{i}\!\uparrow\}$ *and* $\mathsf{SELS} \cup \{\mathsf{1}\!\downarrow\}$ *are strongly equivalent.*

5 Correspondence between MELL and ELS

In this section, I will show the equivalence between the system MELL in the sequent calculus and the systems $\mathsf{SELS} \cup \{\mathsf{1}\!\downarrow\}$ and ELS in the calculus of structures. More precisely, every proof in system $\mathsf{SELS} \cup \{\mathsf{1}\!\downarrow\}$ has a translation in system MELL, and every cut free proof in MELL has a translation in system ELS. As a consequence, we can obtain an (easy) proof of cut elimination for system $\mathsf{SELS} \cup \{\mathsf{1}\!\downarrow\}$, or equivalenty, for system $\mathsf{ELS} \cup \{\mathsf{i}\!\uparrow\}$.

In order to show cut elimination for system $\mathsf{SELS} \cup \{\mathsf{1}\!\downarrow\}$, so as to obtain a system where each rule satisfies the subformula property (in the sense that the premise is built from substructures of the conclusion and there are only finitely many possibilities to apply the rule to a given structure), it would be sufficient to eliminate only the rules $\mathsf{ai}\!\uparrow$ and $\mathsf{w}\!\uparrow$. But we can get more. We can show that the whole up-fragment of system SELS (except for the switch which does also belong to the down-fragment) is admissible. This paper contains two very different proofs of this fact. The first, in this section, uses the cut elimination proof for MELL in the sequent calculus. The second, in Section 9, will be carried out inside the calculus of structures, completely independently from the sequent calculus.

5.1 Theorem *If a given structure R is provable in system* $\mathsf{SELS} \cup \{1\!\downarrow\}$, *then its translation* $\vdash \underline{R}_{\mathsf{L}}$ *is provable in* MELL *(with cut).*

Proof: Suppose, we have a proof Π of R in system $\mathsf{SELS} \cup \{1\!\downarrow\}$. By induction on the length of Π, let us build a proof $\underline{\Pi}_{\mathsf{L}}$ of $\vdash \underline{R}_{\mathsf{L}}$ in MELL.

Base case: Π is $1\!\downarrow \frac{}{1}$: Let $\underline{\Pi}_{\mathsf{L}}$ be the proof $\mathsf{1}\,\frac{}{\vdash 1}$.

Inductive case: Suppose Π is

$$\begin{array}{c} \Pi' \big\| {\scriptstyle \mathsf{SELS} \cup \{1\downarrow\}} \\ \rho\,\dfrac{S\{R\}}{S\{T\}} \end{array} \quad ,$$

where $\rho\,\dfrac{S\{R\}}{S\{T\}}$ is the last rule to be applied in Π. The following MELL proofs show that $\vdash (\underline{R}_{\mathsf{L}})^{\perp}, \underline{T}_{\mathsf{L}}$ is provable in MELL for every rule $\rho\,\dfrac{S\{R\}}{S\{T\}}$ in SELS, i.e. $\underline{R}_{\mathsf{L}} \multimap \underline{T}_{\mathsf{L}}$ is a theorem in MELL:

$$\bot\,\dfrac{⅋\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash a, a^{\perp}}}{\vdash a ⅋ a^{\perp}}}{\vdash \bot, a ⅋ a^{\perp}} \quad , \qquad \mathsf{wk}\,\dfrac{\mathsf{1}\,\dfrac{}{\vdash 1}}{\vdash 1, ?R} \quad , \qquad \mathsf{ct}\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash !R^{\perp}, ?R} \qquad \mathsf{dr}\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R}}{\vdash R^{\perp}, ?R}}{\vdash (!R^{\perp} \otimes R^{\perp}), ?R, ?R}}{\vdash (!R^{\perp} \otimes R^{\perp}), ?R} \quad ,$$

$$⅋\,\dfrac{⅋\,\dfrac{\otimes\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R} \qquad \mathsf{id}\,\dfrac{}{\vdash U^{\perp}, U}}{\vdash R^{\perp}, U^{\perp}, R \otimes U} \qquad \mathsf{id}\,\dfrac{}{\vdash T^{\perp}, T}}{\vdash R^{\perp} \otimes T^{\perp}, U^{\perp}, R \otimes U, T}}{\vdash R^{\perp} \otimes T^{\perp}, U^{\perp}, (R \otimes U) ⅋ T}}{\vdash (R^{\perp} \otimes T^{\perp}) ⅋ U^{\perp}, (R \otimes U) ⅋ T} \quad , \qquad ⅋\,\dfrac{!\,\dfrac{\mathsf{dr}\,\dfrac{\mathsf{dr}\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R} \qquad \mathsf{id}\,\dfrac{}{\vdash T^{\perp}, T}}{\vdash R^{\perp} \otimes T^{\perp}, R, T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), R, T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), ?R, T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), ?R, !T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), ?R ⅋ !T} \quad .$$

This means that for any context $S\{\ \}$, we also have that $\underline{S\{R\}}_{\mathsf{L}} \multimap \underline{S\{T\}}_{\mathsf{L}}$ is a theorem in MELL, i.e. $\vdash (\underline{S\{R\}}_{\mathsf{L}})^{\perp}, \underline{S\{T\}}_{\mathsf{L}}$ is provable in MELL. By induction hypothesis we have a proof $\underline{\Pi'}_{\mathsf{L}}$ of $\vdash \underline{S\{R\}}_{\mathsf{L}}$ in MELL. Now we can get a proof $\underline{\Pi}_{\mathsf{L}}$ of $\vdash \underline{S\{T\}}_{\mathsf{L}}$ by applying the cut rule:

$$\mathsf{cut}\,\dfrac{\vdash \underline{S\{R\}}_{\mathsf{L}} \qquad \vdash (\underline{S\{R\}}_{\mathsf{L}})^{\perp}, \underline{S\{T\}}_{\mathsf{L}}}{\vdash \underline{S\{T\}}_{\mathsf{L}}} \quad .$$

$\square$

5.2 Theorem (a) *If a given sequent* $\vdash \Phi$ *is provable in* MELL *(with cut), then the structure* $\underline{\vdash \Phi}_{\mathsf{S}}$ *is provable in system* $\mathsf{SELS} \cup \{1{\downarrow}\}$. (b) *If a given sequent* $\vdash \Phi$ *is cut free provable in* MELL, *then the structure* $\underline{\vdash \Phi}_{\mathsf{S}}$ *is provable in system* ELS.

Proof: Let Π be the proof of $\vdash \Phi$ in MELL. By structural induction on Π, we will construct a proof $\underline{\Pi}_{\mathsf{S}}$ of $\underline{\vdash \Phi}_{\mathsf{S}}$ in system $\mathsf{SELS} \cup \{1{\downarrow}\}$ (or system ELS if Π is cut free).

- If Π is $\mathsf{id}\,\dfrac{}{\vdash A, A^{\perp}}$ for some formula A, then let $\underline{\Pi}_{\mathsf{S}}$ be the proof obtained via Proposition 4.4 from

$$\mathsf{i}{\downarrow}\,\dfrac{1{\downarrow}\,\dfrac{}{1}}{[\underline{A}_{\mathsf{S}}, \overline{\underline{A}_{\mathsf{S}}}]} \quad .$$

- If $\mathsf{cut}\,\dfrac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$ is the last rule applied in Π, then there are by induction hypothesis two derivations $\begin{array}{c} 1 \\ \Delta_1 \Big\Vert \mathsf{SELS} \\ [\underline{A}_{\mathsf{S}}, \underline{\Phi}_{\mathsf{S}}] \end{array}$ and $\begin{array}{c} 1 \\ \Delta_2 \Big\Vert \mathsf{SELS} \\ [\overline{\underline{A}_{\mathsf{S}}}, \underline{\Psi}_{\mathsf{S}}] \end{array}$. Let $\underline{\Pi}_{\mathsf{S}}$ be the proof obtained via Proposition 4.4 from

$$\mathsf{i}{\uparrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\begin{array}{c} 1{\downarrow}\,\dfrac{}{1} \\ \Delta_1 \Big\Vert \mathsf{SELS} \\ [\underline{A}_{\mathsf{S}}, \underline{\Phi}_{\mathsf{S}}] \\ \Delta_2 \Big\Vert \mathsf{SELS} \\ ([\underline{A}_{\mathsf{S}}, \underline{\Phi}_{\mathsf{S}}], [\overline{\underline{A}_{\mathsf{S}}}, \underline{\Psi}_{\mathsf{S}}]) \end{array}}{[([\underline{A}_{\mathsf{S}}, \underline{\Phi}_{\mathsf{S}}], \overline{\underline{A}_{\mathsf{S}}}), \underline{\Psi}_{\mathsf{S}}]}}{[\underline{\Phi}_{\mathsf{S}}, \underline{\Psi}_{\mathsf{S}}, (\underline{A}_{\mathsf{S}}, \overline{\underline{A}_{\mathsf{S}}})]}}{[\underline{\Phi}_{\mathsf{S}}, \underline{\Psi}_{\mathsf{S}}]} \quad .$$

- If $\mathbin{\bindnasrepma}\,\dfrac{\vdash A, B, \Phi}{\vdash A \mathbin{\bindnasrepma} B, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{S}}$ be the proof of $[\underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}}, \underline{\Phi}_{\mathsf{S}}]$ that exists by induction hypothesis.

- If $\otimes\,\dfrac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi}$ is the last rule applied in Π, then there are by induction hypothesis two derivations $\begin{array}{c} 1 \\ \Delta_1 \Big\Vert \mathsf{SELS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \end{array}$ and $\begin{array}{c} 1 \\ \Delta_2 \Big\Vert \mathsf{SELS} \\ [\underline{B}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}] \end{array}$. Let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\begin{array}{c}
1\!\downarrow \dfrac{}{1} \\
\Delta_1 \Big\Vert \mathsf{SELS} \\
[\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \\
\Delta_2 \Big\Vert \mathsf{SELS} \\
\mathsf{s}\,\dfrac{([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], [\underline{B}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}])}{\mathsf{s}\,\dfrac{[([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], \underline{B}_{\mathsf{s}}), \underline{\Psi}_{\mathsf{s}}]}{[(\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}), \underline{\Phi}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}]}}
\end{array} \quad .$$

- If $\bot\,\dfrac{\vdash \Phi}{\vdash \bot, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof of $\underline{\vdash \Phi}_{\mathsf{s}}$ that exists by induction hypothesis.

- If Π is $1\,\dfrac{}{\vdash 1}$, then let $\underline{\Pi}_{\mathsf{s}}$ be $1\!\downarrow \dfrac{}{1}$.

- If $\mathsf{dr}\,\dfrac{\vdash A, \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\begin{array}{c}
\Pi' \Big\Vert \mathsf{SELS} \cup \{1\!\downarrow\} \\
\mathsf{w}\!\downarrow \dfrac{[\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}{\mathsf{b}\!\downarrow \dfrac{[?\underline{A}_{\mathsf{s}}, \underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}{[?\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}}
\end{array} \quad ,$$

where Π' exists by induction hypothesis.

- If $\mathsf{ct}\,\dfrac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\begin{array}{c}
\Pi' \Big\Vert \mathsf{SELS} \cup \{1\!\downarrow\} \\
\mathsf{b}\!\downarrow \dfrac{[??\underline{A}_{\mathsf{s}}, ?\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}{[??\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}
\end{array} \quad ,$$

where Π' exists by induction hypothesis. (Note that $??\underline{A}_{\mathsf{s}} = ?\underline{A}_{\mathsf{s}}$.)

- If $\mathsf{wk}\,\dfrac{\vdash \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\mathsf{w}{\downarrow}\,\dfrac{\begin{array}{c}\Pi' \Big\| \mathsf{SELS} \cup \{1{\downarrow}\} \\ \underline{\Phi}_{\mathsf{s}}\end{array}}{[?\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]} \quad ,$$

where Π' exists by induction hypothesis.

- If $!\,\dfrac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n}$ is the last rule applied in Π, then there is by induction hypothesis a derivation $\begin{array}{c} 1 \\ \Delta \big\| \mathsf{SELS} \\ [\underline{A}_{\mathsf{s}}, ?\underline{B_1}_{\mathsf{s}}, \ldots, ?\underline{B_n}_{\mathsf{s}}] \end{array}$. Now let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\mathsf{p}{\downarrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\begin{array}{c}1{\downarrow}\,\dfrac{}{!1} \\ \Delta' \big\| \mathsf{SELS} \\ ![\underline{A}_{\mathsf{s}}, ?\underline{B_1}_{\mathsf{s}}, \ldots, ?\underline{B_n}_{\mathsf{s}}]\end{array}}{\vdots}}{[![\underline{A}_{\mathsf{s}}, ?\underline{B_1}_{\mathsf{s}}], ??\underline{B_2}_{\mathsf{s}}, \ldots, ??\underline{B_n}_{\mathsf{s}}]}}{[!\underline{A}_{\mathsf{s}}, ??\underline{B_1}_{\mathsf{s}}, ??\underline{B_2}_{\mathsf{s}}, \ldots, ??\underline{B_n}_{\mathsf{s}}]} \quad .$$

□

5.3 Theorem (Cut Elimination) *The systems* $\mathsf{SELS} \cup \{1{\downarrow}\}$ *and* ELS *are equivalent.*

Proof: Given a proof in $\mathsf{SELS} \cup \{1{\downarrow}\}$, transform it into a proof in MELL (by Theorem 5.1), to which we can apply the cut elimination procedure in the sequent calculus. The cut free proof in MELL can then be transformed into a proof in system ELS by Theorem 5.2. □

5.4 Corollary *The rule* $\mathsf{i}{\uparrow}$ *is admissible for system* ELS.

Proof: Immediate consequence of Theorems 4.10 and 5.3. □

6 Permutability of Rules

The top-down symmetry of derivations in the calculus of structures enables us to study the mutual permutability of rules in a very natural way. This is the starting point for the investigation of several properties of logical systems in the calculus of structures. If we have, for example, a system with three rule ρ, π and σ, and we know that ρ permutes over π and σ, then we can transform every derivation $\begin{array}{c} T \\ \big\|\{\rho,\pi,\sigma\} \\ R \end{array}$ into a derivation

$$\begin{array}{c} T \\ \| \{\rho\} \\ T' \\ \| \{\pi,\sigma\} \\ R \end{array}$$

for some structure T'. This is the basis for the decomposition theorem in Section 8 and the cut elimination proof of Section 9.

6.1 Definition A rule ρ *permutes over* a rule π (or π *permutes under* ρ) if for every derivation $\pi\,\dfrac{Q}{\rho\,\dfrac{U}{P}}$ there is a derivation $\rho\,\dfrac{Q}{\pi\,\dfrac{V}{P}}$ for some structure V.

In order to study the permutation properties of rules, some more definitions are needed. The inference rules of SELS, as it is presented in Figure 4, are all of the kind $\rho\,\dfrac{S\{W\}}{S\{Z\}}$: the structure Z is called the *redex* and W the *contractum* of the rule's instance. A substructure that occurs both in the redex and in the contractum of a rule without changing is called *passive*, and all the substructures of redexes and contracta, that are not passive, (i.e. that change, dissappear or are duplicated) are called *active*. Consider for example the rules

$$\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \text{and} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}} \quad .$$

In $\mathsf{p}{\downarrow}$, the redex is $[!R,?T]$ and the contractum is $![R,T]$; the structures R and T are passive; the structures $[!R,?T]$, $!R$ and $?T$ are active in the redex; and the structures $![R,T]$ and $[R,T]$ are active in the contractum. In $\mathsf{b}{\downarrow}$ there are no passive structures; in the redex the structures $?R$ and R are active and in the contractum $[?R,R], ?R, R$ and R are active (i.e. both occurrences of the structure R are active).

6.2 Definition An application of a rule $\rho\,\dfrac{T}{R}$ will be called *trivial* if $R=T$.

6.3 Case Analysis In order to find out whether a rule ρ permutes over a rule π, we have to consider all possibilities of interference of the redex of π and the contractum of ρ in a situation

$$\pi\,\frac{Q}{\rho\,\dfrac{U}{P}} \quad .$$

Similarly as in the study of critical pairs in term rewriting systems, it can happen that one is inside the other, that they overlap or that they are independent. Although the situation is symmetric with respect to ρ and π, in almost all proofs of this paper, the situation to be considered will be of the shape

$$\pi\,\frac{Q}{\rho\,\dfrac{S\{W\}}{S\{Z\}}} \quad ,$$

$$
(1)\ \dfrac{\dfrac{(d,[a,c],b)}{([b,\bar{b}],d,[a,c],b)}\,\mathsf{ai}{\downarrow}}{([b,\bar{b}],d,[(a,b),c])}\,\mathsf{s}
\qquad
(2)\ \dfrac{\dfrac{(!(a,c),[\bar{a},d])}{[\bar{a},(!(a,c),d)]}\,\mathsf{s}}{[\bar{a},(!(a,[b,\bar{b}],c),d)]}\,\mathsf{ai}{\downarrow}
\qquad
(3)\ \dfrac{\dfrac{([a,c],b)}{([(a,[b,\bar{b}]),c],b)}\,\mathsf{ai}{\downarrow}}{[(a,[b,\bar{b}],b),c]}\,\mathsf{s}
$$

$$
(4)\ \dfrac{\dfrac{(a,![b,(c,d)])}{(a,[!b,?(c,d)])}\,\mathsf{p}{\downarrow}}{[(a,!b),?(c,d)]}\,\mathsf{s}
\qquad
(5)\ \dfrac{\dfrac{[a,b]}{[a,b,?[(c,\bar{c}),\bar{a}]]}\,\mathsf{w}{\downarrow}}{[a,b,?\bar{a}]}\,\mathsf{ai}{\uparrow}
\qquad
(6)\ \dfrac{\dfrac{[?[a,b],a,([b,c],d)]}{[?[a,b],a,b,(c,d)]}\,\mathsf{s}}{[?[a,b],(c,d)]}\,\mathsf{b}{\downarrow}
$$

Figure 6: Possible interferences of redex and contractum of two consecutive rules

where the redex Z and the contractum W of ρ are known and we have to make a case analysis for the position of the redex of π inside the structure $S\{W\}$. Then the following six cases exhaust all possibilities and Figure 6 shows an example for each case:

(1) The redex of π is inside the context $S\{\ \}$ of ρ.

(2) The contractum W of ρ is inside a passive structure of the redex of π.

(3) The redex of π is inside a passive structure of the contractum W of ρ.

(4) The redex of π is inside an active structure of the contractum W of ρ but not inside a passive one.

(5) The contractum W of ρ is inside an active structure of the redex of π but not inside a passive one.

(6) The contractum W of ρ and the redex of π (properly) overlap.

In the first two cases, we have that $Q = S'\{W\}$ for some context $S'\{\ \}$. This means that the derivation above is of the shape

$$
\pi\,\dfrac{\rho\,\dfrac{S'\{W\}}{S\{W\}}}{S\{Z\}}\ ,
$$

where we can permute ρ over π as follows

$$
\pi\,\dfrac{\rho\,\dfrac{S'\{W\}}{S'\{Z\}}}{S\{Z\}}\ .
$$

In the third case, we have that $Z = Z'\{R\}$ and $W = W'\{R\}$ for some contexts $Z'\{\ \}$ and $W'\{\ \}$ and some structure R, and $Q = S\{W'\{R'\}\}$ for some structure R'. This means the derivation is

$$\pi \frac{S\{W'\{R'\}\}}{\rho \frac{S\{W'\{R\}\}}{S\{Z'\{R\}\}}} \quad ,$$

where R is passive for ρ, and we can permute ρ over π as follows

$$\rho \frac{S\{W'\{R'\}\}}{\pi \frac{S\{Z'\{R'\}\}}{S\{Z'\{R\}\}}} \quad .$$

This means that in a proof of a permutation result the cases (1)–(3) are always trivial, whereas for the remaining cases (4)–(6), more elaboration will be necessary.

In every proof concerning a permutation result I will follow this scheme.

6.4 Lemma *The rule* $\mathsf{w}{\downarrow}$ *permutes over the rules* $\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{p}{\downarrow}$ *and* $\mathsf{w}{\uparrow}$*.*

Proof: Consider a derivation $\pi \frac{Q}{\mathsf{w}{\downarrow} \frac{S\{\bot\}}{S\{?R\}}}$, where $\pi \in \{\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{p}{\downarrow}, \mathsf{w}{\uparrow}\}$. Without loss of generality, assume that the application of π is not trivial. According to 6.3, the following cases exhaust all possibilities.

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $\bot$ of $\mathsf{w}{\downarrow}$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum $\bot$ of $\mathsf{w}{\downarrow}$. Not possible because there are no passive structures.

(4) The redex of π is inside the contractum $\bot$ of $\mathsf{w}{\downarrow}$. Not possible because the application of π is not trivial. (Observe that the case $\mathsf{ai}{\uparrow} \frac{S(a, \bar{a})}{\mathsf{w}{\downarrow} \frac{S\{\bot\}}{S\{?R\}}}$ is the same as $\mathsf{ai}{\uparrow} \frac{S[(a, \bar{a}), \bot]}{\mathsf{w}{\downarrow} \frac{S[\bot, \bot]}{S[\bot, ?R]}}$ and is therefore covered by case (1).)

(5) The contractum $\bot$ of $\mathsf{w}{\downarrow}$ is inside an active structure of the redex of π but not inside a passive one. Not possible. (Observe that the case $\mathsf{p}{\downarrow} \frac{S\{![U, T]\}}{\mathsf{w}{\downarrow} \frac{S[!U, ?T]}{S[![U, ?R], ?T]}}$ is covered by (2) and the case $\mathsf{p}{\downarrow} \frac{S\{![U, T]\}}{\mathsf{w}{\downarrow} \frac{S[!U, ?T]}{S[!U, ?R, ?T]}}$ is covered by (1) because $[!U, ?R, ?T] = [?R, [!U, ?T]]$.)

(6) The contractum $\bot$ of $\mathsf{w}\downarrow$ and the redex of π overlap. Not possible, because the structure $\bot$ cannot properly overlap with any other structure. □

6.5 Lemma *The rule* $\mathsf{w}\uparrow$ *permutes under the rules* $\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{p}\downarrow$ *and* $\mathsf{w}\downarrow$.

Proof: Dual to Lemma 6.4. □

6.6 Lemma *The rule* $\mathsf{ai}\downarrow$ *permutes over the rules* $\mathsf{ai}\uparrow, \mathsf{s}, \mathsf{p}\uparrow$ *and* $\mathsf{w}\uparrow$.

Proof: Consider a derivation $\pi\,\dfrac{Q}{\mathsf{ai}\downarrow\,\dfrac{S\{1\}}{S[a,\bar{a}]}}$, where $\pi \in \{\mathsf{ai}\uparrow, \mathsf{s}, \mathsf{p}\uparrow, \mathsf{w}\uparrow\}$. Without loss of generality, assume that the application of π is not trivial. Again, follow 6.3.

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum 1 of $\mathsf{ai}\downarrow$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum 1 of $\mathsf{ai}\downarrow$. Not possible because there are no passive structures.

(4) The redex of π is inside the contractum 1. Not possible because the application of π is not trivial. (Similarly as in the proof of Lemma 6.4, the case $\mathsf{w}\uparrow\,\dfrac{S\{!R\}}{\mathsf{ai}\downarrow\,\dfrac{S\{1\}}{S[a,\bar{a}]}}$ is covered by (1).)

(5) The contractum 1 of $\mathsf{ai}\downarrow$ is inside an active structure of the redex of π, but not inside a passive one. Not possible. (For instance the case $\mathsf{s}\,\dfrac{S([R,T],U)}{\mathsf{ai}\downarrow\,\dfrac{S[(R,U),T]}{S[(R,U,[a,\bar{a}]),T]}}$ is covered by (2) because $S[(R,U,[a,\bar{a}]),T] = S[(R,(U,[a,\bar{a}])),T]$.)

(6) The contractum 1 of $\mathsf{ai}\downarrow$ and the redex of π overlap. Not possible. □

6.7 Lemma *The rule* $\mathsf{ai}\uparrow$ *permutes under the rules* $\mathsf{ai}\downarrow, \mathsf{s}, \mathsf{p}\downarrow$ *and* $\mathsf{w}\downarrow$.

Proof: Dual to Lemma 6.6. □

Observe that the rule $\mathsf{w}\downarrow$ does not permute over $\mathsf{p}\uparrow$. This is easy to see from the derivation

$$\mathsf{p}\uparrow\,\dfrac{S(?U,!V)}{\mathsf{w}\downarrow\,\dfrac{S\{?[(U,V),\bot]\}}{S\{?[(U,V),?R]\}}} \quad .$$

However, with the help of the switch rule, we can get

$$\mathsf{w}\downarrow\,\dfrac{S(?U,!V)}{\mathsf{p}\uparrow\,\dfrac{S(?U,![V,?R])}{\mathsf{s}\,\dfrac{S\{?(U,[V,?R])\}}{S\{?[(U,V),?R]\}}}} \quad .$$

For the rules $\mathsf{ai}\downarrow$ and $\mathsf{p}\downarrow$ the situation is similar. Furthermore, the rule $\mathsf{ai}\downarrow$ does not permute over $\mathsf{w}\downarrow$. For example, in the derivation

$$\mathsf{ai}\downarrow \frac{\mathsf{w}\downarrow \dfrac{S\{\bot\}}{S\{?(a,b)\}}}{S\{?(a,[c,\bar{c}],b)\}} \quad ,$$

we cannot permute $\mathsf{ai}\downarrow$ up, but we could replace the whole derivation by a single application of $\mathsf{w}\downarrow$:

$$\mathsf{w}\downarrow \frac{S\{\bot\}}{S\{?(a,[c,\bar{c}],b)\}} \quad .$$

This leads to the following definition.

6.8 Definition A rule ρ *permutes over* a rule π *by* a rule σ if for every derivation $\rho \dfrac{\pi \dfrac{Q}{U}}{P}$ there is either a derivation $\pi \dfrac{\rho \dfrac{Q}{V}}{P}$ for some structure V or a derivation $\sigma \dfrac{\pi \dfrac{\rho \dfrac{Q}{V}}{V'}}{P}$ for some structures V and V' or a derivation $\pi \dfrac{Q}{P}$ or a derivation $\rho \dfrac{Q}{P}$ or a derivation $\sigma \dfrac{Q}{P}$. Dually, a rule π *permutes under* a rule ρ *by* a rule σ if for every derivation $\rho \dfrac{\pi \dfrac{Q}{U}}{P}$ there is either a derivation $\pi \dfrac{\rho \dfrac{Q}{V}}{P}$ for some structure V or a derivation $\pi \dfrac{\rho \dfrac{\sigma \dfrac{Q}{V}}{V'}}{P}$ for some structures V and V' or a derivation $\pi \dfrac{Q}{P}$ or a derivation $\rho \dfrac{Q}{P}$ or a derivation $\sigma \dfrac{Q}{P}$.

6.9 Lemma (a) *The rule* $\mathsf{w}\downarrow$ *permutes over* $\mathsf{p}\uparrow$ *and* s *by* s. (b) *The rule* $\mathsf{w}\uparrow$ *permutes under* $\mathsf{p}\downarrow$ *and* s *by* s. (c) *The rule* $\mathsf{ai}\downarrow$ *permutes over* $\mathsf{p}\downarrow$ *and* $\mathsf{w}\downarrow$ *by* s. (d) *The rule* $\mathsf{ai}\uparrow$ *permutes under* $\mathsf{p}\uparrow$ *and* $\mathsf{w}\uparrow$ *by* s.

Proof: (a) Consider a derivation $\mathsf{w}\downarrow \dfrac{\pi \dfrac{Q}{S\{\bot\}}}{S\{?R\}}$, where $\pi \in \{\mathsf{p}\uparrow, \mathsf{s}\}$ is not trivial. Then the cases (1)–(4) and (6) are as in the proof of Lemma 6.4. The only non-trivial case is:

(5) The contractum $\bot$ of $\mathsf{w}\downarrow$ is inside an active structure of the redex of π but not inside a passive one. Then there are two subcases

(i) $\pi = \mathsf{p}\uparrow$ and $S\{\bot\} = S'\{?(U, [(U', V), \bot], V')\}$. Then we have

$$\mathsf{w}\downarrow \frac{\mathsf{p}\uparrow \dfrac{S'(?(U,U'),!(V,V'))}{S'\{?(U,U',V,V')\}}}{S'\{?(U,[(U',V),?R],V')\}} \quad ,$$

which yields

$$\mathsf{s}\frac{\mathsf{p}\uparrow\dfrac{\mathsf{w}\downarrow\dfrac{S'(?(U,U'),!(V,V'))}{S'(?(U,[U',?R]),!(V,V'))}}{S'\{?(U,[U',?R],V,V')\}}}{S'\{?(U,[(U',V),?R],V')\}} \quad .$$

(ii) $\pi = \mathsf{s}$ and $S\{\bot\} = S'[(U, [(U', V), \bot], V'), T]$. Then we have

$$\mathsf{w}\downarrow \frac{\mathsf{s}\dfrac{S'([(U;U'),T],V,V')}{S'[(U,U',V,V'),T]}}{S'[(U,[(U',V),?R],V'),T]} \quad ,$$

which yields

$$\mathsf{s}\frac{\mathsf{s}\dfrac{\mathsf{w}\downarrow\dfrac{S'([(U;U'),T],V,V')}{S'([(U,[U',?R]),T],V,V')}}{S'[(U,[U',?R],V,V'),T]}}{S'[(U,[(U',V),?R],V'),T]} \quad .$$

(b) Dual to (a).

(c) Consider a derivation $\mathsf{ai}\downarrow\dfrac{\pi\dfrac{Q}{S\{1\}}}{S[a,\bar{a}]}$, where $\pi \in \{\mathsf{p}\downarrow, \mathsf{w}\downarrow\}$ is not trivial. The cases (1)–(4) and (6) are as in the proof of Lemma 6.6. The only non-trivial case is:

(5) The contractum 1 of $\mathsf{ai}\downarrow$ is inside an active structure of the redex of π, but not inside a passive one. There are three subcases.

(i) $\pi = \mathsf{p}\downarrow$ and $S\{1\} = S'[(!R, 1), ?T]$. Then

$$\mathsf{ai}\downarrow\frac{\mathsf{p}\downarrow\dfrac{S'\{![R,T]\}}{S'[(!R,1),?T]}}{S'[(!R,[a,\bar{a}]),?T]} \qquad \text{yields} \qquad \mathsf{s}\frac{\mathsf{p}\downarrow\dfrac{\mathsf{ai}\downarrow\dfrac{S'\{![R,T]\}}{S'(![R,T],[a,\bar{a}])}}{S'([!R,?T],[a,\bar{a}])}}{S'[(!R,[a,\bar{a}]),?T]} \quad .$$

(ii) $\pi = \mathsf{p}\downarrow$ and $S\{1\} = S'[!R, (?T, 1)]$. Similar to (i).

(iii) $\pi = \mathsf{w}\downarrow$ and $S\{1\} = S'\{?S''\{1\}\}$. Then

$$\mathsf{ai}\downarrow\frac{\mathsf{w}\downarrow\dfrac{S'\{\bot\}}{S'\{?S''\{1\}\}}}{S'\{?S''[a,\bar{a}]\}} \qquad \text{yields} \qquad \mathsf{w}\downarrow\frac{S'\{\bot\}}{S'\{?S''[a,\bar{a}]\}} \quad .$$

(d) Dual to (c). $\square$

This is sufficient to show that in any derivation that does not contain the rules $\mathsf{b}\downarrow$ and $\mathsf{b}\uparrow$, we can permute all instances of $\mathsf{w}\downarrow$ and $\mathsf{ai}\downarrow$ to the top of the derivation and all instances of $\mathsf{w}\uparrow$ and $\mathsf{ai}\uparrow$ to the bottom.

6.10 Proposition *For every derivation* $\begin{array}{c} T \\ \Delta \Vert \mathsf{SELS}\setminus\{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\ R \end{array}$ *there are derivations* Δ_1, Δ_2 *and* Δ_3, *such that*

$$\begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{ai}\downarrow,\mathsf{w}\downarrow\} \\ T' \\ \Delta_2 \Vert \{\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow\} \\ R' \\ \Delta_3 \Vert \{\mathsf{ai}\uparrow,\mathsf{w}\uparrow\} \\ R \end{array}$$

for some structures T' *and* R'.

6.11 Remark The statement of Proposition 6.10 can be strengthened because the derivation $\begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{ai}\downarrow,\mathsf{w}\downarrow\} \\ T' \end{array}$ can be further decomposed into

$$\begin{array}{c} T \\ \Vert \{\mathsf{ai}\downarrow\} \\ T'' \\ \Vert \{\mathsf{w}\downarrow\} \\ T' \end{array} \qquad \text{and} \qquad \begin{array}{c} T \\ \Vert \{\mathsf{w}\downarrow\} \\ T''' \\ \Vert \{\mathsf{ai}\downarrow\} \\ T' \end{array}$$

for some structures T'' and T'''. Dually, $\begin{array}{c} R' \\ \Delta_3 \Vert \{\mathsf{ai}\uparrow,\mathsf{w}\uparrow\} \\ R \end{array}$ can be decomposed into

$$\begin{array}{c} R' \\ \Vert \{\mathsf{ai}\uparrow\} \\ R'' \\ \Vert \{\mathsf{w}\uparrow\} \\ R \end{array} \qquad \text{and} \qquad \begin{array}{c} R' \\ \Vert \{\mathsf{w}\uparrow\} \\ R''' \\ \Vert \{\mathsf{ai}\uparrow\} \\ R \end{array}$$

for some structures R'' and R'''.

Observe that in the sequent calculus the identity rules are at the top of the derivation by default, and the weakening rule can also be pushed up to the top. But it is not possible to permute the cut rule downwards to the bottom of the derivation.

Proposition 6.10 is already half of the decomposition theorem. For the full decomposition theorem it is necessary to handle the rules $\mathsf{b}\downarrow$ and $\mathsf{b}\uparrow$. This is not possible with a trivial permutation argument because they neither permute over nor under any other rule.

7 Cycles in Derivations

In this section, I will provide a tool for dealing with the rules $\mathsf{b}\!\downarrow$ and $\mathsf{b}\!\uparrow$ in the decomposition theorem. The goal is to permute in any derivation all instances of $\mathsf{b}\!\uparrow$ up to the top and all instances of $\mathsf{b}\!\downarrow$ down to the bottom. If we try to permute the rule $\mathsf{b}\!\uparrow$ over the other rules in system SELS applying the schema in 6.3, we encounter (among others) the following case:

$$\mathsf{b}\!\uparrow \frac{\mathsf{p}\!\downarrow \dfrac{S\{![R,T]\}}{S[!R,?T]}}{S[(!R,R),?T]} \quad .$$

It is easy to see that there is no way to permute $\mathsf{b}\!\uparrow$ over $\mathsf{p}\!\downarrow$ in this case. But the derivation can be replaced by

$$\mathsf{b}\!\downarrow \frac{\mathsf{s} \dfrac{\mathsf{s} \dfrac{\mathsf{p}\!\downarrow \dfrac{\mathsf{b}\!\uparrow \dfrac{S\{![R,T]\}}{S(![R,T],[R,T])}}{S([!R,?T],[R,T])}}{S[([!R,?T],R),T]}}{S[(!R,R),?T,T]}}{S[(!R,R),?T]} \quad .$$

This seems to solve the problem because now the instance of $\mathsf{b}\!\uparrow$ is over the instance of $\mathsf{p}\!\downarrow$. However, there is now a new instance of $\mathsf{b}\!\downarrow$ which needs to be permuted down to the bottom of the derivation. Applying the schema in 6.3 again, we encounter the dual case:

$$\mathsf{p}\!\uparrow \frac{\mathsf{b}\!\downarrow \dfrac{S(!R,[?T,T])}{S(!R,?T)}}{S\{?(R,T)\}} \quad .$$

This has now to be replaced by

$$\mathsf{b}\!\downarrow \frac{\mathsf{p}\!\uparrow \dfrac{\mathsf{s} \dfrac{\mathsf{s} \dfrac{\mathsf{b}\!\uparrow \dfrac{S(!R,[?T,T])}{S(!R,R,[?T,T])}}{S([(!R,R),?T],T)}}{S[(!R,?T),(R,T)]}}{S[?(R,T),(R,T)]}}{S\{?(R,T)\}} \quad ,$$

which introduces a new instance of $\mathsf{b}\!\uparrow$. And so on.

The problem is to show that this cannot run forever, but must terminate eventually. In order to do so, we have to inspect the path that is taken by an instance of $\mathsf{b}\!\uparrow$ while it moves up to the top and the path taken by a $\mathsf{b}\!\downarrow$ while it moves down. This is the motivation for the definition of *!-chains* and *?-chains*. These chains can be composed to complex chains. In the next section, I will show that in the process described above, the instances of $\mathsf{b}\uparrow$ and $\mathsf{b}\downarrow$ travel upwards and downwards along such chains. Furthermore, the process will not terminate if such a chain has the form of a cycle. The purpose of this section is to show that there is no such cycle.

In Definition 3.2, I introduced the concept of a context as a structure with a hole. In this section, I also need the concept of a structure with more than one hole. An n-ary context $S\{\ \}\ldots\{\ \}$ is then a context with n holes. For example $[!\{\ \},(a,\{\ \},b)]$ is a 2-ary context.

7.1 Definition A *!-link* is any of-course structure $!R$ that occurs as substructure of any structure S inside a derivation Δ.

In general, in a given derivation Δ, most of the !-links in Δ are uninteresting for the purpose mentioned before. For that reason, I will always mark those !-links that are under discussion with a $!^{\blacktriangle}$.

7.2 Example The derivation

$$
\mathsf{p}\uparrow\frac{\mathsf{s}\frac{\mathsf{p}\downarrow\frac{(!^{\blacktriangle}[(b,!a),\bar{a}],!c)}{([!^{\blacktriangle}(b,!a),?\bar{a}],!c)}}{([!(b,!^{\blacktriangle}a),(?\bar{a},!^{\blacktriangle}c)])}}{([!(b,!a),?(\bar{a},c)])}
$$

contains many !-links, but only four of them are marked.

7.3 Definition Two !-links $!^{\blacktriangle}R$ and $!^{\blacktriangle}R'$ inside a derivation Δ are *connected* if they occur in two consecutive structures, i.e. Δ is of the shape

$$
\begin{array}{c} P \\ \| \\ \rho\,\dfrac{S'\{!^{\blacktriangle}R'\}}{S\{!^{\blacktriangle}R\}} \\ \| \\ Q \end{array}
$$

, such that one of the following cases holds (see Figure 7):

(1) The link $!^{\blacktriangle}R$ is inside the context of ρ, i.e. $R=R'$ and $S\{!^{\blacktriangle}R\}=S''\{!^{\blacktriangle}R\}\{Z\}$ and $S'\{!^{\blacktriangle}R'\}=S''\{!^{\blacktriangle}R\}\{W\}$ for some context $S''\{\ \}\{\ \}$, where Z and W are redex and contractum of ρ.

(2) The link $!^{\blacktriangle}R$ is inside a passive structure of the redex of ρ, i.e. $R=R'$ and there are contexts $S''\{\ \}$, $Z'\{\ \}$ and $W'\{\ \}$ such that $S\{!^{\blacktriangle}R\}=S''\{Z\{!^{\blacktriangle}R\}\}$ and $S'\{!^{\blacktriangle}R'\}=S''\{W\{!^{\blacktriangle}R\}\}$, where $Z\{!^{\blacktriangle}R\}$ and $W\{!^{\blacktriangle}R\}$ are redex and contractum of ρ.

(3) The redex of ρ is inside R, i.e. $S\{\ \}=S'\{\ \}$ and there is a context $R''\{\ \}$ such that $S\{!^{\blacktriangle}R\}=S\{!^{\blacktriangle}R''\{Z\}\}$ and $S'\{!^{\blacktriangle}R'\}=S\{!^{\blacktriangle}R''\{W\}\}$, where Z and W are redex and contractum of ρ.

(4) The link $!^{\blacktriangle}R$ is inside an active structure of the redex of ρ, but not inside a passive one. Then six subcases are possible:

$$
(1)\ \rho\,\frac{S''\{!^{\blacktriangle}R\}\{W\}}{S''\{!^{\blacktriangle}R\}\{Z\}} \qquad (2)\ \rho\,\frac{S''\{W\{!^{\blacktriangle}R\}\}}{S''\{Z\{!^{\blacktriangle}R\}\}} \qquad (3)\ \rho\,\frac{S\{!^{\blacktriangle}R''\{W\}\}}{S\{!^{\blacktriangle}R''\{Z\}\}}
$$

$$
(4.\mathrm{i})\ \mathsf{p}\!\downarrow\frac{S'\{!^{\blacktriangle}[R,T]\}}{S'[!^{\blacktriangle}R,?T]} \qquad (4.\mathrm{ii})\ \mathsf{b}\!\uparrow\frac{S'\{!^{\blacktriangle}R\}}{S'(!^{\blacktriangle}R,R)}
$$

$$
(4.\mathrm{iii})\ \mathsf{b}\!\uparrow\frac{S''\{!V\{!^{\blacktriangle}R\}\}}{S''(!V\{!^{\blacktriangle}R\},V\{!R\})} \qquad (4.\mathrm{iv})\ \mathsf{b}\!\uparrow\frac{S''\{!V\{!^{\blacktriangle}R\}\}}{S''(!V\{!R\},V\{!^{\blacktriangle}R\})}
$$

$$
(4.\mathrm{v})\ \mathsf{b}\!\downarrow\frac{S''[?U\{!^{\blacktriangle}R\},U\{!R\}]}{S''\{?U\{!^{\blacktriangle}R\}\}} \qquad (4.\mathrm{vi})\ \mathsf{b}\!\downarrow\frac{S''[?U\{!R\},U\{!^{\blacktriangle}R\}]}{S''\{?U\{!^{\blacktriangle}R\}\}}
$$

Figure 7: Connection of !-links

(i) $\rho = \mathsf{p}\!\downarrow$ and there is a structure T such that $S\{!^{\blacktriangle}R\} = S'[!^{\blacktriangle}R,?T]$ and $S'\{!^{\blacktriangle}R'\} = S'\{!^{\blacktriangle}[R,T]\}$, i.e. $R' = [R,T]$.

(ii) $\rho = \mathsf{b}\!\uparrow$, $R = R'$, $S\{!^{\blacktriangle}R\} = S'(!^{\blacktriangle}R,R)$ and $S'\{!^{\blacktriangle}R'\} = S'\{!^{\blacktriangle}R\}$.

(iii) $\rho = \mathsf{b}\!\uparrow$, $R = R'$ and there are contexts $S''\{\ \}$ and $V\{\ \}$ such that $S\{!^{\blacktriangle}R\} = S''(!V\{!^{\blacktriangle}R\},V\{!R\})$ and $S'\{!^{\blacktriangle}R'\} = S''\{!V\{!^{\blacktriangle}R\}\}$.

(iv) $\rho = \mathsf{b}\!\uparrow$, $R = R'$ and there are contexts $S''\{\ \}$ and $V\{\ \}$ such that $S\{!^{\blacktriangle}R\} = S''(!V\{!R\},V\{!^{\blacktriangle}R\})$ and $S'\{!^{\blacktriangle}R'\} = S''\{!V\{!^{\blacktriangle}R\}\}$.

(v) $\rho = \mathsf{b}\!\downarrow$, $R = R'$ and there are contexts $S''\{\ \}$ and $U\{\ \}$ such that $S\{!^{\blacktriangle}R\} = S''\{?U\{!^{\blacktriangle}R\}\}$ and $S'\{!^{\blacktriangle}R'\} = S''[?U\{!^{\blacktriangle}R\},U\{!R\}]$.

(vi) $\rho = \mathsf{b}\!\downarrow$, $R = R'$ and there are contexts $S''\{\ \}$ and $U\{\ \}$ such that $S\{!^{\blacktriangle}R\} = S''\{?U\{!^{\blacktriangle}R\}\}$ and $S'\{!^{\blacktriangle}R'\} = S''[?U\{!R\},U\{!^{\blacktriangle}R\}]$.

7.4 Example In the derivation shown in Example 7.2, the two !-links $!^{\blacktriangle}[(b,!a),\bar{a}]$ and $!^{\blacktriangle}(b,!a)$ are connected (by case (4.i)), whereas the !-link $!^{\blacktriangle}a$ is neither connected to $!^{\blacktriangle}(b,!a)$ nor to $!^{\blacktriangle}c$.

7.5 Definition A !-*chain* χ inside a derivation Δ is a sequence of connected !-links. The bottommost !-link of χ is called its *tail* and the topmost !-link of χ is called its *head*.

Throughout this paper, I will visualize !-chains by giving the derivation and marking all !-links of the chain by $!^{\blacktriangle}$. For example the derivation on the left in Figure 8 shows a !-chain with tail $!^{\blacktriangle}(b,?a)$ and head $!^{\blacktriangle}b$.

7.6 Definition The notion of ?-*link* is defined in the same way as the one of !-link. The notion of ?-*chain* is defined dually to !-chain, in particular, the *tail* of a ?-chain is its topmost ?-link and its *head* is its bottommost ?-link.

Similar as !-links, I will mark ?-links that are under discussion with $?^{\blacktriangledown}$.

For convenience, Figure 9 shows the possibilities how ?-links can be connected inside a ?-chain. Observe that cases (4.i) and (4.ii) are the only cases that are different from

$$
\begin{array}{ll}
 & (!^{\blacktriangle}b,!c) \\ \hline
\mathsf{b}\uparrow & (!^{\blacktriangle}b,!c,b) \\ \hline
\mathsf{ai}\downarrow & (!^{\blacktriangle}(b,[a,\bar{a}]),!c,b) \\ \hline
\mathsf{w}\downarrow & (!^{\blacktriangle}(b,[?a,a,\bar{a}]),!c,b) \\ \hline
\mathsf{s} & (!^{\blacktriangle}[(b,[?a,a]),\bar{a}],!c,b) \\ \hline
\mathsf{b}\downarrow & (!^{\blacktriangle}[(b,?a),\bar{a}],!c,b) \\ \hline
\mathsf{p}\downarrow & ([!^{\blacktriangle}(b,?a),?\bar{a}],!c,b) \\ \hline
\mathsf{s} & ([!^{\blacktriangle}(b,?a),(?\bar{a},!c)],b) \\ \hline
\mathsf{p}\uparrow & ([!^{\blacktriangle}(b,?a),?(\bar{a},c)],b)
\end{array}
\qquad
\begin{array}{ll}
 & ![!c,(?^{\blacktriangledown}a,!c),1] \\ \hline
\mathsf{p}\downarrow & [?!c,![(?^{\blacktriangledown}a,!c),1]] \\ \hline
\mathsf{w}\downarrow & [?!([?a,1],!c),![(?^{\blacktriangledown}a,!c),1]] \\ \hline
\mathsf{s} & [?![(?a,!c),1],![(?^{\blacktriangledown}a,!c),1]] \\ \hline
\mathsf{b}\downarrow & ?![(?^{\blacktriangledown}a,!c),1] \\ \hline
\mathsf{ai}\downarrow & ?![(?^{\blacktriangledown}a,!c),b,\bar{b}] \\ \hline
\mathsf{p}\downarrow & ?[![(?^{\blacktriangledown}a,!c),b],?\bar{b}] \\ \hline
\mathsf{b}\uparrow & ?[(![(?^{\blacktriangledown}a,!c),b],[(?a,!c),b]),?\bar{b}] \\ \hline
\mathsf{p}\uparrow & ?[(![?^{\blacktriangledown}(a,c),b],[(?a,!c),b]),?\bar{b}]
\end{array}
$$

Figure 8: A !-chain and a ?-chain

$$
(1)\ \rho\,\frac{S''\{?^{\blacktriangledown}T\}\{W\}}{S''\{?^{\blacktriangledown}T\}\{Z\}}
\qquad
(2)\ \rho\,\frac{S''\{W\{?^{\blacktriangledown}T\}\}}{S''\{Z\{?^{\blacktriangledown}T\}\}}
\qquad
(3)\ \rho\,\frac{S\{?^{\blacktriangledown}T''\{W\}\}}{S\{?^{\blacktriangledown}T''\{Z\}\}}
$$

$$
(4.\mathrm{i})\ \mathsf{p}\uparrow\,\frac{S'(!R,?^{\blacktriangledown}T)}{S'\{?^{\blacktriangledown}(R,T)\}}
\qquad
(4.\mathrm{ii})\ \mathsf{b}\uparrow\,\frac{S'(?^{\blacktriangledown}T,T)}{S'\{?^{\blacktriangledown}T\}}
$$

$$
(4.\mathrm{iii})\ \mathsf{b}\downarrow\,\frac{S''[?U\{?^{\blacktriangledown}T\},U\{?T\}]}{S''\{?U\{?^{\blacktriangledown}T\}\}}
\qquad
(4.\mathrm{iv})\ \mathsf{b}\downarrow\,\frac{S''[?U\{?T\},U\{?^{\blacktriangledown}T\}]}{S''\{?U\{?^{\blacktriangledown}T\}\}}
$$

$$
(4.\mathrm{v})\ \mathsf{b}\uparrow\,\frac{S''\{!V\{?^{\blacktriangledown}T\}\}}{S''(!V\{?^{\blacktriangledown}T\},V\{?T\})}
\qquad
(4.\mathrm{iv})\ \mathsf{b}\uparrow\,\frac{S''\{!V\{?^{\blacktriangledown}T\}\}}{S''(!V\{?T\},V\{?^{\blacktriangledown}T\})}
$$

Figure 9: Connection of ?-links

Figure 7. The cases (1)–(3) are exactly the same as in Figure 7 and the cases (4.iii) and (4.v) as well as the cases (4.iv) and (4.vi) are exchanged in order to maintain the duality.

The derivation on the right in Figure 8 shows an example for a ?-chain with tail $?^{\blacktriangledown}a$ and head $?^{\blacktriangledown}(a,c)$.

7.7 Definition An *upper link* is any structure of the shape $[!R,?T]$ that occurs as substructure of a structure S inside a derivation Δ. Dually, a *lower link* is any structure of the shape $(?T,!R)$ that occurs as substructure of a structure S inside a derivation Δ.

As !-links and ?-links, I will mark upper links as $[!^{\blacktriangle}R,?^{\blacktriangledown}T]$ and lower links as $(?^{\blacktriangledown}T,!^{\blacktriangle}R)$.

7.8 Definition Let Δ be a derivation. The set $X(\Delta)$ of *chains* in Δ is defined

$$
\begin{array}{rc}
 & (![a,(c,\bar{c}),b],!^{\blacktriangle}c) \\ \cline{2-2}
\mathsf{p}\!\downarrow & ([![a,(c,\bar{c})],?b],!^{\blacktriangle}c) \\ \cline{2-2}
\mathsf{ai}\!\uparrow & ([!^{\blacktriangle}a,?^{\blacktriangledown}b],!^{\blacktriangle}c) \\ \cline{2-2}
\mathsf{s} & [!^{\blacktriangle}a,(?^{\blacktriangledown}b,!^{\blacktriangle}c)] \\ \cline{2-2}
\mathsf{p}\!\uparrow & [!^{\blacktriangle}a,?(b,c)] \\ \cline{2-2}
\mathsf{ai}\!\downarrow & [!^{\blacktriangle}(a,[d,\bar{d}]),?(b,c)]
\end{array}
\qquad\qquad
\begin{array}{rc}
 & (a,![c,d]) \\ \cline{2-2}
\mathsf{p}\!\downarrow & (a,[!^{\blacktriangle}c,?^{\blacktriangledown}d]) \\ \cline{2-2}
\mathsf{s} & [(a,!^{\blacktriangle}c),?^{\blacktriangledown}d] \\ \cline{2-2}
\mathsf{w}\!\downarrow & [([a,?^{\blacktriangledown}b],!^{\blacktriangle}c),?^{\blacktriangledown}d] \\ \cline{2-2}
\mathsf{s} & [a,(?^{\blacktriangledown}b,!^{\blacktriangle}c),?^{\blacktriangledown}d] \\ \cline{2-2}
\mathsf{w}\!\uparrow & [a,?b,?^{\blacktriangledown}d] \\ \cline{2-2}
\mathsf{b}\!\downarrow & [?b,?^{\blacktriangledown}d]
\end{array}
$$

Figure 10: Two chains

inductively as follows:

(1) For every !-chain χ in Δ, we have $\chi \in X(\Delta)$.

(2) For every ?-chain χ in Δ, we have $\chi \in X(\Delta)$.

(3) If Δ contains two chains χ_1 and χ_2 and an upper link $[!^{\blacktriangle}R, ?^{\blacktriangledown}T]$ such that $!^{\blacktriangle}R$ is the head of χ_1 and $?^{\blacktriangledown}T$ is the tail of χ_2, then the concatenation of χ_1 and χ_2 forms a chain $\chi_3 \in X(\Delta)$. The tail of χ_3 is the tail of χ_1 and the head of χ_3 is the head of χ_2.

(4) If Δ contains two chains χ_1 and χ_2 and a lower link $(?^{\blacktriangledown}T, !^{\blacktriangle}R)$ such that $?^{\blacktriangledown}T$ is the head of χ_1 and $!^{\blacktriangle}R$ is the tail of χ_2, then the concatenation of χ_1 and χ_2 forms a chain $\chi_3 \in X(\Delta)$. The tail of χ_3 is the tail of χ_1 and the head of χ_3 is the head of χ_2.

(5) There are no other chains in $X(\Delta)$.

7.9 Definition The *length* of a chain χ is the number of !-chains and ?-chains it is composed of.

Figure 10 shows two examples of chains in derivations. In the first chain, the tail is $!^{\blacktriangle}(a,[d,\bar{d}])$ and the head is $!^{\blacktriangle}c$. In the second example the tail is $?^{\blacktriangledown}b$ and the head is $?^{\blacktriangledown}d$. Both have length $l = 3$.

7.10 Definition Let Δ be a derivation. A chain $\chi \in X(\Delta)$ is called a *cycle* if Δ contains an upper link $[!^{\blacktriangle}R, ?^{\blacktriangledown}T]$ such that $!^{\blacktriangle}R$ is the head of χ and $?^{\blacktriangledown}T$ is the tail of χ, or Δ contains a lower link $(?^{\blacktriangledown}T, !^{\blacktriangle}R)$ such that $?^{\blacktriangledown}T$ is the head of χ and $!^{\blacktriangle}R$ its tail.

In other words, a cycle can be seen as a chain without head or tail. Figure 11 shows an example for a cycle. Observe that for every cycle χ there is a number $n = n(\chi) \geqslant 1$ such that χ consists of n !-chains, n ?-chains, n upper links and n lower links. I will call this $n(\chi)$ the *characteristic number* of χ. For the example in Figure 11, we have $n = 2$.

$$
\begin{array}{rc}
 & (![c,d],!a,?b) \\ \cline{2-2}
\mathsf{p}{\downarrow} & ([!^\blacktriangle c,?^\blacktriangledown d],!a,?b) \\ \cline{2-2}
\mathsf{s} & ([!^\blacktriangle c,(!a,?^\blacktriangledown d)],?b) \\ \cline{2-2}
\mathsf{s} & [(!^\blacktriangle c,?b),(!a,?^\blacktriangledown d)] \\ \cline{2-2}
\mathsf{w}{\downarrow} & [?(!c,[!^\blacktriangle a,?^\blacktriangledown b],?d),(!^\blacktriangle c,?b),(!a,?^\blacktriangledown d)] \\ \cline{2-2}
\mathsf{s} & [?(!c,[(!^\blacktriangle a,?d),?^\blacktriangledown b]),(!^\blacktriangle c,?b),(!a,?^\blacktriangledown d)] \\ \cline{2-2}
\mathsf{s} & [?[(!c,?^\blacktriangledown b),(!^\blacktriangle a,?d)],(!^\blacktriangle c,?b),(!a,?^\blacktriangledown d)] \\ \cline{2-2}
\mathsf{b}{\downarrow} & ?[(!^\blacktriangle c,?^\blacktriangledown b),(!^\blacktriangle a,?^\blacktriangledown d)] \\ \cline{2-2}
\mathsf{p}{\uparrow} & ?[?(c,b),(!^\blacktriangle a,?^\blacktriangledown d)] \\ \cline{2-2}
\mathsf{w}{\uparrow} & ?[?(c,b),?d]
\end{array}
$$

Figure 11: A cycle χ with $n(\chi) = 2$

7.11 Definition A cycle χ is called a *promotion cycle* if every upper link of χ is redex of a $\mathsf{p}{\downarrow}$-rule (called *link promotion*) and every lower link of χ is contractum of a $\mathsf{p}{\uparrow}$-rule (called *link copromotion*).

The example in Figure 11 is not a promotion cycle because the upper link $[!^\blacktriangle a, ?^\blacktriangledown b]$ is not redex of a $\mathsf{p}{\downarrow}$-rule and the lower link $(!^\blacktriangle a, ?^\blacktriangledown d)$ is not contractum of a $\mathsf{p}{\uparrow}$-rule. Figure 12 shows an example for a promotion cycle. Observe that it is not necessarily the case that all upper links are above all lower links in the derivation.

7.12 Definition Let χ be a cycle inside a derivation Δ, and let all !-links and ?-links of χ be marked with $!^\blacktriangle$ or $?^\blacktriangledown$, respectively. Then, χ is called *forked* if one of the following holds:

(i) There is an instance of $\mathsf{b}{\downarrow}\,\dfrac{S[?U,U]}{S\{?U\}}$ inside Δ, such that both substructures $?U$ and U of the contractum contain at least one substructure marked by $!^\blacktriangle$ or $?^\blacktriangledown$.

(ii) There is an instance of $\mathsf{b}{\uparrow}\,\dfrac{S\{!V\}}{S(!V,V)}$ inside Δ, such that both substructures $!V$ and V of the redex contain at least one substructure marked by $!^\blacktriangle$ or $?^\blacktriangledown$.

A cycle is called *non-forked* if it is not forked.

Both examples for cycles, that I have shown, are forked cycles. In the remainder of this section, I will show that there are no non-forked cycles.

7.13 Definition If a context can be generated by the syntax

$$S ::= \{\ \} \mid [\underbrace{R,\dots,R}_{\geqslant 0}, S, \underbrace{R,\dots,R}_{\geqslant 0}] \mid (\underbrace{R,\dots,R}_{\geqslant 0}, S, \underbrace{R,\dots,R}_{\geqslant 0}) \quad ,$$

i.e. the hole does not occur inside an !- or ?-structure, then it is called a *basic context*.

$$
\begin{array}{lc}
 & !(![a,b],![c,d],![e,f]) \\ \cline{2-2}
\mathsf{p}{\downarrow} & !([!^{\blacktriangle}a,?^{\blacktriangledown}b],![c,d],![e,f]) \\ \cline{2-2}
\mathsf{p}{\downarrow} & !([!^{\blacktriangle}a,?^{\blacktriangledown}b],![c,d],[!^{\blacktriangle}e,?^{\blacktriangledown}f]) \\ \cline{2-2}
\mathsf{b}{\uparrow} & (!([!^{\blacktriangle}a,?b],![c,d],[!e,?^{\blacktriangledown}f]),[!a,?^{\blacktriangledown}b]![c,d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{w}{\uparrow} & (!([!^{\blacktriangle}a,?b],[!e,?^{\blacktriangledown}f]),[!a,?^{\blacktriangledown}b]![c,d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{s} & (![(!^{\blacktriangle}a,[!e,?^{\blacktriangledown}f]),?b],[!a,?^{\blacktriangledown}b]![c,d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{s} & (![(!^{\blacktriangle}a,?^{\blacktriangledown}f),!e,?b],[!a,?^{\blacktriangledown}b]![c,d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{p}{\uparrow} & (![?(a,f),!e,?b],[!a,?^{\blacktriangledown}b]![c,d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{w}{\uparrow} & ([!a,?^{\blacktriangledown}b]![c,d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{p}{\downarrow} & ([!a,?^{\blacktriangledown}b]\,[!^{\blacktriangle}c,?^{\blacktriangledown}d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{s} & ([(!^{\blacktriangle}c,[!a,?^{\blacktriangledown}b]),?^{\blacktriangledown}d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{s} & ([(!^{\blacktriangle}c,?^{\blacktriangledown}b),!a,?^{\blacktriangledown}d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{p}{\uparrow} & ([?(c,b),!a,?^{\blacktriangledown}d],[!^{\blacktriangle}e,?f]) \\ \cline{2-2}
\mathsf{s} & [?(c,b),!a,(?^{\blacktriangledown}d,[!^{\blacktriangle}e,?f])] \\ \cline{2-2}
\mathsf{s} & [?(c,b),!a,(!^{\blacktriangle}e,?^{\blacktriangledown}d),?f] \\ \cline{2-2}
\mathsf{p}{\uparrow} & [?(c,b),!a,?(e,d),?f]
\end{array}
$$

Figure 12: A promotion cycle χ with $n(\chi) = 3$

7.14 Example The contexts $[a,b,(\bar{a},[c,d,\bar{b},\{\ \},a],?c)]$ and $([!(b,?a),\{\ \}],b)$ are basic, whereas $([!(\{\ \},?a),?(\bar{a},c)],b)$ is not basic.

7.15 Lemma *Let $S\{\ \}$ be a basic context and R and T be any structures. Then there is a derivation*

$$
\begin{array}{c}
S[R,T] \\
\Delta \Big\| \{\mathsf{s}\} \\
[S\{R\},T]
\end{array} .
$$

Proof: By structural induction on $S\{\ \}$.

- $S = \{\ \}$. Trivial because $S[R,T] = [R,T] = [S\{R\},T]$.
- $S = [S', S''\{\ \}]$. Then by induction hypothesis we have $\begin{array}{c} [S',S''[R,T]] \\ \Delta \Big\| \{\mathsf{s}\} \\ [S',S''\{R\},T] \end{array}$.
- $S = (S', S''\{\ \})$. Then let Δ be

$$
\begin{array}{lc}
 & (S',S''[R,T]) \\
 & \Delta' \Big\| \{\mathsf{s}\} \\
 & (S',[S''\{R\},T]) \\ \cline{2-2}
\mathsf{s} & [(S',S''\{R\}),T]
\end{array} ,
$$

where Δ' exists by induction hypothesis. $\square$

7.16 Definition A cycle χ is called *pure* if

(i) for each !-chain and each ?-chain contained in χ, head and tail are equal, and

(ii) all upper links occur in the same structure and all lower links occur in the same structure.

For example, the two cycles in Figures 11 and 12 are not pure. Although in both cases condition (i) is fulfilled, condition (ii) is not. Figure 13 shows an example for a pure cycle.

If a derivation $\begin{array}{c} P \\ \Delta \Vert \mathsf{SELS} \\ Q \end{array}$ contains a pure cycle then there are structures $R_1, \ldots, R_n$ and $T_1, \ldots, T_n$ (for some $n \geqslant 1$) and two n-ary contexts $S\{\ \}\ldots\{\ \}$ and $S'\{\ \}\ldots\{\ \}$, such that Δ is of the shape

$$\begin{array}{c} P \\ \Delta_1 \Vert \mathsf{SELS} \\ S[!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_1][!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_2]\ldots[!^{\blacktriangle}R_n, ?^{\blacktriangledown}T_n] \\ \Delta_2 \Vert \mathsf{SELS} \\ S'(!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_1)(!^{\blacktriangle}R_3, ?^{\blacktriangledown}T_2)\ldots(!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_n) \\ \Delta_3 \Vert \mathsf{SELS} \\ Q \end{array} ,$$

where inside Δ_1 and Δ_3 no structures are marked with $!^{\blacktriangle}$ or $?^{\blacktriangledown}$ because the structure

$$S[!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_1][!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_2]\ldots[!^{\blacktriangle}R_n, ?^{\blacktriangledown}T_n]$$

contains all upper links and

$$S'(!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_1)(!^{\blacktriangle}R_3, ?^{\blacktriangledown}T_2)\ldots(!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_n)$$

contains all lower links of the pure circle.

7.17 Proposition *If there is a derivation* $\begin{array}{c} P \\ \Delta \Vert \mathsf{SELS} \\ Q \end{array}$ *that contains a non-forked promotion cycle, then there is a derivation* $\begin{array}{c} \tilde{P} \\ \tilde{\Delta} \Vert \{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}\} \\ \tilde{Q} \end{array}$ *that contains a pure cycle.*

Proof: Let χ be the non-forked promotion cycle inside Δ and let all !-links and ?-links of χ be marked with $!^{\blacktriangle}$ and $?^{\blacktriangledown}$, respectively (see Figure 14, first derivation). Furthermore, let all instances of a link promotion (Definition 7.10) and all instances of a link copromotion be marked as $\mathsf{p}\downarrow^{\bullet}$ and $\mathsf{p}\uparrow^{\bullet}$, respectively (see Figure 14, second derivation). Now, I will stepwise construct $\tilde{\Delta}$ from Δ by adding some more markings and by permuting, adding and removing rules, until the cycle is pure. Observe that the transformations will not destroy the cycle but might change premise and conclusion of the derivation.

$$
\begin{array}{rc}
 & [?(!c,[!a,?b],?d),(![c,d],!a,?b)] \\ \hline
\mathsf{p}\!\downarrow & [?(!c,[!^{\blacktriangle}a,?^{\blacktriangledown}b],?d),([!^{\blacktriangle}c,?^{\blacktriangledown}d],!a,?b)] \\ \hline
\mathsf{s} & [?(!c,[!^{\blacktriangle}a,?^{\blacktriangledown}b],?d),([!^{\blacktriangle}c,(!a,?^{\blacktriangledown}d)],?b)] \\ \hline
\mathsf{s} & [?(!c,[!^{\blacktriangle}a,?^{\blacktriangledown}b],?d),(!^{\blacktriangle}c,?b),(!a,?^{\blacktriangledown}d)] \\ \hline
\mathsf{s} & [?(!c,[(!^{\blacktriangle}a,?d),?^{\blacktriangledown}b]),(!^{\blacktriangle}c,?b),(!a,?^{\blacktriangledown}d)] \\ \hline
\mathsf{s} & [?[(!c,?^{\blacktriangledown}b),(!^{\blacktriangle}a,?d)],(!^{\blacktriangle}c,?b),(!a,?^{\blacktriangledown}d)] \\ \hline
\mathsf{b}\!\downarrow & ?[(!^{\blacktriangle}c,?^{\blacktriangledown}b),(!^{\blacktriangle}a,?^{\blacktriangledown}d)] \\ \hline
\mathsf{p}\!\uparrow & ?[?(c,b),(!a,?d)]
\end{array}
$$

Figure 13: A pure cycle χ with $n(\chi) = 2$

$$
\begin{array}{c}
P \\ \Vert \\
\mathsf{p}\!\downarrow \dfrac{S_1\{![R_1,T_1]\}}{S_1[!^{\blacktriangle}R_1,?^{\blacktriangledown}T_1]} \\ \Vert \\
\mathsf{p}\!\downarrow \dfrac{S_2\{![R_2,T_2]\}}{S_2[!^{\blacktriangle}R_2,?^{\blacktriangledown}T_2]} \\ \Vert \\
\mathsf{p}\!\uparrow \dfrac{S'_1(!^{\blacktriangle}R'_2,?^{\blacktriangledown}T'_1)}{S'_1\{?(R'_2,T'_1)\}} \\ \Vert \\
\mathsf{p}\!\downarrow \dfrac{S_3\{![R_3,T_3]\}}{S_3[!^{\blacktriangle}R_3,?^{\blacktriangledown}T_3]} \\ \Vert \\
\mathsf{p}\!\uparrow \dfrac{S'_3(!^{\blacktriangle}R'_1,?^{\blacktriangledown}T'_3)}{S'_3\{?(R'_1,T'_3)\}} \\ \Vert \\
\mathsf{p}\!\uparrow \dfrac{S'_2(!^{\blacktriangle}R'_3,?^{\blacktriangledown}T'_2)}{S'_2\{?(R'_3,T'_2)\}} \\ \Vert \\
Q
\end{array}
\quad \leadsto \quad
\begin{array}{c}
P \\ \Vert \\
\mathsf{p}\!\downarrow^{\bullet} \dfrac{S_1\{![R_1,T_1]\}}{S_1[!^{\blacktriangle}R_1,?^{\blacktriangledown}T_1]} \\ \Vert \\
\mathsf{p}\!\downarrow^{\bullet} \dfrac{S_2\{![R_2,T_2]\}}{S_2[!^{\blacktriangle}R_2,?^{\blacktriangledown}T_2]} \\ \Vert \\
\mathsf{p}\!\uparrow^{\bullet} \dfrac{S'_1(!^{\blacktriangle}R'_2,?^{\blacktriangledown}T'_1)}{S'_1\{?(R'_2,T'_1)\}} \\ \Vert \\
\mathsf{p}\!\downarrow^{\bullet} \dfrac{S_3\{![R_3,T_3]\}}{S_3[!^{\blacktriangle}R_3,?^{\blacktriangledown}T_3]} \\ \Vert \\
\mathsf{p}\!\uparrow^{\bullet} \dfrac{S'_3(!^{\blacktriangle}R'_1,?^{\blacktriangledown}T'_3)}{S'_3\{?(R'_1,T'_3)\}} \\ \Vert \\
\mathsf{p}\!\uparrow^{\bullet} \dfrac{S'_2(!^{\blacktriangle}R'_3,?^{\blacktriangledown}T'_2)}{S'_2\{?(R'_3,T'_2)\}} \\ \Vert \\
Q
\end{array}
\quad \leadsto \quad
\begin{array}{c}
P \\ \Vert \\
\mathsf{p}\!\downarrow^{\bullet} \dfrac{S_1\{!^{\blacktriangle}[R_1,T_1]\}}{S_1[!^{\blacktriangle}R_1,?^{\blacktriangledown}T_1]} \\ \Vert \\
\mathsf{p}\!\downarrow^{\bullet} \dfrac{S_2\{!^{\blacktriangle}[R_2,T_2]\}}{S_2[!^{\blacktriangle}R_2,?^{\blacktriangledown}T_2]} \\ \Vert \\
\mathsf{p}\!\uparrow^{\bullet} \dfrac{S'_1(!^{\blacktriangle}R'_2,?^{\blacktriangledown}T'_1)}{S'_1\{?^{\blacktriangledown}(R'_2,T'_1)\}} \\ \Vert \\
\mathsf{p}\!\downarrow^{\bullet} \dfrac{S_3\{!^{\blacktriangle}[R_3,T_3]\}}{S_3[!^{\blacktriangle}R_3,?^{\blacktriangledown}T_3]} \\ \Vert \\
\mathsf{p}\!\uparrow^{\bullet} \dfrac{S'_3(!^{\blacktriangle}R'_1,?^{\blacktriangledown}T'_3)}{S'_3\{?^{\blacktriangledown}(R'_1,T'_3)\}} \\ \Vert \\
\mathsf{p}\!\uparrow^{\bullet} \dfrac{S'_2(!^{\blacktriangle}R'_3,?^{\blacktriangledown}T'_2)}{S'_2\{?^{\blacktriangledown}(R'_3,T'_2)\}} \\ \Vert \\
Q
\end{array}
$$

Figure 14: Example (with $n(\chi) = 3$) for the marking inside Δ

I. Let n be the characteristic number of χ. For each of the n marked instances of $\mathsf{p}\!\downarrow^{\bullet} \dfrac{S\{![R_i,T_i]\}}{S[!^{\blacktriangle}R_i,?^{\blacktriangledown}T_i]}$ proceed as follows: Mark the contractum $![R_i,T_i]$ as $!^{\blacktriangle}[R_i,T_i]$

and continue the marking for all !-links of the (maximal) !-chain that has $!^{\blacktriangle}[R_i, T_i]$ as tail. There is always a unique choice how to continue the marking (see Definition 7.3), except for one case: If the marking reaches a $\mathsf{b}\!\downarrow \dfrac{S[?U,U]}{S\{?U\}}$ and the last marked $!^{\blacktriangle}$-structure is inside the redex $?U$. Then there are two possibilities: either continue inside $?U$ (case (4.v) of Definition 7.3) or continue inside U (case (4.vi) of Definition 7.3). Choose that side that already contains a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure. Since the cycle χ is non-forked, it cannot happen that both sides already contain a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure. If there is no marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure inside the contractum $[?U, U]$ of the $\mathsf{b}\!\downarrow$, then choose either one.

Proceed dually for all marked $\mathsf{p}\!\uparrow^{\bullet} \dfrac{S(!^{\blacktriangle}R'_i, ?^{\blacktriangledown}T'_i)}{S\{?(R'_i, T'_i)\}}$, i.e. mark the redex $?(R'_i, T'_i)$ as $?^{\blacktriangledown}(R'_i, T'_i)$ and mark also all links of the ?-chain that has $?^{\blacktriangledown}(R'_i, T'_i)$ as tail (see Figure 14, third derivation).

II. Now consider all !-substructures and all ?-substructures that occur somewhere in the derivation Δ. They can be divided into three groups:

(a) those which are marked with $!^{\blacktriangle}$ or $?^{\blacktriangledown}$,

(b) those which are a substructure of a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure, and

(c) all the others.

In this step replace all substructures $!R$ and $?T$ that fall in group (c) by R and T respectively, i.e. remove the exponential. This rather drastic step will, of course, yield a non-valid derivation because correct rule applications might become incorrect. Observe that all instance of $\mathsf{ai}\!\downarrow$, $\mathsf{ai}\!\uparrow$ and s inside Δ do not suffer from this step, i.e. they remain valid. Let us now inspect more closely what could happen to the instances of $\mathsf{p}\!\downarrow$, $\mathsf{p}\!\uparrow$, $\mathsf{w}\!\downarrow$, $\mathsf{w}\!\uparrow$, $\mathsf{b}\!\downarrow$ and $\mathsf{b}\!\uparrow$.

- Consider any instance of $\mathsf{p}\!\downarrow \dfrac{S\{![R,T]\}}{S[!R,?T]}$ in Δ. Then the following cases exhaust all possibilities.
  (i) There are two contexts $S'\{\ \}$ and $S''\{\ \}$ such that $S\{\ \} = S'\{!^{\blacktriangle}S''\{\ \}\}$ or $S\{\ \} = S'\{?^{\blacktriangledown}S''\{\ \}\}$. Then redex and contractum of the $\mathsf{p}\!\downarrow$ remain unchanged and the rule remains valid.
  (ii) The $\mathsf{p}\!\downarrow$ is marked as $\mathsf{p}\!\downarrow^{\bullet} \dfrac{S\{!^{\blacktriangle}[R,T]\}}{S[!^{\blacktriangle}R, ?^{\blacktriangledown}T]}$. Then it also remains unchanged.
  (iii) The $\mathsf{p}\!\downarrow$ is marked as $\mathsf{p}\!\downarrow \dfrac{S\{!^{\blacktriangle}[R,T]\}}{S[!^{\blacktriangle}R, ?T]}$. Then the exponentials inside $?T$ are removed, and we obtain an instance $\hat{\mathsf{p}}\!\downarrow \dfrac{S\{!^{\blacktriangle}[R,T]\}}{S[!^{\blacktriangle}R, T']}$. Observe that T' and T might be different because inside T all exponentials remain as they are inside $!^{\blacktriangle}[R,T]$, whereas inside T' some or all exponentials are removed.

(iv) The $\mathsf{p}{\downarrow}$ is not marked and does not occur inside a marked structure. Then it becomes $\mathsf{p}{\downarrow}' \dfrac{S[R',T']}{S[R',T']}$, where R' and T' are obtained from R and T, respectively, by removing some (or all) exponentials.

There are no other cases because there are no other markings possible. Observe that the rule $\mathsf{p}{\downarrow}'$ in case (iv) is vacuous and can therefore be removed in the whole derivation. Hence, it only remains to remove all instances of the rule $\hat{\mathsf{p}}{\downarrow}$ (case (iii)). This will be done in Step V.

- The rule $\mathsf{p}{\uparrow} \dfrac{S(!R,?T)}{S\{?(R,T)\}}$ is dual to the rule $\mathsf{p}{\downarrow}$. Hence the only problem lies in the new rule $\hat{\mathsf{p}}{\uparrow} \dfrac{S(R',?^{\blacktriangledown}T)}{S\{?^{\blacktriangledown}(R,T)\}}$, where R' is obtained from R by removing the exponentials. This rule will also be removed in Step V.

- For the rule $\mathsf{w}{\downarrow} \dfrac{S\{\bot\}}{S\{?T\}}$ only two cases are possible.

  (i) There are two contexts $S'\{\ \}$ and $S''\{\ \}$ such that $S\{\ \} = S'\{!^{\blacktriangle}S''\{\ \}\}$ or $S\{\ \} = S'\{?^{\blacktriangledown}S''\{\ \}\}$. Then redex and contractum of the $\mathsf{w}{\downarrow}$ remain unchanged and the rule remains valid.

  (ii) The rule becomes $\hat{\mathsf{w}}{\downarrow} \dfrac{S\{\bot\}}{S\{T'\}}$, where T' is obtained from T by removing some or all exponentials.

  Observe that the marking $\mathsf{w}{\downarrow} \dfrac{S\{\bot\}}{S\{?^{\blacktriangledown}T\}}$ is not possible.

- For the rule $\mathsf{w}{\uparrow} \dfrac{S\{!R\}}{S\{1\}}$ the situation is dual and we obtain $\hat{\mathsf{w}}{\uparrow} \dfrac{S\{R'\}}{S\{1\}}$, where R' is obtained from R by removing the exponentials. The two rules $\hat{\mathsf{w}}{\downarrow}$ and $\hat{\mathsf{w}}{\uparrow}$ will be removed in Step IV.

- For $\mathsf{b}{\downarrow} \dfrac{S[?T,T]}{S\{?T\}}$ the situation is more complex. The possible cases are

  (i) There are two contexts $S'\{\ \}$ and $S''\{\ \}$ such that $S\{\ \} = S'\{!^{\blacktriangle}S''\{\ \}\}$ or $S\{\ \} = S'\{?^{\blacktriangledown}S''\{\ \}\}$. Then redex and contractum of the $\mathsf{b}{\downarrow}$ remain unchanged and the rule remains valid.

  (ii) The rule is marked as $\mathsf{b}{\downarrow} \dfrac{S[?^{\blacktriangledown}T,T]}{S\{?^{\blacktriangledown}T\}}$. Then it becomes $\mathsf{b}{\downarrow}' \dfrac{S[?^{\blacktriangledown}T,T']}{S\{?^{\blacktriangledown}T\}}$, where T' is obtained from T by removing the exponentials.

  (iii) Neither redex nor contractum of the rule contain any marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure, nor are they contained in a marked structure. Then the rule becomes $\mathsf{b}{\downarrow}'' \dfrac{S[T',T']}{S\{T'\}}$, where T' is obtained form T by removing the exponentials.

(iv) There are marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structures inside the structure T in the redex. Then all those markings reoccur in one of the two substructures T in the contractum whereas the other T does not contain any marking (because the cycle χ is non-forked). Hence the rule becomes $\mathsf{b}{\downarrow}''' \dfrac{S[T'',T']}{S\{T''\}}$, where in T' all exponentials are removed and in T'' some exponentials are removed and some remain.

Observe that all instances of $\mathsf{b}{\downarrow}'$, $\mathsf{b}{\downarrow}''$ and $\mathsf{b}{\downarrow}'''$ are instances of $\hat{\mathsf{b}}{\downarrow} \dfrac{S[T,T']}{S\{T\}}$, where $S\{\ \}$ is a basic context, and T and T' are arbitrary structures.

- Dually, for $\mathsf{b}{\uparrow} \dfrac{S\{!R\}}{S(!R,R)}$, we obtain $\hat{\mathsf{b}}{\uparrow} \dfrac{S\{R\}}{S(R,R')}$, where $S\{\ \}$ is a basic context. The new instances of $\hat{\mathsf{b}}{\downarrow}$ and $\hat{\mathsf{b}}{\uparrow}$ will be removed in the next step.

Let me summarize what is achieved after this step: The original derivation $\begin{array}{c}P\\ \Delta\,\Vert\,\mathsf{SELS}\\ Q\end{array}$ has been transformed into $\begin{array}{c}P\\ \hat{\Delta}\,\Vert\,\mathsf{SELS}\cup\{\mathsf{p}{\downarrow}^{\bullet},\mathsf{p}{\uparrow}^{\bullet},\hat{\mathsf{p}}{\downarrow},\hat{\mathsf{p}}{\uparrow},\hat{\mathsf{w}}{\downarrow},\hat{\mathsf{w}}{\uparrow},\hat{\mathsf{b}}{\downarrow},\hat{\mathsf{b}}{\uparrow}\}\\ Q\end{array}$, where the cycle together with the extentions of its !- and ?-chains is marked. In the following steps, I will remove all rules (including $\hat{\mathsf{p}}{\downarrow}$, $\hat{\mathsf{p}}{\uparrow}$, $\hat{\mathsf{w}}{\downarrow}$, $\hat{\mathsf{w}}{\uparrow}$, $\hat{\mathsf{b}}{\downarrow}$, $\hat{\mathsf{b}}{\uparrow}$) that prevent the cycle from being pure.

III. First, I will remove all instances of $\hat{\mathsf{b}}{\downarrow}$ and $\hat{\mathsf{b}}{\uparrow}$. Consider the bottommost occurrence of $\hat{\mathsf{b}}{\downarrow} \dfrac{S[T,T']}{S\{T\}}$ inside $\hat{\Delta}$. Replace

$$\hat{\Delta} \quad = \quad \begin{array}{c}P\\ \Delta_1\,\Vert\\ \hat{\mathsf{b}}{\downarrow}\dfrac{S[T,T']}{S\{T\}}\\ \Delta_2\,\Vert\\ Q\end{array} \qquad \text{by} \qquad \begin{array}{c}P\\ \Delta_1\,\Vert\\ S[T,T']\\ \Delta_3\,\Vert\,\{\mathsf{s}\}\\ [S\{T\},T']\\ \Delta_2\,\Vert\\ [Q,T']\end{array} \quad ,$$

where Δ_2 does not contain any $\hat{\mathsf{b}}{\downarrow}$ and Δ_3 exists by Lemma 7.15. Repeat this until there are no more $\hat{\mathsf{b}}{\downarrow}$ in the derivation. Then proceed dually to remove all $\hat{\mathsf{b}}{\uparrow}$, i.e. start with the topmost $\hat{\mathsf{b}}{\uparrow}$. This gives us a derivation $\begin{array}{c}P'\\ \hat{\Delta}'\,\Vert\,\mathsf{SELS}\cup\{\mathsf{p}{\downarrow}^{\bullet},\mathsf{p}{\uparrow}^{\bullet},\hat{\mathsf{p}}{\downarrow},\hat{\mathsf{p}}{\uparrow},\hat{\mathsf{w}}{\downarrow},\hat{\mathsf{w}}{\uparrow}\}\\ Q'\end{array}$. Observe that premise and conclusion of the derivation have changed now, but the cycle is still present.

IV. In this step, I will remove all instances of $\hat{\mathsf{w}}{\downarrow}$ and $\hat{\mathsf{w}}{\uparrow}$. For this, observe that the proofs of Lemmata 6.4 and 6.9 (a) do also work for $\hat{\mathsf{w}}{\downarrow}$. Furthermore, observe that it can never happen that the contractum $\bot$ of $\hat{\mathsf{w}}{\downarrow} \dfrac{S\{\bot\}}{S\{T\}}$ is inside an active structure of the redex of $\mathsf{p}{\uparrow}$, $\hat{\mathsf{p}}{\uparrow}$, $\mathsf{b}{\downarrow}$, $\mathsf{b}{\uparrow}$ or $\mathsf{w}{\downarrow}$ because then the redex T would be inside a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure, which is not possible by the construction of $\hat{\mathsf{w}}{\downarrow}$ in Step II. Hence, the rule $\hat{\mathsf{w}}{\downarrow}$ permutes (by s) over all other rules in the derivation $\hat{\Delta}'$. Dually, $\hat{\mathsf{w}}{\uparrow}$ permutes under all other rules in $\hat{\Delta}'$ (by s). This means that $\hat{\Delta}'$ can easily be transformed into

$$\begin{array}{c} P' \\ \Delta'_1 \big\Vert \{\hat{\mathsf{w}}{\downarrow}\} \\ P'' \\ \hat{\Delta}'' \big\Vert \mathsf{SELS} \cup \{\mathsf{p}{\downarrow}^{\bullet}, \mathsf{p}{\uparrow}^{\bullet}, \hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}\} \\ Q'' \\ \Delta'_2 \big\Vert \{\hat{\mathsf{w}}{\uparrow}\} \\ Q' \end{array}$$

by permuting stepwise all $\hat{\mathsf{w}}{\downarrow}$ up and all $\hat{\mathsf{w}}{\uparrow}$ down. Let us now consider only $\begin{array}{c} P'' \\ \hat{\Delta}'' \big\Vert \mathsf{SELS} \cup \{\mathsf{p}{\downarrow}^{\bullet}, \mathsf{p}{\uparrow}^{\bullet}, \hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}\} \\ Q'' \end{array}$ in which the cycle χ is still untouched.

V. Inside $\hat{\Delta}''$ mark all rules ρ whose redex is inside a marked $!^{\blacktriangle}$-structure as $\rho^{\vartriangle}$. Additionally, mark all instances of $\hat{\mathsf{p}}{\downarrow}$ as $\hat{\mathsf{p}}{\downarrow}^{\vartriangle}$. Dually, mark all rules $\hat{\mathsf{p}}{\uparrow}$ as well as all rules ρ whose contractum is inside a marked $?^{\blacktriangledown}$-structure as $\rho^{\triangledown}$. Now mark all remaining, i.e. not yet marked, rules ρ as ρ°. This means, we now have a derivation $\begin{array}{c} P'' \\ \hat{\Delta}'' \big\Vert \{\mathsf{p}{\downarrow}^{\bullet}, \mathsf{p}{\uparrow}^{\bullet}, \rho^{\vartriangle}, \rho^{\triangledown}, \rho^{\circ}\} \\ Q'' \end{array}$, which will in this step be decomposed into

$$\begin{array}{c} P'' \\ \hat{\Delta}''_1 \big\Vert \{\rho^{\vartriangle}\} \\ P''' \\ \hat{\Delta}''_2 \big\Vert \{\mathsf{p}{\downarrow}^{\bullet}\} \\ \tilde{P} \\ \tilde{\Delta} \big\Vert \{\rho^{\circ}\} \\ \tilde{Q} \\ \hat{\Delta}''_3 \big\Vert \{\mathsf{p}{\uparrow}^{\bullet}\} \\ Q''' \\ \hat{\Delta}''_4 \big\Vert \{\rho^{\triangledown}\} \\ Q'' \end{array}$$

only by permutation of rules. In order to obtain this decomposition, we need to show that

(a) all rules marked as $\rho^{\vartriangle}$ permute over all other rules,

(b) all rules marked as $\rho^{\triangledown}$ permute under all other rules,

(c) all rules $\mathsf{p}{\downarrow}^{\bullet}$ permute over all rules marked as ρ° or $\mathsf{p}{\uparrow}^{\bullet}$, and

(d) all rules $\mathsf{p}{\uparrow}^{\bullet}$ permute under all rules marked as ρ° or $\mathsf{p}{\downarrow}^{\bullet}$.

I will apply the scheme of 6.3 to show the four statements.

(a) Consider $\begin{array}{c} \pi \dfrac{Q}{S\{W\}} \\ \rho^{\triangle} \dfrac{}{S\{Z\}} \end{array}$, where π is not marked as $\pi^{\triangle}$ and not trivial. Then the cases are:

(1) The redex of π is inside the context $S\{\ \}$ of $\rho^{\triangle}$. Trivial.

(2) The contractum W of $\rho^{\triangle}$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum W of $\rho^{\triangle}$. Trivial.

(4) The redex of π is inside an active structure of the contractum W of $\rho^{\triangle}$. Not possible because

(i) if the redex of $\rho^{\triangle}$ is inside a $!^{\blacktriangle}$-structure, then the contractum of $\rho^{\triangle}$ is also inside a $!^{\blacktriangle}$-structure, and hence, the redex of π is inside a $!^{\blacktriangle}$-structure, and therefore π is $\pi^{\triangle}$;

(ii) if $\rho^{\triangle} = \hat{\mathsf{p}}{\downarrow}^{\triangle}$, then the redex of π is also inside a $!^{\blacktriangle}$-structure, and therefore π is $\pi^{\triangle}$.

(5) The contractum W of $\rho^{\triangle}$ is inside an active structure of the redex of π but not inside a passive one. There are the following subcases:

(i) The redex of $\rho^{\triangle}$ is inside a $!^{\blacktriangle}$-structure. Not possible because then the contractum of $\rho^{\triangle}$ is also inside a $!^{\blacktriangle}$-structure. Since it is also inside an active structure of the redex of π, we have that either this active structure is a $!^{\blacktriangle}$-structure and therefore $\pi = \hat{\mathsf{p}}{\downarrow}^{\triangle}$, or the whole redex of π is inside a $!^{\blacktriangle}$-structure and therefore π must be marked as $\pi^{\triangle}$.

(ii) $\rho^{\triangle} = \hat{\mathsf{p}}{\downarrow}^{\triangle}$ and $\pi = \hat{\mathsf{p}}{\downarrow}$. Not possible because then π is marked as $\pi^{\triangle}$.

(iii) $\rho^{\triangle} = \hat{\mathsf{p}}{\downarrow}^{\triangle}$ and $\pi = \mathsf{p}{\downarrow}$. Then $\pi = \mathsf{p}{\downarrow}^{\bullet}$ because there are no other $\mathsf{p}{\downarrow}$ that have a marked $!^{\blacktriangle}$-structure in the redex, and we can replace

$$\begin{array}{c} \mathsf{p}{\downarrow}^{\bullet} \dfrac{S'\{!^{\blacktriangle}[R,T_1,T_2]\}}{S'[!^{\blacktriangle}[R,T_1],?^{\blacktriangledown}T_2]} \\ \hat{\mathsf{p}}{\downarrow}^{\triangle} \dfrac{}{S'[!^{\blacktriangle}R,T_1',?^{\blacktriangledown}T_2]} \end{array} \quad \text{by} \quad \begin{array}{c} \hat{\mathsf{p}}{\downarrow}^{\triangle} \dfrac{S'\{!^{\blacktriangle}[R,T_1,T_2]\}}{S'[!^{\blacktriangle}[R,T_2],T_1']} \\ \mathsf{p}{\downarrow}^{\bullet} \dfrac{}{S'[[!^{\blacktriangle}R,?^{\blacktriangledown}T_2],T_1']} \end{array} .$$

(6) The contractum W of ρ° and the redex of π overlap. Not possible.

(b) Dual to (a).

(c) Consider $\begin{array}{c} \pi \dfrac{Q}{S\{!^{\blacktriangle}[R,T]\}} \\ \mathsf{p}{\downarrow}^{\bullet} \dfrac{}{S[!^{\blacktriangle}R,?^{\blacktriangledown}T]} \end{array}$, where $\pi \in \{\rho^{\circ}, \mathsf{p}{\uparrow}^{\bullet}\}$ is not trivial.

(1) The redex of π is inside the context $S\{\ \}$ of $\mathsf{p}{\downarrow}^{\bullet}$. Trivial.

(2) The contractum of $\mathsf{p}{\downarrow}^{\bullet}$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum of $\mathsf{p}{\downarrow}^{\bullet}$. Trivial.

(4) The redex of π is inside an active structure of the contractum of $\mathsf{p}{\downarrow}^{\bullet}$. Not possible because then the redex of π is inside a $!^{\blacktriangle}$-structure, and therefore π is $\pi^{\triangle}$.

(5) The contractum $!^{\blacktriangle}[R,T]$ of $\mathsf{p}{\downarrow}^{\bullet}$ is inside an active structure of the redex of π but not inside a passive one. Not possible because then π were $\hat{\mathsf{p}}{\downarrow}^{\triangle}$ or $\mathsf{p}{\downarrow}^{\bullet}$.

(6) The contractum of $\mathsf{p}{\downarrow}^{\bullet}$ and the redex of π overlap. Not possible.

(d) Dual to (c).

Now it only remains to show that the subderivation $\begin{array}{c}\tilde{P}\\ \tilde{\Delta}\Vert\{\rho^{\circ}\}\\ \tilde{Q}\end{array}$ obtained in the last step has indeed the desired properties (i.e. contains a pure cycle and consists only of the rules $\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow}$ and s). Observe that all rules $\rho \in \{\mathsf{p}{\downarrow},\mathsf{p}{\uparrow},\mathsf{w}{\downarrow},\mathsf{w}{\uparrow},\mathsf{b}{\downarrow},\mathsf{b}{\uparrow}\}$ in Δ

- either have been transformed into $\hat{\rho}$ in Step II and then been removed in the Steps III, IV and V,
- or remained unchanged in Step II (because they occurred inside a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure) and have then been marked as $\rho^{\triangle}$ or $\rho^{\triangledown}$ and removed in Step V.

This means that only the rules $\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow}$ and s are left inside $\tilde{\Delta}$. Now consider the premise $\tilde{P}$ of $\tilde{\Delta}$. Since it is also the conclusion of $\hat{\Delta}_2''$ which consists only of $\mathsf{p}{\downarrow}^{\bullet}$, it is of the shape

$$S[!^{\blacktriangle}R_1,?^{\blacktriangledown}T_1][!^{\blacktriangle}R_2,?^{\blacktriangledown}T_2]\dots[!^{\blacktriangle}R_n,?^{\blacktriangledown}T_n]$$

for some structures $R_1,\dots,R_n,T_1,\dots,T_n$ and some n-ary context $S\{\ \}\{\ \}\dots\{\ \}$. Similarly, we have that

$$\tilde{Q} = S'(!^{\blacktriangle}R_1',?^{\blacktriangledown}T_1')(!^{\blacktriangle}R_2',?^{\blacktriangledown}T_2')\dots(!^{\blacktriangle}R_n',?^{\blacktriangledown}T_n')$$

for some structures $R_1',\dots,R_n',T_1',\dots,T_n'$ and some n-ary context $S'\{\ \}\{\ \}\dots\{\ \}$. Since no transformation in Steps II–V destroyed the cycle, it must still be present in $\tilde{\Delta}$. Since $\tilde{\Delta}$ contains no rule that operates inside a $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure, we have that $R_1' = R_2$, $R_2' = R_3$, $\dots$, $R_n' = R_1$ and $T_1' = T_1$, $T_2' = T_2$, $\dots$, $T_n' = T_n$. This means that $\tilde{\Delta}$ does indeed contain a pure cycle. □

7.18 Definition Let S be a structure and R and T be substructures of S. Then the structures R and T are in *par-relation* in S if there are contexts $S'\{\ \}$, $S''\{\ \}$ and $S'''\{\ \}$ such that $S = S'[S''\{R\},S'''\{T\}]$. Similarly, R and T are in *times-relation* in S if $S = S'(S''\{R\},S'''\{T\})$ for some contexts $S'\{\ \}$, $S''\{\ \}$ and $S'''\{\ \}$.

7.19 Lemma *If there is a derivation* $\begin{array}{c}P\\ \Delta\Vert\{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow},\mathsf{s}\}\\ Q\end{array}$ *that contains a pure cycle* χ*, then* *there is a derivation*

$$\begin{array}{c}([!R_1, ?T_1], [!R_2, ?T_2], \ldots [!R_n, ?T_n]) \\ \tilde{\Delta} \Big\Vert \{\mathsf{s}\} \\ [(!R_2, ?T_1), (!R_3, ?T_2), \ldots (!R_1, ?T_n)]\end{array}$$

for some structures $R_1, \ldots, R_n, T_1, \ldots, T_n$*, where* n *is the characteristic number of* χ*.*

Proof: By Lemma 6.6 and Lemma 6.7, the derivation Δ can be decomposed into

$$\begin{array}{c} P \\ \Delta_1 \Big\Vert \{\mathsf{ai}\!\downarrow\} \\ P' \\ \Delta_2 \Big\Vert \{\mathsf{s}\} \\ Q' \\ \Delta_3 \Big\Vert \{\mathsf{ai}\!\uparrow\} \\ Q \end{array} \quad .$$

This transformation does not destroy the cycle. Hence, the pure cycle is contained in Δ_2. In other words, Δ_2 has a subderivation

$$\begin{array}{c} S[!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_1][!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_2] \ldots [!^{\blacktriangle}R_n, ?^{\blacktriangledown}T_n] \\ \Delta' \Big\Vert \{\mathsf{s}\} \\ S'(!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_1)(!^{\blacktriangle}R_3, ?^{\blacktriangledown}T_2) \ldots (!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_n) \end{array}$$

for some structures $R_1, \ldots, R_n, T_1, \ldots, T_n$ and two n-ary contexts $S\{\ \} \ldots \{\ \}$ and $S'\{\ \} \ldots \{\ \}$. In the premise of Δ', for every $i = 1, \ldots, n$, the substructures $!^{\blacktriangle}R_i$ and $?^{\blacktriangledown}T_i$ are in par-relation. The switch rule is not able (and also no other rule in system SELS) to transform a par-relation into a times-relation while going down in a derivation. Hence, for every $i = 1, \ldots, n$, the substructures $!^{\blacktriangle}R_i$ and $?^{\blacktriangledown}T_i$ are also in par-relation in the conclusion of Δ'. This means that the context $S'\{\ \} \ldots \{\ \} = S'_0[S'_1\{\ \}, \ldots, S'_n\{\ \}]$ for some contexts $S'_0\{\ \}, S'_1\{\ \}, \ldots, S'_n\{\ \}$. Dually, we have that $S\{\ \} \ldots \{\ \} = S_0(S_1\{\ \}, \ldots, S_n\{\ \})$ for some contexts $S_0\{\ \}, S_1\{\ \}, \ldots, S_n\{\ \}$. Hence, the derivation Δ' has the shape

$$\begin{array}{c} S_0(S_1[!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_1], S_2[!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_2], \ldots, S_n[!^{\blacktriangle}R_n, ?^{\blacktriangledown}T_n]) \\ \Delta' \Big\Vert \{\mathsf{s}\} \\ S'_0[S'_1(!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_1), S'_2(!^{\blacktriangle}R_3, ?^{\blacktriangledown}T_2), \ldots, S'_n(!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_n)] \end{array} \quad .$$

Observe that the two contexts $S_0(S_1\{\ \}, \ldots, S_n\{\ \})$ and $S'_0[S'_1\{\ \}, \ldots, S'_n\{\ \}]$ must contain the same atoms because Δ' contains no rules that could create or destroy any atoms. Hence, the derivation Δ' remains valid if those atoms are removed from the derivation, which gives us the derivation

$$\begin{array}{c} ([!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_1], [!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_2], \ldots [!^{\blacktriangle}R_n, ?^{\blacktriangledown}T_n]) \\ \tilde{\Delta} \Big\Vert \{\mathsf{s}\} \\ [(!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_1), (!^{\blacktriangle}R_3, ?^{\blacktriangledown}T_2), \ldots (!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_n)] \end{array} \quad .$$

$\square$

7.20 Lemma *Let $n \geqslant 1$ and $R_1, \ldots, R_n, T_1, \ldots, T_n$ be any structures. Then there is no derivation*

$$\begin{array}{c}([!R_1, ?T_1], [!R_2, ?T_2], \ldots [!R_n, ?T_n]) \\ \Delta \Big\| \{\mathsf{s}\} \\ [(!R_2, ?T_1), (!R_3, ?T_2), \ldots (!R_1, ?T_n)]\end{array} \quad .$$

Proof: By induction on n.

- Base case: Let $n = 1$. Then it is easy to see that there is no derivation

$$\begin{array}{c}[!R_1, ?T_1] \\ \Delta \Big\| \{\mathsf{s}\} \\ (!R_1, ?T_1)\end{array}$$

  because a times-relation can never become a par-relation while going up in a derivation.

- Inductive case: By way of contradiction suppose there is a derivation

$$\begin{array}{c}([!R_1, ?T_1], [!R_2, ?T_2], \ldots [!R_n, ?T_n]) \\ \Delta \Big\| \{\mathsf{s}\} \\ [(!R_2, ?T_1), (!R_3, ?T_2), \ldots (!R_1, ?T_n)]\end{array} \quad .$$

  Now consider the bottommost instance of $\mathsf{s}\,\dfrac{S([\tilde{R}, \tilde{T}], \tilde{U})}{S[(\tilde{R}, \tilde{U}), \tilde{T}]}$ in Δ. Without loss of generality we can assume that $\tilde{R} = !R_2$ and $\tilde{U} = ?T_1$. For matching $\tilde{T}$, we have

$$\tilde{T} \quad = \quad [(!R_{k_1+1}, ?T_{k_1}), (!R_{k_2+1}, ?T_{k_2}), \ldots, (!R_{k_m+1}, ?T_{k_m})] \quad ,$$

  for some $m, k_1, \ldots, k_m$. Hence, we get:

$$\begin{array}{c}([!R_1, ?T_1], [!R_2, ?T_2], \ldots [!R_n, ?T_n]) \\ \Delta' \Big\| \{\mathsf{s}\} \\ \mathsf{s}\,\dfrac{[([!R_2, (!R_{k_1+1}, ?T_{k_1}), \ldots, (!R_{k_m+1}, ?T_{k_m})], ?T_1), \ldots, (!R_1, ?T_n)]}{[(!R_2, ?T_1), (!R_3, ?T_2), (!R_4, ?T_3), \ldots (!R_1, ?T_n)]}\end{array} \quad .$$

  Inside Δ' the structures $!R_1, \ldots, !R_n, ?T_1, \ldots, ?T_n$ occur only inside passive structures of instances of s. Therefore, if we replace inside Δ' any structure $!R_j$ or $?T_j$ by some other structure V, then the derivation Δ' must remain valid. Without loss of generality, assume that $k_1 < k_2 < \cdots < k_m$ and for every $i = 2, \ldots, k_m$ replace inside Δ' the structures $!R_i$ by $\bot$ and the structures $?T_i$ by $\mathsf{1}$, which yields a derivation

$$\begin{array}{c}([!R_1, ?T_1], [\bot, \mathsf{1}], \ldots, [\bot, \mathsf{1}], [!R_{k_m+1}, ?T_{k_m+1}], \ldots, [!R_n, ?T_n]) \\ \Delta'' \Big\| \{\mathsf{s}\} \\ [([\bot, (\bot, \mathsf{1}), \ldots, (\bot, \mathsf{1}), (!R_{k_m+1}, \mathsf{1})], ?T_1), \ldots, (!R_1, ?T_n)]\end{array} \quad ,$$

which is the same as

$$
\begin{array}{c}
([!R_1, ?T_1], [!R_{k_m+1}, ?T_{k_m+1}], \dots [!R_n, ?T_n]) \\
\Delta'' \Big\Vert \{\mathsf{s}\} \\
[(!R_{k_m+1}, ?T_1), (!R_{k_m+2}, ?T_{k_m+1}), \dots, (!R_1, ?T_n)]
\end{array} ,
$$

which cannot exist by induction hypothesis. □

7.21 Theorem *There exists no derivation containing a non-forked promotion cycle.*

Proof: Suppose there is a derivation Δ containing a non-forked promotion cycle χ. By Proposition 7.17, Lemma 7.19 and Lemma 7.20, this is impossible. □

7.22 Corollary *There exists no derivation containing a non-forked cycle.*

Proof: Any (non-forked) cycle can easily be transformed into a (non-forked) promotion cycle by adding instances of $\mathsf{p}\downarrow$ and $\mathsf{p}\uparrow$. □

8 Decomposition of Derivations

In this section, I will use the results of the previous two sections in order to state and prove a decomposition theorem for derivations in system SELS. The theorem states that any derivation $\begin{array}{c} T \\ \Vert \\ R \end{array}$ can be decomposed into five parts:

$$
\begin{array}{l}
T \\
\Vert \textit{non-core} \\
T' \\
\Vert \textit{interaction} \\
T'' \\
\Vert \textit{core} \\
R'' \\
\Vert \textit{interaction} \\
R' \\
\Vert \textit{non-core} \\
R
\end{array} ,
$$

where *interaction* stands for the rules $\mathsf{ai}\downarrow$ and $\mathsf{ai}\uparrow$. The *core* contains that part of the system that is needed to reduce the general interaction rules $\mathsf{i}\downarrow$ (identity) and $\mathsf{i}\uparrow$ (cut) to their atomic versions. In system SELS, the core contains the rules s, $\mathsf{p}\downarrow$ and $\mathsf{p}\uparrow$ (see Definition 4.7). The remaining rules $\mathsf{w}\downarrow$, $\mathsf{w}\uparrow$, $\mathsf{b}\downarrow$ and $\mathsf{b}\uparrow$ are in the *non-core* part.

This decomposition is not restricted to system SELS. The same theorem has also been proved for other systems in the calculus of structures, namely for system SBV in [14] and for a conservative extension of SELS and SBV in [17]. In [7] it is conjectured for a system for classical logic.

It seems to me that there is a very strong connection between this decomposition and cut elimination, in the sense that both are consequences of the same underlying top-down reading

$$\begin{array}{c} T \\ \Big\Downarrow \mathit{creation} \\ T' \\ \Big\Downarrow \mathit{merging} \\ R' \\ \Big\Downarrow \mathit{destruction} \\ R \end{array}$$

bottom-up reading

$$\begin{array}{c} T \\ \Big\Uparrow \mathit{destruction} \\ T' \\ \Big\Uparrow \mathit{merging} \\ R' \\ \Big\Uparrow \mathit{creation} \\ R \end{array}$$

Figure 15: Reading of the decomposition theorem

properties of the logical system. However, the exact nature of these properties is still a mystery.

There is a second reading of the decomposition theorem, namely the decomposition of a derivation into three parts, as already mentioned in the introduction. The three parts can be called *creation*, *merging* and *destruction*. The merging part is in the middle of the derivation, and (depending on your preferred reading of a derivation) the creation and destruction are at the top and at the bottom, as depicted in Figure 15. In system SELS the merging part contains only the rules s, $\mathsf{p}{\downarrow}$ and $\mathsf{p}{\uparrow}$, which do not change the size of the structure. In the top-down reading of a derivation, the creation part (where the size of the structure is increased) contains the rules $\mathsf{b}{\uparrow}$, $\mathsf{w}{\downarrow}$ and $\mathsf{ai}{\downarrow}$, and the destruction part (where the size of the structure decreases) consists of $\mathsf{b}{\downarrow}$, $\mathsf{w}{\uparrow}$ and $\mathsf{ai}{\uparrow}$. In the bottom-up reading, creation and destruction are exchanged.

This decomposition is endorsed by the fact that in the calculus of structures derivations are symmetric objects in the vertical perspective. The statement of the theorem is as follows:

8.1 Theorem *For every derivation* $\begin{array}{c} T \\ \Delta \Vert \mathsf{SELS} \\ R \end{array}$ *there are derivation* $\Delta_1, \ldots, \Delta_7$*, such* *that*

$$\begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}\!\uparrow\} \\ T_1 \\ \Delta_2 \Vert \{\mathsf{w}\!\downarrow\} \\ T_2 \\ \Delta_3 \Vert \{\mathsf{ai}\!\downarrow\} \\ T_3 \\ \Delta_4 \Vert \{\mathsf{s},\mathsf{p}\!\downarrow,\mathsf{p}\!\uparrow\} \\ R_3 \\ \Delta_5 \Vert \{\mathsf{ai}\!\uparrow\} \\ R_2 \\ \Delta_6 \Vert \{\mathsf{w}\!\uparrow\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}\!\downarrow\} \\ R \end{array}$$

for some structures T_1, T_2, T_3, R_1, R_2 *and* R_3.

Proof: The decomposition is done in three steps:

$$\begin{array}{c} T \\ \Delta \Vert \mathsf{SELS} \\ R \end{array} \overset{1}{\rightsquigarrow} \begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}\!\uparrow\} \\ T_1 \\ \Delta' \Vert \{\mathsf{ai}\!\downarrow,\mathsf{ai}\!\uparrow,\mathsf{s},\mathsf{p}\!\downarrow,\mathsf{p}\!\uparrow,\mathsf{w}\!\downarrow,\mathsf{w}\!\uparrow\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}\!\downarrow\} \\ R \end{array} \overset{2}{\rightsquigarrow} \begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}\!\uparrow\} \\ T_1 \\ \Delta_2 \Vert \{\mathsf{w}\!\downarrow\} \\ T_2 \\ \Delta'' \Vert \{\mathsf{ai}\!\downarrow,\mathsf{ai}\!\uparrow,\mathsf{s},\mathsf{p}\!\downarrow,\mathsf{p}\!\uparrow\} \\ R_2 \\ \Delta_6 \Vert \{\mathsf{w}\!\uparrow\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}\!\downarrow\} \\ R \end{array} \overset{3}{\rightsquigarrow} \begin{array}{c} T \\ \Delta_1 \Vert \{\mathsf{b}\!\uparrow\} \\ T_1 \\ \Delta_2 \Vert \{\mathsf{w}\!\downarrow\} \\ T_2 \\ \Delta_3 \Vert \{\mathsf{ai}\!\downarrow\} \\ T_3 \\ \Delta_4 \Vert \{\mathsf{s},\mathsf{p}\!\downarrow,\mathsf{p}\!\uparrow\} \\ R_3 \\ \Delta_5 \Vert \{\mathsf{ai}\!\uparrow\} \\ R_2 \\ \Delta_6 \Vert \{\mathsf{w}\!\uparrow\} \\ R_1 \\ \Delta_7 \Vert \{\mathsf{b}\!\downarrow\} \\ R \end{array} \quad .$$

The first step is done by Proposition 8.8, whose proof is postponed until the end of this section. For the second step, we can repeatedly apply Lemma 6.4, Lemma 6.5 and Lemma 6.9 (a) and (b). For the last step use Lemma 6.6, Lemma 6.7 and Lemma 6.9 (c) and (d). □

Before I complete the proof (i.e. state and prove Proposition 8.8), let me make one more remark about the theorem.

8.2 Remark In the formulation of Theorem 8.1 it is enforced that the instances of $\mathsf{b}\!\uparrow$ and $\mathsf{b}\!\downarrow$ are at the top and the bottom of the derivation. But it is possible to exchange the positions of $\mathsf{w}\!\downarrow$ and $\mathsf{ai}\!\downarrow$ in the derivation (see Section 6). By duality, the same is true for $\mathsf{w}\!\uparrow$ and $\mathsf{ai}\!\uparrow$. It should also be mentioned that it is not possible to further decompose the core-part of the derivation. The rules s, $\mathsf{p}\!\downarrow$ and $\mathsf{p}\!\uparrow$ are entangled in such a way that they cannot be separated.

The remainder of this section is devoted to the proof of Proposition 8.8, which states

I. If there are no subderivations of the shape $\begin{array}{c} \pi\,\dfrac{Q}{U} \\ \mathsf{b}{\uparrow}\,\dfrac{}{P} \end{array}$, where $\pi \neq \mathsf{b}{\uparrow}$, or of the shape $\begin{array}{c} \mathsf{b}{\downarrow}\,\dfrac{Q}{V} \\ \rho\,\dfrac{}{P} \end{array}$, where $\rho \neq \mathsf{b}{\downarrow}$, then terminate.

II. Permute all instances of $\mathsf{b}{\uparrow}$ up by applying $\mathsf{b}{\uparrow}\mathsf{up}$ (shown in Figure 18).

III. Permute all instances of $\mathsf{b}{\downarrow}$ down by applying $\mathsf{b}{\downarrow}\mathsf{down}$ (shown in Figure 19).

IV. Go to step I.

Figure 16: The algorithm $\mathsf{b}{\uparrow}{\downarrow}\mathsf{split}$ for separating $\mathsf{b}{\uparrow}$ and $\mathsf{b}{\downarrow}$

$$
\begin{array}{c} T \\ \Big\Vert\mathsf{SELS} \\ R \end{array}
\overset{\mathsf{b}\uparrow}{\underset{\mathsf{up}}{\leadsto}}
\begin{array}{c} T \\ \Vert\{\mathsf{b}\uparrow\} \\ T_1 \\ \Vert\mathsf{SELS}\setminus\{\mathsf{b}\uparrow\} \\ R \end{array}
\overset{\mathsf{b}\downarrow}{\underset{\mathsf{down}}{\leadsto}}
\begin{array}{c} T \\ \Vert\{\mathsf{b}\uparrow\} \\ T_1 \\ \Vert\mathsf{SELS}\setminus\{\mathsf{b}\downarrow\} \\ R_1 \\ \Vert\{\mathsf{b}\downarrow\} \\ R \end{array}
\overset{\mathsf{b}\uparrow}{\underset{\mathsf{up}}{\leadsto}}
\begin{array}{c} T \\ \Vert\{\mathsf{b}\uparrow\} \\ T_2 \\ \Vert\mathsf{SELS}\setminus\{\mathsf{b}\uparrow\} \\ R_1 \\ \Vert\{\mathsf{b}\downarrow\} \\ R \end{array}
\overset{\mathsf{b}\downarrow}{\underset{\mathsf{down}}{\leadsto}}
\begin{array}{c} T \\ \Vert\{\mathsf{b}\uparrow\} \\ T_2 \\ \Vert\mathsf{SELS}\setminus\{\mathsf{b}\downarrow\} \\ R_2 \\ \Vert\{\mathsf{b}\downarrow\} \\ R \end{array}
\overset{\mathsf{b}\uparrow}{\underset{\mathsf{up}}{\leadsto}} \ldots \leadsto
\begin{array}{c} T \\ \Vert\{\mathsf{b}\uparrow\} \\ T_k \\ \Vert\mathsf{SELS}\setminus\{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\ R_h \\ \Vert\{\mathsf{b}\downarrow\} \\ R \end{array}
$$

Figure 17: Permuting $\mathsf{b}{\uparrow}$ up and $\mathsf{b}{\downarrow}$ down

that for every derivation $\begin{array}{c} T \\ \Delta\Vert\mathsf{SELS} \\ R \end{array}$ there is a derivation $\begin{array}{c} T \\ \Delta_1\Vert\{\mathsf{b}\uparrow\} \\ T' \\ \Delta'\Vert\mathsf{SELS}\setminus\{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\ R' \\ \Delta_2\Vert\{\mathsf{b}\downarrow\} \\ R \end{array}$. The idea of the proof is to permute all instances of $\mathsf{b}{\uparrow}$ up and all instances of $\mathsf{b}{\downarrow}$ down, according to the scheme of 6.3. The problem that occurs is that while permuting the rule $\mathsf{b}{\uparrow}$ over the rule $\mathsf{p}{\downarrow}$, it can happen that new instances of $\mathsf{b}{\downarrow}$ are introduced. Dually, while permuting $\mathsf{b}{\downarrow}$ under $\mathsf{p}{\uparrow}$, new instances of $\mathsf{b}{\uparrow}$ are introduced. The algorithm $\mathsf{b}{\uparrow}{\downarrow}\mathsf{split}$, shown in Figure 16, is used to obtain the desired decomposition. Figure 17 shows its working principle.

The task of the proof is now to show that the process of permuting $\mathsf{b}{\uparrow}$ up and $\mathsf{b}{\downarrow}$ down does terminate. For this, the non-existence of a non-forked promotion cycle plays a crucial role.

The process of permuting up all instances of $\mathsf{b}{\uparrow}$ in a given derivation $\begin{array}{c} T \\ \Delta\Vert\mathsf{SELS} \\ R \end{array}$ is realized by the procedure $\mathsf{b}{\uparrow}\mathsf{up}$ shown in Figure 18. It is easy to see that if this terminates, the resulting derivation has the shape $\begin{array}{c} T \\ \Vert \{\mathsf{b}\!\uparrow\} \\ T' \\ \Vert \mathsf{SELS}\setminus\{\mathsf{b}\!\uparrow\} \\ R \end{array}$. However, it is not obvious that this procedure does indeed terminate, because while permuting the rule $\mathsf{b}\!\uparrow$ up, it might happen that new instances of $\mathsf{b}\!\uparrow$ as well as new instances of $\mathsf{b}\!\downarrow$ are introduced.

8.3 Lemma *For an input derivation*

$$\begin{array}{rc} & T \\ & \Vert \mathsf{SELS}\setminus\{\mathsf{b}\!\uparrow\} \\ \mathsf{b}\!\uparrow & \dfrac{S\{!R\}}{S(!R,R)} \end{array} \quad ,$$

the $\mathsf{b}\!\uparrow\mathsf{up}$ *procedure does terminate.*

Proof: The problem of showing termination is that the number of instances of $\mathsf{b}\!\uparrow$ might increase during the process of permuting up $\mathsf{b}\!\uparrow$. This always happens when an upmoving $\mathsf{b}\!\uparrow$ meets a $\mathsf{b}\!\downarrow$ as in case (5.iii) in Figure 18. Furthermore, the number of instances of $\mathsf{b}\!\downarrow$ inside Δ is not fixed. The number of $\mathsf{b}\!\downarrow$ might increase when an upmoving $\mathsf{b}\!\uparrow$ meets an instance of $\mathsf{p}\!\downarrow$ as in case (5.i) in Figure 18 or an instance of $\mathsf{b}\!\downarrow$ as in case (4). For showing the termination, I will now mark inside Δ all !-chains that have the contractum $!R$ of the $\mathsf{b}\!\uparrow$ instance as tail. But I will mark the links not with $!^{\blacktriangle}$, but with $!^n$ for some $n \geqslant 1$. Start with the contractum $!R$ of the $\mathsf{b}\!\uparrow$ by marking it with $!^1R$. Now continue the marking as indicated in Figure 7 by propagating the number n from conclusion to premise in each rule, with one exception: If in case (2) of Definition 7.3 the rule $\rho = \mathsf{p}\!\downarrow$ and the situation is

$$\mathsf{p}\!\downarrow \frac{S'\{!U, V\{!R'\}\}}{S'[!U, ?V\{!^nR'\}]} \quad ,$$

then continue the marking as follows

$$\mathsf{p}\!\downarrow \frac{S'\{!U, V\{!^{2n}R'\}\}}{S'[!U, ?V\{!^nR'\}]} \quad ,$$

where the marking number is duplicated. For example, the derivation

$$\begin{array}{rc}
 & ![b,(\bar{b},![([!a,?b],[c,\bar{c}]),a])] \\ \hline
\mathsf{p}\!\downarrow & ![b,(\bar{b},[?([!a,?b],[c,\bar{c}]),!a])] \\ \hline
\mathsf{ai}\!\downarrow & ![b,(\bar{b},[?([!a,?b],[c,\bar{c}]),(!a,[c,\bar{c}])])] \\ \hline
\mathsf{w}\!\downarrow & ![b,(\bar{b},[?([!a,?b],[c,\bar{c}]),([!a,?b],[c,\bar{c}])])] \\ \hline
\mathsf{p}\!\downarrow & [!b,?(\bar{b},[?([!a,?b],[c,\bar{c}]),([!a,?b],[c,\bar{c}])])] \\ \hline
\mathsf{b}\!\downarrow & [!b,?(\bar{b},?([!a,?b],[c,\bar{c}]))] \\ \hline
\mathsf{s} & [!b,?(\bar{b},?[!a,(?b,[c,\bar{c}])])] \\ \hline
\mathsf{b}\!\uparrow & [!b,?(\bar{b},?[(!a,a),(?b,[c,\bar{c}])])]
\end{array}$$

is marked as

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{![b,(\bar{b},!^2[([!^4a,?b],[c,\bar{c}]),a])]}{![b,(\bar{b},[?([!^2a,?b],[c,\bar{c}]),!^2a])]}}{![b,(\bar{b},[?([!^2a,?b],[c,\bar{c}]),(!^2a,[c,\bar{c}])])]}}{![b,(\bar{b},[?([!^2a,?b],[c,\bar{c}]),([!^2a,?b],[c,\bar{c}])])]}}{[!b,?(\bar{b},[?([!^1a,?b],[c,\bar{c}]),([!^1a,?b],[c,\bar{c}])])]}}{[!b,?(\bar{b},?([!^1a,?b],[c,\bar{c}]))]}}{[!b,?(\bar{b},?[!^1a,(?b,[c,\bar{c}])])]}}{[!b,?(\bar{b},?[(!a,a),(?b,[c,\bar{c}])])]}\quad.$$

Observe that is might happen that one marking is inside another. But this can only happen if one marking "is pulled inside" another by an instance of $\mathsf{p}{\downarrow}$. In this case the marking that is pulled inside is duplicated. As a consequence we have that whenever there is a marked structure $!^nR$ which has other markings inside, then those markings are even. For notational convenience, let in the following R_* denote the structure R where all markings inside R are divided by two and let $R_\oplus$ denote the structure R where all markings inside R are multiplied by two. During the run of $\mathsf{b}{\uparrow}\mathsf{up}$, the markings are now removed as follows:

- In cases (1) and (2) replace

$$\mathsf{b}{\uparrow}\,\dfrac{\pi\,\dfrac{S'\{!^nR\}}{S\{!^nR\}}}{S(!R_*,R_*)}\qquad\text{by}\qquad \pi\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'\{!^nR\}}{S'(!R_*,R_*)}}{S(!R_*,R_*)}\quad.$$

- In case (4) replace

$$\mathsf{b}{\uparrow}\,\dfrac{\pi\,\dfrac{S\{!^nR'\}}{S\{!^nR\}}}{S(!R_*,R_*)}\qquad\text{by}\qquad \pi\,\dfrac{\pi\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S\{!^nR'\}}{S(!R'_*,R'_*)}}{S(!R'_*,R_*)}}{S(!R_*,R_*)}\quad.$$

- In case (5.i) replace

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S'\{!^n[R,T_\oplus]\}}{S'[!^nR,?T]}}{S'[(!R_*,R_*),?T]}\qquad\text{by}\qquad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'\{!^n[R,T_\oplus]\}}{S'(![R_*,T],[R_*,T])}}{S'([!R_*,?T],[R_*,T])}}{S'[([!R_*,?T],R_*),T]}}{S'[(!R_*,R_*),?T,T]}}{S'[(!R_*,R_*),?T]}\quad.$$

- In case (5.ii) replace

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{S'\{\bot\}}{S'\{?S''\{!^nR\}\}}}{S'\{?S''(!R_*,R_*)\}}\qquad\text{by}\qquad \mathsf{w}{\downarrow}\,\dfrac{S'\{\bot\}}{S'\{?S''(!R_*,R_*)\}}\quad.$$

- In case (5.iii) replace

$$
\mathsf{b}{\downarrow}\,\frac{S'[?S''\{!^n R\}, S''\{!^n R\}]}{\mathsf{b}{\uparrow}\,\dfrac{S'\{?S''\{!^n R\}\}}{S'\{?S''(!R_*, R_*)\}}}
\qquad \text{by} \qquad
\mathsf{b}{\uparrow}\,\frac{S'[?S''\{!^n R\}, S''\{!^n R\}]}{\mathsf{b}{\uparrow}\,\dfrac{S'[?S''(!R_*, R_*), S''\{!^n R\}]}{\mathsf{b}{\downarrow}\,\dfrac{S'[?S''(!R_*, R_*), S''(!R_*, R_*)]}{S'\{?S''(!R_*, R_*)\}}}}
\quad .
$$

All instances of $\mathsf{b}{\uparrow}$ travel up along a !-chain that has been marked in the beginning. Since in the beginning there was only one instance of $\mathsf{b}{\uparrow}$, each marked !-chain can be used only once (and is used exactly once) by an instance of $\mathsf{b}{\uparrow}$, an then the marking is removed. But it might happen that new markings are introduced during the process because the length of the derivation can increase. For a given structure S let $\sigma(S)$ denote the sum of the markings inside S. (For example for $S = ![b, (\bar{b}, !^2[([!^4 a, ?b], [c, \bar{c}]), a])]$ we have $\sigma(S) = 6$.) Then for any two structures S and S' occurring in Δ, such that S' occurs above S, we have $\sigma(S') \geqslant \sigma(S)$. Furthermore, during the process of permuting up $\mathsf{b}{\uparrow}$, the value $\sigma(S)$ never increases for a structure S occurring in Δ. When a new structure S' is inserted (in cases (4), (5.i) and (5.iii)), we have $\sigma(S') = \sigma(S)$ for some structure S occurring below S'. With this observation we can show termination by assigning to Δ a pair $\langle n_\Delta, m_\Delta \rangle \in \mathbf{N} \times \mathbf{N}$, where $\mathbf{N} \times \mathbf{N}$ is endowed with the lexicographic ordering

$$
\langle n, m \rangle < \langle n', m' \rangle \quad \Longleftrightarrow \quad \begin{array}{ll} n < n' & \text{or} \\ n = n' & \text{and} \quad m < m' \quad . \end{array}
$$

and the values of n_Δ and m_Δ are defined as follows: During the process of permuting up $\mathsf{b}{\uparrow}$, the derivation has always the shape

$$
\begin{array}{c}
T \\
\Delta_1 \Big\Vert \{\mathsf{b}{\uparrow}\} \\
T' \\
\Delta_2 \Big\Vert \mathsf{SELS} \setminus \{\mathsf{b}{\uparrow}\} \\
\mathsf{b}{\uparrow}\,\dfrac{S\{!^n R\}}{S(!R, R)} \\
\Delta_3 \Big\Vert \mathsf{SELS} \\
U
\end{array}
\quad ,
$$

where Δ_1 contains the instances of $\mathsf{b}{\uparrow}$ that already have reached the top and Δ_2 is not trivial and the instance of $\mathsf{b}{\uparrow}$ between Δ_2 and Δ_3 is the topmost to be permuted up. Now let $n_\Delta = \sigma(T')$ and m_Δ is the length of Δ_2 (i.e. the number of rule instances in Δ_2). Then we have that $\langle n_\Delta, m_\Delta \rangle$ strictly decreases in each permutation step and we have $\langle n_\Delta, m_\Delta \rangle = \langle 0, 0 \rangle$ when all instances of $\mathsf{b}{\uparrow}$ have reached the top. □

8.4 Lemma *The* $\mathsf{b}{\uparrow}\mathsf{up}$ *algorithm terminates for any input derivation* $\begin{array}{c} T \\ \Delta \Big\Vert \mathsf{SELS} \\ R \end{array}$.

Proof: Apply Lemma 8.3 to every instance of $\mathsf{b}{\uparrow}$ in Δ. □

The dual procedure to $\mathsf{b}{\uparrow}\mathsf{up}$ is $\mathsf{b}{\downarrow}\mathsf{down}$, shown in Figure 19, in which all instances of $\mathsf{b}{\downarrow}$ are moved down in the derivation.

8.5 Lemma *The* $\mathsf{b}{\downarrow}\mathsf{down}$ *procedure terminates for every input derivation* $\begin{array}{c} T \\ \Delta \Big\Vert \mathsf{SELS} \\ R \end{array}$ *and yields a derivation* $\begin{array}{c} T \\ \Delta' \Big\Vert \mathsf{SELS} \setminus \{\mathsf{b}{\downarrow}\} \\ R' \\ \Delta'' \Big\Vert \{\mathsf{b}{\downarrow}\} \\ R \end{array}$.

Proof: Dual to Lemma 8.4. □

Lemma 8.4 and Lemma 8.5 ensure that each step of the procedure depicted in Figure 17 does terminate. It remains to show that the whole algorithm $\mathsf{b}{\uparrow}{\downarrow}\mathsf{split}$ does terminate eventually.

8.6 Lemma *Let* Δ *be a derivation that does not contain a promotion cycle. Then the algorithm* $\mathsf{b}{\uparrow}{\downarrow}\mathsf{split}$ *does terminate for* Δ*.*

Proof: Without loss of generality, let Δ be the outcome of a run of $\mathsf{b}{\downarrow}\mathsf{down}$, i.e. there are no instances of $\mathsf{b}{\downarrow}$ to consider. Since Δ is finite, it contains only finitely many instances of $\mathsf{b}{\uparrow}$. Hence, there are only finitely many chains, say $\chi_1, \ldots, \chi_n$, that have the contractum $!R$ of a $\mathsf{b}{\uparrow}$ as tail. Mark all those chains with $!^{\blacktriangle}$ and $?^{\blacktriangledown}$ as in the previous section, and let $l_i = l(\chi_i)$ be the length of χ_i (see Definition 7.9) for each $i = 1, \ldots, n$. Now run $\mathsf{b}{\uparrow}\mathsf{up}$ and remove the markings $!^{\blacktriangle}$ as in the proof of Lemma 8.3 while the instances of $\mathsf{b}{\uparrow}$ are permuted up. In case (5.i) replace

$$\mathsf{b}{\uparrow}\frac{\mathsf{p}{\downarrow}\dfrac{S'\{!^{\blacktriangle}[R,T]\}}{S'[!^{\blacktriangle}R,?^{\blacktriangledown}T]}}{S'[(!R,R),?^{\blacktriangledown}T]} \quad \text{by} \quad \mathsf{b}{\downarrow}\frac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{\mathsf{p}{\downarrow}\dfrac{\mathsf{b}{\uparrow}\dfrac{S'\{!^{\blacktriangle}[R,T]\}}{S'(![R,T],[R,T])}}{S'([!R,?T],[R,T])}}{S'[([!R,?T],R),T]}}{S'[(!R,R),?T,T]}}{S'[(!R,R),?^{\blacktriangledown}T]} \quad .$$

After this, all chains χ_i with length $l_i = 1$ are no longer marked. If $l(\chi_i) \geqslant 1$, then after the run of $\mathsf{b}{\uparrow}\mathsf{up}$ only a subchain χ'_i of χ_i with length $l'_i = l(\chi'_i) = l(\chi_i) - 1$ remains marked because it is not possible to add links to a chain at the head. (Each of the chains $\chi_1, \ldots, \chi_n$ has a head since there is no promotion cycle inside Δ.) It is only possible to duplicate chains (case (4) of in Figure 18). The situation for $\mathsf{b}{\downarrow}\mathsf{down}$ is dual. Hence the number $l_{max} = \max\{l(\chi_i)\}$ is reduced at each run of $\mathsf{b}{\uparrow}\mathsf{up}$ and $\mathsf{b}{\downarrow}\mathsf{down}$. This ensures the termination. □

8.7 Lemma *Let* Δ *be a derivation that is obtained by a consecutive run of* $\mathsf{b}{\uparrow}\mathsf{up}$ *and* $\mathsf{b}{\downarrow}\mathsf{down}$*. Then* Δ *does not contain a promotion cycle.*

Proof: By way of contradiction, assume Δ contains a promotion cycle χ. Since Δ is the outcome of a run of $\mathsf{b}{\downarrow}\mathsf{down}$, all instances of $\mathsf{b}{\downarrow}$ are at the bottom of Δ. Hence, the cycle χ can only be forked by instances of $\mathsf{b}{\uparrow}$, more precisely, χ is forked by k_χ different instances of $\mathsf{b}{\uparrow}$. If Δ contains more than one promotion cycle, we can, without loss of generality, assume that χ is the one for which k_χ is minimal. I will now proceed by induction on k_χ to show a contradiction.

Base case: If $k_\chi = 0$, then we have an immediate contradiction to Theorem 7.21.

Inductive case: Now let $k_\chi \geqslant 1$ and consider the bottommost instance of $\mathsf{b}\!\uparrow$ that forkes χ and mark it as $\mathsf{b}\!\uparrow^{\blacktriangle}$. It has been introduced during the run of $\mathsf{b}\!\downarrow\!\mathsf{down}$, when

$$\mathsf{p}\!\uparrow \frac{\mathsf{b}\!\downarrow \dfrac{S'(!R,[?T,T])}{S'(!R,?T)}}{S'\{?(R,T)\}} \qquad \text{was replaced by} \qquad \mathsf{b}\!\downarrow^{\blacktriangledown} \frac{\mathsf{p}\!\uparrow \dfrac{\mathsf{s} \dfrac{\mathsf{s} \dfrac{\mathsf{b}\!\uparrow^{\blacktriangle} \dfrac{S'(!R,[?T,T])}{S'(!R^1,R^2,[?T,T])}}{S'([(!R^1,R^2),?T],T)}}{S'[(!R^1,?T),(R^2,T)]}}{S'[?(R^1,T),(R^2,T)]}}{S'\{?(R,T)\}}$$

(see case (5.i) in Algorithm $\mathsf{b}\!\downarrow\!\mathsf{down}$). Here, I have marked the down moving $\mathsf{b}\!\downarrow$ as $\mathsf{b}\!\downarrow^{\blacktriangledown}$ and the two copies of R as R^1 and R^2. By inspecting the cases of $\mathsf{b}\!\downarrow\!\mathsf{down}$ (see Algorithm $\mathsf{b}\!\downarrow\!\mathsf{down}$), it is easy to see that while $\mathsf{b}\!\downarrow^{\blacktriangledown}$ travels further down, the two copies R^1 and R^2 are treated equally, i.e. whenever a rule ρ modifies R^1, then there is another instance of ρ that modifies R^2 in same way. Furthermore, if another $\mathsf{b}\!\downarrow$ is moved down in the same run of $\mathsf{b}\!\downarrow\!\mathsf{down}$ and meets the new $\mathsf{b}\!\uparrow^{\blacktriangle}$ as in case (5.iii) in Algorithm $\mathsf{b}\!\downarrow\!\mathsf{down}$, then it is duplicated into both copies R^1 and R^2. Hence, after finishing the run of $\mathsf{b}\!\downarrow\!\mathsf{down}$, every !-chain with head in R^1 has a counterpart !-chain with head in R^2, and vice versa. Similarly, all ?-chains with tail in R^1 and R^2 correspond to each other. This means that we can construct from χ a new promotion cycle χ' by replacing each subchain of χ with head or tail inside R^1 by the corresponding chain with head or tail in R^2. Then the new cycle is not forked by $\mathsf{b}\!\uparrow^{\blacktriangle}$ since there are no more links inside R^1. Hence, the cycle χ' is forked by $k_{\chi'} = k_\chi - 1$ instances of $\mathsf{b}\!\uparrow$, which is a contradiction to the induction hypothesis. $\square$

8.8 Proposition *For every* $\begin{array}{c} T \\ \Delta \big\Vert \mathsf{SELS} \\ R \end{array}$ *there is a derivation* $\begin{array}{c} T \\ \Delta_1 \big\Vert \{\mathsf{b}\!\uparrow\} \\ T' \\ \Delta' \big\Vert \{\mathsf{ai}\!\downarrow,\mathsf{ai}\!\uparrow,\mathsf{s},\mathsf{p}\!\downarrow,\mathsf{p}\!\uparrow,\mathsf{w}\!\downarrow,\mathsf{w}\!\uparrow\} \\ R' \\ \Delta_2 \big\Vert \{\mathsf{b}\!\downarrow\} \\ R \end{array}$.

Proof: Apply the algorithm $\mathsf{b}\!\uparrow\!\downarrow\!\mathsf{split}$, which terminates by Lemma 8.7 and Lemma 8.6. $\square$

Consider the topmost occurrence of a subderivation $\mathsf{b}{\uparrow}\,\dfrac{\pi\,\dfrac{Q}{S\{!R\}}}{S(!R,R)}$, where $\pi \neq \mathsf{b}{\uparrow}$. According to 6.3 there are the following cases (cases (3) and (6) are not possible):

(1) If the redex of π is inside $S\{\ \}$, or

(2) if the contractum $!R$ of $\mathsf{b}{\uparrow}$ is inside a passive structure of the redex of π, then replace

$$\mathsf{b}{\uparrow}\,\frac{\pi\,\dfrac{S'\{!R\}}{S\{!R\}}}{S(!R,R)} \qquad \text{by} \qquad \pi\,\frac{\mathsf{b}{\uparrow}\,\dfrac{S'\{!R\}}{S'(!R,R)}}{S(!R,R)} \quad .$$

(4) If the redex of π is inside the contractum $!R$ of $\mathsf{b}{\uparrow}$, then replace

$$\mathsf{b}{\uparrow}\,\frac{\pi\,\dfrac{S\{!R'\}}{S\{!R\}}}{S(!R,R)} \qquad \text{by} \qquad \pi\,\frac{\pi\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S\{!R'\}}{S(!R',R')}}{S(!R',R)}}{S(!R,R)} \quad .$$

(5) If the contractum $!R$ of $\mathsf{b}{\uparrow}$ is inside an active structure of the redex of π but not inside a passive one, then there are three subcases:

(i) If $\pi = \mathsf{p}{\downarrow}$ and $S\{!R\} = S'[!R,?T]$, then replace

$$\mathsf{b}{\uparrow}\,\frac{\mathsf{p}{\downarrow}\,\dfrac{S'\{![R,T]\}}{S'[!R,?T]}}{S'[(!R,R),?T]} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\,\frac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'\{![R,T]\}}{S'(![R,T],[R,T])}}{S'([!R,?T],[R,T])}}{S'[([!R,?T],R),T]}}{S'[(!R,R),?T,T]}}{S'[(!R,R),?T]} \quad .$$

(ii) If $\pi = \mathsf{w}{\downarrow}$ and $S\{!R\} = S'\{?S''\{!R\}\}$, then replace

$$\mathsf{b}{\uparrow}\,\frac{\mathsf{w}{\downarrow}\,\dfrac{S'\{\bot\}}{S'\{?S''\{!R\}\}}}{S'\{?S''(!R,R)\}} \qquad \text{by} \qquad \mathsf{w}{\downarrow}\,\frac{S'\{\bot\}}{S'\{?S''(!R,R)\}} \quad .$$

(iii) If $\pi = \mathsf{b}{\downarrow}$ and $S\{!R\} = S'\{?S''\{!R\}\}$, then replace

$$\mathsf{b}{\uparrow}\,\frac{\mathsf{b}{\downarrow}\,\dfrac{S'[?S''\{!R\},S''\{!R\}]}{S'\{?S''\{!R\}\}}}{S'\{?S''(!R,R)\}} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\,\frac{\mathsf{b}{\uparrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'[?S''\{!R\},S''\{!R\}]}{S'[?S''(!R,R),S''\{!R\}]}}{S'[?S''(!R,R),S''(!R,R)]}}{S'\{?S''(!R,R)\}} \quad .$$

Repeat until all instances of $\mathsf{b}{\uparrow}$ are at the top of the derivation.

Figure 18: The $\mathsf{b}{\uparrow}\mathsf{up}$ procedure

Consider the bottommost occurrence of a subderivation $\rho \dfrac{\mathsf{b}{\downarrow} \dfrac{S[?T,T]}{S\{?T\}}}{P}$, where $\rho \neq \mathsf{b}{\downarrow}$, until all instances of $\mathsf{b}{\downarrow}$ are at the bottom of the derivation. The possible cases are:

(1) The contractum of ρ is inside $S\{\ \}$, or

(2) the redex $?T$ of $\mathsf{b}{\downarrow}$ is inside a passive structure of the contractum of ρ. Then replace

$$\rho \frac{\mathsf{b}{\downarrow} \dfrac{S[?T,T]}{S\{?T\}}}{S'\{?T\}} \quad \text{by} \quad \mathsf{b}{\downarrow} \frac{\rho \dfrac{S[?T,T]}{S'[?T,T]}}{S'\{?T\}} \quad .$$

(4) The contractum of ρ is inside the redex $?T$ of $\mathsf{b}{\downarrow}$ Then replace

$$\rho \frac{\mathsf{b}{\downarrow} \dfrac{S[?T,T]}{S\{?T\}}}{S\{?T'\}} \quad \text{by} \quad \mathsf{b}{\downarrow} \frac{\rho \dfrac{\rho \dfrac{S[?T,T]}{S[?T,T']}}{S[?T',T']}}{S\{?T'\}} \quad .$$

(5) The redex $?T$ of $\mathsf{b}{\downarrow}$ is inside an active structure of the contractum of ρ but not inside a passive one. Then there are three cases:

(i) If $\rho = \mathsf{p}{\uparrow}$ and $S\{?T\} = S'(!R, ?T)$, then replace

$$\mathsf{p}{\uparrow} \frac{\mathsf{b}{\downarrow} \dfrac{S'(!R,[?T,T])}{S'(!R,?T)}}{S'\{?(R,T)\}} \quad \text{by} \quad \mathsf{b}{\downarrow} \frac{\mathsf{p}{\uparrow} \dfrac{\mathsf{s} \dfrac{\mathsf{s} \dfrac{\mathsf{b}{\uparrow} \dfrac{S'(!R,[?T,T])}{S'(!R,R,[?T,T])}}{S'([(!R,R),?T],T)}}{S'[(!R,?T),(R,T)]}}{S'[?(R,T),(R,T)]}}{S'\{?(R,T)\}} \quad .$$

(ii) If $\rho = \mathsf{w}{\uparrow}$ and $S\{?T\} = S'\{!S''\{?T\}\}$, then replace

$$\mathsf{w}{\uparrow} \frac{\mathsf{b}{\downarrow} \dfrac{S'\{!S''[?T,T]\}}{S'\{!S''\{?T\}\}}}{S'\{1\}} \quad \text{by} \quad \mathsf{w}{\uparrow} \frac{S'\{!S''[?T,T]\}}{S'\{1\}} \quad .$$

(iii) If $\rho = \mathsf{b}{\uparrow}$ and $S\{?T\} = S'\{!S''\{?T\}\}$, then replace

$$\mathsf{b}{\uparrow} \frac{\mathsf{b}{\downarrow} \dfrac{S'\{!S''[?T,T]\}}{S'\{!S''\{?T\}\}}}{S'(!S''\{?T\},S''\{?T\})} \quad \text{by} \quad \mathsf{b}{\downarrow} \frac{\mathsf{b}{\downarrow} \dfrac{\mathsf{b}{\uparrow} \dfrac{S'\{!S''[?T,T]\}}{S'(!S''[?T,T],S''[?T,T])}}{S'(!S''[?T,T],S''\{?T\})}}{S'(!S''\{?T\},S''\{?T\})} \quad .$$

Remark: Cases (3) and (6) are not possible.

Figure 19: The $\mathsf{b}{\downarrow}\mathsf{down}$ procedure

9 Cut Elimination in the Calculus of Structures

In Section 5, I already presented a proof of cut elimination for system ELS. That proof made use of the cut elimination argument in the sequent calculus. In this section, I will give a new proof of the same theorem which will be very different. This proof will be carried out inside the calculus of structures directly, without the detour of using the sequent calculus presentation of MELL.

There are several reasons to study cut elimination inside the calculus of structures. The first is that we want to investigate systems in the calculus of structures for which no system in the sequent calculus is known or for which it is impossible to give a system in the sequent calculus [14, 16, 30]. This means that we need new methodologies and techniques to prove cut elimination for those systems. One purpose of this paper is to investigate such methodologies. They might be easier to understand if they are first studied for logics that are well-known.

A second important reason to study cut elimination inside the calculus of structures is to obtain new insights on the question why cut elimination works in general, i.e. what are the properties that a logical system must have in order to get cut elimination.

Before I give an overview of what I will do in this section, let me explain why cut elimination in the calculus of structures is much different from cut elimination in the sequent calculus. In the cut elimination proof in the sequent calculus the cut is permuted up and its rank (the size of the cut formula) is reduced by decomposing the cut formula along its main connective. For example, we can replace the derivation

$$\mathsf{cut}\,\dfrac{\mathbin{\bindnasrepma}\,\dfrac{\vdash A, B, \Phi}{\vdash A \mathbin{\bindnasrepma} B, \Phi} \qquad \otimes\,\dfrac{\vdash A^{\perp}, \Psi_1 \quad \vdash B^{\perp}, \Psi_2}{\vdash A^{\perp} \otimes B^{\perp}, \Psi_1, \Psi_2}}{\vdash \Phi, \Psi_1, \Psi_2}$$

by the derivation

$$\mathsf{cut}\,\dfrac{\mathsf{cut}\,\dfrac{\vdash A, B, \Phi \quad \vdash A^{\perp}, \Psi_1}{\vdash B, \Phi, \Psi_1} \qquad \vdash B^{\perp}, \Psi_2}{\vdash \Phi, \Psi_1, \Psi_2}$$

because we *know* that if a rule modifies the cut formula, then that rule deals with the main connective of the cut formula. I will not go into further details here. The important point is (as already observed in [14]) that this method cannot be applied in the calculus of structures, for the following reason. If in a derivation

$$\mathsf{i}{\uparrow}\,\dfrac{\rho\,\dfrac{Q}{S(R, \bar{R})}}{S\{\perp\}}$$

the rule ρ has its redex inside R, we *do not know* how deep inside R the rule is applied. Furthermore, there is no reason to assume that any rule ρ' above ρ does the exact dual of ρ inside the structure $\bar{R}$.

However, in the calculus of structures, we are able to perform a very different procedure: We can reduce the generic cut rule to an atomic version. Showing that the generic cut rule $\mathsf{i}\uparrow$ is admissible is equivalent to showing that the whole up-fragment (except for the switch) is admissible (see Theorem 4.10). For classical logic, this considerably simplifies the cut elimination argument [4]. In [7] Brünnler and Tiu use a semantic argument. In [15], Guglielmi develops a very general techique, called *splitting*, which has also been used in [17] and would therefore also work for system ELS. However, in this paper, I will use the classical technique of permuting rules.

The whole procedure will be carried out in several small steps. The first step uses a version of the decomposition theorem to show that the rules $\mathsf{b}\uparrow$ and $\mathsf{w}\uparrow$ (i.e. the non-core rules) are admissible. In the second step, I will eliminate the rule $\mathsf{p}\uparrow$ (i.e. the up-fragment of the core), and in the last step, I will eliminate the rule $\mathsf{ai}\uparrow$. The rules $\mathsf{p}\uparrow$ and $\mathsf{ai}\uparrow$ are eliminated by using the technique that has already been employed by Gentzen [10]: For both rules, I will give a super-rule that is more general and that helps in the book-keeping of the context. The super-rules are permuted over all other rules until they reach the top of the proof where they disappear.

This permutability is distributed over several lemmata. If new rules are added to the system then those lemmata remain valid: If rule ρ permutes over rule π, then the introduction of a rule σ does not change this fact. This kind of modularity cannot be explored in the sequent calculus.

Let me now start with the first step, which is a corollary of the decomposition theorem.

9.1 Corollary *For every proof* $\begin{array}{c}\Pi\Vert\mathsf{SELS}\cup\{1\downarrow\}\\ R\end{array}$*, there is a proof*

$$
\begin{array}{c}
1\downarrow\dfrac{}{1}\\
\Vert\{\mathsf{w}\downarrow\}\\
R_4\\
\Vert\{\mathsf{ai}\downarrow\}\\
R_3\\
\Vert\{\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow\}\\
R_2\\
\Vert\{\mathsf{ai}\uparrow\}\\
R_1\\
\Vert\{\mathsf{b}\downarrow\}\\
R
\end{array}
$$

for some structures R_1, R_2, R_3 *and* R_4.

Proof: By Propositions 4.4 and 4.6, every occurrence of the rule $\mathsf{w}\uparrow$ can be replaced by a derivation containing only the rules $\mathsf{w}\downarrow, \mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}, \mathsf{p}\downarrow$ and $\mathsf{p}\uparrow$. Apply Theorem 8.1 to the derivation $\begin{array}{c}1\\ \Vert\mathsf{SELS}\setminus\{\mathsf{w}\uparrow\}\\ R\end{array}$ which is obtained from Π by removing the axiom. Now every application of $\mathsf{b}\!\uparrow$ in $\begin{array}{c} 1 \\ \Delta_1 \Vert \{\mathsf{b}\uparrow\} \\ T_1 \end{array}$ must be trivial, i.e. of the shape $\mathsf{b}\!\uparrow \frac{(!1, 1)}{!1}$. Hence $T_1 = 1$. □

This shows that the non-core rules $\mathsf{w}\!\uparrow$ and $\mathsf{b}\!\uparrow$ are admissible. In order to show that the rule $\mathsf{p}\!\uparrow$ is admissible, we need to add the following two rules to system SELS:

$$\mathsf{r}\!\downarrow \frac{S\{?[R,T]\}}{S[?R,?T]} \qquad \text{and} \qquad \mathsf{r}\!\uparrow \frac{S(!R,!T)}{S\{!(R,T)\}} \quad .$$

It is easy to see that both rules are sound.

For technical reasons, I was not able to simply eliminate the rule $\mathsf{p}\!\uparrow$. Instead, I will eliminate the rules $\mathsf{p}\!\uparrow$ and $\mathsf{r}\!\uparrow$ simultaneously, with the result that instances of $\mathsf{r}\!\downarrow$ might be introduced. Those instances will be eliminated afterwards. Finally, the rule $\mathsf{ai}\!\uparrow$ will also be eliminated.

All three rules $\mathsf{p}\!\uparrow$, $\mathsf{r}\!\uparrow$ and $\mathsf{ai}\!\uparrow$ are removed by a method that has already been used in [14] for proving the cut elimination for system BV. Namely, for all three rules $\mathsf{p}\!\uparrow$, $\mathsf{r}\!\uparrow$ and $\mathsf{ai}\!\uparrow$, there are super-rules $\mathsf{sp}\!\uparrow$, $\mathsf{sr}\!\uparrow$ and $\mathsf{sai}\!\uparrow$, respectively:

$$\mathsf{sp}\!\uparrow \frac{S([?R,U],[!T,V])}{S[?(R,T),U,V]} \quad , \quad \mathsf{sr}\!\uparrow \frac{S([!R,U],[!T,V])}{S[!(R,T),U,V]} \quad \text{and} \quad \mathsf{sai}\!\uparrow \frac{S([a,U],[\bar{a},V])}{S[U,V]} \quad .$$

The rules $\mathsf{p}\!\uparrow$, $\mathsf{r}\!\uparrow$ and $\mathsf{ai}\!\uparrow$ are instances of the super-rules $\mathsf{sp}\!\uparrow$, $\mathsf{sr}\!\uparrow$ and $\mathsf{sai}\!\uparrow$, respectively. I will now show that every super-rule can be permuted up in the proof until it disappears or its application becomes trivial.

Before we can start, a few more definitions are necessary.

9.2 Definition A structure R is called a *proper par* if there are two structures R' and R'' with $R = [R', R'']$ and $R' \neq \bot \neq R''$. Similarly, a structure R is a *proper times*, if there are two structures R' and R'' with $R = (R', R'')$ and $R' \neq 1 \neq R''$.

9.3 Definition Let *deep switch* be the rule $\mathsf{ds}\!\downarrow \frac{S([R,T],U)}{S[(R,U),T]}$, where the structure R is not a proper times. The rule $\mathsf{ns}\!\uparrow \frac{S([(R,R'),T],U)}{S[(R,R',U),T]}$, where $R \neq 1 \neq R'$, will be called *non-deep switch*.

Both rules are instances of the switch-rule, and every instance of the switch-rule is either an instance of deep switch or an instance of non-deep switch.

9.4 This is sufficient to outline the scheme (shown in Figure 20) of the full cut elimination proof. We start with a proof obtained by Corollary 9.1. Then, in the first step, all instances of the rule s are replaced by $\mathsf{ds}\!\downarrow$ or $\mathsf{ns}\!\uparrow$, and all instances of $\mathsf{p}\!\uparrow$ and $\mathsf{ai}\!\uparrow$ are replaced by their super rules. While permuting the rules $\mathsf{ns}\!\uparrow$ and $\mathsf{sp}\!\uparrow$ over $\mathsf{ds}\!\downarrow$ and $\mathsf{p}\!\downarrow$ in Step 2, the rules $\mathsf{r}\!\downarrow$ and $\mathsf{sr}\!\uparrow$ are introduced. In Step 3, the rules $\mathsf{ns}\!\uparrow$, $\mathsf{sp}\!\uparrow$ and $\mathsf{sr}\!\uparrow$ are eliminated. Then, the rule $\mathsf{r}\!\downarrow$ is eliminated in Step 4. Finally, the rule $\mathsf{sai}\!\uparrow$ is eliminated.

9.5 Lemma *The rule $\mathsf{ns}\!\uparrow$ permutes over the rules $\mathsf{ds}\!\downarrow$, $\mathsf{p}\!\downarrow$ and $\mathsf{r}\!\downarrow$ by the rule $\mathsf{ds}\!\downarrow$.*

$$
\boxed{
\begin{array}{c}
\mathsf{1}\!\downarrow \dfrac{}{1} \\ \Vert\{\mathsf{w}\!\downarrow\} \\ R_4 \\ \Vert\{\mathsf{ai}\!\downarrow\} \\ R_3 \\ \Vert\{\mathsf{s},\mathsf{p}\!\downarrow,\mathsf{p}\!\uparrow\} \\ R_2 \\ \Vert\{\mathsf{ai}\!\uparrow\} \\ R_1 \\ \Vert\{\mathsf{b}\!\downarrow\} \\ R
\end{array}
\overset{1}{=}
\begin{array}{c}
\mathsf{1}\!\downarrow \dfrac{}{1} \\ \Vert\{\mathsf{w}\!\downarrow\} \\ R_4 \\ \Vert\{\mathsf{ai}\!\downarrow\} \\ R_3 \\ \Vert\{\mathsf{ds}\!\downarrow,\mathsf{ns}\!\uparrow,\mathsf{p}\!\downarrow,\mathsf{sp}\!\uparrow\} \\ R_2 \\ \Vert\{\mathsf{sai}\!\uparrow\} \\ R_1 \\ \Vert\{\mathsf{b}\!\downarrow\} \\ R
\end{array}
\overset{2}{\leadsto}
\begin{array}{c}
\mathsf{1}\!\downarrow \dfrac{}{1} \\ \Vert\{\mathsf{w}\!\downarrow\} \\ R_4 \\ \Vert\{\mathsf{ai}\!\downarrow\} \\ R_3 \\ \Vert\{\mathsf{ns}\!\uparrow,\mathsf{sp}\!\uparrow,\mathsf{sr}\!\uparrow\} \\ R'_3 \\ \Vert\{\mathsf{ds}\!\downarrow,\mathsf{p}\!\downarrow,\mathsf{r}\!\downarrow\} \\ R_2 \\ \Vert\{\mathsf{sai}\!\uparrow\} \\ R_1 \\ \Vert\{\mathsf{b}\!\downarrow\} \\ R
\end{array}
\overset{3}{\leadsto}
\begin{array}{c}
\mathsf{1}\!\downarrow \dfrac{}{1} \\ \Vert\{\mathsf{w}\!\downarrow\} \\ R'_4 \\ \Vert\{\mathsf{ai}\!\downarrow\} \\ R'_3 \\ \Vert\{\mathsf{ds}\!\downarrow,\mathsf{p}\!\downarrow,\mathsf{r}\!\downarrow\} \\ R_2 \\ \Vert\{\mathsf{sai}\!\uparrow\} \\ R_1 \\ \Vert\{\mathsf{b}\!\downarrow\} \\ R
\end{array}
\overset{4}{\leadsto}
\begin{array}{c}
\mathsf{1}\!\downarrow \dfrac{}{1} \\ \Vert\{\mathsf{w}\!\downarrow\} \\ R''_4 \\ \Vert\{\mathsf{ai}\!\downarrow\} \\ R''_3 \\ \Vert\{\mathsf{ds}\!\downarrow,\mathsf{p}\!\downarrow\} \\ R_2 \\ \Vert\{\mathsf{sai}\!\uparrow\} \\ R_1 \\ \Vert\{\mathsf{b}\!\downarrow\} \\ R
\end{array}
\overset{5}{\leadsto}
\begin{array}{c}
\mathsf{1}\!\downarrow \dfrac{}{1} \\ \Vert\{\mathsf{w}\!\downarrow\} \\ R'''_4 \\ \Vert\{\mathsf{ai}\!\downarrow\} \\ R'''_3 \\ \Vert\{\mathsf{ds}\!\downarrow,\mathsf{p}\!\downarrow\} \\ R_1 \\ \Vert\{\mathsf{b}\!\downarrow\} \\ R
\end{array}
}
$$

Figure 20: Cut elimination for system $\mathsf{SELS} \cup \{\mathsf{1}\!\downarrow\}$

Proof: Following the scheme of 6.3, let us consider a derivation

$$\pi \frac{Q}{\mathsf{ns}\!\uparrow \dfrac{S([(R,R'),T],U)}{S[(R,R',U),T]}} \quad ,$$

where the application of $\pi \in \{\mathsf{ds}\!\downarrow, \mathsf{p}\!\downarrow, \mathsf{r}\!\downarrow\}$ is not trivial. Without loss of generality we can assume that R is not a proper times. The cases are:

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $([(R,R'),T],U)$ of $\mathsf{ns}\!\uparrow$ is inside a passive structure in the redex of π. Trivial.

(3) The redex of π is inside one of the passive structures R, R', T or U of the contractum of $\mathsf{ns}\!\uparrow$. Trivial.

(4) The redex of π is inside the contractum $([(R,R'),T],U)$ of $\mathsf{ns}\!\uparrow$, but not inside R, R', T or U. Only one case is possible ($\pi = \mathsf{ds}\!\downarrow$):

$$\mathsf{ds}\!\downarrow \frac{S([R,T],R',U)}{\mathsf{ns}\!\uparrow \dfrac{S([(R,R'),T],U)}{S[(R,R',U),T]}} \quad \text{yields} \quad \mathsf{ds}\!\downarrow \frac{S([R,T],R',U)}{S[(R,R',U),T]} \quad .$$

(5) The contractum $([(R,R'),T],U)$ of $\mathsf{ns}\!\uparrow$ is inside an active structure of the redex of π but not inside a passive one. Then $\pi = \mathsf{ds}\!\downarrow$ and $S([(R,R'),T],U) = S'[([(R,R'),T],U,V),W]$. There are two possibilities:

$$\mathsf{ds}\!\downarrow \frac{S'([(R,R'),T,W],U,V)}{\mathsf{ns}\!\uparrow \dfrac{S'[([(R,R'),T],U,V),W]}{S'[([(R,R',U),T],V),W]}} \quad \text{yields} \quad \mathsf{ns}\!\uparrow \frac{S'([(R,R'),T,W],U,V)}{\mathsf{ds}\!\downarrow \dfrac{S'([(R,R',U),T,W],V)}{S'[([(R,R',U),T],V),W]}} \quad \text{and}$$

$$\mathsf{ds}\!\downarrow\frac{\mathsf{ns}\!\uparrow\dfrac{S'([(R,R'),T],[U,W],V)}{S'[([(R,R'),T],U,V),W]}}{S'[([(R,R',U),T],V),W]}\quad\text{yields}\quad \mathsf{ds}\!\downarrow\frac{\mathsf{ds}\!\downarrow\dfrac{\mathsf{ns}\!\uparrow\dfrac{S'([(R,R'),T],[U,W],V)}{S'([(R,R',[U,W]),T],V)}}{S'([(R,R',U),T,W],V)}}{S'[([(R,R',U),T],V),W]}\quad.$$

Note: The second case is only possible if U is not a proper times.

(6) The redex of π and the contractum $([(R,R'),T],U)$ of $\mathsf{ns}\uparrow$ overlap. Not possible, because the redex of π is always a par-structure which cannot properly overlap with a times-structure. □

9.6 Lemma *The rules* $\mathsf{sai}\uparrow$*,* $\mathsf{sp}\uparrow$ *and* $\mathsf{sr}\uparrow$ *permute over the rule* $\mathsf{ds}\downarrow$*.*

Proof: All three rules are of the shape $\mathsf{sx}\uparrow\dfrac{S([P,U],[P',V])}{S[P'',U,V]}$, where neither P nor P' is a proper par or a proper times. Now consider the derivation $\mathsf{sx}\uparrow\dfrac{\mathsf{ds}\downarrow\dfrac{Q}{S([P,U],[P',V])}}{S[P'',U,V]}$,

where the application of $\mathsf{ds}\downarrow$ is not trivial.

(1) The redex of $\mathsf{ds}\downarrow$ is inside $S\{\ \}$. Trivial.

(2) The contractum $([P,U],[P',V])$ of $\mathsf{sx}\uparrow$ is inside a passive structure of the redex of $\mathsf{ds}\downarrow$. Trivial.

(3) The redex of $\mathsf{ds}\downarrow$ is inside a passive structure of the contractum of $\mathsf{sx}\uparrow$. Trivial. (Remark: If $\mathsf{sx}\uparrow$ is $\mathsf{sai}\uparrow$, then the passive structures are U and V. If $\mathsf{sx}\uparrow$ is $\mathsf{sp}\uparrow$ or $\mathsf{sr}\uparrow$, then U, V, R and T are passive structures.)

(4) The redex of $\mathsf{ds}\downarrow$ is inside the contractum $([P,U],[P',V])$ of $\mathsf{sx}\uparrow$, but not inside a passive structure. Observe that the redex of $\mathsf{ds}\downarrow$ cannot be inside P or P' because they are neither a proper par nor a proper times. Therefore, there are only two remaining cases.

(i) $U=(U',U'')$. Without loss of generality assume that U' is not a proper times. Then

$$\mathsf{sx}\!\uparrow\frac{\mathsf{ds}\!\downarrow\dfrac{S([P,U'],[P',V],U'')}{S([P,(U',U'')],[P',V])}}{S[P'',(U',U''),V]}\quad\text{yields}\quad \mathsf{ds}\!\downarrow\frac{\mathsf{sx}\!\uparrow\dfrac{S([P,U'],[P',V],U'')}{S([P'',U',V],U'')}}{S[P'',(U',U''),V]}\quad.$$

(ii) $V=(V',V'')$. Similar.

(5) The contractum $([P,U],[P',V])$ of $\mathsf{sx}\uparrow$ is inside an active structure of the redex of $\mathsf{ds}\downarrow$ but not inside a passive one. In the most general case we have that $S([P,U],[P',V])=S'[([P,U],[P',V],W),Z]$ for some context $S'\{\ \}$ and

some structures W and Z. Then

$$\dfrac{\mathsf{ds}\!\downarrow \dfrac{S([P,U,Z],[P',V],W)}{S'[([P,U],[P',V],W),Z]}}{\mathsf{sx}\!\uparrow\; S'[([P'',U,V],W),Z]} \quad \text{yields} \quad \dfrac{\mathsf{sx}\!\uparrow \dfrac{S([P,U,Z],[P',V],W)}{S'([P'',U,V,Z],W)}}{\mathsf{ds}\!\downarrow\; S'[([P'',U,V],W),Z]} \quad \text{and}$$

$$\dfrac{\mathsf{ds}\!\downarrow \dfrac{S([P,U],[P',V,Z],W)}{S'[([P,U],[P',V],W),Z]}}{\mathsf{sx}\!\uparrow\; S'[([P'',U,V],W),Z]} \quad \text{yields} \quad \dfrac{\mathsf{sx}\!\uparrow \dfrac{S([P,U],[P',V,Z],W)}{S'([P'',U,V,Z],W)}}{\mathsf{ds}\!\downarrow\; S'[([P'',U,V],W),Z]} \quad .$$

(6) The redex of $\mathsf{ds}\!\downarrow$ and the contractum $([P,U],[P',V])$ of $\mathsf{sx}\!\uparrow$ overlap. Not possible. ☐

Observe that the rules $\mathsf{sai}\!\uparrow$, $\mathsf{sp}\!\uparrow$ and $\mathsf{sr}\!\uparrow$ do not permute over the rule s. For example in the derivation

$$\dfrac{\mathsf{s}\dfrac{S([a,U],[([\bar{a},V],W),Z])}{S[([a,U],[\bar{a},V],W),Z]}}{\mathsf{sai}\!\uparrow\; S[([U,V],W),Z]}$$

the rule $\mathsf{sai}\!\uparrow$ cannot be permuted over the switch. This is the reason why the deep switch has been introduced in [14] in the first place.

9.7 Lemma *For every derivation* $\dfrac{\pi\dfrac{Q}{Z}}{\rho\; P}$ *with* $\rho \in \{\mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow\}$ *and* $\pi \in \{\mathsf{p}\!\downarrow, \mathsf{r}\!\downarrow\}$*, there is either a derivation* $\dfrac{\rho\dfrac{Q}{Z'}}{\pi\; P}$ *for some structure* Z' *or a derivation* $\dfrac{\rho'\dfrac{Q}{Z'}}{\mathsf{s}\dfrac{}{Z''}\;\pi'\; P}$ *for some structures* Z' *and* Z'' *and rules* $\rho' \in \{\mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow\}$ *and* $\pi' \in \{\mathsf{p}\!\downarrow, \mathsf{r}\!\downarrow\}$.

Proof: Consider the derivation $\dfrac{\pi\dfrac{Q}{S([*R,U],[!T,V])}}{\rho\; S[*(R,T),U,V]}$, where $\rho \in \{\mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow\}$, $* \in \{?,!\}$ and the application of $\pi \in \{\mathsf{p}\!\downarrow, \mathsf{r}\!\downarrow\}$ is not trivial. The cases are:

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $([*R,U],[!T,V])$ of ρ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure R, U, T or V of the contractum of ρ. Trivial.

(4) The redex of π is inside the contractum $([*R,U],[!T,V])$ of ρ but not inside R, U, T or V. There are the following five subcases:

(i) $\rho=\mathsf{sp}{\uparrow}, * = ?, \pi=\mathsf{p}{\downarrow}$ and $U=[!U',U'']$. Then

$$\mathsf{sp}{\uparrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S([![R,U'],U''],[!T,V])}{S([?R,!U',U''],[!T,V])}}{S[?(R,T),!U',U'',V]} \quad\text{yields}\quad \mathsf{p}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{sr}{\uparrow}\,\dfrac{S([![R,U'],U''],[!T,V])}{S[!([R,U'],T),U'',V]}}{S[![(R,T),U'],U'',V]}}{S[?(R,T),!U',U'',V]} \quad .$$

(ii) $\rho=\mathsf{sp}{\uparrow}, * = ?, \pi=\mathsf{p}{\downarrow}$ and $V=[?V',V'']$. Then

$$\mathsf{sp}{\uparrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S([?R,U],[![T,V'],V''])}{S([?R,U],[!T,?V',V''])}}{S[?(R,T),U,?V',V'']} \quad\text{yields}\quad \mathsf{r}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{sp}{\uparrow}\,\dfrac{S([?R,U],[![T,V'],V''])}{S[?(R,[T,V']),U,V'']}}{S[?[(R,T),V'],U,V'']}}{S[?(R,T),U,?V',V'']} \quad .$$

(iii) $\rho=\mathsf{sp}{\uparrow}, * = ?, \pi=\mathsf{r}{\downarrow}$ and $U=[?U',U'']$. Then

$$\mathsf{sp}{\uparrow}\,\dfrac{\mathsf{r}{\downarrow}\,\dfrac{S([?[R,U'],U''],[!T,V])}{S([?R,?U',U''],[!T,V])}}{S[?(R,T),?U',U'',V]} \quad\text{yields}\quad \mathsf{r}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{sp}{\uparrow}\,\dfrac{S([?[R,U'],U''],[!T,V])}{S[?([R,U'],T),U'',V]}}{S[?[(R,T),U'],U'',V]}}{S[?(R,T),?U',U'',V]} \quad .$$

(iv) $\rho=\mathsf{sr}{\uparrow}, * = !, \pi=\mathsf{p}{\downarrow}$ and $U=[?U',U'']$. Then

$$\mathsf{sr}{\uparrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S([![R,U'],U''],[!T,V])}{S([!R,?U',U''],[!T,V])}}{S[!(R,T),?U',U'',V]} \quad\text{yields}\quad \mathsf{p}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{sr}{\uparrow}\,\dfrac{S([![R,U'],U''],[!T,V])}{S[!([R,U'],T),U'',V]}}{S[![(R,T),U'],U'',V]}}{S[!(R,T),?U',U'',V]} \quad .$$

(v) $\rho=\mathsf{sr}{\uparrow}, * = !, \pi=\mathsf{p}{\downarrow}$ and $V=[?V',V'']$. Then

$$\mathsf{sr}{\uparrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S([!R,U],[![T,V'],V''])}{S([!R,U],[!T,?V',V''])}}{S[!(R,T),U,?V',V'']} \quad\text{yields}\quad \mathsf{p}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{sr}{\uparrow}\,\dfrac{S([!R,U],[![T,V'],V''])}{S[!(R,[T,V']),U,V'']}}{S[![(R,T),V'],U,V'']}}{S[!(R,T),U,?V',V'']} \quad .$$

(5) The contractum $([*R,U],[!T,V])$ of ρ is inside an active structure of the redex of π, but not inside a passive one. Not possible.

(6) The redex of π and the contractum $([*R,U],[!T,V])$ of ρ overlap. Not possible.

$\square$

9.8 Lemma *For every derivation* $\begin{array}{c}R_3\\ \Vert\{\mathsf{s},\mathsf{p}{\downarrow},\mathsf{p}{\uparrow}\}\\ R_2\end{array}$ *there is a derivation* $\begin{array}{c}R_3\\ \Vert\{\mathsf{ns}{\uparrow},\mathsf{sp}{\uparrow},\mathsf{sr}{\uparrow}\}\\ R'_3\\ \Vert\{\mathsf{ds}{\downarrow},\mathsf{p}{\downarrow},\mathsf{r}{\downarrow}\}\\ R_2\end{array}$.

Proof: All occurrences of the rules $\mathsf{p}\uparrow$ and $\mathsf{r}\uparrow$ are instances of the rules $\mathsf{sp}\uparrow$ and $\mathsf{sr}\uparrow$, respectively; and all occurrences of the rule s are either instances of $\mathsf{ds}\downarrow$ or of $\mathsf{ns}\uparrow$. Now apply the following algorithm:

I. If there is no occurrence of a rule $\mathsf{ns}\uparrow$, $\mathsf{sp}\uparrow$ or $\mathsf{sr}\uparrow$ below a rule $\mathsf{ds}\downarrow$, $\mathsf{p}\downarrow$ or $\mathsf{r}\downarrow$ in the derivation, then terminate.

II. Otherwise, let ρ be the topmost occurrence of a rule $\mathsf{ns}\uparrow$, $\mathsf{sp}\uparrow$ or $\mathsf{sr}\uparrow$ that is below a $\mathsf{ds}\downarrow$, $\mathsf{p}\downarrow$ or $\mathsf{r}\downarrow$.

   (1) If ρ is $\mathsf{ns}\uparrow$, then (by Lemma 9.5) this occurrence can be permuted up (by possibly introducing new instances of $\mathsf{ds}\downarrow$).

   (2) If ρ is $\mathsf{sp}\uparrow$ or $\mathsf{sr}\uparrow$, then (by Lemmata 9.6 and 9.7) it can be permuted over all occurrences of the rules $\mathsf{ds}\downarrow, \mathsf{p}\downarrow$ and $\mathsf{r}\downarrow$ (by possibly introducing new instances of $\mathsf{ds}\downarrow$ and $\mathsf{ns}\uparrow$).

   Go to step I.

It is easy to see that this does indeed terminate. □

9.9 Lemma *The rules $\mathsf{ns}\uparrow$, $\mathsf{sp}\uparrow$ and $\mathsf{sr}\uparrow$ permute over the rule $\mathsf{ai}\downarrow$.*

Proof: Consider the derivation $\rho \dfrac{\mathsf{ai}\downarrow \dfrac{Q}{S\{W\}}}{S\{Z\}}$, where the application of $\rho \in \{\mathsf{ns}\uparrow, \mathsf{sr}\uparrow, \mathsf{sp}\uparrow\}$ is not trivial. The cases are:

(1) The redex of $\mathsf{ai}\downarrow$ is inside the context $S\{\ \}$ of ρ. Trivial.

(2) The contractum of ρ is inside a passive structure of the redex of $\mathsf{ai}\downarrow$. Trivial.

(3) The redex of $\mathsf{ai}\downarrow$ is inside a passive structure of the contractum W of ρ. Trivial.

(4) The redex of $\mathsf{ai}\downarrow$ is inside an active structure of the contractum W of ρ but not inside a passive one. Not possible.

(5) The contractum W of ρ is inside an active structure of the redex of $\mathsf{ai}\downarrow$. Not possible because the application of ρ is not trivial.

(6) The contractum W of ρ and the redex of π overlap. Not possible. □

9.10 Lemma *For every derivation* $\begin{array}{c} 1 \\ \Vert\{\mathsf{w}\downarrow\} \\ \rho \dfrac{P'}{P} \end{array}$ *, where $\rho \in \{\mathsf{ns}\uparrow, \mathsf{sp}\uparrow, \mathsf{sr}\uparrow\}$, there is a derivation* $\begin{array}{c} 1 \\ \Vert\{\mathsf{w}\downarrow\} \\ P \end{array}$ *.*

Proof: Let me introduce the rule $\mathsf{spw}\!\uparrow \dfrac{S(U,[!T,V])}{S[?(R,T),U,V]}$, which is a combination of $\mathsf{sp}\!\uparrow$ and $\mathsf{w}\!\downarrow$. Now consider a derivation $\begin{array}{l} \mathsf{w}\!\downarrow \dfrac{Q}{S\{W\}} \\ \rho \dfrac{\phantom{S\{W\}}}{S\{Z\}} \end{array}$, where $\rho \in \{\mathsf{ns}\!\uparrow, \mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow, \mathsf{spw}\!\uparrow\}$ and permute ρ over $\mathsf{w}\!\downarrow$ by applying the scheme of 6.3:

(1) The redex of $\mathsf{w}\!\downarrow$ is inside the context $S\{\ \}$ of ρ. Trivial.

(2) The contractum of ρ is inside a passive structure of the redex of $\mathsf{w}\!\downarrow$. Trivial.

(3) The redex of $\mathsf{w}\!\downarrow$ is inside a passive structure of the contractum W of ρ. Trivial.

(4) The redex of $\mathsf{w}\!\downarrow$ is inside an active structure of the contractum W of ρ but not inside a passive one. Then $W = ([?R,U],[!T,V])$ and

$$\begin{array}{l} \mathsf{w}\!\downarrow \dfrac{S([\bot,U],[!T,V])}{S([?R,U],[!T,V])} \\ \mathsf{sp}\!\uparrow \dfrac{}{S[?(R,T),U,V]} \end{array} \quad \text{yields} \quad \mathsf{spw}\!\uparrow \frac{S(U,[!T,V])}{S[?(R,T),U,V]} \quad .$$

(5) The contractum W of ρ is inside an active structure of the redex of $\mathsf{w}\!\downarrow$. Then $S\{\ \} = S'\{?S''\{\ \}\}$ and

$$\begin{array}{l} \mathsf{w}\!\downarrow \dfrac{S'\{\bot\}}{S'\{?S''\{W\}\}} \\ \rho \dfrac{\phantom{S'\{?S''\{W\}\}}}{S'\{?S''\{Z\}\}} \end{array} \quad \text{yields} \quad \mathsf{w}\!\downarrow \frac{S'\{\bot\}}{S'\{?S''\{Z\}\}} \quad .$$

(6) The contractum W of ρ and the redex of π overlap. Not possible.

Then, the instance of ρ either disappears, which gives us a derivation $\begin{array}{c} 1 \\ \Big\| \{\mathsf{w}\!\downarrow\} \\ P \end{array}$, or it reaches the top of the derivation, which yields $\begin{array}{c} \rho' \dfrac{1}{P''} \\ \Big\| \{\mathsf{w}\!\downarrow\} \\ P \end{array}$ with $\rho' \in \{\mathsf{ns}\!\uparrow, \mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow, \mathsf{spw}\!\uparrow\}$.

In the former case the proof is finished. In the latter, there are two possibilities.

(1) $\rho' \in \{\mathsf{ns}\!\uparrow, \mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow\}$. Then the application of ρ' must be trivial, because its premise is 1. Hence its conclusion $P'' = 1$ and we have the desired derivation by leaving out ρ'.

(2) $\rho' = \mathsf{spw}\!\uparrow$. Then the application of ρ' must be an instance of $\mathsf{w}\!\downarrow$ because its premise is 1. Hence, it can be replaced by an application of $\mathsf{w}\!\downarrow$. $\square$

9.11 Lemma *For every derivation* $\begin{array}{l} 1 \\ \Vert\{\mathsf{w}\downarrow\} \\ R'_4 \\ \Vert\{\mathsf{ai}\downarrow\} \\ R'_3 \\ \Vert\{\mathsf{ds}\downarrow,\mathsf{p}\downarrow,\mathsf{r}\downarrow\} \\ R_2 \end{array}$ *there is a derivation* $\begin{array}{l} 1 \\ \Vert\{\mathsf{w}\downarrow\} \\ R''_4 \\ \Vert\{\mathsf{ai}\downarrow\} \\ R''_3 \\ \Vert\{\mathsf{ds}\downarrow,\mathsf{p}\downarrow\} \\ R_2 \end{array}$.

Proof: Instead of eliminating the rule $\mathsf{r}\downarrow$, I will eliminate the rule $\mathsf{sr}\downarrow \dfrac{S\{?U\}}{S[?R,?T]}$, where U is any structure such that there is a derivation $\begin{array}{c} U \\ \Delta\Vert\{\mathsf{w}\downarrow,\mathsf{ai}\downarrow,\mathsf{ds}\downarrow,\mathsf{p}\downarrow\} \\ [R,T] \end{array}$. Note that $\mathsf{r}\downarrow$ is an instance of $\mathsf{sr}\downarrow$. Now consider the topmost instance of $\mathsf{sr}\downarrow$ and permute it up by applying the scheme in 6.3, i.e. consider a derivation $\mathsf{sr}\downarrow \dfrac{\pi \dfrac{Q}{S\{?U\}}}{S[?R,?T]}$, where $\pi \in \{\mathsf{w}\downarrow,\mathsf{ai}\downarrow,\mathsf{ds}\downarrow,\mathsf{p}\downarrow\}$ is not trivial. The cases are:

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $?U$ of $\mathsf{sr}\downarrow$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure of the contractum of $\mathsf{sr}\downarrow$. Not possible because there is no passive structure.

(4) The redex of π is inside the contractum $?U$ of $\mathsf{sr}\downarrow$. Then replace

$$\mathsf{sr}\downarrow \frac{\pi \dfrac{S\{?U'\}}{S\{?U\}}}{S[?R,?T]} \qquad \text{by} \qquad \mathsf{sr}\downarrow \frac{S\{?U'\}}{S[?R,?T]} \quad .$$

(5) The contractum $?U$ of $\mathsf{sr}\downarrow$ is inside an active structure of the redex of π but not inside a passive one. Then the following subcases are possible:

(i) $\pi = \mathsf{p}\downarrow$ and $S\{?U\} = S'[!V,?U]$. Then replace

$$\mathsf{sr}\downarrow \frac{\mathsf{p}\downarrow \dfrac{S\{![V,U]\}}{S[!V,?U]}}{S[!V,?R,?T]} \qquad \text{by} \qquad \mathsf{p}\downarrow \frac{\mathsf{p}\downarrow \dfrac{\begin{array}{c} S\{![V,U]\} \\ \Delta\Vert \\ S\{![V,R,T]\} \end{array}}{S[![V,R],?T]}}{S[!V,?R,?T]} \quad ,$$

where Δ is the derivation that exists by definition of $\mathsf{sr}\downarrow$.

(ii) $\pi = \mathsf{w}\downarrow$ and $?U$ is the redex. Then replace

$$\mathsf{sr}\downarrow \frac{\mathsf{w}\downarrow \frac{S\{\bot\}}{S\{?U\}}}{S[?R,?T]} \qquad \text{by} \qquad \mathsf{w}\downarrow \frac{\mathsf{w}\downarrow \frac{S\{\bot\}}{S\{?R\}}}{S[?R,?T]} \quad .$$

(iii) $\pi = \mathsf{w}\downarrow$ and $S\{?U\} = S'\{?S''\{?U\}\}$. Then replace

$$\mathsf{sr}\downarrow \frac{\mathsf{w}\downarrow \frac{S'\{\bot\}}{S'\{?S''\{?U\}\}}}{S'\{?S''[?R,?T]\}} \qquad \text{by} \qquad \mathsf{w}\downarrow \frac{S'\{\bot\}}{S'\{?S''[?R,?T]\}} \quad .$$

(6) The redex of π and the contractum $?U$ of $\mathsf{sr}\downarrow$ overlap. Not possible.

Now there are two possibilities. Either the instance of $\mathsf{sr}\downarrow$ disappeared during the process of permuting up, or it has reached the top of the derivation. Then we have that $S\{?U\} = 1$. This is only possible if $U = \bot$, i.e. $S\{?U\} = S\{\bot\}$. Then

$$\mathsf{sr}\downarrow \frac{S\{\bot\}}{S[?R,?T]} \qquad \text{can be replaced by} \qquad \mathsf{w}\downarrow \frac{\mathsf{w}\downarrow \frac{S\{\bot\}}{S\{?R\}}}{S[?R,?T]} \quad .$$

Repeat this procedure for all instances of $\mathsf{sr}\downarrow$ in the derivation. □

9.12 Lemma (Atomic Cut Elimination) *The rule* $\mathsf{sai}\uparrow$ *permutes over the rules* $\mathsf{w}\downarrow, \mathsf{ai}\downarrow, \mathsf{ds}\downarrow$ *and* $\mathsf{p}\downarrow$ *by the rule* $\mathsf{ds}\downarrow$*.*

Proof: Consider the derivation $\mathsf{sai}\uparrow \dfrac{\pi \dfrac{Q}{S([a,U],[\bar{a},V])}}{S[U,V]}$. If $\pi = \mathsf{ds}\downarrow$, then Lemma 9.6 applies. Now let $\pi \in \{\mathsf{w}\downarrow, \mathsf{ai}\downarrow, \mathsf{p}\downarrow\}$ be not trivial. The cases are:

(1) The redex of π is inside $S\{\ \}$. Trivial.

(2) The contractum $([a,U],[\bar{a},V])$ of $\mathsf{sai}\uparrow$ is inside a passive structure of the redex of π. Trivial.

(3) The redex of π is inside a passive structure U or V of the contractum of $\mathsf{sai}\uparrow$. Trivial.

(4) The redex of π is inside the contractum $([a,U],[\bar{a},V])$ of $\mathsf{sai}\uparrow$ but not inside U or V.

(i) $\pi = \mathsf{ai}\downarrow$ and $U = [\bar{a},U']$. Then

$$\mathsf{sai}\uparrow \frac{\mathsf{ai}\downarrow \frac{S([1,U'],[\bar{a},V])}{S([a,\bar{a},U'],[\bar{a},V])}}{S[\bar{a},U',V]} \qquad \text{yields} \qquad \mathsf{ds}\downarrow \frac{S([1,U'],[\bar{a},V])}{S[(1,[\bar{a},V]),U']} \quad .$$

(ii) $\pi = \mathsf{ai}{\downarrow}$ and $V = [a, V']$. Similar.

(5) The contractum $([a,U],[\bar{a},V])$ of $\mathsf{sai}{\uparrow}$ is inside an active structure of the redex of π but not inside a passive one. The only possible case is $\pi = \mathsf{w}{\downarrow}$ and $S([a,U],[\bar{a},V]) = S'\{?S''([a,U],[\bar{a},V])\}$. Then

$$\mathsf{sai}{\uparrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{S'\{\bot\}}{S'\{?S''([a,U],[\bar{a},V])\}}}{S'\{?S''[U,V]\}} \quad \text{yields} \quad \mathsf{w}{\downarrow}\,\dfrac{S'\{\bot\}}{S'\{?S''[U,V]\}} \quad .$$

(6) The redex of π and the contractum of $\mathsf{sai}{\uparrow}$ overlap. Not possible. □

9.13 Theorem (Cut Elimination) *The systems* $\mathsf{SELS} \cup \{1{\downarrow}\}$ *and* ELS *are equivalent.*

Proof: The proof follows the scheme of 9.4 and Figure 20, where Step 2 is realized by Lemma 9.8, Step 3 by Lemmata 9.9 and 9.10, and Steps 4 and 5 by Lemma 9.11 and Lemma 9.12, respectively. □

9.14 Remark Because of case (4) in the proof of Lemma 9.12, it might happen that after the whole cut elimination process, the obtained proof $\underset{R}{\Pi \Vert \mathsf{ELS}}$, is not of the shape

$$\begin{array}{c} 1{\downarrow}\,\dfrac{}{1} \\ \Vert \{\mathsf{w}{\downarrow}\} \\ R_4''' \\ \Vert \{\mathsf{ai}{\downarrow}\} \\ R_3''' \\ \Vert \{\mathsf{ds}{\downarrow},\mathsf{p}{\downarrow}\} \\ R_1 \\ \Vert \{\mathsf{b}{\downarrow}\} \\ R \end{array}$$

as shown in Figure 20. But it can easily be transformed into such a one by Lemma 6.6.

9.15 Remark The decomposition theorem (Theorem 8.1) is of great value for the proof of cut elimination. First, it shows that the non-core part of the up-fragment is admissible. And second, the rule $\mathsf{b}{\downarrow}$ is moved below the remaining rules of the up-fragment (namely, the rules $\mathsf{p}{\uparrow}$ and $\mathsf{ai}{\uparrow}$). This means that in the cut elimination process we do not have to deal with contraction nor absorption, which are known to be most problematic in cut elimination proofs.

10 Conclusions and Future Work

In this paper, I used a new proof theoretical formalism, the calculus of structures, in order to study a known logic, the multiplicative exponential fragment of linear logic (MELL).

The calculus of structures has originally been developed by Guglielmi for describing system BV [14, 15]. The logic of system BV consists of the multiplicative fragment of linear logic (MLL) plus mix plus one self-dual non-commutative connective. A similar logic, pomset logic [26], defied any sequent calculus presentation so far (it is still open whether both logics are the same or not). Recently it has been shown by Tiu [30] that there is no system in the sequent calculus which is equivalent to BV, by exploring the fact that deep inference is crucial for a deductive system for BV.

This justifies the existence of the calculus of structures. But can we also justify the study of logics, inside the calculus of structures, that have a sequent calculus presentation? I believe that we can answer positively for the following reasons:

Simplicity Although the calculus of structures is more general than the sequent calculus, it is not more complicated. System ELS is simple and elegant, but deeply different from system MELL in the sequent calculus.

Power The calculus of structures unveils new properties, like decomposition, that are not available in the sequent calculus.

Modularity There are two ways in which the calculus of structures presents a new modularity. First, the decomposition theorem allows for a decomposition of a big system into smaller "modules" that can be studied independently. This is important from the viewpoint of denotational semantics. Second, the proof of the cut elimination result becomes modular because the general cut rule is decomposed into several up rules that are shown to be admissible independent from each other. Furthermore, the cut elimination proof is not one big nested induction, but is based on permutation results, which remain valid if new rules are added to the system.

Symmetry In the calculus of structures derivations are not trees as in the sequent calculus but superpositions of trees that can also be flipped upside-down. This gives the calculus of structures a new top-down symmetry. This symmetry is responsible for the power of stating certain properties or conjectures that are unavailable in other proof theoretical formalisms.

There are two main reasons behind these results. The first is that rules can be applied deep inside structures, and the second is the dropping of the main connective. For example, the times rule in the sequent calculus must make an early choice in splitting its context, which is not the case with the switch rule in the calculus of structures. For the promotion rule the situation is similar. In the sequent calculus the rule is global, whereas in the calculus of structures it is local: pieces of context can be brought inside the scope of an of-course one by one.

These two main reasons for the advantages of the calculus of structures do at the same time cause a possible problem, namely proof search can become more non-deterministic. There is research in progress to focus proofs based on the logical relations along the lines of [2] and [21] as well as based on the depth of structures.

There are two immediate possibilities of extending system ELS. First, bringing the self-dual non-commutative connective of system BV to system ELS. In order to do this we have to add the equation $1 = \bot$, which gives us the unit $\circ$ of BV. In the language of the sequent calculus this is equivalent to adding the rules mix and nullary mix [1, 9, 25]. Then we have to add the rule *seq* of BV:

$$\mathsf{q}\!\downarrow \frac{S\langle [R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]} \quad ,$$

where $\langle R_1;\ldots;R_h\rangle$ is the composition of the structures $R_1,\ldots,R_h$ by the new self-dual non-commutative connective. In [17] the new system, called NEL, is discussed and and cut elimination is shown. Because of its self-duality the new non-commutative connective corresponds quite well to the notion of sequentiality in many process algebras. In [8] Bruscoli shows the correspondence to prefixing in CCS [23]. Moreover, recent research has shown that system NEL is Turing-complete [29]. If MELL turns out to be decidable (the problem is still open), the edge is crossed by the self-dual non-commutative connective of system BV, in the sense that we get a very simple propositional system that is undecidable without the use of additives.

The second immediate possibility of extending system ELS is, of course, by the additives of linear logic. I already have two different systems for full propositional linear logic in the calculus of structures. The first is simply system ELS extended by a few rules. The main ingredient is the rule

$$\mathsf{d}\!\downarrow \frac{S(\!\!\{[R,T],[U,V]\}\!\!)}{S[(\!\!\{R,U\}\!\!),\{\!\!\{T,V\}\!\!\}]} \quad ,$$

where the structures $\{\!\!\{R_1,\ldots,R_h\}\!\!\}$ and $(\!\!\{R_1,\ldots,R_h\}\!\!)$ stand for the additive disjunction and additive conjunction, respectively. I will not go into further details here. I just want to draw attention to the similarity of that rule and the rule

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \quad .$$

This maybe unveils a general pattern of philosophical interest.

The second system I have for full linear logic is more complex, because it consists of many more rules, but it has the advantage that all rules are local, in the same sense as in [7], where a local system for classical logic in the calculus of structures is presented. Particularly contraction (as well as absorption) can be reduced to an atomic version in the same way as it has been done for identity and cut in this paper. Both systems for full linear logic can be found in [28].

The calculus of structures is not only suitable for propositional logics, but also for first order logic. In [7, 5] the rules for first order predicative classical logic are shown. The rules for the first order predicative quantifiers in linear logic are very similar:

$$\mathsf{u}\!\downarrow \frac{S\{\forall x.[R,T]\}}{S[\forall x.R,\exists x.T]} \qquad \text{and} \qquad \mathsf{n}\!\downarrow \frac{S\{R\{x \leftarrow t\}\}}{S\{\exists x.R\}} \quad ,$$

where the De Morgan laws and the equations

$$\forall x.R = R = \exists x.R \qquad \text{if } x \text{ is not free in } R$$

are added. Again, observe the similarity between the rule $\mathsf{u}\downarrow$ and the promotion rule $\mathsf{p}\downarrow$. The two advantages over the sequent calculus rules that already occur in the classical case are also there in linear logic: First, there is no need for a proviso saying that the variable x is not free in the conclusion of the rule, and second, in both rules the premise implies the conclusion, without any further quantifications. This pattern can also be ported to the second order propositional quantifiers, where the rules are similar.

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Web Page

Information on the calculus of structures is available from the following web page (maintained by Alessio Guglielmi):

http://www.ki.inf.tu-dresden.de/~guglielm/Research/

References

[1] Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic. *Journal of Symbolic Logic*, 59(2):543–574, 1994.

[2] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.

[3] Jean-Marc Andreoli. Focussing and proof construction. *Annals of Pure and Applied Logic*, 107:131–163, 2001.

[4] Kai Brünnler. Atomic cut elimination for classical logic. Technical Report WV-02-11, Technische Universität Dresden, 2002.

[5] Kai Brünnler. Locality for classical logic. Technical Report WV-03-04, Technische Universität Dresden, 2003. Submitted.

[6] Kai Brünnler. Minimal logic in the calculus of structures, 2003. Note. On the web at: http://www.ki.inf.tu-dresden.de/~kai/minimal.html.

[7] Kai Brünnler and Alwen Fernanto Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347–361. Springer-Verlag, 2001.

[8] Paola Bruscoli. A purely logical account of sequentiality in proof search. In Peter J. Stuckey, editor, *Logic Programming, 18th International Conference*, volume 2401 of *Lecture Notes in Artificial Intelligence*, pages 302–316. Springer-Verlag, 2002.

[9] Arnaud Fleury and Christian Retoré. The mix rule. *Mathematical Structures in Computer Science*, 4(2):273–285, 1994.

[10] Gerhard Gentzen. Untersuchungen über das logische Schließen. I. *Mathematische Zeitschrift*, 39:176–210, 1934.

[11] Gerhard Gentzen. Untersuchungen über das logische Schließen. II. *Mathematische Zeitschrift*, 39:405–431, 1935.

[12] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[13] Jean-Yves Girard. *Proof Theory and Logical Complexity, Volume I*, volume 1 of *Studies in Proof Theory.* Bibliopolis, edizioni di filosofia e scienze, 1987.

[14] Alessio Guglielmi. A calculus of order and interaction. Technical Report WV-99-04, Technische Universität Dresden, 1999. Now obsolete and replaced by [15].

[15] Alessio Guglielmi. A system of interaction and structure. Technical Report WV-02-10, Technische Universität Dresden, 2002. Submitted. On the web at: http://www.ki.inf.tu-dresden.de/~guglielm/Research/Gug/Gug.pdf.

[16] Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In Laurent Fribourg, editor, *Computer Science Logic, CSL 2001*, volume 2142 of *LNCS*, pages 54–68. Springer-Verlag, 2001.

[17] Alessio Guglielmi and Lutz Straßburger. A non-commutative extension of MELL. In Matthias Baaz and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2002*, volume 2514 of *LNAI*, pages 231–246. Springer-Verlag, 2002.

[18] Joshua S. Hodas and Dale Miller. Logic programming in a fragment of intuitionistic linear logic. *Information and Computation*, 110(2):327–365, May 1994.

[19] W. A. Howard. The formulae-as-types notion of construction. In J. P. Seldin and J. R. Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 479–490. Academic Press, 1980.

[20] Joachim Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, 65:154–169, 1958.

[21] Dale Miller. Forum: A multiple-conclusion specification logic. *Theoretical Computer Science*, 165:201–232, 1996.

[22] Dale Miller, Gopalan Nadathur, Frank Pfenning, and Andre Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51:125–157, 1991.

[23] Robin Milner. *Communication and Concurrency*. International Series in Computer Science. Prentice Hall, 1989.

[24] Dag Prawitz. *Natural Deduction, A Proof-Theoretical Study*. Almquist and Wiksell, 1965.

[25] Christian Retoré. *Réseaux et Séquents Ordonnés*. Thèse de Doctorat, spécialité mathématiques, Université Paris 7, February 1993.

[26] Christian Retoré. Pomset logic: A non-commutative extension of classical linear logic. In Ph. de Groote and J. R. Hindley, editors, *Typed Lambda Calculus and Applications, TLCA'97*, volume 1210 of *Lecture Notes in Computer Science*, pages 300–318, 1997.

[27] Charles Alexander Stewart and Phiniki Stouppa. Modal logic in the calculus of structures. In preparation.

[28] Lutz Straßburger. A local system for linear logic. In Matthias Baaz and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2002*, volume 2514 of *LNAI*, pages 388–402. Springer-Verlag, 2002.

[29] Lutz Straßburger. The undecidability of system NEL. Technical Report WV-03-05, Technische Universität Dresden, 2003. On the web at: http://www.ki.inf.tu-dresden.de/~lutz/NELundeci.pdf.

[30] Alwen Fernanto Tiu. Properties of a logical system in the calculus of structures. Technical Report WV-01-06, Technische Universität Dresden, 2001. On the web at: http://www.cse.psu.edu/~tiu/thesisc.pdf.