Linear Logic and Noncommutativity in the Calculus of Structures

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Table of Contents

List of Figures

1

Introduction

1.1 Proof Theory and Declarative Programming

Proof theory is the area of mathematics which studies the concepts of mathematical proof and mathematical provability [Bus98]. It is mainly concerned with the formal syntax of logical formulae and the syntactic presentations of proofs, and can therefore be regarded as "logic from the syntactic viewpoint" [Gir87b]. An important topic of research in proof theory is the relation between finite and infinite objects. In other words, proof theory investigates how infinite mathematical objects are denoted by finite syntactic constructions, and how facts concerning infinite structures are proved by finite proofs. Another important question of proof theoretical research is the investigation of intuitive proofs [Kre68]. More precisely, not only the study of a given formal system is of interest, but also the analysis of the intuitive proofs and the choice of the formal systems needs attention.

These general questions of research are not the only ones that are investigated in proof theory, but they are the most important ones for the application of proof theory in computer science in general, and in declarative programming in particular. By restricting itself to finitary methods, proof theory studies the objects that computers (which are, *per se*, finite) can deal with.

In declarative programming the intention is to describe *what* the user wants to achieve, rather than *how* the machine is accomplishing it. By concerning itself with the relation between intuitive proofs and formal systems, proof theory can help to design declarative programming languages in such a way that the computation of the machine meets the intuition of the user.

This rather high level argumentation is made explicit and put on formal grounds by the two proof theoretical concepts of *proof normalisation* (or *proof reduction*) and *proof search* (or *proof construction*), which provide the theoretical foundations for the two declarative programming paradigms of *functional programming* and *logic programming*, respectively.

Proof normalisation and functional programming. The relation between the functional programming paradigm and proof theory is established by the *Curry-Howard-isomorphism* [CF58, How80, Tai68], which identifies formal logical systems, as they are studied in proof theory, with computational systems, as they are studied in type theory. More

1

precisely, a formula corresponds to a type and a proof of that formula to a term of the corresponding type. For example, natural deduction proofs of intuitionistic logic [Gen34, Pra65] correspond to terms of the simply typed λ-calculus [Chu40]. This bijective mapping between proofs and terms is an isomorphism because a normalisation step of the proof in the logical system corresponds exactly to a normalisation step of the λ-term, which in turn is a computation step in functional programming. Hence, the normalisation (or reduction) of the proof *is* the computation of the corresponding functional program. The isomorphism can be summarised as follows:

$$
\begin{aligned}
\text{formula} &= \text{type} \quad, \\
\text{(correct) proof} &= \text{(well-typed) program} \quad, \\
\text{proof normalisation} &= \text{computation} \quad.
\end{aligned}
$$

The correspondence does not only hold for propositional logic, but also for first order logic, which corresponds to dependent types (without inductive types), and second order logic, which corresponds to polymorphic types. A good introduction and historical overview for this can be found in [Wad00]. A detailed survey is [SU99]. For the relation between classical logic and type theory, the reader is referred to [Ste01].

The importance of the Curry-Howard-isomorphism for computer science is also corroborated by the fact that proofs of termination and correctness in computer science often reduce to known techniques in proof theory. The probably best known example for this is the proof of strong normalisation of the polymorphic λ-calculus (also known as system F) [Gir72], which is the basis for the type systems of functional programming languages like ML or Haskell.

Another application of the Curry-Howard-isomorphism is the possibility of extracting programs from proofs. For example, the Coq proof assistant (see e.g. [BC03]) can extract a certified program from the constructive proof of its formal specification. Another example is the NuPrl development system [CAB+86]. The basic idea is that from a constructive proof of

$$
\Phi \vdash \forall x.\exists y.P(x,y) \quad,
$$

where Φ is a set of hypotheses and P a binary predicate, it is possible to obtain a program that computes from a given input x an output y such that $P(x,y)$ holds. The types corresponding to the formulae in Φ are the types of parameters for the program.

Proof search and logic programming. The relation between logic programming and proof theory is more obvious because the paradigm of logic programming is already defined in terms of logic and proofs. A logic program is a conjunction of formulae, each of which can be seen as an instruction. The input to the program is another formula, called the *goal*. The computation is the search for a proof (also called proof construction) showing that the goal is a logical consequence of the program. We have the following correspondence [And01]:

$$
\begin{aligned}
\text{formula} &= \text{instruction} \quad, \\
\text{(incomplete) proof} &= \text{state} \quad, \\
\text{proof search} &= \text{computation} \quad.
\end{aligned}
$$

One example of a logic programming language is (pure) Prolog, which is based on the Horn fragment of classical logic. However, Horn clauses are not able to support concepts like modular programming or abstract data types, which are common in modern programming languages. For that reason, pure Prolog has been extended in several ways. There are essentially three possible approaches to do so [MNPS91]. The first is to mix concepts of other programming languages into Horn clauses, and the second is to extend a given interpreter by certain nonlogical primitives that provide aspects of the missing features. The third approach is to extend the underlying logic to provide a logical basis for the missing mechanisms. This raises the question in what way the logic should be extended. The solution must be found somewhere between the two extremes of Horn logic, which is weak but proof search can be implemented efficiently, and full first-order or higher-order logic, for which an all-purpose theorem prover has to serve as interpreter. Here, the proof theoretical concept of *uniform provability* [MNPS91] provides a criterion for judging whether a given logical system is an adequate basis for a logic programming language. A *uniform proof* is a proof that can be found by a goal-directed search, i.e. the logical connectives in the goal formula are interpreted as search instructions.

Although the first two approaches lead, in general, to an immediate and efficient extension of the language, they have the disadvantage of cluttering up the semantics and of obscuring the declarative reading of the programs [MNPS91]. The third approach might not immediately lead to an efficient solution, as the other two do, but it has the advantage that the extended language still has a clear semantics.

Through the notion of *uniform proof*, proof theory can help to design logic programming languages that are capable of supporting features like abstract data types without compromising the pure declarative paradigm. An example of such a language is λProlog [MN88, Mil95], which is based on *higher-order hereditary Harrop formulae* [MNPS91]. Their main extension with respect to Horn logic is that implication and quantification is allowed in the body of definite clauses, and that quantification over predicates is possible. λProlog supports modular programming, abstract data types and higher-order programming.

Having a pure declarative paradigm is not only important for having a clear semantics, it also plays a role when security issues are under consideration. An example is the concept of proof-carrying code, where the program carries a proof of its own correctness.

The proof theoretical foundations for both paradigms are provided by the *cut elimination* property, which is one of the most fundamental concepts of proof theory. The notion of cut elimination is tightly connected to the *sequent calculus* [Gen34], which is a proof theoretical formalism for presenting logical systems by specifying their proof rules. For example the rule

$$\wedge \frac{\Phi \vdash A \qquad \Phi \vdash B}{\Phi \vdash A \wedge B}$$

says that if one has a proof of A from hypotheses Φ and a proof of B from hypotheses Φ, then one can obtain a proof of $A \wedge B$ from the same hypotheses. This rule exhibits an important property that rules in the sequent calculus usually have: the premises of a rule contain only subformulae of the conclusion. This is called the *subformula property*. The only (propositional) rule that does not have this property is the *cut rule*. The precise shape of the cut rule varies from logical system to logical system, but it always expresses the transitivity of the logical consequence relation, i.e. it allows to use lemmata inside a proof.

For example in intuitionistic logic, the cut rule is of the following form:

$$\text{cut} \ \frac{\Phi \vdash A \qquad \Phi, A \vdash B}{\Phi \vdash B} \quad .$$

It says that if one can prove A from the hypotheses in Φ and one can prove B from the hypotheses in Φ enriched with A, then one can prove B from Φ directly. In other words, A is used as auxiliary lemma to prove B.

A logical system has the cut elimination property when for every proof in the system that uses the *cut rule*, there is a proof (with the same conclusion) that does not use the cut. In other words, cut elimination says that the transitivity of the logical consequence relation is in some way already contained in the other rules of the system. This rather surprising result has first been shown by G. Gentzen in [Gen34] for ⊢K and ⊢J, which are systems for classical and intuitionistic logic, respectively. An important consequence of cut elimination is the subformula property for proofs: If a formula A is provable, then there is a proof of A that contains only subformulae of A. Another consequence of cut elimination is the consistency of the system [Gen35].

It is important to note that cut elimination is a very fragile property. This means that it easily breaks down if a given system is modified. Consequently, a considerable amount of proof theoretical research is devoted to designing logical systems in such a way that they have the cut elimination property.

It should be mentioned that the construction of a cut-free proof from a proof with cut can have a drastic impact on the size of the proof, because it can lead to a hyperexponential blow-up of the proof.

There is a close relationship between the cut elimination procedure and proof normalisation [Zuc74]. Termination and confluence of cut elimination ensure termination and confluence, respectively, of normalisation. In the case of proof search, the cut elimination property ensures that proof search can make progress and that whenever the proof search fails then there is indeed no proof.

The properties of the functional and logic programming languages depend on the properties of the underlying logical systems. In order to increase the variety and applicability of those languages, it is therefore necessary to design new logical systems and to study their proof theoretical properties.

From the beginning, the main sources of applications of proof theory to computer science were classical and intuitionistic logic. Whereas classical logic is based upon truth values, intuitionistic logic is based upon proofs. In classical logic the meaning of a connective is given by explaining how the truth value of a compound formula can be obtained from the truth values of the constituents. This is usually done by means of truth tables. For example the conjunction $A \wedge B$ is true if A is true and B is true. Similarly, the implication $A \Rightarrow B$ is true if A is false or B is true. Intuitionistic logic is not based on the *a priori* existence of truth values (although it is possible to give a truth values semantics for it, for example, via Heyting algebras or Kripke frames). In intuitionistic logic the meaning of a connective is given by describing how a proof of the compound formula can be obtained from proofs of the constituents. For example a proof of the conjunction $A \wedge B$ consists of a pair of proofs, one for A and one for B. Similarly, a proof of the implication $A \Rightarrow B$ consists of a function mapping a proof of A to a proof of B. This is known as Brouwer-Heyting-Kolmogorov interpretation [Bro24, Hey34, Kol32].

Despite the success that classical and intuitionistic logic have in many areas of computer science, they have certain limitations that prevent them from dealing naturally with certain aspects of computer science, as they occur, for instance, in concurrency theory and in artificial intelligence. Examples of these limitations are their lack of resource consciousness and the inability of dealing with noncommutativity. For that reason, *linear logic* and several other, so called *substructural logics* are being investigated.

1.2 Linear Logic

Linear logic has been conceived by J.-Y. Girard as a refinement of intuitionistic logic [Gir87a]. The two main novelties of linear logic are its resource consciousness and its notion of duality, which is represented by *linear negation*, denoted by $(\cdot)^\perp$.

Resource consciousness means that it matters how often a hypothesis is used inside a proof. In particular, there are two possible conjunctions in linear logic. Proving A "and" B from some hypotheses Φ can mean, either that the hypotheses have to be shared between the proofs of A and B, or that both proofs have access to all of Φ. The former, written as $A \otimes B$, is called *multiplicative conjunction*, and the latter, written as $A \mathbin{\&} B$, is called *additive conjunction*. The corresponding proof rules are (in the one-sided sequent calculus):

$$\otimes \frac{\vdash A, \Phi \qquad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \qquad \text{and} \qquad \mathbin{\&} \frac{\vdash A, \Phi \qquad \vdash B, \Phi}{\vdash A \mathbin{\&} B, \Phi} \quad .$$

The notion of resource consciousness has also been studied in artificial intelligence, in the area of planning [Bib86, HS90], where actions consume and create resources. The relation between planning and a small fragment of linear logic has been investigated in [GHS96]. A more recent work on the use of linear logic for planning problems is [KV01].

An example from the area of planning can be used for better illustrating the difference between the two disjunctions. If we let A stand for having €1 and B for having one loaf of bread and C for having one croissant, then $A \vdash B$ can be used to say that with €1 one can buy a loaf of bread at the bakery, and similarly, $A \vdash C$ can be interpreted as spending €1 for buying a croissant. Then $A \vdash B \otimes C$ is not provable because the money can be spent only for one item. One would need €2 to buy both, bread and croissant, i.e. $A \otimes A \vdash B \otimes C$. However, we have that $A \vdash B \mathbin{\&} C$, with the meaning that with €1 one can buy one of the two items and the customer can choose which he wants.

Although such examples can help to get some intuition for the different connectives—a similar example has been used by J.-Y. Girard in [Gir96]—the reader should be warned that these examples are misleading in the sense that they give a wrong idea of what linear logic is about. In his recent work [Gir01, p. 399], J.-Y. Girard gives a more refined elaboration on the relation between his work and the area of artificial intelligence.

Similar to intuitionistic logic, linear logic is not based on the *a priori* existence of truth values, but it has a truth value semantics, which is given by *phase spaces* [Gir87a, Laf97]. From a proof theoretical viewpoint, (denotational) semantics for proofs are more interesting. This means that the semantics does not ask the question "When is A *true*?" but "What is a *proof* of A?" [GLT89].

An example of such a semantics is *games semantics* [Bla92, Bla96, AJ94, HO93, Lam95, Lam96, LS91], where formulae are interpreted as two-person games, or debates. While the

first player, called proponent, tries to show that the formula is provable, the second player, called opponent, tries to show that there is no proof. A proof of a formula is then given by a winning strategy for proponent. This idea goes back to [Lor61] and [Lor68], where games semantics has been investigated for intuitionistic logic. In [Bla92], A. Blass observed that linear logic is the right context for games semantics because linear negation can be interpreted naturally as exchanging roles between the two players. This means that the duality of linear logic corresponds to the duality between proponent and opponent.

Linear negation can be used to dualise the two conjunctions to obtain two disjunctions:

$$A \oplus B = (A^\perp \mathbin{\&} B^\perp)^\perp \qquad \text{and} \qquad A \mathbin{⅋} B = (A^\perp \otimes B^\perp)^\perp \quad ,$$

where $\oplus$ is called *additive disjunction* and $⅋$ is called *multiplicative disjunction*. In the example above, $A \vdash B \oplus C$ would have the intuitive meaning that the customer gets either the bread or the croissant, but the choice is made by the shop assistant.

The difference between the additive conjunction $\&$ and the additive disjunction $\oplus$ can also be explained from the viewpoint of games semantics. If a formula $A \oplus B$ is considered, then proponent can choose on which constituent the game or debate is continued, and in the case of $A \mathbin{\&} B$, the opponent has the choice [Bla92, Bla96]. Consequently, a winning strategy for $A \mathbin{\&} B$ consists of a pair of winning strategies, one for A and one for B; and a winning strategy for $A \oplus B$ consists of a winning strategy, either for A or for B, together with an indication whether it is for A or for B.

The multiplicative disjunction $⅋$ can be interpreted as playing two games in parallel, where proponent has to win one of them. With this interpretation in mind, it can be intuitively explained why the formula $A \mathbin{⅋} A^\perp$ is provable: The proponent in the game for $A \mathbin{⅋} A^\perp$ plays simultaneously two copies of the game A, one as proponent and one as opponent. The winning strategy is the following: Proponent copies in each game the move of his adversary in the other game. Then he certainly wins in exactly one of the two games (it is assumed that every game is finite and has a winner).

Another possible intuition for linear logic comes from concurrency, where $A \mathbin{⅋} B$ can be seen as parallel composition of processes A and B, that can communicate. The additive disjunction $\oplus$ is then interpreted as choice operator. In this context, linear negation represents the duality between input and output. The formula $A \mathbin{⅋} A^\perp$ can then be interpreted as two parallel processes, one sending message A and the other receiving it. The provability of $A \mathbin{⅋} A^\perp$ resembles the transition $a|\bar{a} \to 0$, as it occurs in process algebras like CCS [Mil89].

Linear logic also has two modalities (called *exponentials*), denoted by ! and ?, which are dual to each other. The formula $!C$ means that C can be used an arbitrary number of times inside a proof. In the example shown above, the formula $!C$ would express an infinite supply of croissants.

In summarising, one can say that linear logic is able to naturally express certain phenomena that occur in computer science. The probably most significant aspect of linear logic is that the concepts of resource consciousness and duality are combined in a single coherent logical system that has the cut elimination property. In fact, the introduction of linear logic led to a considerable increase of research in proof theory and its application to computer science. One hope is, for example, to conceive a computational model (like the λ-calculus), that is based on linear logic, in the same way as the λ-calculus is based on intuitionistic logic [Wad91, Wad92, MOTW95, BW96].

The maybe most harmful drawback of linear logic is that, in the eyes of many, there is no natural semantics, although there exist various kinds of denotational semantics for linear logic. Besides the various forms of games semantics, which I have mentioned before, there are also *coherence spaces* [Gir87a], which provide a natural interpretation of proofs of linear logic. (In fact, linear logic has been discovered on the grounds of coherence spaces.) In addition to that there are many other kinds of semantics including Banach spaces [Gir96] and categorical axiomatisations [Bar91, See89].

Much of the difficulty of finding a natural semantics is probably caused by the fact that one has to switch between various viewpoints in order to grasp the intuition behind the various connectives. For example the multiplicative conjunction $\otimes$ is more intuitive from the viewpoint of actions and resources, whereas linear negation is more intuitive from the viewpoint of games. In particular, there is still no formal semantics that meets the intuition of resources as it has been discussed in the beginning of this section, although most "real-world applications" of linear logic are based on that. For example, linear logic has been used for specifying and verifying the TCP/IP protocols [Sin99].

I believe that one reason for the somewhat experimental stage of the semantics is that the syntax of linear logic is rather underdeveloped and not yet well enough understood. This has consequences for applications of linear logic in computer science. The problems in the presentation of logic in general, and of linear logic in particular, translate into limitations in the applicability of proof theory. Consequently, it is necessary to find new presentations for linear logic, i.e. to develop new logical systems, that are equivalent to linear logic, but have better proof theoretical properties. Positive evidence for this is, for example, Forum [Mil94, Mil96, HP96]. On one side Forum can be seen as just another presentation of linear logic (the set of provable formulae did not change), and on the other side Forum can be seen as a first-class logic programming language because uniform proofs are complete for it. In particular, it is the only logic programming language that can model the concept of state in a natural way and can incorporate simple notions of concurrency. It has been successfully employed as specification language. For example in [Chi95], the sequential and pipelined operational semantics of DLX, a prototypical RISC processor, were specified and shown to be equivalent. In the same work, Forum is used for specifying many of the imperative features of Standard ML, like assignable variables and exceptions, and for proving equivalences of code phrases.

Besides that, it can be argued that for many applications in computer science linear logic is too simple. In particular from the viewpoint of concurrency this is not far-fetched. Compared to the rich syntax of process algebras like CCS or the π-calculus [Mil01], the syntactic possibilities of linear logic seem quite poor. However, there exist preliminary results showing that a relation between process algebras and logical systems can be established [Mil92, Bru02c]. In order to pursue the idea of using proof theoretical methods in concurrency theory, it is therefore necessary to develop new logical systems that go beyond linear logic.

Let me conclude this section by stating the following two research problems which I consider important and which have implicitly already been mentioned.

(1) Find better syntactic presentations for linear logic. More precisely, design new logical systems that are logically equivalent to linear logic and that have better proof theoretical properties. This means, in particular, that they should better express the

mutual relationship between the various connectives of linear logic, and that they should allow for new kinds of normal forms for derivations and proofs.

(2) Design new logical systems that go beyond linear logic, in the sense that they are more expressive and have more syntactic possibilities, such that they are better suited for applications in computer science, without losing good proof theoretical properties, like cut elimination. An example of such a new syntactic possibility is the ability of dealing with *noncommutativity*.

1.3 Noncommutativity

Noncommutative operations occur in many places in computer science. One fundamental example is the sequential composition of processes, which plays an important role in concurrency theory. Similarly, in the context of planning in artificial intelligence, the order in which two actions are carried out is of importance.

Even in the case of logic programming, where commutativity is desired (the goal $\vdash A, B$ holds if and only if the goal $\vdash B, A$ does hold), noncommutative methods are needed to understand different computational behaviours of evaluating A before B and vice versa [BGLM94].

For propagating the idea of using proof theoretical methods for computer science, it is therefore necessary to make the concept of *noncommutativity* accessible to proof theory. This means that logics with noncommutative connectives should be studied. The first logic of this kind was the *Lambek calculus* [Lam58] (introduced by J. Lambek for studying the syntax of natural languages), which contains a noncommutative conjunction together with two implications (left and right). After the introduction of linear logic, several proposals for a noncommutative linear logic have been made: D. Yetter's *cyclic linear logic* [Yet90], M. Abrusci's noncommutative logic with two negations [Abr91] or the noncommutative logic used by P. Lincoln et al. in [LMSS92]. There is more than one version of "a noncommutative linear logic" because forcing the multiplicative conjunction to be noncommutative opens several possibilities for treating linear negation. Without going too much into detail, the important observation to make is that all these logics are purely noncommutative, i.e. contain no (multiplicative) connective that is commutative. But there are various applications in computer science that require commutative as well as noncommutative operations. It is therefore desirable to design logics that contain both, commutative and noncommutative connectives. The problem has been known to be difficult and up to now there exist two different approaches.

The first is to extend the multiplicative conjunction $\otimes$ and the multiplicative disjunction $\invamp$ of linear logic by a self-dual noncommutative connective, that I will here denote by $\lhd$. Self-duality of $\lhd$ means that it is its own dual with respect to linear negation:

$$(A \lhd B)^{\perp} = A^{\perp} \lhd B^{\perp} \quad .$$

This idea did first occur in C. Retoré's *pomset logic* [Ret93] and was then taken up in A. Guglielmi's *system* BV [Gug99]. In fact, I conjecture that both systems are different presentations of the same logic.

The second approach for having commutative and noncommutative connectives inside a single logical system is to have two pairs of multiplicative connectives, one commutative (i.e.

the $\otimes$ and $\wp$ from linear logic) and one noncommutative. The noncommutative conjunction, denoted by $\odot$, and the noncommutative disjunction, denoted by ∇, are dual to each other with respect to linear negation:

$$(A \odot B)^\perp = B^\perp \nabla A^\perp \qquad \text{and} \qquad (A \nabla B)^\perp = B^\perp \odot A^\perp \quad .$$

Observe that the order of A and B is inversed, which was not the case in the first approach of having only one noncommutative connective. This approach has been presented in M. Abrusci's and P. Ruet's *noncommutative logic* [AR00, Rue00], which is a conservative extension of linear logic and cyclic linear logic.

Both approaches can be presented naturally in *proof nets*, which have been introduced by J.-Y. Girard for normalizing proofs of linear logic [Gir87a]. The motivation for this was that the traditional proof theoretical formalism for studying the normalisation of proofs, *natural deduction* [Pra65], was not suited for linear logic.

A similar situation now occurs with another traditional proof theoretical formalism for studying logical systems and their properties, the *sequent calculus* [Gen34], which has problems with both approaches for combining commutative and noncommutative connectives inside a single logical system.

In the case of the noncommutative logic, which is inhabited by two pairs of multiplicative connectives, sequents are no longer sets, multisets or lists of formulae, as it is the case in traditional systems, but *order varieties* of formulae. (An order variety is a set equipped with a ternary relation satisfying certain properties.)

For the other approach, where a single self-dual noncommutative connective is added, the situation is more drastic: for pomset logic, even after 10 years of research no presentation in the sequent calculus could be found; and for BV, A. Tiu has shown that it is impossible to present it in the sequent calculus [Tiu01].

As a consequence, research in proof theory has been confronted with the following problem:

(3) Develop a new proof theoretical formalism that is able to present new logics that defy the sequent calculus. The new formalism should have at least the same flexibility as the sequent calculus and should allow to investigate the same proof theoretical properties. In particular, it should come with methods for proving cut elimination and for doing proof search.

A solution to this problem is given by the *calculus of structures*.

1.4 The Calculus of Structures

The *calculus of structures* has been conceived by A. Guglielmi in [Gug99] for presenting system BV. It is a proof theoretical formalism for presenting logical systems and studying the properties of proofs. By this, it follows the tradition of D. Hilbert's formalism [Hil22, Hil26, HA28], natural deduction [Gen34, Pra65], the sequent calculus [Gen34, Gen35], and proof nets [Gir87a].

Since the calculus of structures can be seen as generalisation of the sequent calculus, I will here explain its basic ideas from the viewpoint of the sequent calculus.

On a technical level, the main difference is that inference rules do not operate on sequents, which are sets, multisets or lists of formulae, but on *structures*, which are equivalence classes of formulae. Structures can be seen as intermediate expressions between formulae and sequents. They are built from atomic expressions via various syntactic constructions as it is usually done for formulae, and they are subject to laws like associativity or commutativity as usually imposed on sequents. Such a notion of structure first occurs in N. Belnap's *display logic* [Bel82] and is established in the tradition of philosophical and substructural logics [Res00]. However, an important difference is that in the calculus of structures, structures are the only kind of expression allowed. They are built from atoms and not from formulae as in [Bel82, Res00]. As a consequence, binary connectives disappear and there is no longer a difference between logical rules and structural rules. All rules (apart from identity and cut) are structural in the sense that they rearrange substructures inside a structure. In particular, the notion of main connective loses its pivotal role.

The second technical difference to the sequent calculus is that in the calculus of structures every rule has only one premise. This means that there is no branching in a proof. Proofs in the calculus of structures are not trees of instances of inference rules as in the sequent calculus, but chains or sequences of instances of inference rules.

On a more fundamental level, the differences to the sequent calculus are more drastic. The calculus of structures draws from the following basic principles:

- *Deep inference:* Inference rules can be applied anywhere deep inside structures.

- *Top-down symmetry:* Derivations are no longer trees but "superpositions" of trees, which are upward- and downward-oriented. This yields a top-down symmetry for derivations.

From a proof theoretical viewpoint, the idea of deep inference did already occur in [Sch60], but the systems presented there are highly asymmetric. A weak form of deep inference is also available in display logic [Bel82], but again, there is no top-down symmetry present. Nevertheless, it is a substantial task for future research to investigate the precise relation between the calculus of structures and display logic.

In computer science, the notion of deep rewriting is well-known from term rewriting (e.g. [BN98]). The resemblance between the calculus of structures and term rewriting systems becomes even more vivid when the inference rules in the calculus of structures are inspected. Usually they are of the shape

$$\rho \, \frac{S\{T\}}{S\{R\}} \quad ,$$

where premise and conclusion are structures. A structure $S\{R\}$ is a structure context $S\{\ \}$, whose hole is filled by the structure R. The rule scheme ρ above specifies that if a structure matches the conclusion, in an arbitrary context $S\{\ \}$, it can be rewritten as in the premise, in the same context $S\{\ \}$ (or vice versa if one reasons top-down). In the terminology of term rewriting this scheme describes a rewriting step modulo an equational theory (since structures are equivalence classes of formulae) where R is the redex and T the contractum.

Because of this resemblance, the calculus of structures might help to establish a bridge between term rewriting and proof theory, in the sense that results from one area can be applied to the other.

However, one should observe that it is not (trivially) possible to present the calculus of structures in general in the terminology of term rewriting (although it might be possible to translate some systems in the calculus of structures into term rewriting systems). There are two main reasons: first, rules in the calculus of structures are not necessarily context independent, and second, negation needs special attention.

To a proof theorist, a probably quite appealing property of calculus of structures is the explicit duality between identity and cut. For example, for linear logic, the two rules are given in the sequent calculus as follows:

$$\text{id} \frac{}{\vdash A, A^\perp} \qquad \text{and} \qquad \text{cut} \frac{\vdash A, \Phi \qquad \vdash A^\perp, \Psi}{\vdash \Phi, \Psi} \quad .$$

It demands some proof theoretical insight to understand the duality between these two rules because from a syntactic viewpoint they are not dual to each other. The syntactic restrictions of the sequent calculus prevent the duality from appearing explicitely in the rules. In the calculus of structures, identity and cut for linear logic are given as follows:

$$\text{i}\downarrow \frac{S\{1\}}{S\{A \,\mathbin{\text{⅋}}\, A^\perp\}} \qquad \text{and} \qquad \text{i}\uparrow \frac{S\{A \otimes A^\perp\}}{S\{\perp\}} \quad ,$$

where 1 and $\perp$ are the units for the binary connectives $\otimes$ and $\mathbin{\text{⅋}}$, respectively. The duality between the two rules is now explicitly reflected in the syntax.

For logics with an involutive negation (i.e. the double negation of a proposition is logically equivalent to the proposition itself), like classical and linear logic, this duality can be observed for all rules, meaning that for every rule

$$\rho\downarrow \frac{S\{T\}}{S\{R\}} \quad \text{(down rule)} \quad ,$$

there is a dual (co-)rule

$$\rho\uparrow \frac{S\{\bar{R}\}}{S\{\bar{T}\}} \quad \text{(up rule)} \quad .$$

This duality derives from the duality between $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$, where $\bar{\cdot}$ and $\Rightarrow$ are the negation and implication, respectively, of the underlying logic. In the case of linear logic these are linear negation, denoted by $(\cdot)^\perp$, and linear implication, denoted by $\multimap$. The cut elimination property can then be generalised: instead of eliminating only the cut rule, it becomes possible to eliminate all up rules. This is the basis for a modular decomposition of the cut elimination argument because the cut rule can be decomposed into several up rules and one can eliminate the up rules independently from each other. This can be seen as another important feature of the calculus of structures because in [Gir87b], J.-Y. Girard has argued that the lack in modularity is one of the two main technical limitations in current proof theory.

The calculus of structures derives its ability of presenting logics that defied the sequent calculus from the new freedom of designing inference rules. In fact, any axiom $T \Rightarrow R$ of a generic Hilbert system, where there is no structural relation between T and R, could be used as an inference rule. But then all the good proof theoretical properties that systems

in the sequent calculus usually have would be lost. This means that the main challenge in developing logical systems in the calculus of structures lies in designing the rules carefully enough such that it is still possible to prove cut elimination and to have (an equivalent of) the subformula property.

Besides linear logic and a form of noncommutativity, which will be studied in this thesis, several other logical systems, including classical logic [BT01, Brü03b, BG03], minimal logic [Brü03c], several modal logics [SS03], and cyclic linear logic [Gia03], have been presented in the calculus of structures.

1.5   Summary of Results

Before I summarise the results that are contained in this thesis, let me recall the three problems mentioned in Sections 1.2 and 1.3:

(1) Design new logical systems that are equivalent to linear logic, but have better proof theoretical properties.

(2) Design new logical systems that are more expressive than linear logic (with respect to certain computational phenomena).

(3) Develop a proof theoretical formalism for presenting these new systems.

The calculus of structures (which is an outcome of research on the third problem), is a good candidate for helping to make progress on the first two problems because of the new proof theoretical properties that are associated to it. Consequently, in this thesis I study several logical systems presented in the calculus of structures and explore their properties. Some of these systems are equivalent to linear logic or its fragments and others go beyond linear logic. The investigation of these systems does not only contribute to problems (1) and (2), but also helps to establish the calculus of structures as a first-class proof theoretical formalism.

In order to give the reader some guidance to the following description of the major results, I have shown in Figure 1.1 the most important logical systems occurring in this thesis. Although not all of them are mentioned in this introduction, I included them in the figure, in case the reader wishes to consult it again later while reading this work. The figure shows a graph, where the nodes are labeled by the names of logical systems in the calculus of structures. The annotations in parentheses are the names of the equivalent sequent systems, where $\varnothing$ means that there is no sequent system. An arrow from one system to another means that the latter one can be obtained from the former one by adding what is written on the arrow. For example NEL is obtained from BV by adding the exponentials. If the arrow is solid, then the second system is a conservative extension of the first one and if the arrow is dotted, then this is not the case. For example, LS and LLS are conservative extensions of ELS, but ELS° is not.

There are three main results contained in this thesis:

- **Linear logic in the calculus of structures.** First, I present system LS in the calculus of structures, which is equivalent to the traditional system LL for linear logic in the sequent calculus. System LS is not a "trivial" translation of LL, but naturally employs the new symmetry of the calculus of structures, and exhibits the mutual
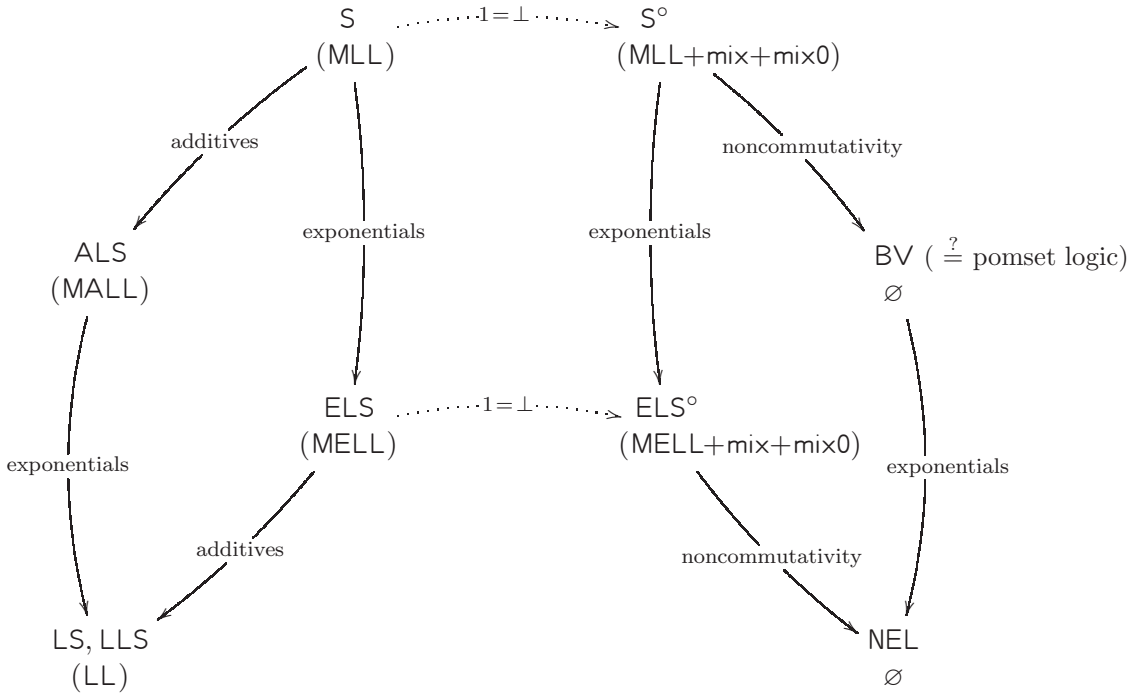
Figure 1.1: Overview of logical systems discussed in this thesis

relationship between the connectives of linear logic better than the sequent calculus system. The two maybe most surprising facts about system LS are the following:

– The global promotion rule of LL (it is called global because it requires inspecting the whole sequent, when applied) is replaced by a local version.

– The additive conjunction rule (see Section 1.2) is decomposed into two rules

  ∗ a "multiplicative" rule, that deals with both additive conjunction and additive disjunction at the same time (the rule is multiplicative in the sense that no context has to be shared), and

  ∗ a contraction rule for dealing with the additive context treatment (i.e. the duplication of formulae).

  This means that the distinction between "multiplicative" and "additive" context treatment is intrinsically connected to the sequent calculus and not to linear logic.

For system LS, I present a cut elimination result and a syntactic proof of it, which is independent from the sequent calculus and which is based on a new method called *splitting* [Gug02e]. This proof shows that the calculus of structures is a proof theoretical formalism with its own syntactic methods for studying cut elimination.

Besides that, I will present a local system for linear logic (system LLS). The system is local in the sense that no rule requires a global knowledge of a certain

substructure. This means that the amount of computational resources (i.e. time and space) that is needed for applying the rules is bounded. This is achieved by having only two kinds kinds of rules in the system:

– Either a rule operates only on atoms, i.e. no generic structures are involved,

– or a rule only rearranges substructures of a structure, without duplicating, deleting, or comparing the contents of a substructure.

This means in particular, that there are no rules like the generic contraction rule

$$?c \frac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi} \quad ,$$

which requires to duplicate a formula of unbounded size while going up in a derivation. In fact, it has been shown in [Brü02b] that in the sequent calculus it is impossible to restrict contraction to an atomic version.

- **Decomposition.** For linear logic and its fragments, I show several decomposition theorems. In general, decomposition is the property of being able to separate any given derivation or proof into several phases, each of which consists of applications of rules coming from mutually disjoint fragments of a given logical system.

  In principle, decomposition is not a new property in proof theory. For example, in natural deduction one can decompose any derivation into an 'introduction' phase and an 'elimination' phase, and this corresponds to a proof in normal form. In the sequent calculus, an example of a decomposition result is Herbrand's theorem that allows a separation between the rules for the quantifiers and the rules for the connectives. But apart from these rather simple consequences of normalisation and cut elimination, decomposition does not receive much attention.

  However, in the calculus of structures, decomposition reaches an unprecedented level of detail. For example, in the case of multiplicative exponential linear logic (MELL), it is possible to separate seven disjoint subsystems.

  The decomposition theorems can be considered as the most fundamental results of this thesis, for the following three reasons:

  – They embody a new kind of normal form for proofs and derivations. Furthermore, because of the new top-down symmetry of the calculus, they are not restricted to derivations without nonlogical axioms (as it is the case for cut elimination), but they do hold for arbitrary derivations, where proper hypotheses are present.

  – They stand in a close relationship to cut elimination. More precisely, cut elimination can be proved by using decomposition. In this case the hyperexponential blow-up of the proof is caused by obtaining the decomposition (i.e. a certain normal form of the proof in which the cut is still present), and not by the elimination of the cut afterwards.

  – They show that the calculus of structures is more powerful than other proof theoretical formalisms because it allows to state properties of logical systems that are not observable in other formalisms.

Besides the general proof theoretical interest, a possible use of the decomposition theorems lies in denotational semantics. Through the decomposition of proofs and derivations into subderivations carried out in smaller subsystems, it becomes possible to study the semantics of those subsystems separately. The results can then be put together. By this, the decomposition theorems can also help in the search for new semantics of proofs.

- **Noncommutativity.** I will present a logical system, called NEL, which is an extension of MELL by A. Guglielmi's self-dual noncommutative connective *seq*. More precisely, it is a conservative extension of A. Guglielmi's system BV [Gug99] and multiplicative exponential linear logic with mix and nullary mix (system ELS°). Since system NEL is a conservative extension of BV, it follows immediately from the result by A. Tiu [Tiu01] that there is no sequent system for NEL. For this reason, system NEL is presented in the calculus of structures.

  For system NEL, I will show cut elimination by combining the techniques of splitting and decomposition. Proving cut elimination for NEL has been difficult because there is no equivalent system in the sequent calculus. As a consequence, completely new techniques had to be developed to prove cut elimination. In fact, the two techniques of decomposition and splitting have originally been invented for that problem, but are now considered as results of independent interest.

  Furthermore, I will show that NEL is Turing-complete by showing that any computation of a two counter machine can be simulated in NEL. This result will gain importance in the case that MELL is decidable (which is still an open problem), because then the border to undecidability is crossed by A. Guglielmi's *seq*.

  Since the *seq* corresponds quite naturally to the sequential composition of processes, system NEL might find applications in concurrency theory, as the work by P. Bruscoli [Bru02c] shows.

Before going on, the reader should be aware of the fact that work in proof theory tends to suffer from long and technical proofs of short and concise theorems. Gentzen's groundbreaking work [Gen34, Gen35] is one example. Unfortunately, this thesis is no exception. Although most of the definitions are simple, the theorems are stated in a concise form and the statements of the theorems are easy to understand, the proofs are quite long and tedious because of complex syntactic constructions.

It is quite paradoxical that on one side syntax is the playground of proof theory and on the other side syntax seems to be the greatest obstacle in proof theoretical research.

1.6 Overview of Contents

In Chapter 2, I will introduce the well-known sequent calculus system LL for linear logic and show the basic ideas of the cut elimination argument in the sequent calculus. The reader who is already familiar with linear logic is invited to skip this chapter.

Then, I will introduce the calculus of structures in Chapter 3, and present system LS for linear logic. I will show the equivalence to the sequent calculus system LL and the cut elimination result.

Chapter 4 is devoted to the multiplicative exponential fragment of linear logic. I will first study the permutation of rules and then show two decomposition results and a proof of

cut elimination based on rule permutation. As a consequence, I will obtain an interpolation theorem for multiplicative exponential linear logic. Referring to Figure 1.1, the systems MELL and ELS are investigated in this chapter.

In Chapter 5, I will present system LLS, the local system for linear logic. In particular, I will explain, how contraction can be reduced to an atomic version. Furthermore, I will show several decomposition results for LLS. The reader who is only interested in locality can just read Sections 3.1 to 3.3, Section 3.5, and Chapter 5. On the other hand, if the reader is not interested in locality, he can skip this chapter entirely.

In Chapter 6, I will move the focus from the left-hand side of Figure 1.1 to the right-hand side. More precisely, I study the systems ELS° and S°, which means that I show how the rules mix and nullary mix are incorporated in the calculus of structures.

Finally, system NEL is investigated in Chapter 7. I will show that it is a conservative extension of ELS° and BV. Then, I will extend the two decomposition theorems of Chapter 4 to system NEL. Further, I will present the cut elimination proof for NEL based on splitting. Finally, the undecidability of NEL is shown via an encoding of two counter machines.

The results of Chapter 4 have partially been published in [GS01, Str01], the results of Chapters 3 and 5 have partially been published in [Str02], and the results of Chapter 7 can be found in [GS02a, GS02b] and [Str03a, Str03b].

2

Linear Logic
and the Sequent Calculus

The sequent calculus has been introduced by G. Gentzen in [Gen34, Gen35] as a tool to reason about proofs as mathematical objects and to manipulate them. It has been used since then by proof theorists as one of the main tools to describe logical systems and to study their properties. In this chapter, I will define linear logic, introduced by J.-Y. Girard in [Gir87a], in the formalism of the sequent calculus. By exploiting the De Morgan dualities, I will restrict myself to the one-sided sequent calculus [Sch50].

The reader who is already familiar with linear logic is invited to skip this chapter.

2.1   Formulae and Sequents

As the name suggests, in the sequent calculus, rules operate on sequents. Sequents are, in the general case, syntactic expressions built from formulae, which in turn are syntactic expressions built from atoms by means of the connectives of the logic to be studied. Since in this thesis I will discuss only propositional logics, the atoms are the smallest syntactic entities to be considered, i.e. there are no terms and relation symbols from which atoms are built.

Here, I will study linear logic, whose atoms and formulae are defined as follows.

2.1.1   Definition   There are countably many *atoms*, denoted by $a, b, c, \ldots$ The set of atoms, denoted by $\mathcal{A}$, is equipped with a bijective function $(\cdot)^{\perp} : \mathcal{A} \to \mathcal{A}$, such that for every $a \in \mathcal{A}$, we have $a^{\perp\perp} = a$ and $a^{\perp} \neq a$. There are four selected atoms, called *constants*, which are denoted by $\perp$, 1, 0, and $\top$ (called *bottom*, *one*, *zero*, and *top*, respectively). The function $(\cdot)^{\perp}$ is defined on them as follows:

$$
\begin{aligned}
1^{\perp} &:= \perp \quad , & \perp^{\perp} &:= 1 \quad , \\
\top^{\perp} &:= 0 \quad , & 0^{\perp} &:= \top \quad .
\end{aligned}
$$

The *formulae of linear logic*, or LL *formulae*, are denoted with $A, B, C, \ldots$, and are built from atoms by means of the (binary) connectives $\parr$, $\otimes$, $\oplus$, and $\&$ (called *par*, *times*, *plus*, and

with, respectively) and the modalities ! and ? (called *of-course* and *why-not*, respectively). *Linear negation* is the extension of the function $(\cdot)^\perp$ to all formulae by De Morgan equations:

$$(A \otimes B)^\perp := A^\perp \mathbin{\invamp} B^\perp \quad, \qquad (A \mathbin{\invamp} B)^\perp := A^\perp \otimes B^\perp \quad,$$
$$(A \mathbin{\&} B)^\perp := A^\perp \oplus B^\perp \quad, \qquad (A \oplus B)^\perp := A^\perp \mathbin{\&} B^\perp \quad,$$
$$(!A)^\perp := ?A^\perp \quad, \qquad (?A)^\perp := !A^\perp \quad.$$

Linear implication $\multimap$ is defined by $A \multimap B = A^\perp \mathbin{\invamp} B$.

2.1.2   Remark   I will say formula instead of LL formula if no ambiguity is possible. However, since more logics will be discussed in this thesis, I will have to define different notions of formula. Then, I will say LL formula in order to avoid ambiguities.

2.1.3   Remark   Since for all atoms we have $a = a^{\perp\perp}$, it follows immediately from Definition 2.1.1 (by induction on formulae and a case analysis) that also $A = A^{\perp\perp}$ for every formula A.

2.1.4   Definition   The connectives $\mathbin{\invamp}$ and $\otimes$, together with the constants $\perp$ and 1, respectively, are called *multiplicatives*. The connectives $\oplus$ and $\mathbin{\&}$, together with the constants 0 and $\top$, respectively, are called *additives*. The modalities ? and ! are called *exponentials* [Gir87a].

2.1.5   Definition   A *sequent* is an expression of the kind $\vdash A_1, \ldots, A_h$, where $h \geqslant 0$ and the comma between the formulae $A_1, \ldots, A_h$ stands for multiset union. Multisets of formulae are denoted with Φ and Ψ.

Observe that the laws of associativity and commutativity are present implicitly in the definition of sequents.

2.2   Rules and Derivations

The general notion of a rule in the sequent calculus is independent from the logic to be specified. For this reason, I will first define the notions of rules and derivations in general and then present the inference rules for linear logic.

2.2.1   Definition   An (*inference*) *rule* in the sequent calculus is a scheme of the shape

$$\rho \, \frac{\vdash \Phi_1 \quad \vdash \Phi_2 \quad \ldots \quad \vdash \Phi_n}{\vdash \Phi} \quad,$$

for some $n \geqslant 0$, where $\vdash \Phi$ is called the *conclusion* and $\vdash \Phi_1$, $\vdash \Phi_2$, $\ldots$, $\vdash \Phi_n$ are the *premises* of the rule. An inference rule is called an *axiom* if it has no premise, i.e. the rule is of the shape

$$\rho \, \frac{}{\vdash \Phi} \quad.$$

A (*formal*) *system* $\mathscr{S}$ is a finite set of rules.

2.2.2   Definition   A *derivation* Δ in a system $\mathscr{S}$ is a tree where the nodes are sequents to which a finite number (possibly zero) of instances of the inference rules in $\mathscr{S}$ are applied. The sequents in the leaves of Δ are called *premises*, and the sequent in the root is the *conclusion*. A derivation with no premises is a *proof*, denoted with Π. A sequent $\vdash \Phi$ is *provable* in $\mathscr{S}$ if there is a proof Π with conclusion $\vdash \Phi$.

Sometimes in the discussion only the premises and the conclusion of a derivation are of importance. In this case, a derivation in the sequent calculus is depicted in the following way:

$$\vdash \Phi_1 \quad \cdots \quad \vdash \Phi_n$$
$$\Delta$$
$$\vdash \Phi \qquad ,$$
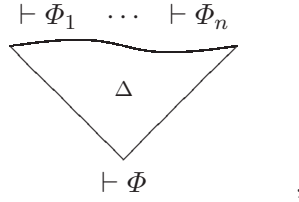
where the sequents $\vdash \Phi_1, \ldots, \vdash \Phi_n$ are the premises and $\vdash \Phi$ is the conclusion. A proof is then depicted as follows:

$$\Pi$$
$$\vdash \Phi \qquad .$$
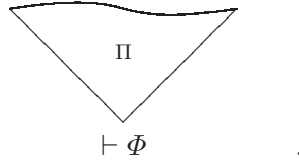
Let me now inspect the rules for linear logic.

2.2.3   Definition   The two rules

$$\mathsf{id} \frac{}{\vdash A, A^{\perp}} \qquad \text{and} \qquad \mathsf{cut} \frac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$$

are called *identity* and *cut*, respectively.

The rules identity and cut form the so-called *identity group*. They are of importance in most sequent calculus systems. The identity rule expresses the fact that from A we can conclude A. The rule is necessary for observing proofs because it allows us to close a branch in the tree. The cut rule expresses the transitivity of the logical consequence relation and is therefore necessary for using lemmata inside a proof.

Opposed to the identity group stands the so called *logical group*, which contains for each connective, modality or constant one or more *logical rules*. In the case of linear logic, this group is subdivided into the multiplicative, additive and exponential groups.

2.2.4   Definition   The *multiplicative group* contains the rules

$$\otimes \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \quad , \quad \Im \frac{\vdash A, B, \Phi}{\vdash A \Im B, \Phi} \quad , \quad \perp \frac{\vdash \Phi}{\vdash \perp, \Phi} \quad , \quad 1 \frac{}{\vdash 1} \quad ,$$

which are called *times*, *par*, *bottom*, and *one*, respectively. The *additive group* contains the rules

$$\& \frac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \& B, \Phi} \quad , \quad \oplus_1 \frac{\vdash A, \Phi}{\vdash A \oplus B, \Phi} \quad , \quad \oplus_2 \frac{\vdash B, \Phi}{\vdash A \oplus B, \Phi} \quad , \quad \top \frac{}{\vdash \top, \Phi} \quad ,$$

$$\text{id} \frac{}{\vdash A, A^\perp} \qquad\qquad \text{cut} \frac{\vdash A, \Phi \quad \vdash A^\perp, \Psi}{\vdash \Phi, \Psi}$$

$$\otimes \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \qquad \mathbin{\mathstrut} \mathord{\mkern-2mu} \frac{\vdash A, B, \Phi}{\vdash A \mathbin{\bindnasrepma} B, \Phi} \qquad \perp \frac{\vdash \Phi}{\vdash \perp, \Phi} \qquad 1 \frac{}{\vdash 1}$$

$$\mathbin{\&} \frac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \mathbin{\&} B, \Phi} \qquad \oplus_1 \frac{\vdash A, \Phi}{\vdash A \oplus B, \Phi} \qquad \oplus_2 \frac{\vdash B, \Phi}{\vdash A \oplus B, \Phi} \qquad \top \frac{}{\vdash \top, \Phi}$$

$$?\text{d} \frac{\vdash A, \Phi}{\vdash ?A, \Phi} \qquad ?\text{c} \frac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi} \qquad ?\text{w} \frac{\vdash \Phi}{\vdash ?A, \Phi} \qquad ! \frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n} \ (n \geqslant 0)$$

Figure 2.1: System LL in the sequent calculus

which are called *with*, *plus left*, *plus right*, and *top*, respectively. (Note that there is no rule for *zero*.) The *exponential group* contains the rules

$$?\text{w} \frac{\vdash \Phi}{\vdash ?A, \Phi} \quad , \quad ?\text{c} \frac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi} ?\text{d} \frac{\vdash A, \Phi}{\vdash ?A, \Phi} \quad , \quad ! \frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n} \ (n \geqslant 0) \quad ,$$

which are called *weakening*, *contraction*, *dereliction*, and *promotion*, respectively.

An important observation is that all logical rules of Definition 2.2.4 have the *subformula property*, that is, the premises of each rule contain only subformulae of the formulae in the conclusion of the rule.

2.2.5   Remark   Usually, in a sequent calculus system, there is a third group of rules, the so-called *structural rules*. As introduced in [Gen34], these rules are usually contraction, weakening and exchange. In the case at hand, the exchange rule is built into the definition of sequents, and the rules of contraction and weakening are not present in linear logic (but they are available in a restricted way in the form of the rules for the exponentials).

2.2.6   Definition   The *system* LL for linear logic in the sequent calculus is shown in Figure 2.1.

2.2.7   Example   Here are two examples for proofs in system LL:

$$\otimes \frac{\text{id} \frac{}{\vdash R^\perp, R} \quad \text{id} \frac{}{\vdash U^\perp, U}}{\otimes \frac{\vdash R^\perp, U^\perp, R \otimes U \quad \text{id} \frac{}{\vdash T^\perp, T}}{\mathbin{\bindnasrepma} \frac{\vdash R^\perp \otimes T^\perp, U^\perp, R \otimes U, T}{\mathbin{\bindnasrepma} \frac{\vdash R^\perp \otimes T^\perp, U^\perp, (R \otimes U) \mathbin{\bindnasrepma} T}{\vdash (R^\perp \otimes T^\perp) \mathbin{\bindnasrepma} U^\perp, (R \otimes U) \mathbin{\bindnasrepma} T}}}} \quad , \quad \mathbin{\bindnasrepma} \frac{!\frac{?\text{d} \frac{?\text{d} \frac{\otimes \frac{\text{id} \frac{}{\vdash R^\perp, R} \quad \text{id} \frac{}{\vdash T^\perp, T}}{\vdash R^\perp \otimes T^\perp, R, T}}{\vdash ?(R^\perp \otimes T^\perp), R, T}}{\vdash ?(R^\perp \otimes T^\perp), ?R, T}}{\vdash ?(R^\perp \otimes T^\perp), ?R, !T}}{\vdash ?(R^\perp \otimes T^\perp), ?R \mathbin{\bindnasrepma} !T} \quad .$$

2.2.8 Remark The identity rule

$$\mathsf{id}\ \frac{}{\vdash A, A^\perp}$$

can be replaced by an atomic version

$$\mathsf{id}\ \frac{}{\vdash a, a^\perp}$$

without affecting provability. This can be shown by using an inductive argument for replacing each general instance of the identity rule by a derivation containing atomic instances of identity and some other rules. For example if $A = B \otimes C$ we can replace

$$\mathsf{id}\ \frac{}{\vdash B \otimes C, B^\perp \otimes C^\perp} \qquad \text{by} \qquad \otimes \frac{\mathsf{id}\ \dfrac{}{\vdash B, B^\perp} \quad \mathsf{id}\ \dfrac{}{\vdash C, C^\perp}}{\dfrac{\vdash B \otimes C, B^\perp, C^\perp}{\vdash B \otimes C, B^\perp \otimes C^\perp}} \ \otimes \ .$$

However, for the cut rule such a reduction is impossible. For example, the instance

$$\mathsf{cut}\ \frac{\vdash B \otimes C, \Phi \quad \vdash B^\perp \otimes C^\perp, \Psi}{\vdash \Phi, \Psi}$$

of the cut cannot be replaced by a derivation

$$\vdash B \otimes C, \Phi \qquad \vdash B^\perp \otimes C^\perp, \Psi$$

$$\vdash \Phi, \Psi \qquad ,$$

using only atomic cuts, because every atomic cut branches the derivation tree. If we try for example

$$\mathsf{cut}\ \frac{\vdash B, \Phi \qquad \mathsf{cut}\ \dfrac{\vdash C \quad \vdash B^\perp, C^\perp, \Psi}{\vdash B^\perp, \Psi}}{\vdash \Phi, \Psi} \qquad ,$$

there is no way to bring the two leftmost branches back together.

2.2.9 Remark Let me draw the attention of the reader to the following well-known fact regarding the connectives of linear logic. If we define a new pair of connectives $\&'$ and $\oplus'$ with the same rules as for the connectives $\&$ and $\oplus$:

$$\&'\ \frac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \&' B, \Phi} \quad , \qquad \oplus'_1\ \frac{\vdash A, \Phi}{\vdash A \oplus' B, \Phi} \quad , \qquad \oplus'_2\ \frac{\vdash B, \Phi}{\vdash A \oplus' B, \Phi} \quad ,$$

then we can show that the connectives $\&'$ and $\&$, as well as $\oplus$ and $\oplus'$ are equivalent. If we do the same to the exponentials, i.e. define two new modalities $?'$ and $!'$ by the same rules as for $?$ and $!$, then there is no possibility to show any relation between $!$ and $!'$, or between $?$ and $?'$.

2.3   Cut Elimination

In system LL all rules, except for the cut, enjoy the subformula property. That means that every derivation that does not use the cut rule does also have the subformula property, i.e. each formula that occurs inside the derivation is a subformula of a formula occurring in the conclusion.

An important consequence of the subformula property is that the number of possibilities in which a rule can be applied to a given sequent is finite. This is of particular importance for proof search because at each step there is only a finite number of possibilities to continue the search.

The cut elimination theorem says that every provable sequent can be proved without using the cut rule. This rather surprising result has first been shown by G. Gentzen in [Gen34] for the two system LK and LJ, which are systems for classical and intuitionistic logic, respectively. In fact, this result was the main purpose for introducing the sequent calculus at all.

2.3.1   Theorem (Cut Elimination)   *Every proof Π of a sequent $\vdash \Phi$ in system* LL *can be transformed into a cut-free proof Π' (i.e. a proof in system* LL *that does not contain the* cut *rule) of the same sequent.* [Gir87a]

I will not show the proof in this thesis, because it is rather long and technical. A very good presentation of the proof can be found in the appendix of [LMSS92] or in [Bra96]. Here I will present only the basic idea, which is that all cuts occurring in the proof are permuted up until they reach the identity axioms where they dissappear as follows:

$$
\mathsf{cut}\ \frac{\mathsf{id}\ \dfrac{}{\vdash A, A^\perp}\quad \overset{\Pi}{\vdash A, \Phi}}{\vdash A, \Phi}
\qquad \text{is replaced by} \qquad
\overset{\Pi}{\vdash A, \Phi}\quad .
$$

In order to permute up an instance of the cut, it has to be matched against each other rule in the system. There are two types of cases: The first type, called *commutative case*, occurs if the principal formula of one of the two rules above the cut is not the cut formula. Then the situation is as follows:

$$
\mathsf{cut}\ \frac{\otimes\ \dfrac{\overset{\Pi_1}{\vdash A, \Phi}\quad \overset{\Pi_2}{\vdash B, \Psi, C}}{\vdash A \otimes B, \Phi, \Psi, C}\quad \overset{\Pi_3}{\vdash C^\perp, \Sigma}}{\vdash A \otimes B, \Phi, \Psi, \Sigma}\quad .
$$

This can be replaced by

$$
\otimes\ \frac{\overset{\Pi_1}{\vdash A, \Phi}\quad \mathsf{cut}\ \dfrac{\overset{\Pi_2}{\vdash B, \Psi, C}\quad \overset{\Pi_3}{\vdash C^\perp, \Sigma}}{\vdash B, \Psi, \Sigma}}{\vdash A \otimes B, \Phi, \Psi, \Sigma}\quad .
$$

The cases for the other rules are similar. But it should be noted that it can happen that the cut is duplicated:

$$
\mathrm{cut}\;\dfrac{
\&\;\dfrac{
\dfrac{\Pi_1}{\vdash A,\Phi,C} \quad \dfrac{\Pi_2}{\vdash B,\Phi,C}
}{\vdash A\,\&\,B,\Phi,C}
\quad
\dfrac{\Pi_3}{\vdash C^{\perp},\Psi}
}{\vdash A\,\&\,B,\Phi,\Psi}
$$

is replaced by

$$
\&\;\dfrac{
\mathrm{cut}\;\dfrac{
\dfrac{\Pi_1}{\vdash A,\Phi,C} \quad \dfrac{\Pi_3}{\vdash C^{\perp},\Psi}
}{\vdash A,\Phi,\Psi}
\quad
\mathrm{cut}\;\dfrac{
\dfrac{\Pi_2}{\vdash B,\Phi,C} \quad \dfrac{\Pi_3}{\vdash C^{\perp},\Psi}
}{\vdash B,\Phi,\Psi}
}{\vdash A\,\&\,B,\Phi,\Psi}
$$

This can blow-up of the proof exponentially.

The second type of case, called *key case*, occurs when both cut formulas coincide with the principal formulae of the two rules above the cut. Then the situation is as follows:

$$
\mathrm{cut}\;\dfrac{
\otimes\!\!\!\!\!\!\!\;\dfrac{\Pi_1}{\vdash A,B,\Phi}\;\;\;
\otimes\;\dfrac{
\dfrac{\Pi_2}{\vdash A^{\perp},\Psi}\quad \dfrac{\Pi_3}{\vdash B^{\perp},\Sigma}
}{\vdash A^{\perp}\otimes B^{\perp},\Psi,\Sigma}
}{\vdash \Phi,\Psi,\Sigma}\quad,
$$

which can be replaced by

$$
\mathrm{cut}\;\dfrac{
\mathrm{cut}\;\dfrac{
\dfrac{\Pi_1}{\vdash A,B,\Phi}\quad \dfrac{\Pi_2}{\vdash A^{\perp},\Psi}
}{\vdash B,\Phi,\Psi}
\quad
\dfrac{\Pi_3}{\vdash B^{\perp},\Sigma}
}{\vdash \Phi,\Psi,\Sigma}\quad.
$$

Observe that the number of cuts has increased, but their rank (the size of the cut formula) has decreased. This is used in an induction to show the termination of the whole procedure. But there is also the following case:

$$
\mathrm{cut}\;\dfrac{
!\;\dfrac{\dfrac{\Pi_1}{\vdash A,?B_1,\ldots,?B_n}}{\vdash\,!A,?B_1,\ldots,?B_n}
\quad
?\mathrm{c}\;\dfrac{\dfrac{\Pi_2}{\vdash\,?A^{\perp},?A^{\perp},\Phi}}{\vdash\,?A^{\perp},\Phi}
}{\vdash\,?B_1,\ldots,?B_n,\Phi}\quad,
$$

which can be replaced by

$$
\mathsf{?c}\cfrac{\mathsf{?c}\cfrac{\mathsf{cut}\cfrac{\mathsf{!}\cfrac{\overbrace{\Pi_1}\quad\vdash A,?B_1,\ldots,?B_n}{\vdash\,!A,?B_1,\ldots,?B_n}\qquad\mathsf{cut}\cfrac{\mathsf{!}\cfrac{\overbrace{\Pi_1}\quad\vdash A,?B_1,\ldots,?B_n}{\vdash\,!A,?B_1,\ldots,?B_n}\qquad\overbrace{\Pi_2}\quad\vdash\,?A^\perp,?A^\perp,\Phi}{\vdash\,?A^\perp,?B_1,\ldots,?B_n,\Phi}}{\vdash\,?B_1,\ldots,?B_n,?B_1,\ldots,?B_n,\Phi}}{\vdots}}{\vdash\,?B_1,\ldots,?B_n,\Phi}\quad.
$$

Observe that in this case the rank of the cut does not decrease. For this reason, this case is also called a *pseudo key case* [Bra96]. In order to ensure termination, we can use the fact that the number of contractions above the cut has decreased.

It should be mentioned here that removing all cuts from a proof can increase its size hyperexponentially.

2.4   Discussion

In this chapter, I have defined the system for linear logic in the sequent calculus and presented the basic idea of the cut elimination procedure. For further details, the reader is referred to the extensive literature on the subject, e.g. [Gir87a, Gir95, Bra96, Tro92].

3

Linear Logic
and the Calculus of Structures

Similar to the sequent calculus, the calculus of structures, introduced in [Gug99], is a formalism for describing logical systems and for studying the properties of proofs. Its basic principles are described in Section 1.4.

The purpose of this chapter is to introduce the calculus of structures at a technical level. In Sections 3.1 and 3.2, I will show how linear logic can be presented in the calculus of structures. In Section 3.3, I will show the relation to the sequent system discussed in the previous chapter. In Section 3.4, I will show how one can prove cut elimination within the calculus of structures, without using the sequent calculus or semantics. Finally, Section 3.5 contains a discussion on some interesting features of the new system for linear logic and its major differences to the sequent system.

3.1 Structures for Linear Logic

In the calculus of structures, rules operate on structures, which are syntactic expressions intermediate between formulae and sequents. More precisely, structures can be seen as equivalence classes of formulae, where the equivalence relation is based on laws like associativity and commutativity which are usually imposed on sequents.

In this section I will define the structures for linear logic. In later chapters I will discuss other logics, and therefore, shall define new structures.

Structures are built from atoms, in the same way as the formulae defined in the previous chapter. However, here I will use the bar $\bar{\cdot}$ to denote negation.

Instead of using the infix notation for binary connectives, as it is done in formulae, I will employ the comma as it is done in sequents. For example, the structure $[R_1, \ldots, R_h]$ corresponds to a sequent $\vdash R_1, \ldots, R_h$ in linear logic, whose formulae are essentially connected by pars. For the other connectives, I will do the same. I order to distinguish between the connectives, different pairs of parentheses are used.

3.1.1 **Definition** Let $\mathcal{A}$ be a countable set equipped with a bijective function $\bar{\cdot} : \mathcal{A} \to \mathcal{A}$, such that $\bar{\bar{a}} = a$ and $\bar{a} \neq a$ for every $a \in \mathcal{A}$. The elements of $\mathcal{A}$ are called *atoms* (denoted

Associativity	Exponentials	Negation
$[R,[T,U]] \;=\; [[R,T],U]$	$??R \;=\; ?R$	$\overline{[R,T]} \;=\; (\bar{R},\bar{T})$
$(R,(T,U)) \;=\; ((R,T),U)$	$!!R \;=\; !R$	$\overline{(R,T)} \;=\; [\bar{R},\bar{T}]$
$\{R,\{T,U\}\} \;=\; \{\{R,T\},U\}$	**Units**	$\overline{\{R,T\}} \;=\; \langle\bar{R},\bar{T}\rangle$
$\langle R,\langle T,U\rangle\rangle \;=\; \langle\langle R,T\rangle,U\rangle$		$\overline{\langle R,T\rangle} \;=\; \{\bar{R},\bar{T}\}$
Commutativity	$[\bot,R] \;=\; R$	$\overline{?R} \;=\; !\bar{R}$
	$(1,R) \;=\; R$	$\overline{!R} \;=\; ?\bar{R}$
$[R,T] \;=\; [T,R]$	$\{0,R\} \;=\; R$	$\bar{\bar{R}} \;=\; R$
$(R,T) \;=\; (T,R)$	$\langle\top,R\rangle \;=\; R$	
$\{R,T\} \;=\; \{T,R\}$		
$\langle R,T\rangle \;=\; \langle T,R\rangle$	$\{\bot,\bot\} \;=\; \bot \;=\; ?\bot$	
	$\langle 1,1\rangle \;=\; 1 \;=\; !1$	

Figure 3.1: Basic equations for the syntactic congruence for LS structures

with a, b, c, …). The set $\mathcal{A}$ contains four selected elements, called *constants*, which are denoted by $\bot$, 1, 0, and $\top$ (called *bottom*, *one*, *zero*, and *top*, respectively). The function $\bar{\cdot}$ is defined on them as follows:

$$\bar{1} := \bot \quad , \qquad \bar{\bot} := 1 \quad , $$
$$\bar{\top} := 0 \quad , \qquad \bar{0} := \top \quad . $$

Let $\mathcal{R}$ be the set of expressions generated by the following syntax:

$$R ::= a \mid [R,R] \mid (R,R) \mid \{R,R\} \mid \langle R,R\rangle \mid !R \mid ?R \mid \bar{R} \quad ,$$

where a stands for any atom. On the set $\mathcal{R}$, the relation $=$ is defined to be the smallest congruence relation induced by the equations shown in Figure 3.1. An LS *structure* (denoted by P, Q, R, S, …) is an element of $\mathcal{R}/=$, i.e. an equivalence class of expressions. For a given LS structure R, the LS structure $\bar{R}$ is called its *negation*.

3.1.2   Remark   To be formally precise, the set of equations shown in Figure 3.1 should also contain the equation $\bar{a} = \bar{a}$, where the $\bar{\cdot}$ on the left-hand side stands for the syntactic negation and the $\bar{\cdot}$ on the right-hand side for the involution function defined on atoms. In other words, on atoms the negation is defined to be the involution.

3.1.3   Remark   The set of equations shown in Figure 3.1 is not minimal. It was not my goal to produce a minimal set because in this case minimality does not contribute to clarity.

3.1.4   Notation   In order to denote a structure, I will pick one expression from the equivalence class. Because of the associativity, superfluous parentheses can be omitted. For example, $[a,b,c,d]$ will be written for $[[[a,b],c],d]$ and $[[a,b],[c,d]]$. This notation

is closer to the notation for sequents than to the notation for formulae. Because of this notation, it was under consideration to use $[\,]$, $(\,)$, $\lfloor\rfloor$ and $\langle\rangle$ instead of $\perp$, $\mathsf{1}$, $\mathsf{0}$ and $\top$, respectively, to denote the units. But for the following two reasons I chose not to do so: First, Girard's notation for the units is now standard and I did not want to make the life of the reader even more difficult, and second, for reasons of personal taste, I prefer not to denote a unit by two separate symbols.

3.1.5   Remark   Because of the equational theory, there are many (in fact, countable many) expressions denoting the same structure. This is not problematic because one can consider structures to be in a certain normalform, where negation is pushed to the atoms and superfluous units and parentheses are omitted. For example, the expressions $[a, \perp, \bar{b}]$, $(\overline{[\perp, \bar{b}]}, \lfloor\bar{a}, \mathsf{0}\rfloor)$ and $(\bar{a}, \mathsf{1}, b)$ denote the same structure, but none of them is in normalform. The two expressions $(\bar{a}, b)$ and $(b, \bar{a})$ are both equivalent to the previous three and both are in normalform. Observe that because of commutativity the normalform is not necessarily unique, but there is only finite number of them for a given structure.

3.1.6   Definition   A structure $[R_1, \ldots, R_h]$ is called a *par structure*, $(R_1, \ldots, R_h)$ is called a *times structure*, $\lfloor R_1, \ldots, R_h \rfloor$ is called a *plus structure*, $\langle R_1, \ldots, R_h \rangle$ is called a *with structure*, $!R$ is called an *of-course structure*, and $?R$ is called a *why-not structure*.

3.1.7   Definition   We can define LS *contexts*, to be LS structures with a hole. Formally, they are generated by

$$S ::= \{\ \} \mid [R, S] \mid (R, S) \mid \lfloor R, S \rfloor \mid \langle R, S \rangle \mid !S \mid ?S \quad .$$

3.1.8   Remark   The idea of Remark 2.1.2 does also apply here: I will use the term structure (or context) if no ambiguity is possible, but I will say LS structure (or LS context) to distinguish between other types of structures or contexts to be defined in later chapters.

Because of the De Morgan laws there is no need to include the negation into the definition of the context, which means that the structure that is plugged into the hole of a context will always be positive. Contexts will be denoted with $S\{\ \}$, $T\{\ \}$, $\ldots$. Then, $S\{R\}$ denotes the structure that is obtained by replacing the hole $\{\ \}$ in the context $S\{\ \}$ by the structure R. The structure R is a *substructure* of $S\{R\}$ and $S\{\ \}$ is its *context*. For a better readability, I will omit the context braces if no ambiguity is possible, e.g. I will write $S[R, T]$ instead of $S\{[R, T]\}$.

3.1.9   Example   Let $S\{\ \} = [(a, ![\{\ \}, ?a], \bar{b}), b]$ and $R = c$ and $T = (\bar{b}, \bar{c})$. Then

$$S[R, T] = [(a, ![c, (\bar{b}, \bar{c}), ?a], \bar{b}), b] \quad .$$

3.1.10   Definition   The function $\underline{\cdot}_{\mathsf{s}}$ defines the obvious translation from LL formulae into LS structures:

$$
\begin{array}{rclcrclcrcl}
\underline{\perp}_{\mathsf{s}} & = & \perp & , & \underline{A \mathbin{\bindnasrepma} B}_{\mathsf{s}} & = & [\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}] & , & \underline{a}_{\mathsf{s}} & = & a & , \\
\underline{\mathsf{1}}_{\mathsf{s}} & = & \mathsf{1} & , & \underline{A \otimes B}_{\mathsf{s}} & = & (\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}) & , & \underline{?A}_{\mathsf{s}} & = & ?\underline{A}_{\mathsf{s}} & , \\
\underline{\mathsf{0}}_{\mathsf{s}} & = & \mathsf{0} & , & \underline{A \oplus B}_{\mathsf{s}} & = & \lfloor\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}\rfloor & , & \underline{!A}_{\mathsf{s}} & = & !\underline{A}_{\mathsf{s}} & , \\
\underline{\top}_{\mathsf{s}} & = & \top & , & \underline{A \mathbin{\&} B}_{\mathsf{s}} & = & \langle\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}\rangle & , & \underline{A^{\perp}}_{\mathsf{s}} & = & \overline{\underline{A}_{\mathsf{s}}} & .
\end{array}
$$

The domain of $\underline{\cdot}_{\mathsf{s}}$ is extended to sequents by

$$
\begin{aligned}
\underline{\vdash}_{\mathsf{s}} &= \bot \quad \text{and} \\
\underline{\vdash A_1,\dots,A_h}_{\mathsf{s}} &= [\underline{A_1}_{\mathsf{s}},\dots,\underline{A_h}_{\mathsf{s}}] \quad , \quad \text{for } h \geqslant 0 \quad .
\end{aligned}
$$

3.2 Rules and Derivations

In this section, I will define the general notions of rules, derivations and proofs. Those definitions are independent from linear logic and are the same for all logical systems in the calculus of structures discussed in this thesis.

Afterwards, I will show the rules for linear logic.

3.2.1 Definition In the calculus of structures, an *inference rule* is a scheme of the kind

$$
\rho \frac{T}{R} \quad ,
$$

where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion*. An inference rule is called an *axiom* if its premise is empty, i.e. the rule is of the shape

$$
\rho \frac{}{R} \quad .
$$

3.2.2 Remark A typical rule has the shape

$$
\rho \frac{S\{T\}}{S\{R\}}
$$

and specifies a step of rewriting inside a generic context $S\{\ \}$. But in general it is not forbidden to put constraints on the context. In particular, it is possible to have rules without context. This would then correspond to the to the case of the sequent calculus.

3.2.3 Remark It has already been mentioned in the introduction that it is not (trivially) possible to present the calculus of structures in general in the terminology of term rewriting. At this point I will explain the reasons in more detail.

- There are logical systems in the calculus of structures, where the rules are not independent from the context $S\{\ \}$. For example, one could imagine a quantifier rule like

$$
\forall \frac{S\{R\}}{S\{\forall x.R\}} \quad ,
$$

  where x is not allowed to be free in the context $S\{\ \}$. It is also possible, to translate sequent calculus rules directly into the calculus of structures, which would yield rules without generic context. For example

$$
\wedge \frac{(\varPhi \vee A) \wedge (\varPhi \vee B)}{\varPhi \vee (A \wedge B)} \quad .
$$

  Both cases are difficult to accommodate in a general term rewriting system.

- Many systems in the calculus of structures contain rules whose application is restricted to atoms. For example the atomic contraction rule

$$\mathsf{ac}{\downarrow}\,\frac{S\{a \vee a\}}{S\{a\}}$$

  allows to duplicate only atoms, whereas the generic contraction rule

$$\mathsf{c}{\downarrow}\,\frac{S\{R \vee R\}}{S\{R\}}$$

  allows to duplicate arbitrary structures. To make this distinction in a term rewriting system, one would need two different kind of variables.

- A more difficult issue is probably the notion of negation. The first observation to make is that negation cannot simply be encoded as unary function symbol $not(\cdot)$, because in the rule scheme

$$\rho\,\frac{S\{T\}}{S\{R\}}\quad,$$

  the rewriting is not allowed to be inside a negation. This condition would be difficult to check in a term rewriting system. On the other hand, in many systems in the calculus of structures (in particular, in all systems discussed in this thesis) negation is via the De Morgan laws pushed to the atoms. So, one could imagine to have positive and negative atoms. But then, it becomes difficult to translate the generic identity rule

$$\mathsf{id}\,\frac{S\{\mathbf{true}\}}{S\{A \vee \bar{A}\}}\quad,$$

  where $\bar{A}$ stands for the negation of A.

However, it is possible to translate some systems in the calculus of structures into term rewriting systems.

3.2.4 **Definition** A (*formal*) *system* $\mathscr{S}$ is a set of inference rules.

3.2.5 **Definition** A *derivation* Δ in a certain formal system is a finite chain of instances of inference rules in the system:

$$\rho''\,\frac{\rho'\,\dfrac{\rho\,\dfrac{R}{R'}}{\vdots}}{R''}\quad.$$

A derivation can consist of just one structure. The topmost structure in a derivation, if present, is called the *premise* of the derivation, and the bottommost structure is called its *conclusion*. A derivation Δ whose premise is T, whose conclusion is R, and whose inference rules are in $\mathscr{S}$ will be indicated with

$$\begin{array}{c} T \\ \Delta \| \mathscr{S} \\ R \end{array}\quad.$$

A *proof* Π in the calculus of structures is a derivation whose topmost inference rule is an axiom. It will be denoted by

$$\Pi \Vert_{\mathscr{S}} \atop R \quad .$$

The *length* of a derivation Δ is the number of instances of inference rules used in Δ. It is denoted by length(Δ). The *length* of a proof Π, denoted by length(Π), is defined in the same way.

3.2.6   Definition   A rule ρ is *derivable* in a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every instance of $\rho \dfrac{T}{R}$, there is a derivation $\dfrac{T}{\Delta\Vert_{\mathscr{S}}} \atop R$.

3.2.7   Definition   Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if for every derivation $\dfrac{T}{\Delta\Vert_{\mathscr{S}}} \atop R$ there is a derivation $\dfrac{T}{\Delta'\Vert_{\mathscr{S}'}} \atop R$, and vice versa.

3.2.8   Definition   A rule ρ is *admissible* for a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every proof $\Pi\Vert_{\mathscr{S}\cup\{\rho\}} \atop R$ there is a proof $\Pi'\Vert_{\mathscr{S}} \atop R$.

3.2.9   Definition   Two systems $\mathscr{S}$ and $\mathscr{S}'$ are (*weakly*) *equivalent* if for every proof $\Pi\Vert_{\mathscr{S}} \atop R$ there is a proof $\Pi'\Vert_{\mathscr{S}'} \atop R$, and vice versa.

   In the following, I will show the inference rules for linear logic in the calculus of structures. The proof of the equivalence to the rules in the sequent calculus, which have been presented in the previous chapter, will follow in Section 3.3. Before I start, let me recall that in the sequent calculus, rules can be divided into three groups: identity rules, logical rules and structural rules. Because there is no longer a difference between formulae and sequents, there is also no difference between structural and logical rules. Therefore, there are only two groups of rules in the calculus of structures: the so called *interaction rules*, which correspond to the identity rules in the sequent calculus, and the *structural rules*, which correspond to the structural and logical rules in the sequent calculus.

   Let me now begin with the interaction rules.

3.2.10   Definition   The rules

$$\mathsf{i}{\downarrow}\,\frac{S\{1\}}{S[R,\bar{R}]} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\,\frac{S(R,\bar{R})}{S\{\bot\}}$$

are called *interaction* and *cut* (or *cointeraction*), respectively.

   These rules correspond to the identity and cut rule in the sequent calculus (the technical details of this correspondence are shown in the proof of Theorem 3.3.2 on page 38). But in the calculus of structures, the duality between the two rules is more obvious than in the sequent calculus. Furthermore, in the calculus of structures, the duality between two rules can be defined formally in the following way: A rule ρ' is *dual* to a rule ρ if it is the contrapositive of ρ, i.e. it is obtained from ρ by exchanging and negating premise and conclusion.

3.2.11   Definition   The structural rule

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

is called *switch*.

Observe that the switch rule is self-dual, i.e. if premise and conclusion are negated and exchanged, we obtain again an instance of switch. This can be seen easily as follows:

$$
\begin{aligned}
\overline{S[(R,T),U]} \;\; &= \;\; S'\{\overline{[(R,T),U]}\} \quad,\quad \text{for some context } S'\{\ \} \\
&= \;\; S'(\overline{(R,T)},\bar{U}) \\
&= \;\; S'([\bar{R},\bar{T}],\bar{U}) \\
&= \;\; S'([R',U'],T') \quad,\quad \text{for some structures } R',\, T' \text{ and } U' \quad.
\end{aligned}
$$

Similarly, we can get $\overline{S([R,U],T)} = S'[(R',T'),U']$ for the same context $S'\{\ \}$ and structures $R',\, T'$ and U'.

The switch rule stands for the linear implication $A \otimes (B \,\mathbin{\rotatebox[origin=c]{180}{\&}}\, C) \multimap (A \otimes B) \,\mathbin{\rotatebox[origin=c]{180}{\&}}\, C$, which is also known as weak distributivity [CS97]. In [Gug99], the switch rule has been conceived by enforcing cut elimination in a certain combinatorial computational model, entirely independent from weakly distributive categories.

The two interaction rules and the switch rule have together a remarkable property (as already observed in [Gug99]), namely that they make for every rule ρ its dual rule derivable.

3.2.12   Proposition   *Let ρ and ρ' be two rules that are dual to each other. Then ρ' is derivable in $\{\mathsf{i}{\downarrow},\mathsf{i}{\uparrow},\mathsf{s},\rho\}$.*

Before I will show the proof, let me introduce two notational conventions whose solely purpose is to ease the checking of such proofs for the reader:

(1) Sometimes I will insert the (pseudo-)rule

$$=\frac{T}{R} \quad,$$

where $R = T$. This will help checking the correctness of a derivation when R and T are quite different expressions for the same structure.

(2) Sometimes I will use a light grey background to mark the place of the redex of a rewriting step when the derivation is read *bottom-up*. (The notion of redex will be defined precisely in Section 4.2, when the permutation of rules is studied.)

Although derivations are top-down symmetric objects that can be read top-down as well as bottom-up, the reader is therefore advised to read them bottom-up in case he wants to verify them.

Proof of Proposition 3.2.12:   Let $\rho \dfrac{S\{P\}}{S\{Q\}}$ be given. Then any instance of $\rho' \dfrac{S\{\bar{Q}\}}{S\{\bar{P}\}}$ can be replaced by the following derivation:

$$
\begin{array}{c}
= \dfrac{S\{\bar{Q}\}}{S(\bar{Q},1)} \\[4pt]
\mathsf{i}\!\downarrow \dfrac{\phantom{S\{\bar{Q}\}}}{S(\bar{Q},[P,\bar{P}])} \\[4pt]
\mathsf{s} \dfrac{\phantom{S(\bar{Q},[P,\bar{P}])}}{S[(\bar{Q},P),\bar{P}]} \\[4pt]
\rho \dfrac{\phantom{S[(\bar{Q},P),\bar{P}]}}{S[(\bar{Q},Q),\bar{P}]} \\[4pt]
\mathsf{i}\!\uparrow \dfrac{\phantom{S[(\bar{Q},Q),\bar{P}]}}{S[\bot,\bar{P}]} \\[4pt]
= \dfrac{\phantom{S[\bot,\bar{P}]}}{S\{\bar{P}\}}
\end{array} \quad .
$$

$\square$

3.2.13   Definition   The structural rules

$$
\mathsf{d}\!\downarrow \frac{S([R,U],[T,V])}{S[(R,T),(U,V)]} \quad , \quad
\mathsf{d}\!\uparrow \frac{S((R,U),(T,V))}{S[(R,T),(U,V)]} \quad ,
$$

$$
\mathsf{t}\!\downarrow \frac{S\{0\}}{S\{R\}} \quad , \quad
\mathsf{t}\!\uparrow \frac{S\{R\}}{S\{\top\}} \quad , \quad
\mathsf{c}\!\downarrow \frac{S(R,R)}{S\{R\}} \quad , \quad
\mathsf{c}\!\uparrow \frac{S\{R\}}{S(R,R)}
$$

are called *additive* ($\mathsf{d}\!\downarrow$), *coadditive* ($\mathsf{d}\!\uparrow$), *thinning* ($\mathsf{t}\!\downarrow$), *cothinning* ($\mathsf{t}\!\uparrow$), *contraction* ($\mathsf{c}\!\downarrow$), and *cocontraction* ($\mathsf{c}\!\uparrow$), respectively. They form the *additive group*.

Although those rules and the rules for the additives in the sequent calculus (see Chapter 2) have a very different shape, they are equivalent. This is shown in detail in Section 3.3.

3.2.14   Definition   The *exponential group* is formed by the structural rules

$$
\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \quad , \quad
\mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}} \quad ,
$$

$$
\mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \quad , \quad
\mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{1\}} \quad , \quad
\mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \quad , \quad
\mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R,R)} \quad ,
$$

which are called *promotion* ($\mathsf{p}\!\downarrow$), *copromotion* ($\mathsf{p}\!\uparrow$), *weakening* ($\mathsf{w}\!\downarrow$), *coweakening* ($\mathsf{w}\!\uparrow$), *absorption* ($\mathsf{b}\!\downarrow$), and *coabsorption* ($\mathsf{b}\!\uparrow$), respectively.

These rules capture exactly the behaviour of the rules for the exponentials in the sequent calculus. For the weakening and absorption rules this is not surprising. The surprising fact is, that the promotion rule is no longer global, i.e. it does not need to check an unbounded number of formulae in the context. Again, the exact correspondence is shown in in Section 3.3.

As in the sequent system for linear logic, there is also a *multiplicative group*: its only member is the switch rule.

Since for every rule in the calculus of structures we can give a dual rule, we can also give for every derivation a dual derivation. For example,

$$
\begin{array}{c}
\mathsf{i}\!\downarrow \dfrac{\langle[a,\bar{b}],1\rangle}{\langle[a,\bar{b}],[b,\bar{b}]\rangle} \\
\mathsf{d}\!\downarrow \dfrac{}{[\langle a,b\rangle,\{\bar{b},\bar{b}\}]} \\
\mathsf{c}\!\downarrow \dfrac{}{[\langle a,b\rangle,\bar{b}]} \\
\mathsf{w}\!\downarrow \dfrac{}{[\langle a,b\rangle,?\bar{c},\bar{b}]}
\end{array}
\qquad \text{is dual to} \qquad
\begin{array}{c}
\mathsf{w}\!\uparrow \dfrac{(\{\bar{a},\bar{b}\},!c,b)}{(\{\bar{a},\bar{b}\},b)} \\
\mathsf{c}\!\uparrow \dfrac{}{(\{\bar{a},\bar{b}\},\langle b,b\rangle)} \\
\mathsf{d}\!\uparrow \dfrac{}{\{(\bar{a},b),(\bar{b},b)\}} \\
\mathsf{i}\!\uparrow \dfrac{}{\{(\bar{a},b),\bot\}}
\end{array}
\;.
$$

3.2.15 Definition The rules

$$
\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \text{and} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\bot\}}
$$

are called *atomic interaction* and *atomic cut* (or *atomic cointeraction*), respectively.

The rules $\mathsf{ai}\!\downarrow$ and $\mathsf{ai}\!\uparrow$ are obviously instances of the rules $\mathsf{i}\!\downarrow$ and $\mathsf{i}\!\uparrow$ above. It is well known that in many systems in the sequent calculus, the identity rule can be reduced to its atomic version. In the calculus of structures we can do the same. But furthermore, by duality, we can do the same to the cut rule, which is impossible in the sequent calculus (see Remark 2.2.8).

3.2.16 Proposition *The rule $\mathsf{i}\!\downarrow$ is derivable in the system $\{\mathsf{ai}\!\downarrow,\mathsf{s},\mathsf{d}\!\downarrow,\mathsf{p}\!\downarrow\}$. Dually, the rule $\mathsf{i}\!\uparrow$ is derivable in $\{\mathsf{ai}\!\uparrow,\mathsf{s},\mathsf{d}\!\uparrow,\mathsf{p}\!\uparrow\}$.*

Proof: For a given application of $\mathsf{i}\!\downarrow \dfrac{S\{1\}}{S[R,\bar{R}]}$, by structural induction on R, I will construct an equivalent derivation that contains only $\mathsf{ai}\!\downarrow$, s, $\mathsf{d}\!\downarrow$ and $\mathsf{p}\!\downarrow$.

- R is an atom: Then the given instance of $\mathsf{i}\!\downarrow$ is an instance of $\mathsf{ai}\!\downarrow$.

- $R = [P,Q]$, where $P \neq \bot \neq Q$: Apply the induction hypothesis to

$$
\begin{array}{c}
\mathsf{i}\!\downarrow \dfrac{S\{1\}}{S[Q,\bar{Q}]} \\
= \dfrac{}{S(1,[Q,\bar{Q}])} \\
\mathsf{i}\!\downarrow \dfrac{}{S([P,\bar{P}],[Q,\bar{Q}])} \\
\mathsf{s} \dfrac{}{S[P,(\bar{P},[Q,\bar{Q}])]} \\
\mathsf{s} \dfrac{}{S[P,Q,(\bar{P},\bar{Q})]}
\end{array}
\;.
$$

- $R = (P,Q)$, where $P \neq 1 \neq Q$: Similar to the previous case.

- $R = \{P,Q\}$, where $P \neq 0 \neq Q$: Apply the induction hypothesis to

$$
\begin{array}{c}
= \dfrac{S\{1\}}{S\langle 1,1\rangle} \\
\mathsf{i}\!\downarrow \dfrac{}{S\langle 1,[Q,\bar{Q}]\rangle} \\
\mathsf{i}\!\downarrow \dfrac{}{S\langle [P,\bar{P}],[Q,\bar{Q}]\rangle} \\
\mathsf{d}\!\downarrow \dfrac{}{S[\{P,Q\},\langle \bar{P},\bar{Q}\rangle]}
\end{array}
\;.
$$

- $R = \langle P, Q \rangle$, where $P \neq \top \neq Q$: Similar to the previous case.
- $R = ?P$, where $P \neq \bot$: Apply the induction hypothesis to

$$
\mathsf{p}\downarrow \frac{\mathsf{i}\downarrow \dfrac{= \dfrac{S\{1\}}{S\{!1\}}}{S\{![P, \bar{P}]\}}}{S[?P, !\bar{P}]} \quad .
$$

- $R = !P$, where $P \neq 1$: Similar to the previous case.

The second statement is dual to the first. $\square$

3.2.17 Definition   The system

$$\{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}, \mathsf{d}\downarrow, \mathsf{d}\uparrow, \mathsf{t}\downarrow, \mathsf{t}\uparrow, \mathsf{c}\downarrow, \mathsf{c}\uparrow, \mathsf{p}\downarrow, \mathsf{p}\uparrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow, \mathsf{b}\downarrow, \mathsf{b}\uparrow\} \quad ,$$

shown in Figure 3.2, is called *symmetric (or self-dual) linear logic in the calculus of structures*, or *system* SLS. The set $\{\mathsf{ai}\downarrow, \mathsf{s}, \mathsf{d}\downarrow, \mathsf{t}\downarrow, \mathsf{c}\downarrow, \mathsf{p}\downarrow, \mathsf{w}\downarrow, \mathsf{b}\downarrow\}$ is called the *down fragment* and $\{\mathsf{ai}\uparrow, \mathsf{s}, \mathsf{d}\uparrow, \mathsf{t}\uparrow, \mathsf{c}\uparrow, \mathsf{p}\uparrow, \mathsf{w}\uparrow, \mathsf{b}\uparrow\}$ is called the *up fragment*. Further, the subsystem

$$\{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}, \mathsf{d}\downarrow, \mathsf{d}\uparrow, \mathsf{t}\downarrow, \mathsf{t}\uparrow, \mathsf{c}\downarrow, \mathsf{c}\uparrow\}$$

is called the *multiplicative additive fragment*, or *system* SALS. Similarly, the system

$$\{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}, \mathsf{p}\downarrow, \mathsf{p}\uparrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow, \mathsf{b}\downarrow, \mathsf{b}\uparrow\}$$

is called the *multiplicative exponential fragment*, or *system* SELS. The system

$$\{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}\}$$

is called the *multiplicative fragment*, or *system* SS.

System SELS will be investigated in more detail in Chapter 4.

Observe that in Proposition 3.2.16 only the rules s, $\mathsf{d}\downarrow$, $\mathsf{d}\uparrow$, $\mathsf{p}\downarrow$ and $\mathsf{p}\uparrow$ are used to reduce the general interaction and the general cut to their atomic versions, whereas all other rules are not used. This motivates the following definition.

3.2.18 Definition   The *core* of a system in the calculus of structures is the set of structural rules, that are needed to reduce interaction and cut to atomic form. All other structural rules are called *noncore*. In system SLS, the core is the set $\{\mathsf{s}, \mathsf{d}\downarrow, \mathsf{d}\uparrow, \mathsf{p}\downarrow, \mathsf{p}\uparrow\}$ and is denoted by SLSc.

So far we are only able to write derivations. In order to present proofs, we need an axiom.

3.2.19 Definition   The following rule is called *one*:

$$\mathsf{1}\downarrow \frac{}{1} \quad .$$

In the language of the sequent calculus it simply says that $\vdash 1$ is provable. I will put this rule to the down fragment of system SLS and by this break the top-down symmetry of derivations and observe proofs.

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad\qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\bot\}} \qquad\Bigg\}\ \text{Interaction}$$

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]} \qquad\Bigg\}\ \text{Multiplicatives}$$

$$\mathsf{d}\!\downarrow \frac{S\langle[R,U],[T,V]\rangle}{S[\langle R,T\rangle,\langle U,V\rangle]} \qquad \mathsf{d}\!\uparrow \frac{S(\langle R,U\rangle,\langle T,V\rangle)}{S\langle(R,T),(U,V)\rangle}$$

$$\mathsf{t}\!\downarrow \frac{S\{0\}}{S\{R\}} \qquad\qquad \mathsf{t}\!\uparrow \frac{S\{R\}}{S\{\top\}} \qquad\Bigg\}\ \text{Additives}$$

$$\mathsf{c}\!\downarrow \frac{S\langle R,R\rangle}{S\{R\}} \qquad\qquad \mathsf{c}\!\uparrow \frac{S\{R\}}{S\langle R,R\rangle}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}} \qquad\Bigg\}$$

$$\mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad\qquad \mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{1\}} \qquad\Bigg\}\ \text{Exponentials}$$

$$\mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \qquad\qquad \mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R,R)}$$
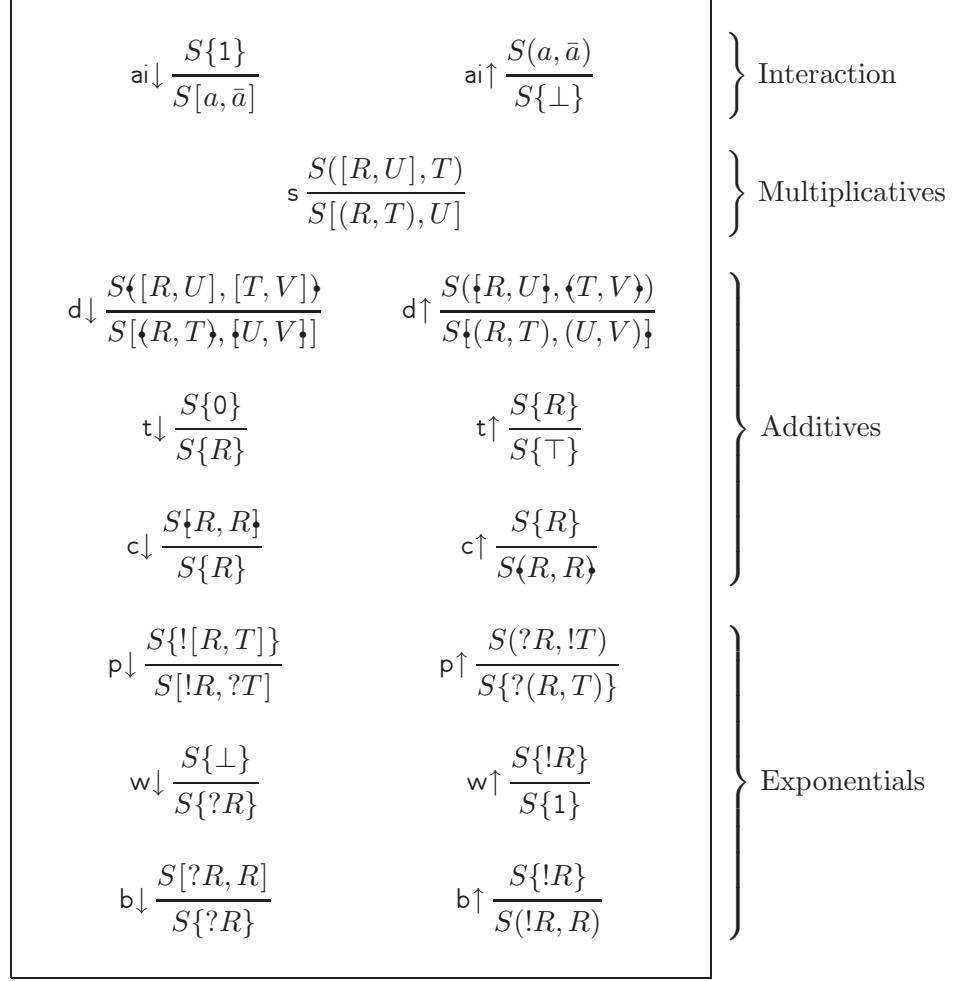
Figure 3.2: System SLS

3.2.20 Definition The system $\{1\!\downarrow, \mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{d}\!\downarrow, \mathsf{t}\!\downarrow, \mathsf{c}\!\downarrow, \mathsf{p}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow\}$, shown in Figure 3.3, that is obtained from the down fragment of system SLS together with the axiom, is called *linear logic in the calculus of structures*, or *system* LS. The restriction to the multiplicative additive fragment $\{1\!\downarrow, \mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{d}\!\downarrow, \mathsf{t}\!\downarrow, \mathsf{c}\!\downarrow\}$ is called *system* ALS, the restriction to the multiplicative exponential fragment $\{1\!\downarrow, \mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{p}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow\}$ is called *system* ELS, and the restriction to the multiplicative fragment $\{1\!\downarrow, \mathsf{ai}\!\downarrow, \mathsf{s}\}$ is called *system* S.

In every proof in system LS, the rule $1\!\downarrow$ occurs exactly once, namely as the topmost rule of the proof.

The following theorem says, that the general cut rule $\mathsf{i}\!\uparrow$ is as powerful as the whole up fragment of the system SLS, and vice versa.

3.2.21 Theorem *The systems* SLS $\cup\, \{1\!\downarrow\}$ *and* LS $\cup\, \{\mathsf{i}\!\uparrow\}$ *are strongly equivalent.*

Proof: Immediate consequence of Propositions 3.2.16 and 3.2.12. □

$$1\downarrow \frac{}{1} \qquad \text{ai}\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \text{s}\, \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\text{t}\downarrow \frac{S\{0\}}{S\{R\}} \qquad \text{c}\downarrow \frac{S\{\!\langle R,R\rangle\!\}}{S\{R\}} \qquad \text{d}\downarrow \frac{S\langle\!\langle[R,U],[T,V]\rangle\!\rangle}{S[\langle R,T\rangle,\langle U,V\rangle]}$$

$$\text{w}\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad \text{b}\downarrow \frac{S[?R,R]}{S\{?R\}} \qquad \text{p}\downarrow \frac{S\{![R,T]\}}{S[!R,?T]}$$

Figure 3.3: System LS

3.3   Equivalence to the Sequent Calculus System

In this section, I will show the exact correspondence between the sequent calculus system LL for linear logic (shown in Figure 2.1 on page 20) and the two systems in the calculus of structures shown in the previous section.

More precisely, every proof in system $\text{SLS} \cup \{1\downarrow\}$ has a translation in system LL, and every cut-free proof in LL has a translation in system LS. As a consequence, we can obtain an (easy) proof of cut elimination for system $\text{LS} \cup \{i\uparrow\}$ (or equivalently for system $\text{SLS} \cup \{1\downarrow\}$).

First, it is important to note that under the translation $\underline{\cdot}_{\text{s}}$ (Definition 3.1.10) the equations shown in Figure 3.1 are logical equivalences of linear logic. Therefore, we have for any two formulae A and B, if $\underline{A}_{\text{s}} = \underline{B}_{\text{s}}$ then A and B are logically equivalent (i.e. $A \multimap B$ and $B \multimap A$ are provable).

It is possible to invert the function $\underline{\cdot}_{\text{s}}$, which is done by the function $\underline{\cdot}_{\text{L}}$:

$$\begin{array}{rclcrclcrcl}
\underline{\bot}_{\text{L}} &=& \bot &,& \underline{[R,T]}_{\text{L}} &=& \underline{R}_{\text{L}} \,\bindnasrepma\, \underline{T}_{\text{L}} &,& \underline{a}_{\text{L}} &=& a &,\\
\underline{1}_{\text{L}} &=& 1 &,& \underline{(R,T)}_{\text{L}} &=& \underline{R}_{\text{L}} \otimes \underline{T}_{\text{L}} &,& \underline{?R}_{\text{L}} &=& ?\underline{R}_{\text{L}} &,\\
\underline{0}_{\text{L}} &=& 0 &,& \underline{\{R,T\}}_{\text{L}} &=& \underline{R}_{\text{L}} \oplus \underline{T}_{\text{L}} &,& \underline{!R}_{\text{L}} &=& !\underline{R}_{\text{L}} &,\\
\underline{\top}_{\text{L}} &=& \top &,& \underline{\langle R,T\rangle}_{\text{L}} &=& \underline{R}_{\text{L}} \,\&\, \underline{T}_{\text{L}} &,& \underline{\bar{R}}_{\text{L}} &=& (\underline{R}_{\text{L}})^{\bot} &.
\end{array}$$

Observe that the formula $\underline{R}_{\text{L}}$ depends on the chosen representation of the structure R. Therefore, the mapping $\underline{\cdot}_{\text{L}}$ does not map structures uniquely to formulae. This is not problematic since the mapping $\underline{\cdot}_{\text{L}}$ is only used in the following theorem and its proof, where each statement concerning a formula $\underline{R}_{\text{L}}$ is independent from the chosen representation of R, because whenever $R = T$, then $\underline{R}_{\text{L}}$ and $\underline{T}_{\text{L}}$ are logically equivalent formulae.

3.3.1   Theorem   *If a given structure P is provable in system $\text{SLS} \cup \{1\downarrow\}$, then $\vdash \underline{P}_{\text{L}}$ is provable in LL (with cut).*

Proof: Suppose, we have a proof Π of P in system $\text{SLS} \cup \{1\downarrow\}$. By induction on the length of Π, let us build a proof $\underline{\Pi}_{\text{L}}$ of $\vdash \underline{P}_{\text{L}}$ in LL.

Base case: Π is $1{\downarrow} \dfrac{}{1}$: Let $\underline{\Pi}_{\mathsf{L}}$ be the proof $1 \dfrac{}{\vdash 1}$.

Inductive case: Suppose Π is

$$\rho \, \dfrac{\overset{\Pi'\big\|\mathsf{SLS}\cup\{1{\downarrow}\}}{S\{W\}}}{S\{Z\}} \quad .$$

In other words, $\rho \, \dfrac{S\{W\}}{S\{Z\}}$ is the last rule applied in Π. The following LL proofs show that $\vdash (\underline{W}_{\mathsf{L}})^{\perp}, \underline{Z}_{\mathsf{L}}$ is provable in LL for every rule $\rho \, \dfrac{S\{W\}}{S\{Z\}}$ in SLS, i.e. $\underline{W}_{\mathsf{L}} \multimap \underline{Z}_{\mathsf{L}}$ is a theorem in LL:

- For the rules $\mathsf{ai}{\downarrow}$ and $\mathsf{ai}{\uparrow}$:

$$\perp \dfrac{\mathord{\otimes} \dfrac{\mathsf{id} \dfrac{}{\vdash a, a^{\perp}}}{\vdash a \mathbin{\bindnasrepma} a^{\perp}}}{\vdash \perp, a \mathbin{\bindnasrepma} a^{\perp}} \quad .$$

- For the rule s:

$$\mathord{\bindnasrepma} \dfrac{\mathord{\bindnasrepma} \dfrac{\mathord{\otimes} \dfrac{\mathord{\otimes} \dfrac{\mathsf{id}\dfrac{}{\vdash R^{\perp}, R} \quad \mathsf{id}\dfrac{}{\vdash T^{\perp}, T}}{\vdash R^{\perp}, T^{\perp}, R \otimes T} \quad \mathsf{id}\dfrac{}{\vdash U^{\perp}, U}}{\vdash R^{\perp} \otimes U^{\perp}, T^{\perp}, R \otimes T, U}}{\vdash R^{\perp} \otimes U^{\perp}, T^{\perp}, (R \otimes T) \mathbin{\bindnasrepma} U}}{\vdash (R^{\perp} \otimes U^{\perp}) \mathbin{\bindnasrepma} T^{\perp}, (R \otimes T) \mathbin{\bindnasrepma} U} \quad .$$

- For the rules $\mathsf{d}{\downarrow}$ and $\mathsf{d}{\uparrow}$:

$$\mathord{\bindnasrepma} \dfrac{\mathord{\&} \dfrac{\mathord{\oplus} \dfrac{\mathord{\oplus}\dfrac{\mathord{\otimes}\dfrac{\mathsf{id}\dfrac{}{\vdash R^{\perp}, R}\;\mathsf{id}\dfrac{}{\vdash U^{\perp}, U}}{\vdash R^{\perp}\otimes U^{\perp}, R, U}}{\vdash R^{\perp}\otimes U^{\perp}, R, U\oplus V}}{\vdash (R^{\perp}\otimes U^{\perp})\oplus(T^{\perp}\otimes V^{\perp}), R, U\oplus V} \;\; \mathord{\oplus}\dfrac{\mathord{\oplus}\dfrac{\mathord{\otimes}\dfrac{\mathsf{id}\dfrac{}{\vdash T^{\perp}, T}\;\mathsf{id}\dfrac{}{\vdash V^{\perp}, V}}{\vdash T^{\perp}\otimes V^{\perp}, T, V}}{\vdash T^{\perp}\otimes V^{\perp}, T, U\oplus V}}{\vdash (R^{\perp}\otimes U^{\perp})\oplus(T^{\perp}\otimes V^{\perp}), T, U\oplus V}}{\vdash (R^{\perp}\otimes U^{\perp})\oplus(T^{\perp}\otimes V^{\perp}), R\mathbin{\&} T, U\oplus V}}{\vdash (R^{\perp}\otimes U^{\perp})\oplus(T^{\perp}\otimes V^{\perp}), (R\mathbin{\&} T)\mathbin{\bindnasrepma}(U\oplus V)} \quad .$$

- For the rules $\mathsf{t}{\downarrow}$ and $\mathsf{t}{\uparrow}$:

$$\top \dfrac{}{\vdash \top, R} \quad .$$

- For the rules $\mathsf{c}{\downarrow}$ and $\mathsf{c}{\uparrow}$:

$$\mathord{\&} \dfrac{\mathsf{id}\dfrac{}{\vdash R^{\perp}, R} \quad \mathsf{id}\dfrac{}{\vdash R^{\perp}, R}}{\vdash R^{\perp} \mathbin{\&} R^{\perp}, R} \quad .$$

- For the rules p↓ and p↑:

$$
\cfrac{
\cfrac{
\mathsf{id}\ \cfrac{}{\vdash R^\perp, R}
\quad
\mathsf{id}\ \cfrac{}{\vdash T^\perp, T}
}{
\otimes\ \cfrac{}{\vdash R^\perp \otimes T^\perp, R, T}
}
}{
\begin{array}{l}
\mathsf{?d}\ \cfrac{}{\vdash ?(R^\perp \otimes T^\perp), R, T} \\
\mathsf{?d}\ \cfrac{}{\vdash ?(R^\perp \otimes T^\perp), ?R, T} \\
!\ \cfrac{}{\vdash ?(R^\perp \otimes T^\perp), ?R, !T} \\
\mathbin{\text{⅋}}\ \cfrac{}{\vdash ?(R^\perp \otimes T^\perp), ?R \mathbin{⅋} !T}
\end{array}
}
\ .
$$

- For the rules w↓ and w↑:

$$
\mathsf{?w}\ \cfrac{1\ \cfrac{}{\vdash 1}}{\vdash 1, ?R}\quad ,
$$

- For the rules b↓ and b↓:

$$
\otimes\ \cfrac{
\mathsf{id}\ \cfrac{}{\vdash !R^\perp, ?R}
\qquad
\mathsf{?d}\ \cfrac{\mathsf{id}\ \cfrac{}{\vdash R^\perp, R}}{\vdash R^\perp, ?R}
}{
\mathsf{?c}\ \cfrac{\vdash !R^\perp \otimes R^\perp, ?R, ?R}{\vdash !R^\perp \otimes R^\perp, ?R}
}
\ .
$$

This means that for any context $S\{\ \}$, we also have that $\underline{S\{W\}}_{\!\mathsf{L}} \multimap \underline{S\{Z\}}_{\!\mathsf{L}}$ is a theorem in LL, i.e. $\vdash (\underline{S\{W\}}_{\!\mathsf{L}})^\perp, \underline{S\{Z\}}_{\!\mathsf{L}}$ has a proof Π' in LL. By induction hypothesis we also have a proof $\underline{\Pi''}_{\!\mathsf{L}}$ of $\vdash \underline{S\{W\}}_{\!\mathsf{L}}$ in LL. By applying the cut rule

$$
\mathsf{cut}\ \cfrac{
\overset{\Pi''}{\vdash \underline{S\{W\}}_{\!\mathsf{L}}}
\qquad
\overset{\Pi'}{\vdash (\underline{S\{W\}}_{\!\mathsf{L}})^\perp, \underline{S\{Z\}}_{\!\mathsf{L}}}
}{
\vdash \underline{S\{Z\}}_{\!\mathsf{L}}
}\quad ,
$$

we get a proof $\underline{\Pi}_{\!\mathsf{L}}$ of $\vdash \underline{S\{Z\}}_{\!\mathsf{L}}$.   □

3.3.2   Theorem   (a) *If a given sequent $\vdash \Phi$ is provable in LL (with cut), then the structure $\vdash \underline{\Phi}_{\!\mathsf{S}}$ is provable in system $\mathsf{SLS} \cup \{1{\downarrow}\}$. (b) If a given sequent $\vdash \Phi$ is cut-free provable in LL, then the structure $\vdash \underline{\Phi}_{\!\mathsf{S}}$ is provable in system LS.*

Proof: Let Π be the proof of $\vdash \Phi$ in LL. By structural induction on Π, we will construct a proof $\underline{\Pi}_{\!\mathsf{S}}$ of $\vdash \underline{\Phi}_{\!\mathsf{S}}$ in system $\mathsf{SLS} \cup \{1{\downarrow}\}$ (or in system LS if Π is cut-free).

- If Π is $\mathsf{id}\ \cfrac{}{\vdash A, A^\perp}$ for some formula A, then let $\underline{\Pi}_{\!\mathsf{S}}$ be the proof obtained via Proposition 3.2.16 from

$$
\mathsf{i}{\downarrow}\ \cfrac{1{\downarrow}\ \cfrac{}{1}}{[\underline{A}_{\mathsf{s}}, \overline{\underline{A}_{\mathsf{s}}}]}\ .
$$

- If $\mathsf{cut}\dfrac{\vdash A,\Phi \quad \vdash A^{\perp},\Psi}{\vdash \Phi,\Psi}$ is the last rule applied in Π, then there are by induction

  hypothesis two derivations $\overset{1}{\underset{[\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}]}{\Delta_1\|\mathsf{SLS}}}$ and $\overset{1}{\underset{[\overline{\underline{A}_{\mathsf{s}}},\underline{\Psi}_{\mathsf{s}}]}{\Delta_2\|\mathsf{SLS}}}$ .Let $\underline{\Pi}_{\mathsf{s}}$ be the proof obtained via

  Proposition 3.2.16 from

$$
\begin{array}{c}
1{\downarrow}\,\dfrac{\ }{1}\\[4pt]
\Delta_1\|\mathsf{SLS}\\[2pt]
=\dfrac{[\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}]}{([\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}],1)}\\[6pt]
\Delta_2\|\mathsf{SLS}\\[2pt]
\mathsf{s}\,\dfrac{([\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}],[\overline{\underline{A}_{\mathsf{s}}},\underline{\Psi}_{\mathsf{s}}])}{\mathsf{s}\,\dfrac{[(([\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}],\overline{\underline{A}_{\mathsf{s}}}),\underline{\Psi}_{\mathsf{s}}]}{\mathsf{i}{\uparrow}\,\dfrac{[(\underline{A}_{\mathsf{s}},\overline{\underline{A}_{\mathsf{s}}}),\underline{\Phi}_{\mathsf{s}},\underline{\Psi}_{\mathsf{s}}]}{=\dfrac{[\perp,\underline{\Phi}_{\mathsf{s}},\underline{\Psi}_{\mathsf{s}}]}{[\underline{\Phi}_{\mathsf{s}},\underline{\Psi}_{\mathsf{s}}]}}}}\quad.
\end{array}
$$

- If $\mathbin{⅋}\dfrac{\vdash A,B,\Phi}{\vdash A\mathbin{⅋}B,\Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof of $[\underline{A}_{\mathsf{s}},\underline{B}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}]$ that exists by induction hypothesis.

- If $\otimes\dfrac{\vdash A,\Phi \quad \vdash B,\Psi}{\vdash A\otimes B,\Phi,\Psi}$ is the last rule applied in Π, then there are by induction hy-

  pothesis two derivations $\overset{1}{\underset{[\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}]}{\Delta_1\|\mathsf{SLS}}}$ and $\overset{1}{\underset{[\underline{B}_{\mathsf{s}},\underline{\Psi}_{\mathsf{s}}]}{\Delta_2\|\mathsf{SLS}}}$. Let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$
\begin{array}{c}
1{\downarrow}\,\dfrac{\ }{1}\\[4pt]
\Delta_1\|\mathsf{SLS}\\[2pt]
=\dfrac{[\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}]}{([\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}],1)}\\[6pt]
\Delta_2\|\mathsf{SLS}\\[2pt]
\mathsf{s}\,\dfrac{([\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}],[\underline{B}_{\mathsf{s}},\underline{\Psi}_{\mathsf{s}}])}{\mathsf{s}\,\dfrac{[(([\underline{A}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}}],\underline{B}_{\mathsf{s}}),\underline{\Psi}_{\mathsf{s}}]}{[(\underline{A}_{\mathsf{s}},\underline{B}_{\mathsf{s}}),\underline{\Phi}_{\mathsf{s}},\underline{\Psi}_{\mathsf{s}}]}}\quad.
\end{array}
$$

- If $\perp\dfrac{\vdash \Phi}{\vdash \perp,\Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof of $\vdash \underline{\Phi}_{\mathsf{s}}$ that exists by induction hypothesis (since $\underline{\vdash \perp,\Phi}_{\mathsf{s}} = \underline{\vdash \Phi}_{\mathsf{s}}$).

- If Π is $1\dfrac{\ }{\vdash 1}$, then let $\underline{\Pi}_{\mathsf{s}}$ be $1{\downarrow}\dfrac{\ }{1}$.

- If $\&\dfrac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \& B, \Phi}$ is the last rule applied in Π, then there are by induction hypothesis two derivations $\overset{1}{\underset{[\underline{A}_\mathsf{s}, \underline{\Phi}_\mathsf{s}]}{\Delta_1 \| \mathsf{SLS}}}$ and $\overset{1}{\underset{[\underline{B}_\mathsf{s}, \underline{\Phi}_\mathsf{s}]}{\Delta_2 \| \mathsf{SLS}}}$. Let $\underline{\Pi}_\mathsf{s}$ be the proof

$$
= \cfrac{1\downarrow \cfrac{}{1}}{\cfrac{\langle 1, 1 \rangle}{\begin{array}{c}\Delta_1 \| \mathsf{SLS} \\ \langle [\underline{A}_\mathsf{s}, \underline{\Phi}_\mathsf{s}], 1 \rangle \\ \Delta_2 \| \mathsf{SLS} \\ \mathsf{d}\downarrow \cfrac{\langle [\underline{A}_\mathsf{s}, \underline{\Phi}_\mathsf{s}], [\underline{B}_\mathsf{s}, \underline{\Phi}_\mathsf{s}] \rangle}{\mathsf{c}\downarrow \cfrac{[\langle \underline{A}_\mathsf{s}, \underline{B}_\mathsf{s} \rangle, \langle \underline{\Phi}_\mathsf{s}, \underline{\Phi}_\mathsf{s} \rangle]}{[\langle \underline{A}_\mathsf{s}, \underline{B}_\mathsf{s} \rangle, \underline{\Phi}_\mathsf{s}]}}\end{array}}} \quad .
$$

- If $\oplus_1 \dfrac{\vdash A, \Phi}{\vdash A \oplus B, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_\mathsf{s}$ be the proof

$$
= \cfrac{\Pi' \| \mathsf{SLS} \cup \{1\downarrow\}}{\cfrac{[\underline{A}_\mathsf{s}, \underline{\Phi}_\mathsf{s}]}{\mathsf{t}\downarrow \cfrac{[\langle \underline{A}_\mathsf{s}, 0 \rangle, \underline{\Phi}_\mathsf{s}]}{[\langle \underline{A}_\mathsf{s}, \underline{B}_\mathsf{s} \rangle, \underline{\Phi}_\mathsf{s}]}}} \quad ,
$$

where Π' exists by induction hypothesis.

- The case for the rule $\oplus_2 \dfrac{\vdash B, \Phi}{\vdash A \oplus B, \Phi}$ is similar.

- If Π is $\top \dfrac{}{\vdash \top, \Phi}$, then let $\underline{\Pi}_\mathsf{s}$ be the proof

$$
\cfrac{1\downarrow \cfrac{}{1}}{\mathsf{ai}\downarrow \cfrac{[\top, 0]}{\mathsf{t}\downarrow \cfrac{}{[\top, \underline{\Phi}_\mathsf{s}]}}} \quad .
$$

- If $?\mathsf{d}\dfrac{\vdash A, \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_\mathsf{s}$ be the proof

$$
\cfrac{\Pi' \| \mathsf{SLS} \cup \{1\downarrow\}}{\mathsf{w}\downarrow \cfrac{[\underline{A}_\mathsf{s}, \underline{\Phi}_\mathsf{s}]}{\mathsf{b}\downarrow \cfrac{[?\underline{A}_\mathsf{s}, \underline{A}_\mathsf{s}, \underline{\Phi}_\mathsf{s}]}{[?\underline{A}_\mathsf{s}, \underline{\Phi}_\mathsf{s}]}}} \quad ,
$$

where Π' exists by induction hypothesis.

- If $?c \dfrac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_s$ be the proof

$$
= \dfrac{
\dfrac{
\dfrac{
\Pi' \big\| \mathsf{SLS} \cup \{1\downarrow\}
}{
[?\underline{A}_s, ?\underline{A}_s, \underline{\Phi}_s]
}
}{
\mathsf{b}\downarrow \dfrac{[??\underline{A}_s, ?\underline{A}_s, \underline{\Phi}_s]}{[??\underline{A}_s, \underline{\Phi}_s]}
}
}{
[?\underline{A}_s, \underline{\Phi}_s]
} \quad ,
$$

where Π' exists by induction hypothesis.

- If $?w \dfrac{\vdash \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_s$ be the proof

$$
= \dfrac{
\dfrac{
\dfrac{\Pi' \big\| \mathsf{SLS} \cup \{1\downarrow\}}{\underline{\Phi}_s}
}{
\mathsf{w}\downarrow \dfrac{[\bot, \underline{\Phi}_s]}{[?\underline{A}_s, \underline{\Phi}_s]}
}
}{}
\quad ,
$$

where Π' exists by induction hypothesis.

- If $! \dfrac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n}$ is the last rule applied in Π, then there exists by induction hypothesis a derivation $\dfrac{1}{[\underline{A}_s, ?\underline{B_1}_s, \ldots, ?\underline{B_n}_s]} \Delta \big\| \mathsf{SLS}$. Now let $\underline{\Pi}_s$ be the proof

$$
= \dfrac{
\mathsf{p}\downarrow \dfrac{
\mathsf{p}\downarrow \dfrac{
\mathsf{p}\downarrow \dfrac{
\dfrac{
\Delta \big\| \mathsf{SLS}
}{
!{}[\underline{A}_s, ?\underline{B_1}_s, \ldots, ?\underline{B_n}_s]
}
\dfrac{1\downarrow \dfrac{}{1}}{!1}
}{
\vdots
}
}{
[!{}[\underline{A}_s, ?\underline{B_1}_s], ??\underline{B_2}_s, \ldots, ??\underline{B_n}_s]
}
}{
[!\underline{A}_s, ??\underline{B_1}_s, ??\underline{B_2}_s, \ldots, ??\underline{B_n}_s]
}
}{
[!\underline{A}_s, ?\underline{B_1}_s, ?\underline{B_2}_s, \ldots, ?\underline{B_n}_s]
} \quad . \qquad \square
$$

3.4 Cut Elimination

One of the reasons for doing cut elimination in the sequent calculus is to eliminate the only rule that violates the subformula property. Since in the calculus of structures we do no longer distinguish between sequents and formulae, it is no longer clear what the subformula property is. In the sequent calculus, the most important consequence of the subformula property is that for every rule (except the cut) there is, for a given sequent, only a finite number of possibilities to apply that rule. This property is still present in the calculus of

structures. Hence, in the calculus of structures the subformula property could be formulated as follows: A rule has the subformula property if its premise is built of substructures of its conclusion, such that there is only a finite number of possibilities to apply the rule (in a nontrivial way).

This property holds for most of the rules of system $\mathsf{SLS} \cup \{1{\downarrow}\}$. The only rules which violate this property are the rules $\mathsf{ai}{\uparrow}$, $\mathsf{t}{\uparrow}$ and $\mathsf{w}{\uparrow}$. It would therefore be sufficient to eliminate those three rules in order to get a system in which each rule enjoys the subformula property, which essentially would be a cut elimination result.

But we can get more. We can show that the whole up fragment of system SLS is admissible. This section contains two very different proofs of this fact.

The first uses the cut elimination proof for LL in the sequent calculus:

3.4.1 Theorem (Cut Elimination) *System LS is equivalent to every subsystem of the system $\mathsf{SLS} \cup \{1{\downarrow}\}$ containing LS.*

Proof: Given a proof in $\mathsf{SLS} \cup \{1{\downarrow}\}$, transform it into a proof in LL (by Theorem 3.3.1), to which we can apply the cut elimination procedure in the sequent calculus (Theorem 2.3.1). The cut-free proof in LL can then be transformed into a proof in system LS by Theorem 3.3.2. $\qquad\square$

3.4.2 Corollary *The rule $\mathsf{i}{\uparrow}$ is admissible for system LS.*

Proof: Immediate consequence of Theorems 3.2.21 and 3.4.1. $\qquad\square$

Another immediate consequence of the cut elimination theorem is the following corollary, that makes explicit the relation between derivations and proofs. In particular, it shows that system SLS can be seen as a system for derivations and LS as a system for proofs.

3.4.3 Corollary *Let R and T be two structures. Then we have*

$$
\begin{array}{ccc}
\begin{array}{c} T \\ \|\mathsf{SLS} \\ R \end{array} & \textit{if and only if} & \begin{array}{c} \|\mathsf{LS} \\ [\bar{T}, R] \end{array}
\end{array} \quad .
$$

Proof: For the first direction, perform the following transformations:

$$
\begin{array}{c} T \\ {\scriptstyle\Delta}\|\mathsf{SLS} \\ R \end{array}
\;\overset{1}{\leadsto}\;
\begin{array}{c} [\bar{T}, T] \\ {\scriptstyle\Delta}\|\mathsf{SLS} \\ [\bar{T}, R] \end{array}
\;\overset{2}{\leadsto}\;
\begin{array}{c} \mathsf{i}{\downarrow}\dfrac{\;\mathsf{1}{\downarrow}\dfrac{}{1}\;}{[\bar{T}, T]} \\ {\scriptstyle\Delta}\|\mathsf{SLS} \\ [\bar{T}, R] \end{array}
\;\overset{3}{\leadsto}\;
\begin{array}{c} {\scriptstyle\Pi}\|\mathsf{LS} \\ [\bar{T}, R] \end{array}
\quad .
$$

In the first step we replace each structure S occurring inside Δ by $[\bar{T}, S]$, which is then transformed into a proof by adding an instance of $\mathsf{i}{\downarrow}$ and $\mathsf{1}{\downarrow}$. Then we apply Proposition 3.2.16 and cut elimination (Theorem 3.4.1) to obtain a proof in system LS. For the other direction, let $\begin{array}{c} {\scriptstyle\Pi}\|\mathsf{LS} \\ [\bar{T}, R] \end{array}$ be given. Then there is a derivation $\begin{array}{c} 1 \\ {\scriptstyle\Delta}\|\mathsf{LS}\backslash\{\mathsf{1}{\downarrow}\} \\ [\bar{T}, R] \end{array}$, from which

we can construct the derivation

$$
\begin{array}{c}
= \dfrac{T}{(T,1)} \\[4pt]
\Delta \Big\| \mathsf{LS}\backslash\{1\downarrow\} \\[2pt]
\mathsf{s}\,\dfrac{(T,[\bar{T},R])}{[(T,\bar{T}),R]} \\[2pt]
\mathsf{i}\uparrow\,\dfrac{}{[\bot,R]} \\[2pt]
= \dfrac{}{R}
\end{array}
\qquad ,
$$

to which we apply Proposition 3.2.16 to get a derivation $\begin{array}{c} T \\ \|\mathsf{SLS} \\ R \end{array}$.   □

3.4.4   Remark   From Corollary 3.4.3, we immediately get the correspondence between derivability in system SLS and linear implication. For any two formulae A and B, we have that

$$
\begin{array}{c}
\underline{A}_{\,\mathsf{s}} \\
\|\mathsf{SLS} \\
\underline{B}_{\,\mathsf{s}}
\end{array}
\qquad \text{if and only if} \qquad A \multimap B \quad .
$$

The second proof of cut elimination will be carried out inside the calculus of structures, completely independent from the sequent calculus, and by using methods that are not available in the sequent calculus. Before I will go into the details of this proof, let me explain why it is indeed a challenge to prove cut elimination in the calculus of structures and why the methods of the sequent calculus cannot be applied.

In the cut elimination process in the sequent calculus (see Section 2.3), we *know* at each step whether we have a commutative case or a key case, and if we have a key case, we *know* (because of the principle of the main connective) how to decompose the cut formula and reduce the rank of the cut. Because there is no main connective in the calculus of structures and rules can applied anywhere deep inside structures, we cannot easily permute up the cut. For instance, if in a derivation

$$
\rho\,\dfrac{Q}{\mathsf{i}\uparrow\,\dfrac{S(R,\bar{R})}{S\{\bot\}}}
$$

the rule ρ modifies some substructure of R, we *do not know* how deep inside R the rule is applied. Furthermore, there is no reason to assume that any rule ρ' above ρ does the exact dual of ρ inside the structure $\bar{R}$. On the other hand, a great simplification is made possible in the calculus of structures by the reduction of cut to its atomic form. The remaining difficulty is actually understanding what happens, while going up in a proof, *around* the atoms produced by an atomic cut. The two atoms of an atomic cut can be produced inside any structure, and they do not belong to distinct branches, as in the sequent calculus. This means that complex interactions with their context are possible. As a consequence, the techniques for showing cut elimination inside the calculus of structures are quite different from the traditional ones.

In the following, I will present a proof of Theorem 3.4.1, that uses the techniques of *splitting* and *context reduction* that have been introduced in [Gug02e]. Because the whole proof is rather long and technical and uses some concepts that are introduced only in later chapters, the reader is invited to skip it in the first reading and jump immediately to Section 3.5 on page 80. Furthermore, this thesis contains in Chapter 7 another cut elimination proof based on splitting, which is much simpler. Therefore the reader interested in splitting is strongly advised not to read the splitting proof in this chapter until he has mastered the proof in Section 7.3. For that reason I also postponed the introductory notes on splitting and context reduction to that section, where I will explain the basic ideas on an informal level.

3.4.1 Splitting

In this section, I will prove the splitting lemma for system LS, which is stated as follows:

3.4.5   Lemma (Splitting)   *Let R, T, P be any structures, and let a be an atom.*

(a) *If $[\langle R,T \rangle, P]$ is provable in LS, then*

$$\overset{\displaystyle \| \mathsf{LS}}{[R,P]} \quad and \quad \overset{\displaystyle \| \mathsf{LS}}{[T,P]} \quad .$$

(b) *If $[\{R,T\}, P]$ is provable in LS, then there are structures P_R and P_T, such that*

$$\overset{\langle P_R, P_T \rangle}{\underset{P}{\| \mathsf{LS}}} \quad and \quad \overset{\displaystyle \| \mathsf{LS}}{[R,P_R]} \quad and \quad \overset{\displaystyle \| \mathsf{LS}}{[T,P_T]} \quad .$$

(c) *If $[(R,T),P]$ is provable in LS, then there is an $n \geqslant 0$ and there are structures $P_{R1},\ldots,P_{Rn}$ and $P_{T1},\ldots,P_{Tn}$, such that*

$$\overset{\langle [0,\top],\ldots,[0,\top],[P_{R1},P_{T1}],\ldots,[P_{Rn},P_{Tn}] \rangle}{\underset{P}{\| \mathsf{LS}}} \quad and$$

$$\overset{\displaystyle \| \mathsf{LS}}{[R,P_{Ri}]} \quad and \quad \overset{\displaystyle \| \mathsf{LS}}{[T,P_{Ti}]}$$

for every $i \in \{1,\ldots,n\}$.

(d) *If $[!R,P]$ is provable in LS, then there are $n \geqslant 0$ and $k_1,\ldots,k_n \geqslant 0$ and structures $P_{R11},\ldots,P_{R1k_1},P_{R21},\ldots,P_{R2k_2},\ldots,P_{Rn1},\ldots,P_{Rnk_n}$, such that*

$$\overset{\langle [0,\top],\ldots,[0,\top],[?P_{R11},\ldots,?P_{R1k_1}],\ldots,[?P_{Rn1},\ldots,?P_{Rnk_n}] \rangle}{\underset{P}{\| \mathsf{LS}}} \quad and$$

$$\overset{\displaystyle \| \mathsf{LS}}{[R,P_{Ri1},\ldots,P_{Rik_i}]}$$

for every $i \in \{1,\ldots,n\}$.

(f) *If $[a, P]$ is provable in* LS, *then there is a derivation*

$$\langle[0, \top], \ldots, [0, \top], \bar{a}, \ldots, \bar{a}\rangle$$
$$\|\text{LS}$$
$$P$$

.

3.4.6   Remark   In the statement of Lemma 3.4.5, the structure $\langle[0, \top], \ldots, [0, \top]\rangle$ consists of an arbitrary number (including zero) of copies of the structure $[0, \top]$. In other words, $\langle[0, \top], \ldots, [0, \top]\rangle$ can stand for any of the structures

$$\top \quad , \quad [0, \top] \quad , \quad \langle[0, \top], [0, \top]\rangle \quad , \quad \langle[0, \top], [0, \top], [0, \top]\rangle \quad , \quad \ldots$$

This convention is used throughout the remainder of this chapter.

The proof of the splitting lemma is rather technical. Before studying it, the reader should have developed some familiarity with the calculus of structures. In particular, the reader should have mastered at least Sections 4.2 and 7.3. Since this is the only place in this thesis where I use concepts that I introduce only later, one might argue about putting this proof into an appendix. This is not done for the following reasons: (1) conceptually this proof belongs to this chapter, and (2) for reasons of personal taste I do not like the idea of putting proofs into an appendix.

As mentioned in Section 7.3.1, the biggest difficulty in proving the splitting lemma is finding the right induction measure. The presence of the rules $c\!\downarrow$ and $b\!\downarrow$ makes this even more difficult for system LS. In order to simplify the situation, the following rules are introduced:

$$\text{cd}\!\downarrow \frac{S\langle[R, U], [T, U]\rangle}{S[\langle R, T\rangle, U]} \quad , \qquad \text{cb}\!\downarrow \frac{S[?R, ?R]}{S\{?R\}} \qquad \text{and} \qquad \text{cr}\!\downarrow \frac{S\{R\}}{S\{?R\}} \quad .$$

The following derivations show that the rules $\text{cd}\!\downarrow$, $\text{cb}\!\downarrow$, and $\text{cr}\!\downarrow$, are all derivable in LS.

$$\text{c}\!\downarrow \frac{\text{d}\!\downarrow \dfrac{S\langle[R, U], [T, U]\rangle}{S[\langle R, T\rangle, \langle U, U\rangle]}}{S[\langle R, T\rangle, U]} \quad , \qquad \begin{array}{c} = \dfrac{S[?R, ?R]}{\text{b}\!\downarrow \dfrac{S[??R, ?R]}{= \dfrac{S\{??R\}}{S\{?R\}}}} \end{array} \qquad \text{and} \qquad \text{b}\!\downarrow \frac{\text{w}\!\downarrow \dfrac{S\{R\}}{S[?R, R]}}{S\{?R\}} \quad .$$

The rule $\text{cd}\!\downarrow$ will simplify the case analysis because it cannot be applied as freely as the rule $c\!\downarrow$ can. Through the presence of the rule $\text{cb}\!\downarrow$ the induction measure can be simplified because $\text{cb}\!\downarrow$ can be permuted down (which will be shown in Lemma 3.4.9) but $b\!\downarrow$ cannot.

Another (technical) difficulty for determining the proper induction measure is caused by the presence of the constants. For that reason, the rules $\text{cd}'\!\downarrow$ and s' are introduced, which are identical to $\text{cd}\!\downarrow$ and s, respectively, with the difference that in the case of $\text{cd}'\!\downarrow$ we have $R \neq \top \neq T$, and in the case of s' we have that $R \neq 1$. The two rules are special cases of $\text{cd}\!\downarrow$ and s, respectively, and therefore derivable in LS. Lemma 3.4.8 will show that completeness is not lost when they replace the general rules. It will also be necessary to restrict the applicability of the rule $t\!\downarrow$. Let

$$\text{t}'\!\downarrow \frac{S[0, T]}{S[R, T]} \quad \text{(where } R \neq \bot \neq T\text{)} \qquad \text{and} \qquad \text{t}''\!\downarrow \frac{S\{T\}}{S\langle R, T\rangle} \quad \text{(where } T \neq 0\text{)} \quad .$$

$$1{\downarrow}\ \frac{}{1} \qquad\qquad {\top}{\downarrow}\ \frac{}{\top} \qquad\qquad \mathsf{ai}{\downarrow}\ \frac{S\{1\}}{S[a,\bar a]}$$

$$\mathsf{cd'}{\downarrow}\ \frac{S\langle[R,U],[T,U]\rangle}{S[\langle R,T\rangle,U]}\ \ (\text{where } R\neq\top\neq T) \qquad \mathsf{s'}\ \frac{S([R,U],T)}{S[(R,T),U]}\ \ (\text{where } R\neq 1)$$

$$\mathsf{t'}{\downarrow}\ \frac{S[0,T]}{S[R,T]}\ \ (\text{where } R\neq\bot\neq T) \qquad \mathsf{t''}{\downarrow}\ \frac{S\{T\}}{S\langle R,T\rangle}\ \ (\text{where } T\neq 0)$$

$$\mathsf{p}{\downarrow}\ \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{w}{\downarrow}\ \frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{cb}{\downarrow}\ \frac{S[?R,?R]}{S\{?R\}} \qquad \mathsf{cr}{\downarrow}\ \frac{S\{R\}}{S\{?R\}}$$

Figure 3.4: System $\mathsf{LS'}$

both are derivable in LS because they are special cases of $\mathsf{t}{\downarrow}$. However, if they are used to replace $\mathsf{t}{\downarrow}$, then completeness is lost, but can be regained by adding another axiom

$$\top{\downarrow}\ \frac{}{\top}\qquad ,$$

which is derivable in LS:

$$\mathsf{ai}{\downarrow}\ \frac{1{\downarrow}\ \dfrac{}{1}}{\underline{[0,\top]}}$$
$$\mathsf{t}{\downarrow}\ \frac{\underline{[0,\top]}}{\underline{[\bot,\top]}}$$
$$=\ \frac{}{\top}\qquad .$$

3.4.7   Definition   The system $\{1{\downarrow},\top{\downarrow},\mathsf{ai}{\downarrow},\mathsf{s'},\mathsf{cd'}{\downarrow},\mathsf{t'}{\downarrow},\mathsf{t''}{\downarrow},\mathsf{p}{\downarrow},\mathsf{w}{\downarrow},\mathsf{cb}{\downarrow},\mathsf{cr}{\downarrow}\}$, shown in Figure 3.4, is called *system* $\mathsf{LS'}$.

It should be obvious that system $\mathsf{LS'}$ is equivalent to LS because its rules are closer to the sequent system LL for linear logic. The precise relation between system $\mathsf{LS'}$ and system LS is given by the following lemma.

3.4.8   Lemma   *Let R and T be structures.*

(a) *If there is a derivation* $\ \overset{T}{\underset{R}{\Delta'\|\mathsf{LS'}}}$ *, then there is a derivation* $\ \overset{T}{\underset{R}{\Delta\|\mathsf{LS}}}$ *.*

(b) *If there is a proof* $\ \overset{}{\underset{R}{\Pi\|\mathsf{LS}}}$ *, then there is a proof* $\ \overset{}{\underset{R}{\Pi'\|\mathsf{LS'}}}$ *.*

Proof: (a) As mentioned before, the rules $\top{\downarrow}$, $\mathsf{cd'}{\downarrow}$, $\mathsf{s'}$, $\mathsf{t'}{\downarrow}$, $\mathsf{t''}{\downarrow}$, $\mathsf{cb}{\downarrow}$ and $\mathsf{cr}{\downarrow}$ are all derivable in LS.

(b) A given proof Π in LS will be transformed into a proof Π' in LS' according to the following steps. First replace all instances of the rules $\mathsf{d}{\downarrow}$ and $\mathsf{b}{\downarrow}$ by the following derivations:

$$
\mathsf{cd}{\downarrow}\,\frac{\mathsf{t}{\downarrow}\,\dfrac{\mathsf{t}{\downarrow}\,\dfrac{S\langle[R,U],[T,V]\rangle}{S\langle[R,U],[T,\langle U,V\rangle]\rangle}}{S\langle[R,\langle U,V\rangle],[T,\langle U,V\rangle]\rangle}}{S[\langle R,T\rangle,\langle U,V\rangle]}
\qquad\text{and}\qquad
\mathsf{cb}{\downarrow}\,\frac{\mathsf{cr}{\downarrow}\,\dfrac{S[?R,R]}{S[?R,?R]}}{S\{?R\}}\quad.
$$

In the second step, the instances of $\mathsf{t}{\downarrow}$ are permuted up until they disappear or become instances of $\mathsf{t}'{\downarrow}$ or $\mathsf{t}''{\downarrow}$. Following the case analysis in 4.2.3, we have that the cases (1) to (3) are trivial, and cases (4) and (6) are impossible. If case (5) occurs we have one of the following:

(i) The instance of $\mathsf{t}{\downarrow}$ is duplicated by an instance of $\mathsf{cd}{\downarrow}$, $\mathsf{c}{\downarrow}$ or $\mathsf{cb}{\downarrow}$, or it is eliminated by $\mathsf{w}{\downarrow}$, $\mathsf{t}'{\downarrow}$ or $\mathsf{t}''{\downarrow}$.

(ii) The 0 in the contractum of $\mathsf{t}{\downarrow}$ appears actively in the redex of the rule above (as in case (5) in the proof of Lemma 5.3.18). Then we have already an instance of $\mathsf{t}''{\downarrow}$.

(iii) The 0 in the contractum of $\mathsf{t}{\downarrow}$ appears in the redex of $\mathsf{i}{\uparrow}$. Then we either have already an instance of $\mathsf{t}'{\downarrow}$, or we have the situation

$$
\mathsf{t}{\downarrow}\,\frac{\mathsf{ai}{\downarrow}\,\dfrac{S\{1\}}{S[0,\top]}}{S[\bot,\top]}\quad.
$$

In this case remove the subderivation and replace the 1 occurring in the premise everywhere by $\top$, which either disappears somewhere as unit in a context $\langle\ldots\rangle$ or remains until the very top of the proof where the rule $\top{\downarrow}$ can be used.

In the third step, eliminate all instances of $\mathsf{cd}{\downarrow}$ that are not instances of $\mathsf{cd}'{\downarrow}$, i.e. they are of the form

$$
\mathsf{cd}{\downarrow}\,\frac{S\langle[R,U],[\top,U]\rangle}{S[R,U]}\quad.
$$

Consider the topmost such instance and replace the substructure $[\top,U]$ everywhere above by $\top$. (By this every rule modifying $[\top,U]$ becomes trivial and will be eliminated too.) Repeat this for every instance of $\mathsf{cd}{\downarrow}$ that is not an instance of $\mathsf{cd}'{\downarrow}$.

Now, all instances of s that are not instances of s' are eliminated. For this, consider the topmost instance of

$$
\mathsf{s}\,\frac{S([1,U],T)}{S[T,U]}\quad,
$$

which can be permuted up in the proof until it disappears (i.e. $U=\bot$ or $T=1$). As before, follow 4.2.3. The situation

$$
\mathsf{s}\,\frac{\mathsf{s}'\,\dfrac{S([1,U,W],T)}{S[([1,U],T),W]}}{S[T,U,W]}\quad,
$$

which can occur in case (5), is replaced by

$$\mathsf{s}\,\frac{S([1,U,W],T)}{S[T,U,W]}\quad.$$

It remains to eliminate the rule $\mathsf{c}{\downarrow}$. For doing so, let me introduce the super rule

$$\mathsf{sc}{\downarrow}\,\frac{S\{R_1,R_2\}}{S\{R\}}\quad,\quad\text{where there are derivations}\quad \begin{array}{c}R_1\\\Delta_1\|\mathsf{LS}'\\R\end{array}\quad\text{and}\quad\begin{array}{c}R_2\\\Delta_2\|\mathsf{LS}'\\R\end{array}\quad.$$

Observe that $\mathsf{c}{\downarrow}$ is an instance of $\mathsf{sc}{\downarrow}$. I will now show that the rule $\mathsf{sc}{\downarrow}$ is admissible for LS'. This will be done by permuting all instances of $\mathsf{sc}{\downarrow}$ to the top where they disappear eventually. More precisely, if an instance of $\mathsf{sc}{\downarrow}$ appears at the top of a proof, i.e. we have the situation

$$\mathsf{sc}{\downarrow}\,\frac{1{\downarrow}\,\overline{S\{R_1,R_2\}}}{S\{R\}}\quad,$$

then $S\{R_1,R_2\} = 1$. Therefore one of R_1 and R_2 must be equal to 0. Without loss of generality, assume $R_1 = 0$. Then we can replace the derivation by

$$\begin{array}{c}1{\downarrow}\,\overline{S\{R_2\}}\\\Delta_2\|\mathsf{LS}'\\S\{R\}\end{array}\quad.$$

Now consider

$$\mathsf{sc}{\downarrow}\,\frac{\pi\,\dfrac{Q}{S\{R_1,R_2\}}}{S\{R\}}\quad,$$

where $\pi \in \mathsf{LS}'$ is nontrivial. According to 4.2.3, there are the following cases to consider

(4) The redex of π is inside an active structure of the contractum of $\mathsf{sc}{\downarrow}$. We have the following subcases:

(i) The redex of π is inside R_1. Then

$$\mathsf{sc}{\downarrow}\,\frac{\pi\,\dfrac{S\{R_1',R_2\}}{S\{R_1,R_2\}}}{S\{R\}}\quad\text{yields}\quad\mathsf{sc}{\downarrow}\,\frac{S\{R_1',R_2\}}{S\{R\}}\quad,$$

because there is a derivation $\begin{array}{c}R_1'\\\|\mathsf{LS}'\\R\end{array}$.

(ii) The redex of π is inside R_2. This is similar to (i).

(iii) The redex of π overlaps with both, R_1 and R_2. Then $\pi = \mathsf{t}''{\downarrow}$ and $R_1 = \{R_1',R_1''\}$ and $R_2 = \{R_2',R_2''\}$. We have

$$\mathsf{sc}{\downarrow}\,\frac{\mathsf{t}''{\downarrow}\,\dfrac{S\{R_1',R_2''\}}{S\{R_1',R_1'',R_2',R_2''\}}}{S\{R\}}\quad\text{yields}\quad\mathsf{sc}{\downarrow}\,\frac{S\{R_1',R_2''\}}{S\{R\}}\quad.$$

(5) The contractum of $\mathsf{sc}{\downarrow}$ is inside an active structure of the redex of π. There are four subcases:

   (i) $\pi = \mathsf{cd}{\downarrow}$ and the contractum $\{R_1, R_2\}$ of $\mathsf{sc}{\downarrow}$ occurs inside the substructure which is duplicated in the instance of $\mathsf{cd}{\downarrow}$. In other words, we have

$$
\mathsf{sc}{\downarrow}\,\cfrac{\mathsf{cd}{\downarrow}\,\cfrac{S\langle [V, U\{R_1, R_2\}], [T, U\{R_1, R_2\}]\rangle}{S[\langle V, T\rangle, U\{R_1, R_2\}]}}{S[\langle V, T\rangle, U\{R\}]}\quad,
$$

   which can be replaced by

$$
\mathsf{sc}{\downarrow}\,\cfrac{S\langle [V, U\{R_1, R_2\}], [T, U\{R_1, R_2\}]\rangle}{\mathsf{sc}{\downarrow}\,\cfrac{S\langle [V, U\{R_1, R_2\}], [T, U\{R\}]\rangle}{\mathsf{cd}{\downarrow}\,\cfrac{S\langle [V, U\{R\}], [T, U\{R\}]\rangle}{S[\langle V, T\rangle, U\{R\}]}}}\quad.
$$

   (ii) $\pi = \mathsf{cb}{\downarrow}$. Similar to (i).

   (iii) $\pi = \mathsf{t}'{\downarrow}$ or $\pi = \mathsf{t}''{\downarrow}$. Then

$$
\mathsf{sc}{\downarrow}\,\cfrac{\mathsf{t}'{\downarrow}\,\cfrac{S\{0\}}{S\{U\{R_1, R_2\}\}}}{S\{U\{R\}\}}\quad\text{yields}\quad \mathsf{t}'{\downarrow}\,\cfrac{S\{0\}}{S\{U\{R\}\}}\quad.
$$

   (iv) $\pi = \mathsf{w}{\downarrow}$. Similar to (iii).

(6) The redex of π and the contractum $\{R_1, R_2\}$ of $\mathsf{sc}{\downarrow}$ overlap. This is only possible, if $\pi = \mathsf{t}''{\downarrow}$. In the most general case we have $S\{\ \} = S'\{\{\ \}, T\}$ and $R_1 = \{R'_1, R''_1\}$ and $R_2 = \{R'_2, R''_2\}$. Then

$$
\mathsf{sc}{\downarrow}\,\cfrac{\mathsf{t}''{\downarrow}\,\cfrac{S'\{R'_1, R''_2\}}{S'\{R'_1, R''_1, R'_2, R''_2, T\}}}{S'\{R, T\}}\quad\text{yields}\quad \mathsf{t}''{\downarrow}\,\cfrac{\mathsf{sc}{\downarrow}\,\cfrac{S\{R'_1, R''_2\}}{S\{R\}}}{S'\{R, T\}}\quad,
$$

   which is similar to case (4.iii). $\qquad\square$

The following lemma says that the rule $\mathsf{cb}{\downarrow}$ can be separated from the other rules in system LS'. This property contributes considerably for simplifying the induction measure for the proof of the splitting lemma because it will suffice to find a induction measure for the system $\mathsf{LS}' \setminus \{\mathsf{cb}{\downarrow}\}$.

3.4.9 Lemma *For every proof* $\begin{array}{c} \\ \|\mathsf{LS}' \\ R \end{array}$ *, there is a proof*

$$
\begin{array}{c}
\\
\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
R' \\
\|\{\mathsf{cb}{\downarrow}\} \\
R
\end{array}\quad,
$$

for some structure R'.

Proof: All instance of the rule $\mathsf{cb}{\downarrow}$ can be permuted down. To see this, consider

$$\mathsf{cb}{\downarrow}\frac{S[?Z,?Z]}{\rho\dfrac{S\{?Z\}}{P}} \quad ,$$

where $\rho \in \mathsf{LS}' \setminus \{\mathsf{cb}{\downarrow}\}$ and $\mathsf{cb}{\downarrow}$ are nontrivial. According to the case analysis in 4.2.3, we have the following cases to consider

(4) The redex $?Z$ of $\mathsf{cb}{\downarrow}$ is inside an active structure of the contractum of ρ. This is only possible if $\rho = \mathsf{cd}'{\downarrow}$ and we have

$$\mathsf{cb}{\downarrow}\frac{S\langle[R,U[?Z,?Z]],[T,U\{?Z\}]\rangle}{\mathsf{cd}'{\downarrow}\dfrac{S\langle[R,U\{?Z\}],[T,U\{?Z\}]\rangle}{S[\langle R,T\rangle,U\{?Z\}]}} \quad ,$$

which can be replaced by

$$\mathsf{w}{\downarrow}\frac{S\langle[R,U[?Z,?Z]],[T,U\{?Z\}]\rangle}{\mathsf{cd}'{\downarrow}\dfrac{S\langle[R,U[?Z,?Z]],[T,U[?Z,?Z]]\rangle}{\mathsf{cb}{\downarrow}\dfrac{S[\langle R,T\rangle,U[?Z,?Z]]}{S[\langle R,T\rangle,U\{?Z\}]}}} \quad .$$

(5) The contractum of ρ is inside the redex $?Z$ of $\mathsf{cb}{\downarrow}$. Then we have

$$\mathsf{cb}{\downarrow}\frac{S[?Z,?Z]}{\rho\dfrac{S\{?Z\}}{S\{?Z'\}}} \quad , \quad \text{which can be replaced by} \quad \mathsf{cb}{\downarrow}\frac{\rho\dfrac{\rho\dfrac{S[?Z,?Z]}{S[?Z,?Z']}}{S[?Z',?Z']}}{S\{?Z'\}} \quad .$$

(6) The redex $?Z$ of $\mathsf{cb}{\downarrow}$ and the contractum of ρ overlap. This is impossible.

According to this, all instances of $\mathsf{cb}{\downarrow}$ can be permuted down until the derivation has the desired shape. $\square$

In the following, I will show the splitting lemma for system $\mathsf{LS}' \setminus \{\mathsf{cb}{\downarrow}\}$ (Lemma 3.4.19), from which the splitting lemma for system LS (Lemma 3.4.5) will easily follow. Before I show the (rather technical) proof of splitting, let me explain the induction measure.

Consider the finite multisets of natural numbers, or equivalently, the elements of the free commutative monoid generated by $\mathbf{N}$. Let M and N be two such multisets. Then define

$$M + N = \{\, n + m \mid n \in N \;\; \text{and} \;\; m \in N \,\} \quad .$$

This sum is the sum of the free commutative monoid, and distributes over the union: Given three such multisets M, N, and O, then

$$(M \cup N) + O = (M + O) \cup (N + O) \quad .$$

On the set of multisets of natural numbers, let me define the binary relation $<$ as follows: We have $M < N$ iff there exists an injective $f : M \to N$ such that for all $n \in M$ we have $m \leqslant f(m)$ and there is an $n \in N$ such that $f^{-1}(n)$ is undefined or $f^{-1}(n) < n$. Transitivity of $<$ is obvious. Since we consider only finite multisets, $<$ is also irreflexive. Hence $<$ is a strict partial order. That $<$ is well-founded follows immediately. I will write $M \leqslant N$ if $M < N$ or $M = N$.

In the following, I will define the size of a structure to be a multiset of natural numbers (or, equivalently, an element of the free commutative monoid generated by $\mathbf{N}$). For this, I consider structures to be in normal form (see Remark 3.1.5), in particular, superfluous units are omitted. The *size* of an LS structure R, denoted by $\mathrm{size}(R)$, is inductively defined as follows:

$$
\begin{aligned}
\mathrm{size}(a) &= \{1\} \quad \text{(for every atom, including } \bot, \text{1, 0, and } \top) \quad, \\
\mathrm{size}([R, T]) &= \mathrm{size}(R) + \mathrm{size}(T) \quad, \\
\mathrm{size}((R, T)) &= \mathrm{size}(R) \cup \mathrm{size}(T) \quad, \\
\mathrm{size}(\langle R, T \rangle) &= \mathrm{size}(R) \cup \mathrm{size}(T) \quad, \\
\mathrm{size}(\langle R, T \rangle) &= \mathrm{size}(R) \cup \mathrm{size}(T) \quad, \\
\mathrm{size}(!R) &= \mathrm{size}(R) \quad, \\
\mathrm{size}(?R) &= \mathrm{size}(R) \quad.
\end{aligned}
$$

3.4.10 Example Let $R = [!(1, a, [c, \bar{c}], \langle ?\bar{a}, \top, b, \bot \rangle), c]$. Then we have that $R = [!(a, [c, \bar{c}], \langle ?\bar{a}, b, \bot \rangle), c]$, which has

$$
\mathrm{size}(R) = (\{1\} \cup (\{1\} + \{1\}) \cup \{1\} \cup \{1\} \cup \{1\}) + \{1\} = \{2, 3, 2, 2, 2\} \quad.
$$

3.4.11 Observation Let $\rho \dfrac{T}{R}$ be an application of a rule $\rho \in \mathsf{LS}' \setminus \{\mathsf{cb}\!\downarrow\}$. Then we have $\mathrm{size}(T) \leqslant \mathrm{size}(R)$. This is the reason, why the rule s' and $\mathsf{cd}'\!\downarrow$ have been introduced and the restriction of $R \neq \bot$ has been put on $\mathsf{t}'\!\downarrow$.

3.4.12 Observation Let R and T be structures and $S\{\ \}$ be a context. If $\mathrm{size}(T) < \mathrm{size}(R)$, then $\mathrm{size}(S\{T\}) < \mathrm{size}(S\{R\})$. (This follows immediately from the definitions by using induction on $S\{\ \}$).

3.4.13 Observation Let R and T be structures. If T is a proper substructure of R (i.e. $R = S\{T\}$ for some nontrivial context $S\{\ \}$), then $\mathrm{size}(T) < \mathrm{size}(R)$.

Observations 3.4.11 to 3.4.13 ensure that in the proof of Lemma 3.4.19, which will follow below, the induction hypothesis can indeed be applied in every case where it is applied. Let me use two typical examples.

If R, P_1, P_2 and P_3 are given such that $P_1 \neq \top \neq P_2$, then

$$
\begin{aligned}
\mathrm{size}([R, P_1, P_3]) &< \mathrm{size}([R, \langle P_1, P_2 \rangle, P_3]) \quad \text{and} \\
\mathrm{size}([R, P_2, P_3]) &< \mathrm{size}([R, \langle P_1, P_2 \rangle, P_3]) \quad.
\end{aligned}
$$

If R, P_1, P_2, P_3, Q and $S\{\ \}$ are given such that $P_1 \neq 1 \neq P_2$ and $\begin{array}{c} S\{Q\} \\ \|\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \\ P_3 \end{array}$, then

$$\mathrm{size}\big([R, P_1, Q]\big) \quad < \quad \mathrm{size}\big([R, (P_1, P_2), P_3]\big) \quad .$$

The full induction measure in the proof of Lemma 3.4.19 will be the size of the proofs, which is defined as follows:

3.4.14   Definition   The *size* of a proof $\begin{array}{c}\Pi\| \\ R\end{array}$ is the pair

$$\mathrm{size}\big(\Pi\big) = \big\langle \mathrm{size}\big(R\big), \mathrm{length}(\Pi) \big\rangle \quad .$$

Given two proofs $\begin{array}{c}\Pi\| \\ R\end{array}$ and $\begin{array}{c}\Pi'\| \\ R'\end{array}$, then define

$$\begin{aligned} \mathrm{size}\big(\Pi\big) < \mathrm{size}\big(\Pi'\big) \quad \Longleftrightarrow \quad & \mathrm{size}\big(R\big) < \mathrm{size}\big(R'\big) \quad \text{or} \\ & \mathrm{size}\big(R\big) = \mathrm{size}\big(R'\big) \quad \text{and} \quad \mathrm{length}(\Pi) < \mathrm{length}(\Pi') \quad . \end{aligned}$$

I will make heavy use of the following lemmata (without explicitly mentioning them all the time).

3.4.15   Lemma   *Let $n, m \geqslant 0$ and let $R_1, \dots, R_n$ and $T_1, \dots, T_m$ be any structures. Then there is are derivations*

$$\begin{array}{c} \langle[R_1, T_1], [R_1, T_2], \dots, [R_1, T_m], \dots, [R_n, T_1], [R_n, T_2], \dots, [R_n, T_m]\rangle \\ \Delta\big\|\{\mathsf{c}\downarrow, \mathsf{d}\downarrow\} \\ [\langle R_1, \dots, R_n\rangle, \langle T_1, \dots, T_m\rangle] \end{array}$$

and

$$\begin{array}{c} \langle[R_1, T_1], [R_1, T_2], \dots, [R_1, T_m], \dots, [R_n, T_1], [R_n, T_2], \dots, [R_n, T_m]\rangle \\ \Delta'\big\|\{\mathsf{cd}'\downarrow\} \\ [\langle R_1, \dots, R_n\rangle, \langle T_1, \dots, T_m\rangle] \end{array} \quad .$$

Proof: The derivation Δ consists of $(n-1)(m-1)$ instances of $\mathsf{c}\downarrow$ and $\mathsf{d}\downarrow$.   □

3.4.16   Lemma   *Let $n \geqslant 0$ and let $R_1, \dots, R_n$ structures that are provable in LS. Then $\langle R_1, \dots, R_n\rangle$ is also provable in LS.*

Proof: Immediate from the definitions (use $\langle 1, 1 \rangle = 1$).   □

3.4.17   Lemma   *Let $n, m \geqslant 0$ and let $R_1, \dots, R_n$ and $T_1, \dots, T_m$ be any structures. If for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, the structure $[R_i, T_j]$ is provable in LS, then $[\langle R_1, \dots, R_n\rangle, \langle T_1, \dots, T_m\rangle]$ is also provable in LS.*

Proof: Immediate consequence of the previous two lemmata.   □

3.4.18 Lemma *Let R be any structures. Then $[R, 0, \top]$ is provable in* LS.

Proof: Use

$$
\mathsf{t}\!\downarrow \frac{\mathsf{ai}\!\downarrow \dfrac{\mathsf{1}\!\downarrow \dfrac{}{1}}{[0, \top]}}{[R, 0, \top]} \quad .
$$

$\square$

Observe that Lemmata 3.4.16 to 3.4.18 do also hold for system $\mathsf{LS'} \setminus \{\mathsf{cb}\!\downarrow\}$. It follows the splitting lemma for the system $\mathsf{LS'} \setminus \{\mathsf{cb}\!\downarrow\}$. Its proof is essentially the same as the proof of splitting for system NELm in Chapter 7. But in the presence of the additives, the cases are more complex.

3.4.19 Lemma *Let R, T, P be any structures, and let a be an atom.*

(a) *If $[\langle R, T\rangle, P]$ is provable in $\mathsf{LS'} \setminus \{\mathsf{cb}\!\downarrow\}$, then*

$$
\overset{\displaystyle\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}{[R, P]} \quad \text{and} \quad \overset{\displaystyle\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}{[T, P]} \quad .
$$

(b) *If $[\{R, T\}, P]$ is provable in $\mathsf{LS'} \setminus \{\mathsf{cb}\!\downarrow\}$, then there are structures P_R and P_T, such that*

$$
\overset{\langle P_R, P_T\rangle}{\underset{P}{\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}} \quad \text{and} \quad \overset{\displaystyle\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}{[R, P_R]} \quad \text{and} \quad \overset{\displaystyle\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}{[T, P_T]} \quad .
$$

(c) *If $[(R, T), P]$ is provable in $\mathsf{LS'} \setminus \{\mathsf{cb}\!\downarrow\}$, then there is an $n \geqslant 0$ and there are structures $P_{R1}, \ldots, P_{Rn}$ and $P_{T1}, \ldots, P_{Tn}$, such that*

$$
\overset{\langle [0, \top], \ldots, [0, \top], [P_{R1}, P_{T1}], \ldots, [P_{Rn}, P_{Tn}]\rangle}{\underset{P}{\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}} \quad \text{and}
$$

$$
\overset{\displaystyle\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}{[R, P_{Ri}]} \quad \text{and} \quad \overset{\displaystyle\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}{[T, P_{Ti}]} \quad ,
$$

for every $i \in \{1, \ldots, n\}$.

(d) *If $[!R, P]$ is provable in $\mathsf{LS'} \setminus \{\mathsf{cb}\!\downarrow\}$, then there are $n \geqslant 0$ and $k_1, \ldots, k_n \geqslant 0$ and structures $P_{R11}, \ldots, P_{R1k_1}, P_{R21}, \ldots, P_{R2k_2}, \ldots, P_{Rn1}, \ldots, P_{Rnk_n}$, such that*

$$
\overset{\langle [0, \top], \ldots, [0, \top], [?P_{R11}, \ldots, ?P_{R1k_1}], \ldots, [?P_{Rn1}, \ldots, ?P_{Rnk_n}]\rangle}{\underset{P}{\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}} \quad \text{and}
$$

$$
\overset{\displaystyle\|\mathsf{LS'}\setminus\{\mathsf{cb}\!\downarrow\}}{[R, P_{Ri1}, \ldots, P_{Rik_i}]}
$$

for every $i \in \{1, \ldots, n\}$.

(e) *If $[?R, P]$ is provable in $\mathsf{LS}' \setminus \{\mathsf{cb}\!\downarrow\}$, then either P is provable in $\mathsf{LS}' \setminus \{\mathsf{cb}\!\downarrow\}$, or there are $n \geqslant 0$ and $k \leqslant n$ and structures $P_{R1}, \ldots, P_{Rn}$, such that*

$$\langle [0, \top], \ldots, [0, \top], !P_{R1}, \ldots, !P_{Rk}, P_{Rk+1}, \ldots, P_{Rn} \rangle$$
$$\big\| \mathsf{LS}' \setminus \{\mathsf{cb}\!\downarrow\} \qquad\qquad and$$
$$P$$

$$\big\| \mathsf{LS}' \setminus \{\mathsf{cb}\!\downarrow\}$$
$$[R, P_{Ri}]$$

for every $i \in \{1, \ldots, n\}$.

(f) *If $[a, P]$ is provable in LS, then there is a derivation*

$$\langle [0, \top], \ldots, [0, \top], \bar{a}, \ldots, \bar{a} \rangle$$
$$\big\| \mathsf{LS}' \setminus \{\mathsf{cb}\!\downarrow\} \qquad .$$
$$P$$

Proof: All six statements are proved simultaneously by the induction on the size of the proof as defined before. What follows is a case analysis which is conceptually similar to the cut elimination proof in the sequent calculus.

(a) I will consider only the case where $R \neq \top \neq T$ (otherwise the statement is trivially true). Observe that also in the case $R = 1 = T$ the statement is trivially true.

Consider now the bottommost rule instance ρ in the proof $\prod \big\| \mathsf{LS}' \setminus \{\mathsf{cb}\!\downarrow\}$. Without
$$[\langle R, T \rangle, P]$$
loss of generality, we can assume that the application of ρ is nontrivial. As in the splitting proof in Chapter 7, we can distinguish between three conceptually different cases:

(1) The redex of ρ is inside R, T or P.

(2) The substructure $\langle R, T \rangle$ is inside the redex of ρ, but the application of ρ does not decompose the structure $\langle R, T \rangle$.

(3) The application of ρ decomposes the substructure $\langle R, T \rangle$.

Case (2) can be compared to a commutative case in a cut elimination proof in the sequent calculus (see Section 2.3), and case (3) to a key case. Case (1) has no counterpart in the sequent calculus because there is no possibility of deep inference.

(1) The redex of ρ is inside R, T or P. Then there are three subcases of which I show only the first, the other two being similar (compare the proof of Lemma 7.3.5 on page 217).

(i) The redex of ρ is inside R. Then the proof Π has the shape

$$\Pi' \big\| \mathsf{LS}' \setminus \{\mathsf{cb}\!\downarrow\}$$
$$\rho \, \frac{[\langle R', T \rangle, P]}{[\langle R, T \rangle, P]} \qquad .$$

By applying the induction hypothesis, we get

$$\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad,$$
$$[R',P] \qquad\qquad\qquad [T,P]$$

from which we get

$$\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$\rho\,\frac{[R',P]}{[R,P]}\quad.$$

(2) The substructure $\langle R,T\rangle$ is inside the redex of ρ, but the application of ρ does not decompose the structure $\langle R,T\rangle$. We have the following subcases:

(i) $\rho = \mathsf{s'}$ and $P = [(P_1,P_2),P_3,P_4]$ (and $P_1 \neq 1 \neq P_2$) and Π is

$$\Pi'\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$\mathsf{s'}\,\frac{[([\langle R,T\rangle,P_1,P_3],P_2),P_4]}{[\langle R,T\rangle,(P_1,P_2),P_3,P_4]}\quad.$$

By applying the induction hypothesis to Π', we get an $n \geqslant 0$ and structures $Q_{11},\ldots,Q_{1n}$ and $Q_{21},\ldots,Q_{2n}$, such that

$$\langle[0,\top],\ldots,[0,\top],[Q_{11},Q_{21}],\ldots,[Q_{1n},Q_{2n}]\rangle$$
$$\Delta_P\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad\qquad \text{and}$$
$$P_4$$

$$\Pi_{1i}\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \Pi_{2i}\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$[\langle R,T\rangle,P_1,P_3,Q_{1i}] \qquad\qquad\qquad [P_2,Q_{2i}]$$

for every $i \in \{1,\ldots,n\}$. By applying the induction hypothesis again to Π_{1i}, we get

$$\Pi_{Ri}\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \Pi_{Ti}\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad.$$
$$[R,P_1,P_3,Q_{1i}] \qquad\qquad\qquad [T,P_1,P_3,Q_{1i}]$$

From which we can build for every $i \in \{1,\ldots,n\}$:

$$\Pi_{Ri}\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad\qquad\qquad\qquad \Pi_{Ti}\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$[R,P_1,P_3,Q_{1i}] \qquad\qquad\qquad\qquad\qquad [T,P_1,P_3,Q_{1i}]$$
$$\Pi_{2i}\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad\qquad\qquad\qquad \Pi_{2i}\overset{\|}{\|}\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$\mathsf{s'}\,\frac{[R,(P_1,[P_2,Q_{2i}]),P_3,Q_{1i}]}{[R,(P_1,P_2),P_3,Q_{1i},Q_{2i}]} \quad \text{and} \quad \mathsf{s'}\,\frac{[T,(P_1,[P_2,Q_{2i}]),P_3,Q_{1i}]}{[T,(P_1,P_2),P_3,Q_{1i},Q_{2i}]} \quad.$$

We can now obtain the proof

$$\underset{\Vert \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}}{}$$
$$\langle [R,(P_1,P_2),P_3,0,\top],\ldots,[R,(P_1,P_2),P_3,0,\top],$$
$$[R,(P_1,P_2),P_3,Q_{11},Q_{21}],\ldots,[R,(P_1,P_2),P_3,Q_{1n},Q_{2n}]\rangle$$
$$\Big\Vert\{\mathsf{cd'}\downarrow\}$$
$$[R,(P_1,P_2),P_3,\langle[0,\top],\ldots,[0,\top],[Q_{11},Q_{21}],\ldots,[Q_{1n},Q_{2n}]\rangle]$$
$$\Delta_P \Big\Vert \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$= \frac{[R,(P_1,P_2),P_3,P_4]}{[R,P]} \quad .$$

Similarly, we can get a proof of $[T,P]$.

(ii) $\rho = \mathsf{cd'}\downarrow$ and $P = [\langle P_1,P_2\rangle,P_3,P_4]$ and Π is

$$\Pi' \Big\Vert \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$\mathsf{cd'}\downarrow \frac{[\langle[\langle R,T\rangle,P_1,P_3],[\langle R,T\rangle,P_2,P_3]\rangle,P_4]}{[\langle R,T\rangle,\langle P_1,P_2\rangle,P_3,P_4]} \quad .$$

By applying the induction hypothesis to Π', we get

$$\Pi_1\Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \Pi_2\Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \quad .$$
$$[\langle R,T\rangle,P_1,P_3,P_4] \qquad\qquad\qquad [\langle R,T\rangle,P_2,P_3,P_4]$$

Applying the induction hypothesis again to Π_1 and Π_2 yields the four proofs:

$$\Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \quad \text{and} \qquad \Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \quad \text{and}$$
$$[R,P_1,P_3,P_4] \qquad\qquad\qquad [T,P_1,P_3,P_4]$$

$$\Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \quad \text{and} \qquad \Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \quad .$$
$$[R,P_2,P_3,P_4] \qquad\qquad\qquad [T,P_2,P_3,P_4]$$

We can now obtain

$$\Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$\mathsf{cd'}\downarrow \frac{\langle[R,P_1,P_3,P_4],[R,P_2,P_3,P_4]\rangle}{= \dfrac{[R,\langle P_1,P_2\rangle,P_3,P_4]}{[R,P]}} \quad ,$$

and a similar proof for $[T,P]$.

(iii) $\rho = \mathsf{t'}\downarrow$ and $P = [P_1,P_2]$ and Π is

$$\Pi'\Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$\mathsf{t'}\downarrow \frac{[0,P_2]}{[\langle R,T\rangle,P_1,P_2]} \quad .$$

We can get:

$$\Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad\qquad \Big\Vert\mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$\mathsf{t'}\downarrow\frac{[0,P_2]}{[R,P_1,P_2]} \qquad \text{and} \qquad \mathsf{t'}\downarrow\frac{[0,P_2]}{[T,P_1,P_2]} \quad .$$

(3) The application of ρ decomposes the substructure $\langle R, T \rangle$. This case is the crucial one. We have only one subcase:

(i) $\rho = \mathsf{cd'}{\downarrow}$ and $R = \langle R_1, R_2 \rangle$ and $T = \langle T_1, T_2 \rangle$ and $P = [P_1, P_2]$ and Π is

$$\mathsf{cd'}{\downarrow}\ \frac{\Pi' \big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\}}{[\langle [\langle R_1, T_1\rangle, P_1], [\langle R_2, T_2\rangle, P_1]\rangle, P_2]}{[\langle R_1, R_2, T_1, T_2\rangle, P_1, P_2]}\ .$$

By applying the induction hypothesis to Π', we get

$$\begin{array}{ccc}\Pi_1 \big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\} & \text{and} & \Pi_2 \big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\}\\ [\langle R_1, T_1\rangle, P_1, P_2] & & [\langle R_2, T_2\rangle, P_1, P_2]\end{array}\ .$$

Applying the induction hypothesis again to Π_1 and Π_2 yields the four proofs:

$$\begin{array}{ccc}\big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\} & \text{and} & \big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\} & \text{and}\\ [R_1, P_1, P_2] & & [T_1, P_1, P_2]\end{array}$$

$$\begin{array}{ccc}\big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\} & \text{and} & \big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\}\\ [R_2, P_1, P_2] & & [T_2, P_1, P_2]\end{array}\ .$$

We can now obtain

$$\mathsf{cd'}{\downarrow}\ \frac{\dfrac{\big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\}}{\langle [R_1, P_1, P_2], [R_2, P_1, P_2]\rangle}}{\dfrac{[\langle R_1, R_2\rangle, P_1, P_2]}{[R, P]}}\ ,$$

and a similar proof for $[T, P]$.

(b) I will consider only the case where $R \neq 0 \neq T$ because otherwise the statement is trivially true. (If $R = 0$ let $P_R = \top$.) Also the case where $R = \bot = T$ is trivially true (let $P_R = P_T = 1$). Consider now the bottommost rule instance ρ in the proof $\Pi \big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\}$. Without loss of generality, we can assume that the application of ρ
$[\{R, T\}, P]$
is nontrivial. As in case (a), we can distinguish between the following three cases:

(1) The redex of ρ is inside R, T or P. This is similar to case (a.i).

(2) The substructure $\{R, T\}$ is inside the redex of ρ, but the application of ρ does not decompose the structure $\{R, T\}$. This case is also similar to (a), but there are some differences.

(i) $\rho = \mathsf{s'}$ and $P = [(P_1, P_2), P_3, P_4]$ and Π is

$$\mathsf{s'}\ \frac{\Pi' \big\| \mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\}}{[([\{R, T\}, P_1, P_3], P_2), P_4]}{[\{R, T\}, (P_1, P_2), P_3, P_4]}\ .$$

By applying the induction hypothesis to Π', we get an $n \geqslant 0$ and structures $Q_{11}, \ldots, Q_{1n}$ and $Q_{21}, \ldots, Q_{2n}$, such that

$$
\langle [0, \top], \ldots, [0, \top], [Q_{11}, Q_{21}], \ldots, [Q_{1n}, Q_{2n}] \rangle
$$
$$
\Delta_P \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \qquad\qquad \text{and}
$$
$$
P_4
$$

$$
\Pi_{1i} \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \Pi_{2i} \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}
$$
$$
[\langle R, T \rangle, P_1, P_3, Q_{1i}] \qquad\qquad\qquad [P_2, Q_{2i}]
$$

for every $i \in \{1, \ldots, n\}$. By applying the induction hypothesis again to Π_{1i}, we get structures P_{Ri} and P_{Ti}, such that

$$
\langle P_{Ri}, P_{Ti} \rangle
$$
$$
\Delta_{Qi} \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \qquad .
$$
$$
[P_1, P_3, Q_{1i}] \qquad\qquad\qquad [R, P_{Ri}] \qquad\qquad\qquad [T, P_{Ti}]
$$

Now let $P_R = \langle P_{R1}, \ldots, P_{Rn} \rangle$ and $P_T = \langle P_{T1}, \ldots, P_{Tn}, [0, \top], \ldots, [0, \top] \rangle$, and build

$$
= \frac{\langle P_R, P_T \rangle}{\langle [0, \top], \ldots, [0, \top], P_{R1}, P_{T1}, \ldots, P_{Rn}, P_{Tn} \rangle}
$$
$$
\Delta_{Qi} \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}
$$
$$
\langle [0, \top], \ldots, [0, \top], [P_1, P_3, Q_{11}] \rangle, \ldots, [P_1, P_3, Q_{1n}]
$$
$$
\Pi_{2i} \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}
$$
$$
\langle [0, \top], \ldots, [0, \top], [(P_1, [P_2, Q_{21}], P_3, Q_{11})], \ldots, [(P_1, [P_2, Q_{2n}], P_3, Q_{1n})] \rangle
$$
$$
\Big\| \{\mathsf{t'}\downarrow, \mathsf{s'}\}
$$
$$
\langle [(P_1, P_2), P_3, 0, \top], \ldots, [(P_1, P_2), P_3, 0, \top],
$$
$$
[(P_1, P_2), P_3, Q_{11}, Q_{21}], \ldots, [(P_1, P_2), P_3, Q_{1n}, Q_{2n}] \rangle
$$
$$
\Big\| \{\mathsf{cd'}\downarrow\}
$$
$$
[(P_1, P_2), P_3, \langle [0, \top], \ldots, [0, \top], [Q_{11}, Q_{21}], \ldots, [Q_{1n}, Q_{2n}] \rangle]
$$
$$
\Delta_P \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}
$$
$$
= \frac{[(P_1, P_2), P_3, P_4]}{P}
$$

and

$$
\Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}
$$
$$
\langle [R, P_{R1}], \ldots, [R, P_{Rn}] \rangle
$$
$$
\Big\| \{\mathsf{cd'}\downarrow\}
$$
$$
= \frac{[R, \langle P_{R1}, \ldots, P_{Rn} \rangle]}{[R, P_R]}
$$

and

$$
\Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}
$$
$$
\langle [T, P_{T1}], \ldots, [T, P_{Tn}], [T, 0, \top], \ldots, [T, 0, \top] \rangle
$$
$$
\Big\| \{\mathsf{cd'}\downarrow\}
$$
$$
= \frac{[T, \langle P_{T1}, \ldots, P_{Tn}, [0, \top], \ldots, [0, \top] \rangle]}{[T, P_T]} \qquad .
$$

(ii) $\rho = \mathsf{cd}'\!\downarrow$ and $P = [\langle P_1, P_2\rangle, P_3, P_4]$ and Π is

$$
\mathsf{cd}'\!\downarrow \frac{\overset{\Pi'\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[\langle[\langle R,T\rangle, P_1, P_3], [\langle R,T\rangle, P_2, P_3]\rangle, P_4]}}{[\langle R,T\rangle, \langle P_1, P_2\rangle, P_3, P_4]} \quad .
$$

By applying the induction hypothesis to Π', we get

$$
\overset{\Pi_1\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[\langle R,T\rangle, P_1, P_3, P_4]} \qquad \text{and} \qquad \overset{\Pi_2\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[\langle R,T\rangle, P_2, P_3, P_4]} \quad .
$$

Applying the induction hypothesis again to Π_1 and Π_2 yields structures P_{R1}, P_{R2}, P_{T1}, and P_{T2}, such that

$$
\overset{\langle P_{R1}, P_{T1}\rangle}{\underset{[P_1, P_3, P_4]}{\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}} \quad \text{and} \quad \overset{\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[R, P_{R1}]} \quad \text{and} \quad \overset{\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[T, P_{T1}]}
$$

and

$$
\overset{\langle P_{R2}, P_{T2}\rangle}{\underset{[P_2, P_3, P_4]}{\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}} \quad \text{and} \quad \overset{\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[R, P_{R2}]} \quad \text{and} \quad \overset{\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[T, P_{T2}]} \quad .
$$

Now let $P_R = \langle P_{R1}, P_{R2}\rangle$ and $P_T = \langle P_{T1}, P_{T2}\rangle$, and build

$$
\mathsf{cd}'\!\downarrow \frac{= \dfrac{\langle P_R, P_T\rangle}{\langle P_{R1}, P_{T1}, P_{R2}, P_{T2}\rangle} \Big\| \mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{= \dfrac{\langle[P_1, P_3, P_4], [P_2, P_3, P_4]\rangle}{\dfrac{[\langle P_1, P_2\rangle, P_3, P_4]}{P}}}
$$

and

$$
\mathsf{cd}'\!\downarrow \frac{= \dfrac{\overset{\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{\langle[R, P_{R1}], [R, P_{R2}]\rangle}}{[R, \langle P_{R1}, P_{R2}\rangle]}}{[R, P_R]} \qquad \text{and} \qquad \mathsf{cd}'\!\downarrow \frac{= \dfrac{\overset{\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{\langle[T, P_{T1}], [T, P_{T2}]\rangle}}{[T, \langle P_{T1}, P_{T2}\rangle]}}{[T, P_T]} \quad .
$$

(iii) $\rho = \mathsf{t}'\!\downarrow$ and $P = [P_1, P_2]$ and Π is

$$
\mathsf{t}'\!\downarrow \frac{\overset{\Pi'\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[0, P_2]}}{[\langle R,T\rangle, P_1, P_2]} \quad .
$$

Now let $P_R = P = [P_1, P_2]$ and $P_T = \top$. We can get:

$$
= \frac{\langle P_R, P_T\rangle}{P} \qquad \text{and} \qquad \mathsf{t}'\!\downarrow \frac{\overset{\|\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[0, P_2]}}{[R, P_1, P_2]} \qquad \text{and} \qquad \mathsf{t}'\!\downarrow \frac{\mathsf{ai}\!\downarrow \dfrac{\mathsf{1}\!\downarrow \dfrac{}{1}}{[0, \top]}}{[T, \top]} \quad .
$$

(Remark: If $[R, P_1] = \bot$ then exchange P_R and P_T. If $T = \bot$, then use the axiom $\top\!\downarrow$ for proving $[T, \top]$. It cannot be the case that $[R, P_1] = \bot = T$ because then $[\{R, T\}, P_1] = \bot$.)

(3) The application of ρ decomposes the substructure $\{R, T\}$. Again, we have only one possible case:

(i) $\rho = t''\!\downarrow$ and $R = \{R_1, R_2\}$ and $T = \{T_1, T_2\}$ and Π is

$$t''\!\downarrow \frac{\overset{\displaystyle \Pi' \,\Big\| \, \mathsf{LS'}\backslash\{\mathsf{cb}\!\downarrow\}}{[\{R_1, T_1\}, P]}}{[\{R_1, R_2, T_2, T_1\}, P]} \quad .$$

By applying the induction hypothesis to Π', we get structures P_R and P_T, such that

$$\begin{array}{c} \langle P_R, P_T \rangle \\ \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\!\downarrow\} \\ P \end{array} \quad \text{and} \quad \Pi_{R_1} \Big\| \mathsf{LS'}\backslash\{\mathsf{cb}\!\downarrow\} \\ [R_1, P_R] \quad \text{and} \quad \Pi_{T_1} \Big\| \mathsf{LS'}\backslash\{\mathsf{cb}\!\downarrow\} \\ [T_1, P_T] \quad .$$

In the case where $R_1 \neq 0$ we can get

$$t''\!\downarrow \frac{\overset{\displaystyle \Big\| \mathsf{LS'}\backslash\{\mathsf{cb}\!\downarrow\}}{[R_1, P_R]}}{[\{R_1, R_2\}, P_R]} \quad .$$

If $R_1 = 0$ and $R_2 \neq \bot$, then use

$$t'\!\downarrow \frac{\overset{\displaystyle \Big\| \mathsf{LS'}\backslash\{\mathsf{cb}\!\downarrow\}}{[R_1, P_R]}}{[\{R_1, R_2\}, P_R]} \quad .$$

If $R_1 = 0$ and $R_2 = \bot$, then apply the induction hypothesis to Π_{R_1}, which yields

$$\begin{array}{c} \langle [0, \top], \dots, [0, \top], \top, \dots, \top \rangle \\ \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\!\downarrow\} \\ P_R \end{array} \quad ,$$

from which we can get

$$\begin{array}{c} \Big\| \mathsf{LS'}\backslash\{\mathsf{cb}\!\downarrow\} \\ \langle [0, \top], \dots, [0, \top], \top, \dots, \top \rangle \\ \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\!\downarrow\} \\ = \dfrac{P_R}{[\{0, \bot\}, P_R]} \\ = \dfrac{}{[\{R_1, R_2\}, P_R]} \end{array} \quad .$$

Similarly, we get a proof of $[T, P_T]$.

(c) I will consider only the case where $R \neq 1 \neq T$ because otherwise the statement is trivially true. (If $R = 1$, let $n = 1$ and $P_{R1} = \bot$ and $P_{T1} = P$. For $T = 1$ the situation is similar.) As before, consider the bottommost rule instance ρ in the proof
$$\Pi \| \mathsf{LS}' \backslash \{\mathsf{cb}\downarrow\}$$
$$[(R,T),P]$$
. Without loss of generality, we can assume that the application of ρ is nontrivial. As in case (a), we can distinguish between the following three cases:

(1) The redex of ρ is inside R, T or P. This is similar to case (a.i).

(2) The substructure (R,T) is inside the redex of ρ, but the application of ρ does not decompose the structure (R,T). Again, this case is similar to (a), but there are some differences.

   (i) $\rho = \mathsf{s}'$ and $P = [(P_1, P_2), P_3, P_4]$ and Π is

$$\mathsf{s}' \frac{\Pi' \| \mathsf{LS}' \backslash \{\mathsf{cb}\downarrow\}}{[([(R,T), P_1, P_3], P_2), P_4]}{[(R,T), (P_1, P_2), P_3, P_4]} \quad .$$

By applying the induction hypothesis to Π', we get an $h \geqslant 0$ and structures $Q_{11}, \ldots, Q_{1n}$ and $Q_{21}, \ldots, Q_{2h}$, such that

$$\langle [0, \top], \ldots, [0, \top], [Q_{11}, Q_{21}], \ldots, [Q_{1h}, Q_{2h}] \rangle$$
$$\Delta_P \| \mathsf{LS}' \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and}$$
$$P_4$$

$$\Pi_{1i} \| \mathsf{LS}' \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \Pi_{2i} \| \mathsf{LS}' \backslash \{\mathsf{cb}\downarrow\}$$
$$[(R,T), P_1, P_3, Q_{1i}] \qquad\qquad [P_2, Q_{2i}]$$

for every $i \in \{1, \ldots, h\}$. By applying the induction hypothesis again to Π_{1i}, we get an $m_i \geqslant 0$ and structures $P_{Ri1}, \ldots, P_{Rim_i}$ and $P_{Ti1}, \ldots, P_{Tim_i}$, such that

$$\langle [0, \top], \ldots, [0, \top], [P_{Ri1}, P_{Ti1}], \ldots, [P_{Rim_i}, P_{Tim_i}] \rangle$$
$$\Delta_{Q_i} \| \mathsf{LS}' \backslash \{\mathsf{cb}\downarrow\} \qquad\qquad \text{and}$$
$$[P_1, P_3, Q_{1i}]$$

$$\| \mathsf{LS}' \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \| \mathsf{LS}' \backslash \{\mathsf{cb}\downarrow\} \quad ,$$
$$[R, P_{Rij}] \qquad\qquad\qquad [T, P_{Tij}]$$

for every $j \in \{1, \dots, m_i\}$. Now let $n = m_1 + \cdots + m_h$. We can get

$$\langle [0, \top], \dots, [0, \top], [P_{R11}, P_{T11}], \dots, [P_{R1m_1}, P_{T1m_1}], \dots,$$
$$[P_{Rh1}, P_{Th1}], \dots, [P_{Rhm_h}, P_{Thm_h}] \rangle$$
$$\Delta_{Qi} \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}$$
$$\langle [0, \top], \dots, [0, \top], [P_1, P_3, Q_{11}], \dots, [P_1, P_3, Q_{1h}] \rangle$$
$$\Pi_{2i} \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}$$
$$\langle [0, \top], \dots, [0, \top], [(P_1, [P_2, Q_{21}]), P_3, Q_{11}], \dots, [(P_1, [P_2, Q_{2h}]), P_3, Q_{1h}] \rangle$$
$$\Big\| \{\mathsf{t'}\downarrow, \mathsf{s'}\}$$
$$\langle [(P_1, P_2), P_3, 0, \top], \dots, [(P_1, P_2), P_3, 0, \top],$$
$$[(P_1, P_2), P_3, Q_{11}, Q_{21}], \dots, [(P_1, P_2), P_3, Q_{1h}, Q_{2h}] \rangle$$
$$\Big\| \{\mathsf{cd'}\downarrow\}$$
$$[(P_1, P_2), P_3, \langle [0, \top], \dots, [0, \top], [Q_{11}, Q_{21}], \dots, [Q_{1h}, Q_{2h}] \rangle]$$
$$\Delta_P \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}$$
$$= \frac{[(P_1, P_2), P_3, P_4]}{P} \quad .$$

(ii) $\rho = \mathsf{cd'}\downarrow$ and $P = [\langle P_1, P_2 \rangle, P_3, P_4]$ and Π is

$$\Pi' \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}$$
$$\mathsf{cd'}\downarrow \frac{[\langle [(R, T), P_1, P_3], [(R, T), P_2, P_3] \rangle, P_4]}{[(R, T), \langle P_1, P_2 \rangle, P_3, P_4]} \quad .$$

By applying the induction hypothesis to Π', we get

$$\Pi_1 \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \Pi_2 \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \quad .$$
$$[(R, T), P_1, P_3, P_4] \qquad\qquad\qquad [(R, T), P_2, P_3, P_4]$$

Applying the induction hypothesis again to Π_1 and Π_2 yields an $h \geqslant 0$ and structures $P_{R1}, \dots, P_{Rh}$ and $P_{T1}, \dots, P_{Th}$, such that

$$\langle [0, \top], \dots, [0, \top], [P_{R1}, P_{T1}], \dots, [P_{Rh}, P_{Th}] \rangle$$
$$\Delta_{P1} \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and}$$
$$[P_1, P_3, P_4]$$

$$\Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \quad \text{and} \quad \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \quad ,$$
$$[R, P_{Ri}] \qquad\qquad\qquad [T, P_{Ti}]$$

for every $i \in \{1, \dots, h\}$, and an $h' \geqslant 0$ and structures $P'_{R1}, \dots, P'_{Rh'}$ and $P'_{T1}, \dots, P'_{Th'}$, such that

$$\langle [0, \top], \dots, [0, \top], [P'_{R1}, P'_{T1}], \dots, [P'_{Rh'}, P'_{Th'}] \rangle$$
$$\Delta_{P2} \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and}$$
$$[P_1, P_3, P_4]$$

$$\Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \quad \text{and} \quad \Big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \quad ,$$
$$[R, P'_{Rj}] \qquad\qquad\qquad [T, P'_{Tj}]$$

for every $j \in \{1, \ldots, h'\}$. Now let $n = h + h'$, we can get

$$\langle [0, \top], \ldots, [0, \top], [P_{R1}, P_{T1}], \ldots, [P_{Rh}, P_{Th}], [P'_{R1}, P'_{T1}], \ldots, [P'_{Rh'}, P'_{Th'}] \rangle$$

$$\mathsf{cd'} \downarrow \frac{\Delta_{P1}, \Delta_{P2} \big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}}{= \frac{\langle [P_1, P_3, P_4], [P_2, P_3, P_4] \rangle}{[\langle P_1, P_2 \rangle, P_3, P_4]}}{P} \quad .$$

(iii) $\rho = \mathsf{t'}\downarrow$ and $P = [P_1, P_2]$ and Π is

$$\mathsf{t'}\downarrow \frac{\Pi' \big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}}{[(R, T), P_1, P_2]} \quad .$$

By applying the induction hypothesis to Π', we get

$$\langle [0, \top], \ldots, [0, \top], \top, \ldots, \top \rangle \atop \big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \atop P_2 \quad ,$$

from which we can get the following (let $n = 0$):

$$\langle [0, \top], \ldots, [0, \top] \rangle$$
$$\big\| \{\mathsf{t'}\downarrow\}$$
$$\langle [P_1, 0, \top], \ldots, [P_1, 0, \top] \rangle$$
$$\big\| \{\mathsf{cd'}\downarrow\}$$
$$= \frac{[P_1, \langle [0, \top], \ldots, [0, \top] \rangle]}{[P_1, \langle [0, \top], \ldots, [0, \top], \top, \ldots, \top \rangle]} \quad .$$
$$\big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}$$
$$= \frac{[P_1, P_2]}{P} \quad ,$$

(3) The application of ρ decomposes the substructure (R, T). As before, this case is the crucial one and there is one subcase:

(i) $\rho = \mathsf{s'}$ and $R = (R_1, R_2)$ and $T = (T_1, T_2)$ and $P = [P_1, P_2]$ and Π is

$$\mathsf{s'} \frac{\Pi' \big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\}}{[([(R_1, T_1), P_1], R_2, T_2), P_2]}{[(R_1, R_2, T_1, T_2), P_1, P_2]} \quad .$$

By applying the induction hypothesis to Π', we get an $h \geqslant 0$ and structures $Q_{11}, \ldots, Q_{1h}$ and $Q_{21}, \ldots, Q_{2h}$, such that

$$\langle [0, \top], \ldots, [0, \top], [Q_{11}, Q_{21}], \ldots, [Q_{1h}, Q_{2h}] \rangle \atop \Delta_P \big\| \mathsf{LS'} \backslash \{\mathsf{cb}\downarrow\} \qquad \text{and} \atop P_2$$

$$\Pi_{1i} \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \Pi_{2i} \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$[(R_1, T_1), P_1, Q_{1i}] \qquad\qquad\qquad [(R_2, T_2), Q_{2i}]$$

for every $i \in \{1, \dots, h\}$. Applying the induction hypothesis again to Π_{1i} and Π_{2i} yields an $m_i \geqslant 0$ and structures $P_{Ri1}, \dots, P_{Rim_i}$ and $P_{Ti1}, \dots, P_{Tim_i}$, such that

$$\langle\!\langle [0, \top], \dots, [0, \top], [P_{Ri1}, P_{Ti1}], \dots, [P_{Rim_i}, P_{Tim_i}] \rangle\!\rangle$$
$$\Delta_{Q_{1i}} \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad\qquad \text{and}$$
$$[P_1, Q_{1i}]$$

$$\big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \quad ,$$
$$[R, P_{Rij}] \qquad\qquad\qquad [T, P_{Tij}]$$

for every $j \in \{1, \dots, m_i\}$, and an $m_i' \geqslant 0$ and structures $P_{Ri1}', \dots, P_{Rim_i'}'$ and $P_{Ti1}', \dots, P_{Tim_i'}'$, such that

$$\langle\!\langle [0, \top], \dots, [0, \top], [P_{Ri1}', P_{Ti1}'], \dots, [P_{Rim_i'}', P_{Tim_i'}'] \rangle\!\rangle$$
$$\Delta_{Q_{2i}} \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad\qquad \text{and}$$
$$Q_{2i}$$

$$\big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \qquad \text{and} \qquad \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \quad ,$$
$$[R, P_{Rik}'] \qquad\qquad\qquad [T, P_{Tik}']$$

for every $k \in \{1, \dots, m_i'\}$. Now let $n = m_1 m_1' + \cdots + m_h m_h'$. We can get

$$\langle\!\langle [0, \top], \dots, [0, \top],$$
$$[P_{R11}, P_{R11}', P_{T11}, P_{T11}'], \dots, [P_{R1m_1}, P_{R1m_1'}', P_{T1m_1}, P_{T1m_1'}'], \dots,$$
$$[P_{Rh1}, P_{Rh1}', P_{Th1}, P_{Th1}'], \dots, [P_{Rhm_h}, P_{Rhm_h'}', P_{Thm_h}, P_{Thm_h'}'] \rangle\!\rangle$$
$$\big\| \{\mathsf{t'}\downarrow, \mathsf{cd'}\downarrow\}$$
$$\langle\!\langle [0, \top], \dots, [0, \top],$$
$$[\langle\!\langle [0, \top], \dots, [0, \top], [P_{R11}, P_{T11}], \dots, [P_{R1m_1}, P_{T1m_1}] \rangle\!\rangle,$$
$$\langle\!\langle [0, \top], \dots, [0, \top], [P_{R11}', P_{T11}'], \dots, [P_{R1m_1'}', P_{T1m_1'}'] \rangle\!\rangle], \dots,$$
$$[\langle\!\langle [0, \top], \dots, [0, \top], [P_{Rh1}, P_{Th1}], \dots, [P_{Rhm_h}, P_{Thm_h}] \rangle\!\rangle,$$
$$\langle\!\langle [0, \top], \dots, [0, \top], [P_{Rh1}', P_{Th1}'], \dots, [P_{Rhm_h'}', P_{Thm_h'}'] \rangle\!\rangle] \rangle\!\rangle$$
$$\Delta_{Q_{1i}}, \Delta_{Q_{2i}} \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$\langle\!\langle [P_1, 0, \top], \dots, [P_1, 0, \top], [P_1, Q_{11}, Q_{21}], \dots, [P_1, Q_{1h}, Q_{2h}] \rangle\!\rangle \quad .$$
$$\big\| \{\mathsf{cd'}\downarrow\}$$
$$[P_1, \langle\!\langle [0, \top], \dots, [0, \top], [Q_{11}, Q_{21}], \dots, [Q_{1h}, Q_{2h}] \rangle\!\rangle]$$
$$\Delta_P \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\}$$
$$= \frac{[P_1, P_2]}{P}$$

(d) Consider the bottommost rule instance ρ in the proof $\; \Pi \big\| \mathsf{LS'}\backslash\{\mathsf{cb}\downarrow\} \;$. Without loss of
$$[!R, P]$$
generality, we can assume that the application of ρ is nontrivial. As in case (a), we can distinguish between the following three cases:

(1) The redex of ρ is inside R or P. This is similar to case (a.i).

(2) The substructure $!R$ is inside the redex of ρ, but the application of ρ does not decompose the structure $!R$. This case is similar to (c.2).

(3) The application of ρ decomposes the substructure $!R$. This is only possible if $\rho = \mathsf{p}{\downarrow}$. Hence, there is only one subcase:

(i) $\rho = \mathsf{p}{\downarrow}$, $P = [?P_1, P_2]$ and Π is

$$
\mathsf{p}{\downarrow}\,\frac{\Pi'\big\|\mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\}}{\dfrac{[![R,P_1],P_2]}{[!R,?P_1,P_2]}} \quad .
$$

By applying the induction hypothesis to Π', we get $n \geqslant 0$ and $l_1, \ldots, l_n \geqslant 0$ and structures $P_{R11}, \ldots, P_{R1l_1}, P_{R21}, \ldots, P_{R2l_2}, \ldots, P_{Rn1}, \ldots, P_{Rnl_n}$, such that

$$
\frac{\langle[0,\top],\ldots,[0,\top],[?P_{R11},\ldots,?P_{R1l_1}],\ldots,[?P_{Rn1},\ldots,?P_{Rnl_n}]\rangle}{\Delta_P\big\|\mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\}}
$$

and
$$
\begin{array}{c}\big\|\mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\} \\ [R, P_1, P_{Ri1}, \ldots, P_{Ril_i}]\end{array} \quad ,
$$

for every $i \in \{1, \ldots, n\}$. Now let $k_i = l_i + 1$ and $P_{Rik_i} = P_1$. We can get

$$
\frac{\langle[0,\top],\ldots,[0,\top],[?P_{R11},\ldots,?P_{R1k_1}],\ldots,[?P_{Rn1},\ldots,?P_{Rnk_n}]\rangle}{\Big\|\{\mathsf{t'}{\downarrow}\}}
$$
$$
\frac{\langle[?P_1,0,\top],\ldots,[?P_1,0,\top],}{\qquad\qquad[?P_1,?P_{R11},\ldots,?P_{R1l_1}],\ldots,[?P_1,?P_{Rn1},\ldots,?P_{Rnl_n}]\rangle}
$$
$$
\Big\|\{\mathsf{cd'}{\downarrow}\}
$$
$$
[?P_1,\langle[0,\top],\ldots,[0,\top],[?P_{R11},\ldots,?P_{R1l_1}],\ldots,[?P_{Rn1},\ldots,?P_{Rnl_n}]\rangle] \quad .
$$
$$
\frac{\Delta_P\big\|\mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\}}{=\dfrac{[?P_1,P_2]}{P}}
$$

(e) Consider the bottommost rule instance ρ in the proof $\begin{array}{c}\Pi\big\|\mathsf{LS'}\backslash\{\mathsf{cb}{\downarrow}\} \\ [?R,P]\end{array}$. Without loss of generality, we can assume that the application of ρ is nontrivial. As in case (a), we can distinguish between the following three cases:

(1) The redex of ρ is inside R or P. This is similar to case (a.i).

(2) The substructure $?R$ is inside the redex of ρ, but the application of ρ does not decompose the structure $?R$. As in the previous case, this is similar to (c.2).

(3) The application of ρ decomposes the substructure $?R$. We have the following subcases:

(i) $\rho = \mathsf{w}\downarrow$, and Π is

$$\mathsf{w}\downarrow\frac{\overset{\Pi'\big\Vert\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{P}}{[?R,P]}\quad.$$

Then P is provable in $\mathsf{LS}' \setminus \{\mathsf{cb}\downarrow\}$.

(ii) $\rho = \mathsf{cr}\downarrow$, and Π is

$$\mathsf{cr}\downarrow\frac{\overset{\Pi'\big\Vert\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[R,P]}}{[?R,P]}\quad.$$

Then let $n = 1$ and $k = 0$ and $P_{R1} = P$.

(iii) $\rho = \mathsf{p}\downarrow$, $P = [!P_1, P_2]$ and Π is

$$\mathsf{p}\downarrow\frac{\overset{\Pi'\big\Vert\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{[![R,P_1],P_2]}}{[?R,!P_1,P_2]}\quad.$$

By applying the induction hypothesis to Π', we get $n \geqslant 0$ and $l_1, \ldots, l_n \geqslant 0$ and structures $P_{R11}, \ldots, P_{R1l_1}, P_{R21}, \ldots, P_{R2l_2}, \ldots, P_{Rn1}, \ldots, P_{Rnl_n}$, such that

$$\frac{\langle[0,\top],\ldots,[0,\top],[?P_{R11},\ldots,?P_{R1l_1}],\ldots,[?P_{Rn1},\ldots,?P_{Rnl_n}]\rangle}{\underset{\Delta_P\big\Vert\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{P_2}}$$

and $\quad \underset{[R,P_1,P_{Ri1},\ldots,P_{Ril_i}]}{\big\Vert\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}\quad$,

for every $i \in \{1, \ldots, n\}$. Now let $k = n$ and $P_{Ri} = [P_1, P_{Ri1}, \ldots, ?P_{Ril_i}]$. We can get

$$\frac{\langle[0,\top],\ldots,[0,\top],![P_1,P_{R11},\ldots,?P_{R1l_1}],\ldots,![P_1,P_{Rn1},\ldots,?P_{Rnl_n}]\rangle}{\underset{\{\mathsf{t}'\downarrow,\mathsf{p}\downarrow\}}{\big\Vert}}$$

$$\frac{\langle[!P_1,0,\top],\ldots,[!P_1,0,\top],}{[!P_1,?P_{R11},\ldots,?P_{R1l_1}],\ldots,[!P_1,?P_{Rn1},\ldots,?P_{Rnl_n}]\rangle}$$

$$\big\Vert\{\mathsf{cd}'\downarrow\}$$

$$\frac{[!P_1,\langle[0,\top],\ldots,[0,\top],[?P_{R11},\ldots,?P_{R1l_1}],\ldots,[?P_{Rn1},\ldots,?P_{Rnl_n}]\rangle]}{\underset{\Delta_P\big\Vert\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}{}}\quad.$$

$$=\frac{[!P_1,P_2]}{P}$$

(f) Consider the bottommost rule instance ρ in the proof $\quad\underset{[a,P]}{\Pi\big\Vert\mathsf{LS}'\backslash\{\mathsf{cb}\downarrow\}}\quad$. Without loss of generality, we can assume that the application of ρ is nontrivial. As in case (a), we can distinguish between the following three cases:

(1) The redex of ρ is inside P. This is similar to case (a.i).

(2) The atom a is inside the redex of ρ, but the application of ρ does not "decompose" the atom a. As before, this is similar to (c.2).

(3) The application of ρ "decomposes" the atom a: In other words, a is used in an application of the rule $\mathsf{ai}\!\downarrow$, i.e. there is only one case:

(i) $\rho = \mathsf{ai}\!\downarrow$, $P = [\bar{a}, P_1]$ and Π is

$$\mathsf{ai}\!\downarrow \frac{\Pi' \big\| \mathsf{LS}'\backslash\{\mathsf{cb}\!\downarrow\}}{[1, P_1]}{[a, \bar{a}, P_1]} \quad .$$

By applying the induction hypothesis to Π', we get

$$\frac{\langle [0, \top], \ldots, [0, \top], \bot, \ldots, \bot \rangle}{\Delta_P \big\| \mathsf{LS}'\backslash\{\mathsf{cb}\!\downarrow\}}{P_1}$$

From this we can get

$$\frac{\langle [0, \top], \ldots, [0, \top], \bar{a} \ldots, \bar{a} \rangle}{\big\| \{\mathsf{t}'\!\downarrow\}}{\langle [\bar{a}, 0, \top], \ldots, [\bar{a}, 0, \top], [\bar{a}, \bot], \ldots, [\bar{a}, \bot] \rangle}$$
$$\frac{\big\| \{\mathsf{cd}'\!\downarrow\}}{[\bar{a}, \langle [0, \top], \ldots, [0, \top], \bot, \ldots, \bot \rangle]} \quad .$$
$$\frac{\Delta_P \big\| \mathsf{LS}'\backslash\{\mathsf{cb}\!\downarrow\}}{= \frac{[\bar{a}, P_1]}{P}}$$

$\square$

From splitting for $\mathsf{LS}' \backslash \{\mathsf{cb}\!\downarrow\}$, we can immediately obtain a proof of the splitting statement for system LS:

Proof of Lemma 3.4.5: I will show only the case (c). The others are similar. Let

$$\frac{\big\| \mathsf{LS}}{[\langle R, T \rangle, P]}$$

be given. By Lemma 3.4.8 there is a proof

$$\frac{\big\| \mathsf{LS}'}{[\langle R, T \rangle, P]} \quad ,$$

which can by Lemma 3.4.9 be decomposed into

$$\frac{\big\| \mathsf{LS}'\backslash\{\mathsf{cb}\!\downarrow\}}{U}{\big\| \{\mathsf{cb}\!\downarrow\}}{[\langle R, T \rangle, P]} \quad ,$$

for some structure U. Since the redex of cb↓ is always a why-not structure, we have that $U = [\{R', T'\}, P']$ for some structures R', T' and P', such that

$$
\begin{array}{ccccc}
R' & & T' & & P' \\
\|\{\text{cb}\downarrow\} & \text{and} & \|\{\text{cb}\downarrow\} & \text{and} & \|\{\text{cb}\downarrow\} \\
R & & T & & P
\end{array}
\quad .
$$

We can now apply splitting for $\mathsf{LS}' \setminus \{\text{cb}\downarrow\}$ (Lemma 3.4.19). This yields structures P_R and P_T, such that

$$
\begin{array}{ccccc}
\langle P_R, P_T \rangle & & & & \\
\|\mathsf{LS}'\setminus\{\text{cb}\downarrow\} & \text{and} & \overline{\overline{\|}}\mathsf{LS}'\setminus\{\text{cb}\downarrow\} & \text{and} & \overline{\overline{\|}}\mathsf{LS}'\setminus\{\text{cb}\downarrow\} \\
P' & & [R', P_R] & & [T', P_T]
\end{array}
\quad .
$$

with the derivations above, we get

$$
\begin{array}{ccccc}
\langle P_R, P_T \rangle & & & & \\
\|\mathsf{LS}'\setminus\{\text{cb}\downarrow\} & & \overline{\overline{\|}}\mathsf{LS}'\setminus\{\text{cb}\downarrow\} & & \overline{\overline{\|}}\mathsf{LS}'\setminus\{\text{cb}\downarrow\} \\
P' & \text{and} & [R', P_R] & \text{and} & [T', P_T] \\
\|\{\text{cb}\downarrow\} & & \overline{\overline{\|}}\{\text{cb}\downarrow\} & & \overline{\overline{\|}}\{\text{cb}\downarrow\} \\
P & & [R, P_R] & & [T, P_T]
\end{array}
\quad .
$$

By applying Lemma 3.4.8 we get

$$
\begin{array}{ccccc}
\langle P_R, P_T \rangle & & & & \\
\|\mathsf{LS} & \text{and} & \overline{\overline{\|}}\mathsf{LS} & \text{and} & \overline{\overline{\|}}\mathsf{LS} \\
P & & [R, P_R] & & [T, P_T]
\end{array}
\quad . \qquad \square
$$

The splitting lemma can also be proved directly for system LS, without introducing LS'. In this presentation I used the detour over system LS' for the following reasons:

(1) For system LS the induction measure is more complicated because it has to be defined such that $\text{size}([?R, R]) \leqslant \text{size}(?R)$.

(2) Because of the behaviour of the units, the size of a structure must be such that $\text{size}(a) = 0$ for $a \in \{\bot, 0, 1, \top\}$. This causes some problems in the applicability of the induction hypothesis.

(3) Because of the rule c↓, the case analysis for system LS is more tedious than for $\mathsf{LS}' \setminus \{\text{cb}\downarrow\}$ in the proof of Lemma 3.4.19.

(4) I was not able to find the right splitting statement for why-not structures for system LS, as it has been done for $\mathsf{LS}' \setminus \{\text{cb}\downarrow\}$ (case (e) in Lemma 3.4.19). But such a statement is needed for the elimination of the rule p↑ (see Section 3.4.3). Therefore, splitting for $\mathsf{LS}' \setminus \{\text{cb}\downarrow\}$ is employed.

3.4.2 Context Reduction

The idea of context reduction is to reduce a problem that concerns an arbitrary (deep) context $S\{\ \}$ to a problem that concerns only a shallow context $[\{\ \}, P]$. In the case of cut elimination, for example, we will then be able to apply splitting.

Due to the presence of the additives and the exponentials, the situation is not as simple as in [Gug02e].

3.4.20 **Definition** Let $\mathcal{S}$ be a finite (possibly empty) set of structures. The set $\mathcal{W}$ of *woo structures* over $\mathcal{S}$ is the smallest set such that

- $1 \in \mathcal{W}$,

- if $R \in \mathcal{S}$, then $R \in \mathcal{W}$,

- if $R_1, R_2 \in \mathcal{W}$, then $\langle R_1, R_2 \rangle \in \mathcal{W}$,

- if $R \in \mathcal{W}$, then $!R \in \mathcal{W}$.

The name woo comes from <u>w</u>ith, <u>o</u>f-course and <u>o</u>ne. The following lemma shows a crucial property of woo structures.

3.4.21 **Lemma** *Let $\mathcal{S}$ be a set of structures and let $\mathcal{W}$ be the set of woo structures over $\mathcal{S}$. If for every $R \in \mathcal{S}$ there is a proof $\underset{R}{\overset{}{\parallel}}\mathsf{LS}$, then for every $T \in \mathcal{W}$ there is a proof $\underset{T}{\overset{}{\parallel}}\mathsf{LS}$.*

Proof: Immediate from the definition and the fact that $\langle 1, 1 \rangle = 1 = !1$. □

3.4.22 **Lemma (Context Reduction)** *Let R be a structure and $S\{ \}$ be a context where the hole is not inside a why-not structure. If $S\{R\}$ is provable in LS, then there is an $n \geqslant 0$ and structures $P_{R1}, \ldots, P_{Rn}$, such that such that $[R, P_{Ri}]$ is provable in LS for every $i \in \{1, \ldots, n\}$, and and such that for every structure X, we have*

$$\begin{array}{c} W \\ \parallel\mathsf{LS} \\ S\{X\} \end{array} \quad ,$$

where W is a woo structure over the set $\big\{[X, P_{R1}], \ldots, [X, P_{Rn}]\big\}$.

Proof: The proof will be carried out by induction on the context $S\{ \}$.

(1) $S\{ \} = \{ \}$. Then the lemma is trivially true for $n = 1$, $P_{R1} = \bot$ and $W = [X, \bot]$.

(2) $S\{ \} = [S'\{ \}, P]$ for some P, such that $S\{R\}$ is not a proper par structure.

   (i) $S'\{ \} = \{ \}$. Then the lemma is trivially true for $n = 1$ and $P_{R1} = P$ and $W = [X, \bot]$.

   (ii) $S'\{ \} = \langle S''\{ \}, T \rangle$ for some context $S''\{ \}$ and structure $T \neq \top$. Then we can apply splitting (Lemma 3.4.5) to the proof of $[\langle S''\{R\}, T \rangle, P]$ and get:

$$\begin{array}{ccc} \Pi_S \parallel\mathsf{LS} & \text{and} & \Pi_T \parallel\mathsf{LS} \\ [S''\{R\}, P] & & [T, P] \end{array} \quad .$$

By applying the induction hypothesis to Π_S we get $n \geqslant 0$ and $P_{R1}, \ldots, P_{Rn}$ such that $\underset{[R, P_{Ri}]}{\overset{}{\parallel}}\mathsf{LS}$ for every $i \in \{1, \ldots, n\}$ and for every X

$$\begin{array}{c} W' \\ \parallel\mathsf{LS} \\ [S''\{X\}, P] \end{array} \quad .$$

where W' is a woo structure over the set $\big\{[X, P_{R1}], \ldots, [X, P_{Rn}]\big\}$. Now let $W = \langle W', 1\rangle$ and build:

$$
\begin{array}{c}
\langle W', 1\rangle \\
\Big\|\mathsf{LS} \\
\langle [S''\{X\}, P], 1\rangle \\
\Pi_T\Big\|\mathsf{LS} \\
\mathsf{d}{\downarrow}\dfrac{\langle [S''\{X\}, P], [T, P]\rangle}{[\langle S''\{X\}, T\rangle, \{P, P\}]} \\
\mathsf{c}{\downarrow}\dfrac{}{[\langle S''\{X\}, T\rangle, P]}
\end{array}
\qquad .
$$

(iii) $S'\{\ \} = \{S''\{\ \}; T\}$ for some context $S''\{\ \}$ and structure $T \neq 0$. Then we can apply splitting (Lemma 3.4.5) to the proof of $[\{S''\{R\}; T\}, P]$ and get:

$$
\begin{array}{ccccc}
\langle P_S, P_T\rangle & & & & \\
\Delta_P\Big\|\mathsf{LS} & \text{and} & \Pi_S\Big\|\mathsf{LS} & \text{and} & \Pi_T\Big\|\mathsf{LS} \\
P & & [S''\{R\}, P_S] & & [T, P_T]
\end{array}
\qquad .
$$

By applying the induction hypothesis to Π_S we get $n \geqslant 0$ and $P_{R1}, \ldots, P_{Rn}$ such that $\begin{array}{c}\Big\|\mathsf{LS} \\ [R, P_{Ri}]\end{array}$ for every $i \in \{1, \ldots, n\}$ and for every X

$$
\begin{array}{c}
W' \\
\Big\|\mathsf{LS} \\
[S''\{X\}, P]
\end{array}
\qquad .
$$

where W' is a woo structure over the set $\big\{[X, P_{R1}], \ldots, [X, P_{Rn}]\big\}$. Now let $W = \langle W', 1\rangle$ and build:

$$
\begin{array}{c}
\langle W', 1\rangle \\
\Big\|\mathsf{LS} \\
\langle [S''\{X\}, P], 1\rangle \\
\Pi_T\Big\|\mathsf{LS} \\
\mathsf{d}{\downarrow}\dfrac{\langle [S''\{X\}, P_S], [T, P_T]\rangle}{[\{S''\{X\}, T\}, \langle P_S, P_T\rangle]} \\
\Delta_P\Big\|\mathsf{LS} \\
[\{S''\{X\}, T\}, P]
\end{array}
\qquad .
$$

(iv) $S'\{\ \} = (S''\{\ \}, T)$ for some context $S''\{\ \}$ and structure $T \neq 1$. Then we can apply splitting (Lemma 3.4.5) to the proof of $[(S''\{R\}, T), P]$ and get: an $h \geqslant 0$ and structures $P_{S1}, \ldots, P_{Sh}$ and $P_{T1}, \ldots, P_{Th}$, such that

$$
\begin{array}{cc}
\langle [0, \top], \ldots, [0, \top], [P_{S1}, P_{T1}], \ldots, [P_{Sh}, P_{Th}]\rangle & \\
\Big\|\mathsf{LS} & \text{and} \\
P &
\end{array}
$$

$$
\begin{array}{ccc}
\Pi_{Sj}\Big\|\mathsf{LS} & \text{and} & \Pi_{Tj}\Big\|\mathsf{LS} \\
[S''\{R\}, P_{Sj}] & & [T, P_{Tj}]
\end{array}
\qquad .
$$

for every $j \in \{1, \ldots, h\}$. By applying the induction hypothesis to every Π_{Sj}, we get $n_j \geqslant 0$ and $P_{Rj1}, \ldots, P_{Rjn_j}$ such that $\overset{\parallel\!\parallel\text{LS}}{[R, P_{Rji}]}$ for every $i \in \{1, \ldots, n_j\}$ and for every X

$$\begin{array}{c} W_j \\ \parallel\text{LS} \\ [S''\{X\}, P_{Sj}] \end{array} \quad .$$

where W_j is a woo structure over the set $\{[X, P_{Rj1}], \ldots, [X, P_{Rjn_j}]\}$. Now let $W = \langle 1, W_1, \ldots, W_h \rangle$ and $n = n_1 + \cdots + n_h$ and build:

$$\begin{array}{c} \langle 1, W_1, \ldots, W_h \rangle \\ \parallel\text{LS} \\ \langle 1, [S''\{X\}, P_{S1}], \ldots, [S''\{X\}, P_{Sh}] \rangle \\ {}_{\Pi_{T1}, \ldots, \Pi_{Th}}\parallel\text{LS} \\ \langle 1, [(S''\{X\}, [T, P_{T1}]), P_{S1}], \ldots, [(S''\{X\}, [T, P_{Th}]), P_{Sh}] \rangle \\ \parallel\text{\{s\}} \\ \langle 1, [(S''\{X\}, T), P_{T1}, P_{S1}], \ldots, [(S''\{X\}, T), P_{Th}, P_{Sh}] \rangle \\ \parallel\text{\{ai↓,t↓\}} \\ \langle\, [(S''\{X\}, T), 0, \top], \ldots, [(S''\{X\}, T), 0, \top], \\ [(S''\{X\}, T), P_{T1}, P_{S1}], \ldots, [(S''\{X\}, T), P_{Th}, P_{Sh}]\, \rangle \\ \parallel\text{\{d↓,c↓\}} \\ [(S''\{X\}, T), \langle [0, \top], \ldots, [0, \top], [P_{S1}, P_{T1}], \ldots, [P_{Sh}, P_{Th}] \rangle] \quad . \\ {}_{\Delta_P}\parallel\text{LS} \\ [(S''\{X\}, T), P] \end{array}$$

(v) $S'\{\ \} = !S''\{\ \}$ for some context $S''\{\ \}$. Then we can apply the splitting lemma (Lemma 3.4.5) to the proof of $[!S''\{R\}, P]$ and get: $h \geqslant 0$ and $k_1, \ldots, k_h \geqslant 0$ and structures $P_{S11}, \ldots, P_{S1k_1}, P_{S21}, \ldots, P_{S2k_2}, \ldots, P_{Sh1}, \ldots, P_{Shk_h}$, such that

$$\begin{array}{c} \langle [0, \top], \ldots, [0, \top], [?P_{S11}, \ldots, ?P_{S1k_1}], \ldots, [?P_{Sh1}, \ldots, ?P_{Shk_h}] \rangle \\ \parallel\text{LS} \\ P \end{array} \qquad \text{and}$$

$$\begin{array}{c} {}_{\Pi_{Sj}}\parallel\text{LS} \\ [S''\{R\}, P_{Sj1}, \ldots, P_{Sjk_j}] \end{array} \quad .$$

for every $j \in \{1, \ldots, h\}$. By applying the induction hypothesis to every Π_{Sj} we get we get $n_j \geqslant 0$ and $P_{Rj1}, \ldots, P_{Rjn_j}$ such that $\overset{\parallel\!\parallel\text{LS}}{[R, P_{Rji}]}$ for every $i \in \{1, \ldots, n_j\}$ and for every X

$$\begin{array}{c} W_j \\ \parallel\text{LS} \\ [S''\{X\}, P_{Sj1}, \ldots, P_{Sjk_j}] \end{array} \quad .$$

where W_j is a woo structure over the set $\{[X, P_{Rj1}], \ldots, [X, P_{Rjn_j}]\}$. Now let $W = \langle 1, !W_1, \ldots, !W_h \rangle$ and $n = n_1 + \cdots + n_h$ and build:

$$
\begin{array}{c}
\langle 1, !W_1, \ldots, !W_h \rangle \\
\Big\| \mathsf{LS} \\
\langle 1, ![S''\{X\}, P_{S11}, \ldots, P_{S1k_1}], \ldots, ![S''\{X\}, P_{Sh1}, \ldots, P_{Shk_h}] \rangle \\
\Big\| \{\mathsf{p}\downarrow\} \\
\langle 1, [!S''\{X\}, ?P_{S11}, \ldots, ?P_{S1k_1}], \ldots, [!S''\{X\}, ?P_{Sh1}, \ldots, ?P_{Shk_h}] \rangle \\
\Big\| \{\mathsf{ai}\downarrow, \mathsf{t}\downarrow\} \\
\langle [!S''\{X\}, 0, \top], \ldots, [!S''\{X\}, 0, \top], \\
\qquad [!S''\{X\}, ?P_{S11}, \ldots, ?P_{S1k_1}], \ldots, [!S''\{X\}, ?P_{Sh1}, \ldots, ?P_{Shk_h}] \rangle \\
\Big\| \{\mathsf{d}\downarrow, \mathsf{c}\downarrow\} \\
[!S''\{X\}, \langle [0, \top], \ldots, [0, \top], [?P_{S11}, \ldots, ?P_{S1k_1}], \ldots, [?P_{Sh1}, \ldots, ?P_{Shk_h}] \rangle] \quad . \\
\Delta_P \Big\| \mathsf{LS} \\
[!S''\{X\}, P]
\end{array}
$$

(3) $S\{\ \} = \langle S'\{\ \}, T \rangle$ for some context $S'\{\ \}$ and $T \neq \top$. This is a special case of (2.ii) for $P = \bot$.

(4) $S\{\ \} = \{S'\{\ \}, T\}$ for some context $S'\{\ \}$ and $T \neq 0$. This is a special case of (2.iii) for $P = \bot$.

(5) $S\{\ \} = (S'\{\ \}, T)$ for some context $S'\{\ \}$ and $T \neq 1$. This is a special case of (2.iv) for $P = \bot$.

(6) $S\{\ \} = !S'\{\ \}$ for some context $S'\{\ \}$. This is a special case of (2.v) for $P = \bot$. $\square$

3.4.3 Elimination of the Up Fragment

In this section, I will first show a series of lemmata, which are all easy consequences of splitting and which say that the up rules of system SLS are admissible if they are applied in a shallow context $[\{\ \}, P]$. Then I will show how context reduction is used to extend these lemmata to any context. As a result we get a modular proof of cut elimination.

3.4.23 Lemma *Let P be a structure and let a be an atom. If $[(a, \bar{a}), P]$ is provable in* LS, *then P is also provable in* LS.

Proof: Apply splitting (Lemma 3.4.5) to the proof

$$
\begin{array}{c}
\Big\| \mathsf{LS} \\
[(a, \bar{a}), P]
\end{array} \quad .
$$

We get an $n \geqslant 0$ and structures $P_{a1}, \ldots, P_{an}$ and $P_{\bar{a}1}, \ldots, P_{\bar{a}n}$, such that

$$
\begin{array}{c}
\langle [0, \top], \ldots, [0, \top], [P_{a1}, P_{\bar{a}1}], \ldots, [P_{an}, P_{\bar{a}n}] \rangle \\
\Delta_P \Big\| \mathsf{LS} \\
P
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Pi_1 \Big\| \mathsf{LS} \\
[a, P_{ai}]
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Pi_2 \Big\| \mathsf{LS} \\
[\bar{a}, P_{\bar{a}i}]
\end{array}
$$

for every $i \in \{1, \ldots, n\}$. By applying splitting to Π_1 and Π_2, we obtain

$$\langle [0, \top], \ldots, [0, \top], \bar{a}, \ldots, \bar{a} \rangle \qquad \text{and} \qquad \langle [0, \top], \ldots, [0, \top], a, \ldots, a \rangle$$
$$\| \mathsf{LS} \qquad\qquad\qquad\qquad\qquad\qquad \| \mathsf{LS}$$
$$P_{ai} \qquad\qquad\qquad\qquad\qquad\qquad\qquad P_{\bar{a}i}$$.

From this we can build for every $i \in \{1, \ldots, n\}$ (by applying Lemmata 3.4.17 and 3.4.18):

$$\| \mathsf{LS}$$
$$[\langle [0, \top], \ldots, [0, \top], \bar{a}, \ldots, \bar{a} \rangle, \langle [0, \top], \ldots, [0, \top], a, \ldots, a \rangle]$$
$$\| \mathsf{LS}$$
$$[P_{ai}, P_{\bar{a}i}]$$.

Plugging this into the derivation Δ_P yields a proof of P. □

3.4.24 Lemma *Let R, T, U, V and P be structures. If $[(\langle R, U \rangle, \langle T, V \rangle), P]$ is provable in LS, then $[\langle (R, T), (U, V) \rangle, P]$ is also provable in LS.*

Proof: Apply splitting (Lemma 3.4.5) to the proof

$$\| \mathsf{LS}$$
$$[(\langle R, U \rangle, \langle T, V \rangle), P]$$.

We get an $n \geqslant 0$ and structures $P_{RU1}, \ldots, P_{RUn}$ and $P_{VT1}, \ldots, P_{VTn}$, such that

$$\langle [0, \top], \ldots, [0, \top], [P_{RU1}, P_{VT1}], \ldots, [P_{RUn}, P_{VTn}] \rangle$$
$$\Delta_P \| \mathsf{LS} \qquad\qquad\qquad\qquad\qquad \text{and}$$
$$P$$

$$\Pi_{RUi} \| \mathsf{LS} \qquad\qquad \text{and} \qquad\qquad \Pi_{TVi} \| \mathsf{LS}$$
$$[\langle R, U \rangle, P_{RUi}] \qquad\qquad\qquad\qquad [\langle T, V \rangle, P_{TVi}]$$

for every $i \in \{1, \ldots, n\}$. By applying splitting again to every Π_{RUi} and Π_{TVi}, we get

$$\langle P_{Ri}, P_{Ui} \rangle$$
$$\Delta_{RUi} \| \mathsf{LS} \qquad \text{and} \qquad \Pi_{Ri} \| \mathsf{LS} \qquad \text{and} \qquad \Pi_{Ui} \| \mathsf{LS} \qquad \text{and}$$
$$P_{RUi} \qquad\qquad\qquad [R, P_{Ri}] \qquad\qquad\qquad [U, P_{Ui}]$$

$$\Pi_{Vi} \| \mathsf{LS} \qquad\qquad \text{and} \qquad\qquad \Pi_{Ti} \| \mathsf{LS}$$
$$[V, P_{TVi}] \qquad\qquad\qquad\qquad [T, P_{TVi}]$$.

From this we can build for every $i \in \{1, \ldots, n\}$:

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\Pi_{Vi}, \Pi_{Ti} \Big\| \mathsf{LS} \atop \langle [T, P_{TVi}], [V, P_{TVi}] \rangle
}{
\Pi_{Ri}, \Pi_{Ui} \Big\| \mathsf{LS} \atop \langle [([R, P_{Ri}], T), P_{TVi}], [([U, P_{Ui}], V), P_{TVi}] \rangle
}
}{
\langle [([R, P_{Ri}], T), P_{TVi}], [(U, V), P_{Ui}, P_{TVi}] \rangle
} \mathsf{s}
}{
\langle [(R, T), P_{Ri}, P_{TVi}], [(U, V), P_{Ui}, P_{TVi}] \rangle
} \mathsf{s}
}{
[\{(R, T), (U, V)\}, \langle [P_{Ri}, P_{TVi}], [P_{Ui}, P_{TVi}] \rangle]
} \mathsf{d}\!\downarrow
}{
[\{(R, T), (U, V)\}, \langle P_{Ri}, P_{Ui} \rangle, \{P_{TVi}, P_{TVi}\}]
} \mathsf{d}\!\downarrow
}{
[\{(R, T), (U, V)\}, \langle P_{Ri}, P_{Ui} \rangle, P_{TVi}]
} \mathsf{c}\!\downarrow
}{
[\{(R, T), (U, V)\}, P_{RUi}, P_{TVi}]
} \Delta_{RUi} \Big\| \mathsf{LS}
\quad .
$$

Now we can build (by applying Lemma 3.4.15 and Lemma 3.4.18):

$$
\cfrac{
\cfrac{
\cfrac{
\Big\| \mathsf{LS} \atop \langle [\{(R, T), (U, V)\}, 0, \top], \ldots, [\{(R, T), (U, V)\}, 0, \top], \atop [\{(R, T), (U, V)\}, P_{RU1}, P_{VT1}], \ldots, [\{(R, T), (U, V)\}, P_{RUn}, P_{VTn}] \rangle
}{
[\{(R, T), (U, V)\}, \langle [0, \top], \ldots, [0, \top], [P_{RU1}, P_{VT1}], \ldots, [P_{RUn}, P_{VTn}] \rangle]
} \Big\| \{\mathsf{c}\!\downarrow, \mathsf{d}\!\downarrow\}
}{
[\{(R, T), (U, V)\}, P]
} \Delta_P \Big\| \mathsf{LS}
\qquad \square
$$

3.4.25 Lemma *Let R and P be structures. If $[R, P]$ is provable in LS, then $[\top, P]$ and $[\langle R, R \rangle, P]$ are also provable in LS.*

Proof: Use the proofs

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{}{1} 1\!\downarrow
}{
[\top, 0]
} \mathsf{ai}\!\downarrow
}{
[\top, P]
} \mathsf{t}\!\downarrow
\qquad \text{and} \qquad
\cfrac{
\cfrac{
\cfrac{
\Big\| \mathsf{LS} \atop \langle [R, P], [R, P] \rangle
}{
[\langle R, R \rangle, \{P, P\}]
} \mathsf{d}\!\downarrow
}{
[\langle R, R \rangle, P]
} \mathsf{c}\!\downarrow
}{}
}{}
\qquad . \qquad \square
$$

3.4.26 Lemma *Let R and P be structures. If $[!R, P]$ is provable in LS, then $[1, P]$ and $[(!R, R), P]$ are also provable in LS.*

Proof: Apply splitting (Lemma 3.4.5) to the proof

$$
\Big\| \mathsf{LS} \atop [!R, P] \qquad .
$$

We get an $n \geqslant 0$ and $k_1, \ldots, k_n \geqslant 0$, and structures $P_{R11}, \ldots, P_{R1k_1}$, $P_{R21}, \ldots, P_{R2k_2}$, $\ldots$, $P_{Rn1}, \ldots, P_{Rnk_n}$, such that

$$
\cfrac{
\langle [0, \top], \ldots, [0, \top], [?P_{R11}, \ldots, ?P_{R1k_1}], \ldots, [?P_{Rn1}, \ldots, ?P_{Rnk_n}] \rangle
}{
P
} \Delta_P \Big\| \mathsf{LS}
$$

$$\overset{\parallel\mathsf{LS}}{[R, P_{Ri1}, \ldots, P_{Rik_i}]}$$

for every $i \in \{1, \ldots, n\}$. We can build:

$$\overset{\parallel\mathsf{LS}}{(![R, P_{Ri1}, \ldots, P_{Rik_i}], [R, P_{Ri1}, \ldots, P_{Rik_i}])}$$
$$\Big\|_{\{\mathsf{p}\downarrow\}}$$
$$([!R, ?P_{Ri1}, \ldots, ?P_{Rik_i}], [R, P_{Ri1}, \ldots, P_{Rik_i}])$$
$$\Big\|_{\{\mathsf{s}\}}$$

$$1\downarrow \frac{}{1} \qquad\qquad [(!R, R), ?P_{Ri1}, P_{Ri1}, \ldots, ?P_{Rik_i}, P_{Rik_i}]$$
$$\Big\|_{\{\mathsf{w}\downarrow\}} \qquad \text{and} \qquad \Big\|_{\{\mathsf{b}\downarrow\}}$$
$$[1, ?P_{Ri1}, \ldots, ?P_{Rik_i}] \qquad\qquad [(!R, R), ?P_{Ri1}, \ldots, ?P_{Rik_i}]$$

From this we get:

$$\overset{\parallel\mathsf{LS}}{\langle[1, 0, \top], \ldots, [1, 0, \top], [1, ?P_{R11}, \ldots, ?P_{R1k_1}], \ldots, [1, ?P_{Rn1}, \ldots, ?P_{Rnk_n}]\rangle}$$
$$\Big\|_{\{\mathsf{c}\downarrow,\mathsf{d}\downarrow\}}$$
$$[1, \langle[0, \top], \ldots, [0, \top], [?P_{R11}, \ldots, ?P_{R1k_1}], \ldots, [?P_{Rn1}, \ldots, ?P_{Rnk_n}]\rangle]$$
$$\Delta_P \Big\|\mathsf{LS}$$
$$[1, P]$$

and

$$\overset{\parallel\mathsf{LS}}{\langle[(!R, R), 0, \top], \ldots, [(!R, R), 0, \top],}$$
$$[(!R, R), ?P_{R11}, \ldots, ?P_{R1k_1}], \ldots, [(!R, R), ?P_{Rn1}, \ldots, ?P_{Rnk_n}]\rangle$$
$$\Big\|_{\{\mathsf{c}\downarrow,\mathsf{d}\downarrow\}}$$
$$[(!R, R), \langle[0, \top], \ldots, [0, \top], [?P_{R11}, \ldots, ?P_{R1k_1}], \ldots, [?P_{Rn1}, \ldots, ?P_{Rnk_n}]\rangle]$$
$$\Delta_P \Big\|\mathsf{LS}$$
$$[(!R, R), P] \qquad\qquad\qquad \square$$

3.4.27 Lemma *Let R, T and P be structures. If $[(?R, !T), P]$ is provable in* LS*, then $[?(R, T), P]$ is also provable in* LS*.*

Although the statement and the purpose of this lemma is quite similar to the previous lemmata, its proof is a little different because in the case of system LS we do not have a splitting statement for why-not structures (case (e) is missing in Lemma 3.4.5). However, we have such a statement for system $\mathsf{LS}' \setminus \{\mathsf{cb}\downarrow\}$ (see case (e) in Lemma 3.4.19). This will be employed here.

3.4.28 Lemma *Let $h \geqslant 0$ and let $R, T, P,$ and $V_1, \ldots, V_h$ be structures. If $[?R, P]$ and $[T, V_1, \ldots, V_h]$ are provable in* $\mathsf{LS}' \setminus \{\mathsf{cb}\downarrow\}$*, then $[?(R, T), P, ?V_1, \ldots, ?V_h]$ is also provable in* $\mathsf{LS}' \setminus \{\mathsf{cb}\downarrow\}$*.*

Proof: By applying splitting for $\mathsf{LS}' \setminus \{\mathsf{cb}{\downarrow}\}$ (Lemma 3.4.19) to the proof of $[?R, P]$, we either have that P is provable in $\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\}$, or we get some $n \geqslant 0$ and $k \leqslant n$ and structures $P_{R1}, \ldots, P_{Rn}$ such that

$$
\begin{array}{c}
\langle[0, \top], \ldots, [0, \top], !P_{R1}, \ldots, !P_{Rk}, P_{Rk+1}, \ldots, P_{Rn}\rangle \\
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
P
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
[R, P_{Ri}]
\end{array}
$$

for every $i = 1, \ldots, n$. If we have a proof of P, then we can obtain

$$
\begin{array}{c}
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
P \\
\big\|\{\mathsf{w}{\downarrow}\} \\
[?(R, T), P, ?V_1, \ldots, ?V_h]
\end{array} \quad .
$$

Otherwise we can obtain

$$
\begin{array}{c}
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
[T, V_1, \ldots, V_h] \\
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
\mathsf{s}\dfrac{[([R, P_{Rj}], T), V_1, \ldots, V_h]}{[(R, T), P_{Rj}, V_1, \ldots, V_h]} \\
\big\|\{\mathsf{cr}{\downarrow}\} \\
[?(R, T), P_{Rj}, ?V_1, \ldots, ?V_h]
\end{array}
\quad \text{and} \quad
\begin{array}{c}
1{\downarrow}\,\dfrac{}{!1} \\
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
![T, V_1, \ldots, V_h] \\
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
\mathsf{s}\dfrac{![([R, P_{Ri}], T), V_1, \ldots, V_h]}{![(R, T), P_{Ri}, V_1, \ldots, V_h]} \\
\big\|\{\mathsf{p}{\downarrow}\} \\
[?(R, T), !P_{Ri}, ?V_1, \ldots, ?V_h]
\end{array} \quad ,
$$

for every $i = 1, \ldots, k$ and $j = k+1, \ldots, n$. From this we can build the following proof, where $?V$ stands for $[?V_1, \ldots, ?V_h]$:

$$
\begin{array}{c}
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
\langle[?(R, T), ?V, 0, \top], \ldots, [?(R, T), ?V, 0, \top], \\
[?(R, T), !P_{R1}, ?V], \ldots, [?(R, T), !P_{Rk}, ?V], \\
[?(R, T), P_{Rk+1}, ?V], \ldots, [?(R, T), P_{Rn}, ?V]\rangle \\
\big\|\{\mathsf{cd}{\downarrow}\} \\
[?(R, T), \langle[0, \top], \ldots, [0, \top], !P_{R1}, \ldots, !P_{Rk}, P_{Rk+1}, \ldots, P_{Rn}\rangle, ?V] \\
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
[?(R, T), P, ?V]
\end{array} \quad . \qquad \qquad \square
$$

3.4.29 Lemma *Let $h \geqslant 0$ and let $R, T, P,$ and $V_1, \ldots, V_h$ be structures. If $[?R, P]$ and $[T, V_1, \ldots, V_h]$ are provable in LS, then $[?(R, T), P, ?V_1, \ldots, ?V_h]$ is also provable in LS.*

Proof: By Lemma 3.4.8 (b) and Lemma 3.4.9, there is a proof

$$
\begin{array}{c}
\big\|\mathsf{LS}'\setminus\{\mathsf{cb}{\downarrow}\} \\
U \\
\big\|\{\mathsf{cb}{\downarrow}\} \\
[?R, P]
\end{array} \quad ,
$$

for some structure U. Since the redex of cb↓ is always a why-not structure, we have $k \geqslant 1$ and structures $R_1, \ldots, R_k$ and P', such that $U = [?R_1, \ldots, ?R_k, P']$ and there are derivations

$$
\begin{array}{ccccccccc}
R_1 & & R_2 & & & & R_k & & P' \\
\big\|\{\text{cb}\downarrow\} & , & \big\|\{\text{cb}\downarrow\} & , & \cdots & , & \big\|\{\text{cb}\downarrow\} & , \text{ and} & \big\|\{\text{cb}\downarrow\} \\
R & & R & & & & R & & P
\end{array} \quad .
$$

Now apply the previous lemma k-times to the proof $\begin{array}{c} \big\|\text{LS}'\backslash\{\text{cb}\downarrow\} \\ {[?R_1, \ldots, ?R_k, P']} \end{array}$. This yields:

$$
\begin{array}{c}
\Pi \big\| \text{LS}'\backslash\{\text{cb}\downarrow\} \\
{[?(R_1, T), \ldots, ?(R_k, T), P', ?V, \ldots, ?V]}
\end{array} \quad ,
$$

where $?V$ stands for $[?V_1, \ldots, ?V_h]$, which occurs k times in the conclusion of Π. Further, we can construct a derivation

$$
\begin{array}{c}
{[?(R_1, T), \ldots, ?(R_k, T), P', ?V, \ldots, ?V]} \\
\big\|\{\text{cb}\downarrow\} \\
{[?(R, T), P, ?V]}
\end{array} \quad .
$$

By Lemma 3.4.8 (a), we get the desired proof

$$
\begin{array}{c}
\big\|\text{LS} \\
{[?(R, T), P, ?V_1, \ldots, ?V_h]}
\end{array} \quad . \qquad \square
$$

Proof of Lemma 3.4.27: By splitting (Lemma 3.4.5), we get an $n \geqslant 0$ and structures $P_{R1}, \ldots, P_{Rn}$ and $P_{T1}, \ldots, P_{Tn}$, such that

$$
\begin{array}{c}
\langle[0, \top], \ldots, [0, \top], [P_{R1}, P_{T1}], \ldots, [P_{Rn}, P_{Tn}]\rangle \\
\Delta_P \big\| \text{LS} \\
P
\end{array} \quad \text{and}
$$

$$
\begin{array}{ccc}
\Pi_{Ri} \big\| \text{LS} & \text{and} & \Pi_{Ti} \big\| \text{LS} \\
{[?R, P_{Ri}]} & & {[!T, P_{Ti}]}
\end{array}
$$

for every $i \in \{1, \ldots, n\}$. By applying splitting again to every Π_{Ti}, we get an $m_i \geqslant 0$ and $k_{i1}, \ldots, k_{im_i} \geqslant 0$ and structures $V_{i11}, \ldots, V_{i1k_{i1}}, V_{i21}, \ldots, V_{i2k_{i2}}, \ldots, V_{im_i1}, \ldots, V_{im_ik_{im_i}}$, such that

$$
\begin{array}{c}
\langle[0, \top], \ldots, [0, \top], [?V_{i11}, \ldots, ?V_{iR1k_{i1}}], \ldots, [?V_{im_i1}, \ldots, ?V_{im_ik_{im_i}}]\rangle \\
\Delta_{Ti} \big\| \text{LS} \\
P_{Ti}
\end{array} \quad \text{and}
$$

$$
\begin{array}{c}
\big\| \text{LS} \\
{[T, V_{ij1}, \ldots, V_{ijk_{ij}}]}
\end{array}
$$

for every $j \in \{1, \ldots, m_i\}$. By Lemma 3.4.29, we get for every $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$ a proof

$$
\begin{array}{c}
\big\| \text{LS} \\
{[?(R, T), P_{Ri}, ?V_{ij1}, \ldots, ?V_{ijk_{ij}}]}
\end{array} \quad .
$$

Now we can build for every $i \in \{1, \ldots, n\}$ the following proof, where $?V_{ij}$ stands for $[?V_{ij1}, \ldots, ?V_{ijk_{ij}}]$.

$$
\begin{array}{c}
\text{\rotatebox{90}{$\|$}LS} \\
\langle[?(R,T), P_{Ri}, 0, \top], \ldots, [?(R,T), P_{Ri}, 0, \top], [?(R,T), P_{Ri}, ?V_{i1}], \ldots, [?(R,T), P_{Ri}, ?V_{im_i}]\rangle \\
\Big\|\{\mathsf{c}{\downarrow},\mathsf{d}{\downarrow}\} \\
[?(R,T), P_{Ri}, \langle[0, \top], \ldots, [0, \top], ?V_{i1}, \ldots, ?V_{im_i}\rangle] \quad . \\
\Delta_{Ti}\Big\|\mathsf{LS} \\
[?(R,T), P_{Ri}, P_{Ti}]
\end{array}
$$

From this we can now build:

$$
\begin{array}{c}
\text{\rotatebox{90}{$\|$}LS} \\
\langle[?(R,T), 0, \top], \ldots, [?(R,T), 0, \top], [?(R,T), P_{R1}, P_{T1}], \ldots, [?(R,T), P_{Rn}, P_{Tn}]\rangle \\
\Big\|\{\mathsf{c}{\downarrow},\mathsf{d}{\downarrow}\} \\
[?(R,T), \langle[0, \top], \ldots, [0, \top], [P_{R1}, P_{T1}], \ldots, [P_{Rn}, P_{Tn}]\rangle] \quad . \\
\Delta_P\Big\|\mathsf{LS} \\
[?(R,T), P]
\end{array}
$$
                                                                                       □

3.4.30   Lemma   *Let* $\rho \in \{\mathsf{ai}{\uparrow}, \mathsf{d}{\uparrow}, \mathsf{t}{\uparrow}, \mathsf{c}{\uparrow}, \mathsf{p}{\uparrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\uparrow}\}$ *be one of the up rules in system* SLS *and let* $S\{\ \}$ *be a context in which the hole does not occur inside a why-not structure. Then*

$$
\text{for every proof} \quad \rho\frac{\overset{\text{\rotatebox{90}{$\|$}LS}}{S\{V\}}}{S\{Z\}} \quad , \quad \text{there is a proof} \quad \overset{\text{\rotatebox{90}{$\|$}LS}}{S\{Z\}} \quad .
$$

Proof: Apply context reduction to the proof

$$
\overset{\text{\rotatebox{90}{$\|$}LS}}{S\{V\}} \quad .
$$

This yields some $n \geqslant 0$ and structures $P_{V1}, \ldots, P_{Vn}$, such that for every $i \in \{1, \ldots, n\}$, the structure $[V, P_{Vi}]$ is provable in LS, and such that there is a derivation

$$
\begin{array}{c}
W \\
\Big\|\mathsf{LS} \quad , \\
S\{Z\}
\end{array}
$$

where W is a woo structure over the set $\{[Z, P_{V1}], \ldots, [Z, P_{Vn}]\}$. By Lemmata 3.4.23 to 3.4.26, we have that $[Z, P_{Vi}]$ is provable in LS for every $i \in \{1, \ldots, n\}$. By Lemma 3.4.21, the structure W is also provable in LS. Hence, we have

$$
\begin{array}{c}
\text{\rotatebox{90}{$\|$}LS} \\
W \\
\Big\|\mathsf{LS} \quad . \\
S\{Z\}
\end{array}
$$
                                                                                       □

It remains to show how to handle the situation when an up rule that should be eliminated is applied inside a why-not structure. This is the purpose of the following definition and lemma.

3.4.31 Definition An application of a rule $\rho \dfrac{S\{T\}}{S\{R\}}$ is called *semishallow*, if the hole of the context $S\{\ \}$ does not occur inside a why-not structure (i.e. there are no contexts $S'\{\ \}$ and $S''\{\ \}$, such that $S\{\ \} = S'\{?S''\{\ \}\}$. A derivation Δ is called *semishallow*, every rule application occurring in Δ is semishallow. A proof Π is called *semishallow*, every rule application occurring in Π, except the axiom, is semishallow.

3.4.32 Observation An inspection of the proofs for splitting and context reduction shows that they preserve the property of being semishallow. More precisely, splitting (Lemma 3.4.5) and context reduction (Lemma 3.4.22) do also hold for semishallow LS, i.e. system LS where each rule is replaced by its semishallow version.

3.4.33 Lemma *Let* $\rho \in \{\mathsf{ai}{\uparrow}, \mathsf{d}{\uparrow}, \mathsf{t}{\uparrow}, \mathsf{c}{\uparrow}, \mathsf{p}{\uparrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\uparrow}\}$ *be one of the up rules in system* SLS *and let* $S\{\ \}$ *be an arbitrary context. Then*

$$
\textit{for every proof} \quad \rho\,\dfrac{\overset{\big\|\mathsf{LS}}{S\{V\}}}{S\{Z\}} \quad , \quad \textit{there is a semishallow proof} \quad \dfrac{\Pi\big\|\mathsf{LS}\cup\{\rho\}}{S\{Z\}} \quad .
$$

Proof: The proof Π can be obtained by simple permutation of rules, i.e. all nonsemishallow rules are permuted up until they disappear or become semishallow. To see this, consider the topmost instance of a rule σ that is not semishallow:

$$
\sigma\,\dfrac{\pi\,\dfrac{Q}{S\{W\}}}{S\{P\}} \quad ,
$$

where $\pi \in \mathsf{LS} \cup \{\rho\}$ is semishallow and nontrivial. According to 4.2.3, the cases to consider are:

(4) The redex of π is inside an active structure of the contractum of σ. This is impossible because π is semishallow and σ is not.

(5) The contractum of σ is inside an active structure of the redex of π. Since π is semishallow and σ is not, only the following four cases are possible:

(i) $\pi = \mathsf{c}{\downarrow}$. Then

$$
\sigma\,\dfrac{\mathsf{c}{\downarrow}\,\dfrac{S\{R\{W\}, R\{W\}\}}{S\{R\{W\}\}}}{S\{R\{P\}\}} \quad \text{yields} \quad \mathsf{c}{\downarrow}\,\dfrac{\sigma\,\dfrac{\sigma\,\dfrac{S\{R\{W\}, R\{W\}\}}{S\{R\{W\}, R\{P\}\}}}{S\{R\{P\}, R\{P\}\}}}{S\{R\{P\}\}} \quad .
$$

(ii) $\pi = \mathsf{b}{\downarrow}$. Similar to (i).

(iii) $\pi = \mathsf{t}{\downarrow}$. Then

$$
\sigma\,\dfrac{\mathsf{t}{\downarrow}\,\dfrac{S\{0\}}{S\{R\{W\}\}}}{S\{R\{P\}\}} \quad \text{yields} \quad \mathsf{t}{\downarrow}\,\dfrac{S\{0\}}{S\{R\{P\}\}} \quad .
$$

(iv) $\pi = \mathsf{w}{\downarrow}$. Similar to (iii).

(6) The redex of π and the contractum of σ (properly) overlap. This is impossible because π is semishallow and σ is not.

The following lemma says that Lemma 3.4.30 does also hold for arbitrary contexts $S\{\ \}$.

3.4.34   Lemma   *Let $\rho \in \{\mathsf{ai}{\uparrow}, \mathsf{d}{\uparrow}, \mathsf{t}{\uparrow}, \mathsf{c}{\uparrow}, \mathsf{p}{\uparrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\uparrow}\}$ be one of the up rules in system* SLS *and let $S\{\ \}$ be a context. Then*

$$
\text{for every proof} \quad \rho\,\dfrac{\overset{\displaystyle \parallel\mathsf{LS}}{S\{V\}}}{S\{Z\}} \quad , \quad \text{there is a proof} \quad \overset{\displaystyle \parallel\mathsf{LS}}{S\{Z\}} \quad .
$$

Proof: Let

$$
\rho\,\dfrac{\overset{\displaystyle \parallel\mathsf{LS}}{S\{V\}}}{S\{Z\}}
$$

be given. Transform it into a semishallow proof

$$
\overset{\displaystyle \Pi\parallel\mathsf{LS}\cup\{\rho\}}{S\{Z\}} \quad ,
$$

which exists by Lemma 3.4.33. From Π eliminate all instances of ρ, starting with the topmost one, by repeatedly applying Lemma 3.4.30.   □

The cut elimination result is now an easy consequence.

Proof of Theorem 3.4.1:   Let

$$
\overset{\displaystyle \Pi\parallel\mathsf{SLS}\cup\{1{\downarrow}\}}{R}
$$

be given. Remove each instance of an up rule inside Π (starting with the topmost) by applying Lemma 3.4.34. The result is a proof $\overset{\displaystyle \parallel\mathsf{LS}}{R}$.   □

3.5   Discussion

The main purpose of this chapter was to introduce the calculus of structures and to show that linear logic can benefit from its presentation in the calculus of structures. In particular, there are the following surprising phenomena to observe:

- The global promotion rule in the sequent calculus

$$
!\,\dfrac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash\, !A, ?B_1, \ldots, ?B_n} \quad (\text{for } n \geqslant 0) \quad ,
$$

has been replaced by a local version

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \quad,$$

in the sense that is is no longer necessary to check a context of unbounded size. This shows that it is not linear logic that causes the !-rule to be global, but the restrictions of the sequent calculus. (See also the introductory part of Chapter 5 for a discussion on locality.)

- The rule for the additive conjunction in the sequent calculus

$$\&\ \frac{\vdash A,\Phi \quad \vdash B,\Phi}{\vdash A \,\&\, B,\Phi}$$

has been replaced by two rules,

  - the rule

$$\mathsf{d}\!\downarrow \frac{S\langle[R,U],[T,V]\rangle}{S[\langle R,T\rangle,\langle U,V\rangle]} \quad,$$

    which is purely multiplicative, in the sense that no context or substructure is shared or duplicated,

  - and a contraction rule

$$\mathsf{c}\!\downarrow \frac{S\langle R,R\rangle}{S\{R\}} \quad,$$

    which is responsible for the duplication.

This means that the distinction between additive and multiplicative context treatment is intrinsically connected to the sequent calculus and not to the connectives of linear logic.

- The rule for the multiplicative conjunction in the sequent calculus

$$\otimes\ \frac{\vdash A,\Phi \quad \vdash B,\Psi}{\vdash A \otimes B,\Phi,\Psi}$$

has from the point of view of proof search the problem that one has to decide how to split the context of the formula $A \otimes B$ at the moment the rule is applied. For n formulas in Φ,Ψ, there are 2^n possibilities. Of course, there are methods, like lazy evaluation, that can circumvent this problem inside an implementation [HM94], but the switch rule

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

is able to handle this problem inside the logical system, as already observed in [Gug99].

- The duality between the two rules

$$\mathsf{id}\ \frac{}{\vdash A,A^\perp} \qquad \text{and} \qquad \mathsf{cut}\ \frac{\vdash A,\Phi \quad \vdash A^\perp,\Psi}{\vdash \Phi,\Psi} \quad,$$

which is hidden in the sequent calculus, is made obvious in the calculus of structures
by the two rules

$$\mathsf{i}{\downarrow}\ \frac{S\{1\}}{S[R,\bar{R}]} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\ \frac{S(R,\bar{R})}{S\{\bot\}} \quad .$$

In particular, both are reducible to their atomic version (as already observed in
[Gug99]), and not only one, as in the sequent calculus.

- If we define another pair of modalities $!'$ and $?'$ by the same rules as for $!$ and $?$ (i.e. we
  do the same experiment as in Remark 2.2.9), then (as in the sequent calculus) there
  is no relation between $!$ and $!'$, or between $?$ and $?'$. The interesting observation is
  now, that if we define a new pair of structural contexts $\{'\ldots\}'$ and $('\ldots)'$ via the same
  rules as for $\{\ldots\}$ and $(\ldots)$, then there is also no relation between $\{\ldots\}$ and $\{'\ldots\}'$, or
  between $(\ldots)$ and $('\ldots)'$. This means that the calculus of structures provides a certain
  uniformity, which is not present in the sequent calculus.

Furthermore, this chapter shows that it is possible to do proof theory with the calculus
of structures, in the sense that we have a notion of cut and can provide a cut elimination
procedure. Since the general cut rule $\mathsf{i}{\uparrow}$, which corresponds to the sequent calculus cut,
can be decomposed into several up rules, which can be eliminated independently from each
other, we have a wide variety of equivalent systems with different degrees of expressiveness
in proof construction. For example, in the case presented in this chapter, we have seven
up rules, which means that there are $2^7 = 128$ different systems that are all equivalent to
system LS. This means that the user has a large amount of possibilities to adapt the system
for his application, without need to worry about soundness or completeness of the system.
Such a modularity is not available in the sequent calculus.

4

The Multiplicative Exponential Fragment of Linear Logic

As the name suggests, the multiplicative exponential fragment of linear logic (MELL) is the restriction of linear logic to the multiplicatives and the exponentials. This fragment deserves particular attention because, on one hand, it incorporates the resource sensitiveness of linear logic with a controlled way of contraction, which makes it suitable to a wide range of applications, and on the other hand, it is the only fragment of linear logic whose decidability is still not known.

In the first section of this chapter, I will show system MELL in the sequent calculus and then discuss the two systems SELS and ELS in the calculus of structures that have already been defined in Section 3.2. Then, in Section 4.2, I will show some permutation results that will be needed in the other sections and also in later chapters.

In Section 4.3, I will show two decomposition theorems for system SELS. These two theorems can be seen as the central results of this chapter (and, in some sense, also of this thesis). Section 4.3 is rather long because of the technical effort that has to be made to prove the decomposition theorems. The impatient reader who is not interested in the proofs might read only the introductory part of this section, which also gives a sketch of the proofs.

Section 4.4, contains another proof of the cut elimination theorem for system ELS. This proof will be based on decomposition and the permutation of rules. Using the permutation of rules for proving cut elimination is conceptually similar to the sequent calculus. However, due to the new top-down symmetry of the calculus of structures, it is possible to obtain an interpolation theorem, which can be seen as a top-down symmetric version of cut elimination. This interpolation theorem is discussed in Section 4.5.

4.1 Sequents, Structures and Rules

In this section, I will define the systems for MELL in the sequent calculus and the calculus of structures. They will, in fact, be restrictions of the systems already defined in Chapters 2 and 3.

$$
\mathsf{id}\ \frac{}{\vdash A, A^{\perp}}
\qquad\qquad
\mathsf{cut}\ \frac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}
$$

$$
\otimes\ \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi}
\qquad
\bindnasrepma\ \frac{\vdash A, B, \Phi}{\vdash A \bindnasrepma B, \Phi}
\qquad
\perp\ \frac{\vdash \Phi}{\vdash \perp, \Phi}
\qquad
1\ \frac{}{\vdash 1}
$$

$$
\mathsf{?d}\ \frac{\vdash A, \Phi}{\vdash\ ?A, \Phi}
\qquad
\mathsf{?c}\ \frac{\vdash\ ?A, ?A, \Phi}{\vdash\ ?A, \Phi}
\qquad
\mathsf{?w}\ \frac{\vdash \Phi}{\vdash\ ?A, \Phi}
\qquad
!\ \frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash\ !A, ?B_1, \ldots, ?B_n}\quad (n \geqslant 0)
$$

Figure 4.1: System MELL in the sequent calculus

4.1.1   Definition   A MELL *formula* is an LL formula that does not contain an additive connective $\oplus$, $\&$, or constant 0, $\top$. The rules for *system* MELL are shown in Figure 4.1.

Of course, the cut elimination result does also hold for MELL. This can be seen as a consequence of the cut elimination result for full linear logic (Theorem 2.3.1) because the rules for the additives, which are missing in system MELL are not needed in a cut-free proof of a MELL formula in which the additive connectives do not occur. But it is also easily possible to obtain a direct proof from the proof of the cut elimination result for full linear logic by omitting all the cases that involve the additives.

4.1.2   Theorem (Cut Elimination)   *Every proof Π of a sequent $\vdash \Phi$ in system* MELL *can be transformed into a cut-free proof Π' in system* MELL *of the same sequent.*

In the calculus of structures the situation is similar.

4.1.3   Definition   An ELS *structure* is an LS structure that does not contain the constants 0, $\top$, or a substructure of the shape $[R_1, \ldots, R_h]$ or $\langle R_1, \ldots, R_h \rangle$.

4.1.4   Remark   In the case of ELS structures, the equations concerning the additives can be removed from the underlying equational theory. For convenience, the remaining equations are show in Figure 4.2.

System SELS $= \{\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{p}{\downarrow}, \mathsf{p}{\uparrow}, \mathsf{w}{\downarrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}\}$ (see Definition 3.2.17) is shown in Figure 4.3. The core (see Definition 3.2.18) of system SELS is SELSc $= \{\mathsf{s}, \mathsf{p}{\downarrow}, \mathsf{p}{\uparrow}\}$. System ELS $= \{1{\downarrow}, \mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{p}{\downarrow}, \mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}\}$, which is obtained from the down fragment of system SELS together with the axiom, is shown in Figure 4.4.

All results concerning systems SLS and LS presented in Chapter 3 remain true for systems SELS and ELS. The most important are listed below.

4.1.5   Theorem   *The systems* SELS $\cup \{1{\downarrow}\}$ *and* ELS $\cup \{\mathsf{i}{\uparrow}\}$ *are strongly equivalent.*

Proof: Similar to the proof of Theorem 3.2.21. Proposition 3.2.12 is independent from the additives and in the proof of Proposition 3.2.16, the rules $\mathsf{d}{\downarrow}$ and $\mathsf{d}{\uparrow}$ are needed only for decomposing structures of the shape $[R_1, \ldots, R_h]$ or $\langle R_1, \ldots, R_h \rangle$.   □

$$
\begin{array}{ccc}
\textbf{Associativity} & \textbf{Exponentials} & \textbf{Negation} \\[4pt]
\begin{aligned}
[R,[T,U]] &= [[R,T],U] \\
(R,(T,U)) &= ((R,T),U)
\end{aligned}
&
\begin{aligned}
??R &= ?R \\
!!R &= !R
\end{aligned}
&
\begin{aligned}
\overline{[R,T]} &= (\bar{R},\bar{T}) \\
\overline{(R,T)} &= [\bar{R},\bar{T}] \\
\overline{?R} &= !\bar{R} \\
\overline{!R} &= ?\bar{R} \\
\bar{\bar{R}} &= R
\end{aligned}
\\[10pt]
\textbf{Commutativity} & \textbf{Units} & \\[4pt]
\begin{aligned}
[R,T] &= [T,R] \\
(R,T) &= (T,R)
\end{aligned}
&
\begin{aligned}
[\bot,R] &= R \\
(1,R) &= R \\
?\bot &= \bot \\
!1 &= 1
\end{aligned}
&
\end{array}
$$

Figure 4.2: Basic equations for the syntactic congruence of ELS structures

4.1.6   Theorem   *If a given structure R is provable in system* $\mathsf{SELS} \cup \{1{\downarrow}\}$, *then its translation* $\vdash \underline{R}_{\mathsf{L}}$ *is provable in* MELL *(with cut).*

Proof: The proof is literally the same as the proof of Theorem 3.3.1, except that the cases for the rules $\mathsf{d}{\downarrow}, \mathsf{d}{\uparrow}, \mathsf{t}{\downarrow}, \mathsf{t}{\uparrow}, \mathsf{c}{\downarrow}$ and $\mathsf{c}{\uparrow}$ are not needed, which means that the sequent calculus rules for $\&, \oplus$ and $\top$ are not needed. $\qquad\square$

4.1.7   Theorem   (a) *If a given sequent $\vdash \Phi$ is provable in* MELL *(with cut), then the structure $\vdash \underline{\Phi}_{\mathsf{S}}$ is provable in system* $\mathsf{SELS} \cup \{1{\downarrow}\}$. (b) *If a given sequent $\vdash \Phi$ is cut-free provable in* MELL, *then the structure $\vdash \underline{\Phi}_{\mathsf{S}}$ is provable in system* ELS.

Proof: The proof is literally the same as the proof of Theorem 3.3.2, except that the cases for the rules $\&, \oplus_1, \oplus_2$ and $\top$ are omitted, which means that the rules $\mathsf{d}{\downarrow}, \mathsf{t}{\downarrow}$, and $\mathsf{c}{\downarrow}$ are not needed. $\qquad\square$

As a consequence we have already two different proofs for the cut elimination theorem for system ELS at hand.

4.1.8   Theorem (Cut Elimination)   *System* ELS *is equivalent to every subsystem of the system* $\mathsf{SELS} \cup \{1{\downarrow}\}$ *containing* ELS.

The first proof uses the translation to the sequent calculus system, and the second one uses the technique of splitting and context reduction (see Section 3.4, observe that the statements of splitting and context reduction become much simpler in the case of ELS). In this chapter, in Section 4.4, I will present a third proof of the same theorem, using the techniques of decomposition and permutation.

We immediately get the following two corollaries, whose proofs are the same as for full linear logic.

4.1.9   Corollary   *The rule $\mathsf{i}{\uparrow}$ is admissible for system* ELS.

$$
\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad\qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\bot\}}
$$

$$
\mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]}
$$

$$
\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad\qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}}
$$

$$
\mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad\qquad \mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{1\}}
$$

$$
\mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \qquad\qquad \mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R,R)}
$$

Figure 4.3: System SELS

$$
\mathsf{1}\!\downarrow \frac{}{1} \qquad\qquad \mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad\qquad \mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]}
$$

$$
\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad\qquad \mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad\qquad \mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}}
$$

Figure 4.4: System ELS

4.1.10 Corollary   *Let R and T be two structures. Then we have*

$$
\begin{array}{c} T \\ \|\mathsf{SELS} \\ R \end{array} \qquad \textit{if and only if} \qquad \begin{array}{c} \ \\ \|\mathsf{ELS} \\ [\bar{T},R] \end{array} \quad.
$$

4.2 Permutation of Rules

Various applications of proof theory are based on the permutability of rules [Cur52]. For example the idea of focusing proofs [And92] relies on it. In the case of Forum [Mil94, Mil96], a logic programming language based on linear logic, the permutability of rules has been used for adapting the notion of uniform proof to multiple conclusion sequents. The observation, that some rules permute over each other and others do not, offers the possibility of defining new inference rules of a larger granularity, whose application can then be seen as the

application of several rules (that permute over each other) in parallel. For example the development of *ludics* in [Gir01] has been influenced by this idea.

The top-down symmetry of derivations in the calculus of structures enables us to study the mutual permutability of rules in a very natural way. This is the starting point for the investigation of several properties of logical systems in the calculus of structures. If we have, for example, a system with three rule ρ, π and σ, and we know that ρ permutes over π and σ, then we can transform every derivation

$$
\begin{array}{c}
T \\
\parallel{}_{\{\rho,\pi,\sigma\}} \\
R
\end{array}
$$

into a derivation

$$
\begin{array}{c}
T \\
\parallel{}_{\{\rho\}} \\
T' \\
\parallel{}_{\{\pi,\sigma\}} \\
R
\end{array}
$$

for some structure T'. This is the basis for most of the decomposition theorems in this thesis (see Sections 4.3, 5.3 and 7.2). Also the cut elimination proof of Section 4.4 is based on the permutability of rules.

4.2.1 Definition A rule ρ *permutes over* a rule π (or π *permutes under* ρ) if for every derivation $\begin{array}{c}\pi\dfrac{Q}{U}\\[2pt]\rho\dfrac{}{P}\end{array}$ there is a derivation $\begin{array}{c}\rho\dfrac{Q}{V}\\[2pt]\pi\dfrac{}{P}\end{array}$ for some structure V.

In order to study the permutation properties of rules, some more definitions are needed. The inference rules of the systems SLS, SELS (shown in Figures 3.2 and 4.3, respectively) and all other systems in the calculus of structures, that will be discussed in this thesis, are of the kind

$$
\rho\,\frac{S\{W\}}{S\{Z\}} \quad,
$$

where the structure Z is called the *redex* and W the *contractum* of the rule's instance. A substructure that occurs exactly once in the redex as well as in the contractum of a rule without changing is called *passive*, and all substructures of redexes and contracta, that are not passive, (i.e. that change, disappear or are duplicated) are called *active*. More precisely, let R be a substructure of both, Z and W, i.e. we have that $Z = Z'\{R\}$ and $W = W'\{R\}$ for some contexts $Z'\{\ \}$ and $W'\{\ \}$. Then R is called *passive*, if for every structure X, we have that

$$
\rho\,\frac{S\{W'\{X\}\}}{S\{Z'\{X\}\}}
$$

is a valid instance of the rule ρ. Otherwise R is called *active*. Consider for example the rules

$$
\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \text{and} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}} \quad.
$$
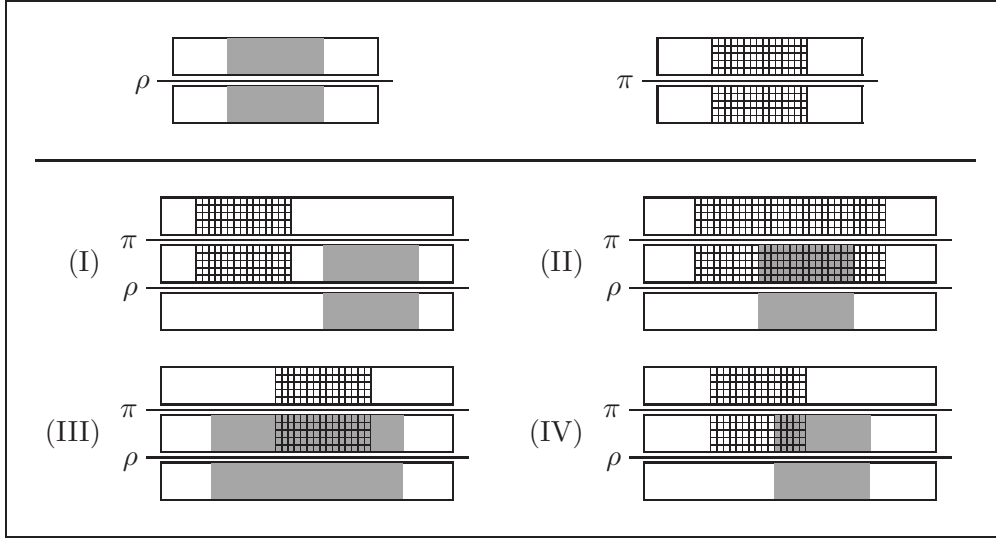
Figure 4.5: Possible interferences of redex and contractum of two consecutive rules

In $\mathsf{p}{\downarrow}$, the redex is $[!R, ?T]$ and the contractum is $![R, T]$; the structures R and T are passive; the structures $[!R, ?T]$, $!R$ and $?T$ are active in the redex; and the structures $![R, T]$ and $[R, T]$ are active in the contractum. In $\mathsf{b}{\downarrow}$ there are no passive structures; in the redex the structures $?R$ and R are active and in the contractum $[?R, R]$, $?R$, R and R are active (i.e. both occurrences of the structure R are active).

4.2.2   Definition   An application of a rule $\rho \dfrac{T}{R}$ will be called *trivial* if $R = T$. Otherwise, it is called *nontrivial*.

For example, an application of the rule

$$\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]}$$

becomes trivial if $R = \mathbf{1}$ and $T = \bot$, because then $S[!R, ?T] = S\{\mathbf{1}\} = S\{![R, T]\}$.

4.2.3   Case Analysis   In order to find out whether a rule ρ permutes over a rule π, we have to consider all possibilities of interference of the redex of π and the contractum of ρ in a situation

$$\pi\,\frac{Q}{U}\\\rho\,\frac{}{P}\quad.$$

It can happen that one is inside the other, that they overlap or that they are independent. The four possible interferences are shown Figure 4.5. Although the situation is symmetric with respect to ρ and π, in most cases the situation to be considered will be of the shape

$$\pi\,\frac{Q}{S\{W\}}\\\rho\,\frac{}{S\{Z\}}\quad,$$

$$
\begin{array}{ll}
(1) \quad \text{ai}\downarrow \dfrac{(1,d,[a,c],b)}{([b,\bar{b}],d,[a,c],b)} \\
\qquad \text{s} \dfrac{\phantom{([b,\bar{b}],d,[a,c],b)}}{([b,\bar{b}],d,[(a,b),c])}
\end{array}
\qquad
\begin{array}{ll}
(2) \quad \text{s} \dfrac{(!(a,1,c),[\bar{a},d])}{[\bar{a},(!(a,1,c),d)]} \\
\qquad \text{ai}\downarrow \dfrac{\phantom{[\bar{a},(!(a,1,c),d)]}}{[\bar{a},(!(a,[b,\bar{b}],c),d)]}
\end{array}
$$

$$
\begin{array}{ll}
(3) \quad \text{ai}\downarrow \dfrac{([a,1,c],b)}{([(a,[b,\bar{b}]),c],b)} \\
\qquad \text{s} \dfrac{\phantom{([(a,[b,\bar{b}]),c],b)}}{[(a,[b,\bar{b}],b),c]}
\end{array}
\qquad
\begin{array}{ll}
(4) \quad \text{p}\downarrow \dfrac{(a,![b,(c,d)])}{(a,[!b,?(c,d)])} \\
\qquad \text{s} \dfrac{}{[(a,!b),?(c,d)]}
\end{array}
$$

$$
\begin{array}{ll}
(5) \quad \text{w}\downarrow \dfrac{[a,b,\bot]}{[a,b,?[(c,\bar{c}),\bar{a}]]} \\
\qquad \text{ai}\uparrow \dfrac{\phantom{[a,b,?[(c,\bar{c}),\bar{a}]]}}{[a,b,?[\bot,\bar{a}]]}
\end{array}
\qquad
\begin{array}{ll}
(6) \quad \text{s} \dfrac{[?[a,b],a,([b,c],d)]}{[?[a,b],a,b,(c,d)]} \\
\qquad \text{b}\downarrow \dfrac{}{[?[a,b],(c,d)]}
\end{array}
$$

Figure 4.6: Examples for interferences between two consecutive rules

where the redex Z and the contractum W of ρ are known and we have to make a case analysis for the position of the redex of π inside the structure $S\{W\}$. There are altogether six cases, which are listed below. Figure 4.6 shows an example for each case. In Figure 4.5 only four cases occur because the there is no distinction between active and passive structures.

(1) The redex of π is inside the context $S\{\ \}$ of ρ. This corresponds to case (I) in Figure 4.5.

(2) The contractum W of ρ is inside a passive structure of the redex of π. This is a subcase of (II) in Figure 4.5.

(3) The redex of π is inside a passive structure of the contractum W of ρ. This is a subcase of (III) in Figure 4.5.

(4) The redex of π is inside an active structure of the contractum W of ρ but not inside a passive one. This is a subcase of (III) in Figure 4.5.

(5) The contractum W of ρ is inside an active structure of the redex of π but not inside a passive one. This is a subcase of (II) in Figure 4.5.

(6) The contractum W of ρ and the redex of π (properly) overlap. This corresponds to case (IV) in Figure 4.5. Observe that this case can happen because of associativity.

In the first two cases, we have that $Q = S'\{W\}$ for some context $S'\{\ \}$. This means that the derivation above is of the shape

$$
\pi\dfrac{S'\{W\}}{\rho\dfrac{S\{W\}}{S\{Z\}}}\quad,
$$

where we can permute ρ over π as follows

$$\pi \cfrac{\rho \cfrac{S'\{W\}}{S'\{Z\}}}{S\{Z\}} \quad .$$

In the third case, we have that $Z = Z'\{R\}$ and $W = W'\{R\}$ for some contexts $Z'\{\ \}$ and $W'\{\ \}$ and some structure R, and $Q = S\{W'\{R'\}\}$ for some structure R'. This means the derivation is

$$\rho \cfrac{\pi \cfrac{S\{W'\{R'\}\}}{S\{W'\{R\}\}}}{S\{Z'\{R\}\}} \quad ,$$

where R is passive for ρ, and we can permute ρ over π as follows

$$\pi \cfrac{\rho \cfrac{S\{W'\{R'\}\}}{S\{Z'\{R'\}\}}}{S\{Z'\{R\}\}} \quad .$$

This means that in a proof of a permutation result the cases (1) to (3) are always trivial, whereas for the remaining cases (4) to (6), more elaboration will be necessary. In every proof concerning a permutation result, I will follow this scheme.

In the remainder of this section, I will show some permutation results that will be needed for the decomposition theorems of the next section.

4.2.4   Lemma   *The rule* w↓ *permutes over the rules* ai↓, ai↑, p↓ *and* w↑.

Proof: Consider a derivation

$$\mathsf{w}{\downarrow} \cfrac{\pi \cfrac{Q}{S\{\bot\}}}{S\{?Z\}} \quad ,$$

where $\pi \in \{\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{p}{\downarrow}, \mathsf{w}{\uparrow}\}$. Without loss of generality, assume that the application of π is nontrivial. According to the case analysis 4.2.3, there are six cases, where the first three do not need to be considered. The remaining cases are:

(4) The redex of π is inside (an active structure of) the contractum $\bot$ of w↓. This is impossible because $\bot$ cannot have a proper substructure. If the redex of π is equal to $\bot$, then this case can be reduced to case (1) by using the fact that $\bot = [\bot, \bot]$.

(5) The contractum $\bot$ of w↓ is inside an active structure of the redex of π but not inside a passive one. This case is impossible, as the following case analysis shows:

   (i) If π is w↑ then the redex is 1, and $\bot$ cannot be inside it.

   (ii) If π is ai↑ then the redex is $\bot$, and we can reduce this case to case (1) by using the fact that $\bot = [\bot, \bot]$.

(iii) If π is $\mathsf{p}{\downarrow}$, then the active structures of the redex are $[!R, ?T]$, $!R$ and $?T$. If $\perp$ is inside $!R$ or $?T$, then it can be considered inside R or T, respectively, which are passive. Hence, this is covered by case (2). If $\perp$ is inside $[!R, ?T]$ but not inside $!R$ or $?T$, then we can use $[!R, \perp, ?T] = [[!R, ?T], \perp]$ to reduce this case to (1).

(iv) For $\pi = \mathsf{ai}{\downarrow}$, the situation is similar. The active structures of the redex are $[a, \bar{a}]$, a and $\bar{a}$. If $\perp$ is inside $[a, \bar{a}]$, then use $[a, \perp, \bar{a}] = [[a, \bar{a}], \perp]$ to reduce this case to (1). If one of a or $\bar{a}$ is $\perp$, then the application of $\mathsf{ai}{\downarrow}$ is trivial.

(6) The contractum $\perp$ of $\mathsf{w}{\downarrow}$ and the redex of π overlap. This case is impossible because the structure $\perp$ cannot properly overlap with any other structure. □

4.2.5 Lemma *The rule $\mathsf{w}{\uparrow}$ permutes under the rules $\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{p}{\uparrow}$ and $\mathsf{w}{\downarrow}$.*

Proof: Dual to Lemma 4.2.4. □

4.2.6 Lemma *The rule $\mathsf{ai}{\downarrow}$ permutes over the rules $\mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{p}{\uparrow}$ and $\mathsf{w}{\uparrow}$.*

Proof: Consider a derivation

$$\mathsf{ai}{\downarrow}\frac{\pi\dfrac{Q}{S\{1\}}}{S[a, \bar{a}]} \quad,$$

where $\pi \in \{\mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{p}{\uparrow}, \mathsf{w}{\uparrow}\}$. Without loss of generality, assume that the application of π is nontrivial. Again, follow 4.2.3:

(4) The redex of π is inside (an active structure of) the contractum 1 of $\mathsf{ai}{\downarrow}$. This is impossible because 1 cannot have a proper substructure. If the redex of π is equal to 1, then this case can be reduced to case (1) by using the fact that $1 = (1, 1)$.

(5) The contractum 1 of $\mathsf{ai}{\downarrow}$ is inside an active structure of the redex of π, but not inside a passive one. This case is not possible. This can be seen from a similar analysis as in the proof of Lemma 4.2.4.

(6) The contractum 1 of $\mathsf{ai}{\downarrow}$ and the redex of π overlap. This case is impossible because the structure 1 cannot properly overlap with any other structure. □

4.2.7 Lemma *The rule $\mathsf{ai}{\uparrow}$ permutes under the rules $\mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{p}{\downarrow}$ and $\mathsf{w}{\downarrow}$.*

Proof: Dual to Lemma 4.2.6. □

Observe that the rule $\mathsf{w}{\downarrow}$ does not permute over $\mathsf{p}{\uparrow}$. For example in the derivation

$$= \frac{\mathsf{p}{\uparrow}\dfrac{S(?R, !T)}{S\{?(R, T)\}}}{\mathsf{w}{\downarrow}\dfrac{S\{?[(R, T), \perp]\}}{S\{?[(R, T), ?Z]\}}} \quad.$$

it is unclear how the instance of w↓ could be permuted up. However, with the help of the switch rule, we can get

$$
\mathsf{s}\cfrac{\mathsf{p}\uparrow\cfrac{\mathsf{w}\downarrow\cfrac{=\cfrac{S(?R,!T)}{S(?R,![T,\bot])}}{S(?R,![T,?Z])}}{S\{?(R,[T,?Z])\}}}{S\{?[(R,T),?Z]\}}\quad.
$$

In other words, the switch is used to move the structure $?Z$ inside a passive structure of the redex of p↑. For the rules ai↓ and p↓ the situation is similar:

$$
\mathsf{ai}\downarrow\cfrac{\mathsf{p}\downarrow\cfrac{S'\{![R,T]\}}{S'[!R,?T]}}{=\cfrac{}{S'[(!R,1),?T]}}{S'[(!R,[a,\bar a]),?T]}
\qquad\text{yields}\qquad
\mathsf{s}\cfrac{\mathsf{p}\downarrow\cfrac{\mathsf{ai}\downarrow\cfrac{=\cfrac{S'\{![R,T]\}}{S'(![R,T],1)}}{S'(![R,T],[a,\bar a])}}{S'(([!R,?T],[a,\bar a])}}{S'[(!R,[a,\bar a]),?T]}\quad.
$$

This time the switch is used to move the disturbing structure $[a,\bar a]$ outside the redex of p↓.

Another example is the situation

$$
\mathsf{ai}\downarrow\cfrac{\mathsf{w}\downarrow\cfrac{S\{\bot\}}{S\{?(a,b)\}}}{=\cfrac{}{S\{?(a,1,b)\}}}{S\{?(a,[c,\bar c],b)\}}\quad,
$$

where it is unclear, how the rule ai↓ could be permuted over w↓. But we can replace the whole derivation by a single application of w↓:

$$
\mathsf{w}\downarrow\cfrac{S\{\bot\}}{S\{?(a,[c,\bar c],b)\}}\quad.
$$

This leads to the following definition.

4.2.8   Definition   A rule ρ *permutes over* a rule π *by* a rule σ if for every derivation

$$
\rho\cfrac{\pi\cfrac{Q}{U}}{P}\quad,
$$

there exists one of the derivations

$$
\pi\cfrac{\rho\cfrac{Q}{V}}{P}\quad,\qquad
\sigma\cfrac{\pi\cfrac{\rho\cfrac{Q}{V}}{V'}}{P}\quad,\qquad
\pi\cfrac{Q}{P}\quad,\qquad
\rho\cfrac{Q}{P}\qquad\text{or}\qquad
\sigma\cfrac{Q}{P}\quad,
$$

for some structures V and V'. Dually, a rule π *permutes under* a rule ρ by a rule σ if for every derivation

$$\pi \frac{Q}{\dfrac{U}{\rho \, \dfrac{}{P}}} \quad ,$$

there exists one of following the derivations (for some structures V and V'):

$$\rho \frac{\dfrac{Q}{V}}{\pi \, \dfrac{}{P}} \quad , \qquad \rho \frac{\sigma \dfrac{Q}{V}}{\pi \, \dfrac{V'}{P}} \quad , \qquad \pi \frac{Q}{P} \quad , \qquad \rho \frac{Q}{P} \qquad \text{or} \qquad \sigma \frac{Q}{P} \quad .$$

4.2.9 Lemma (a) *The rule* $\mathsf{w}{\downarrow}$ *permutes over* $\mathsf{p}{\uparrow}$ *and* s *by* s. (b) *The rule* $\mathsf{w}{\uparrow}$ *permutes under* $\mathsf{p}{\downarrow}$ *and* s *by* s. (c) *The rule* $\mathsf{ai}{\downarrow}$ *permutes over* $\mathsf{p}{\downarrow}$ *and* $\mathsf{w}{\downarrow}$ *by* s. (d) *The rule* $\mathsf{ai}{\uparrow}$ *permutes under* $\mathsf{p}{\uparrow}$ *and* $\mathsf{w}{\uparrow}$ *by* s.

Proof: (a) Consider a derivation

$$\mathsf{w}{\downarrow} \frac{\pi \dfrac{Q}{S\{\bot\}}}{S\{?Z\}} \quad ,$$

where $\pi \in \{\mathsf{p}{\uparrow}, \mathsf{s}\}$ is nontrivial. Then the cases (1)–(4) and (6) are as in the proof of Lemma 4.2.4. The remaining case is:

(5) The contractum $\bot$ of $\mathsf{w}{\downarrow}$ is inside an active structure of the redex of π but not inside a passive one. Then there are two subcases

    (i) If $\pi = \mathsf{p}{\uparrow}$, then the redex is $?(R, T)$. The active structures are $?(R, T)$ and (R, T). There are several possibilities for $\bot$ to be inside. They are all covered by the general case where $R = (R_1, R_2)$ and $T = (T_1, T_2)$, for some R_1, R_2, T_1, and T_2, and we have

$$?(R, T) \;=\; ?(R_1, R_2, T_1, T_2) \;=\; ?(R_1, [(R_2, T_1), \bot], T_2) \quad .$$

In the situation above, we then get $S\{\bot\} = S'\{?(R_1, [(R_2, T_1), \bot], T_2)\}$ for some context $S'\{\ \}$. Then

$$\mathsf{w}{\downarrow} \frac{\mathsf{p}{\uparrow} \dfrac{S'(?(R_1, R_2), !(T_1, T_2))}{S'\{?(R_1, R_2, T_1, T_2)\}}}{\dfrac{S'\{?(R_1, [(R_2, T_1), \bot], T_2)\}}{S'\{?(R_1, [(R_2, T_1), ?Z], T_2)\}}} \quad \text{yields} \quad \mathsf{s} \frac{\mathsf{p}{\uparrow} \dfrac{\mathsf{w}{\downarrow} \dfrac{\dfrac{S'(?(R_1, R_2), !(T_1, T_2))}{S'(?(R_1, [R_2, \bot]), !(T_1, T_2))}}{S'(?(R_1, [R_2, ?Z]), !(T_1, T_2))}}{S'\{?(R_1, [R_2, ?Z], T_1, T_2)\}}}{S'\{?(R_1, [(R_2, T_1), ?Z], T_2)\}} \quad .$$

    (ii) If $\pi = \mathsf{s}$, the situation is similar. The most general case for $\bot$ occurring inside an active structure of the redex $[(R, T), U]$ is when $R = (R_1, R_2)$ and $T = (T_1, T_2)$, for some R_1, R_2, T_1, and T_2, and

$$[(R, T), U] \;=\; [(R_1, R_2, T_1, T_2), U] \;=\; [(R_1, [(R_2, T_1), \bot], T_2), U] \quad .$$

Then we have $S\{\bot\} = S'[(R_1, [(R_2, T_1), \bot], T_2), U]$ for some context $S'\{\ \}$:

$$
\cfrac{\mathsf{s}\ \cfrac{S'([(R_1; R_2), U], T_1, T_2)}{S'[(R_1, R_2, T_1, T_2), U]}}{\mathsf{w}\downarrow\ \cfrac{S'[(R_1, [(R_2, T_1), \bot], T_2), U]}{S'[(R_1, [(R_2, T_1), ?Z], T_2), U]}}
\qquad \text{yields} \qquad
\cfrac{\mathsf{s}\ \cfrac{\mathsf{w}\downarrow\ \cfrac{=\ \cfrac{S'([(R_1; R_2), U], T_1, T_2)}{S'([(R_1, [R_2, \bot]), U], T_1, T_2)}}{S'([(R_1, [R_2, ?Z]), U], T_1, T_2)}}{S'[(R_1, [R_2, ?Z], T_1, T_2), U]}}{\mathsf{s}\ \cfrac{}{S'[(R_1, [(R_2, T_1), ?Z], T_2), U]}}
\qquad .
$$

(b) Dual to (a).

(c) Consider a derivation

$$
\mathsf{ai}\downarrow\ \cfrac{\pi\ \cfrac{Q}{S\{1\}}}{S[a, \bar{a}]} \qquad ,
$$

where $\pi \in \{\mathsf{p}\downarrow, \mathsf{w}\downarrow\}$ is nontrivial.   The cases (1)–(4) and (6) are as in the proof of Lemma 4.2.6. The remaining case is:

(5) The contractum 1 of $\mathsf{ai}\downarrow$ is inside an active structure of the redex of π, but not inside a passive one. There are the following subcases:

(i) It $\pi = \mathsf{p}\downarrow$, then the redex is $[!R, ?T]$. The possibilities for 1 to occur inside are $[(!R, 1), ?T]$ and $[!R, (?T, 1)]$. Both are similar and I show only the first, where $S\{1\} = S'[(!R, 1), ?T]$ for some $S'\{\ \}$. Then

$$
\mathsf{ai}\downarrow\ \cfrac{\mathsf{p}\downarrow\ \cfrac{=\ \cfrac{S'\{![R, T]\}}{S'[!R, ?T]}}{S'[(!R, 1), ?T]}}{S'[(!R, [a, \bar{a}]), ?T]}
\qquad \text{yields} \qquad
\cfrac{\mathsf{s}\ \cfrac{\mathsf{p}\downarrow\ \cfrac{\mathsf{ai}\downarrow\ \cfrac{=\ \cfrac{S'\{![R, T]\}}{S'(![R, T], 1)}}{S'(![R, T], [a, \bar{a}])}}{S'([!R, ?T], [a, \bar{a}])}}{S'[(!R, [a, \bar{a}]), ?T]}}{}
\qquad .
$$

(ii) If $\pi = \mathsf{w}\downarrow$, then the redex is $?R$. If 1 occurs inside, we have $?R = ?R'\{1\}$ for some context $R'\{\ \}$. Then $S\{1\} = S'\{?R'\{1\}\}$ for some $S'\{\ \}$, and

$$
\mathsf{ai}\downarrow\ \cfrac{\mathsf{w}\downarrow\ \cfrac{S'\{\bot\}}{S'\{?R'\{1\}\}}}{S'\{?R'[a, \bar{a}]\}}
\qquad \text{yields} \qquad
\mathsf{w}\downarrow\ \cfrac{S'\{\bot\}}{S'\{?R'[a, \bar{a}]\}}
\qquad .
$$

(d) Dual to (c).   □

This is sufficient to show that in any derivation that does not contain the rules $\mathsf{b}\downarrow$ and $\mathsf{b}\uparrow$, we can permute all instances of $\mathsf{w}\downarrow$ and $\mathsf{ai}\downarrow$ to the top of the derivation and all instances of $\mathsf{w}\uparrow$ and $\mathsf{ai}\uparrow$ to the bottom.

4.2.10 Proposition *For every derivation* $\overset{T}{\underset{R}{\Delta\|}}$SELS\\{b↓,b↑}$ *there are structures T' and R' and derivations Δ_1, Δ_2, and Δ_3, such that*

$$
\begin{array}{c}
T \\
\Delta_1\|\{\text{ai}\downarrow,\text{w}\downarrow\} \\
T' \\
\Delta_2\|\{\text{s},\text{p}\downarrow,\text{p}\uparrow\} \\
R' \\
\Delta_3\|\{\text{ai}\uparrow,\text{w}\uparrow\} \\
R
\end{array}
\quad .
$$

4.2.11 Remark The statement of Proposition 4.2.10 can be strengthened because the derivation $\overset{T}{\underset{T'}{\Delta_1\|}}\{\text{ai}\downarrow,\text{w}\downarrow\}$ can be further decomposed into

$$
\begin{array}{c}
T \\
\|\{\text{ai}\downarrow\} \\
T'' \\
\|\{\text{w}\downarrow\} \\
T'
\end{array}
\quad \text{and} \quad
\begin{array}{c}
T \\
\|\{\text{w}\downarrow\} \\
T''' \\
\|\{\text{ai}\downarrow\} \\
T'
\end{array}
\quad ,
$$

for some T'' and T'''. Dually, the derivation $\overset{R'}{\underset{R}{\Delta_3\|}}\{\text{ai}\uparrow,\text{w}\uparrow\}$ can be be decomposed into

$$
\begin{array}{c}
R' \\
\|\{\text{ai}\uparrow\} \\
R'' \\
\|\{\text{w}\uparrow\} \\
R
\end{array}
\quad \text{and} \quad
\begin{array}{c}
R' \\
\|\{\text{w}\uparrow\} \\
R''' \\
\|\{\text{ai}\uparrow\} \\
R
\end{array}
\quad ,
$$

for some structures R'' and R'''.

Proposition 4.2.10 gives a first idea of how a decomposition theorem does look like. For the full decomposition theorems for system SELS to be discussed in the following section, it is necessary to handle the rules b↓ and b↑. Unfortunately, this is impossible with such easy permutation arguments that have been shown in this section. For example, permuting b↑ over another rule π can cause the duplication of π:

$$
\text{b}\uparrow\frac{\pi\dfrac{S\{!R'\}}{S\{!R\}}}{S(!R,R)}
\quad \rightsquigarrow \quad
\pi\dfrac{\text{b}\uparrow\dfrac{S\{!R'\}}{S(!R',R')}}{\pi\dfrac{S(!R',R)}{S(!R,R)}}
\quad .
$$

Permuting b↑ under another rule ρ is impossible. This can be seen from the derivation

$$\mathsf{b\uparrow}\frac{S\{!R\}}{\rho\,\dfrac{S(!R,R)}{S(!R,R')}}\quad,$$

where b↑ cannot be applied to $(!R,R')$ because R and R' are in general not equal. For b↓, the situation is dual.

4.3   Decomposition

The new top-down symmetry of derivations in the calculus of structures allows to study properties that are not observable in the sequent calculus. The most remarkable results so far are decomposition theorems. In general, a decomposition theorem says that a given system $\mathscr{S}$ can be divided into n pairwise disjoint subsystems $\mathscr{S}_1, \ldots, \mathscr{S}_n$ such that every derivation Δ in system $\mathscr{S}$ can be rearranged as composition of n derivations $\Delta_1, \ldots, \Delta_n$, where Δ_i uses only rules of $\mathscr{S}_i$, for every $1 \leqslant i \leqslant n$.

   In this section, I will state and prove two such decomposition theorems for derivations in system SELS. Both of them allow a decomposition of system SELS into seven disjoint subsystems.

4.3.1   Theorem (First Decomposition)   *For every derivation* $\Delta\overset{T}{\underset{R}{\|}}\mathsf{SELS}$ *there are structures* $T_1, T_2, T_3, R_1, R_2, R_3,$ *and derivations* $\Delta_1, \ldots, \Delta_7,$ *such that*

$$
\begin{array}{c}
T \\
\Delta_1 \big\| \{\mathsf{b\uparrow}\} \\
T_1 \\
\Delta_2 \big\| \{\mathsf{w\downarrow}\} \\
T_2 \\
\Delta_3 \big\| \{\mathsf{ai\downarrow}\} \\
T_3 \\
\Delta_4 \big\| \{\mathsf{s,p\downarrow,p\uparrow}\} \\
R_3 \\
\Delta_5 \big\| \{\mathsf{ai\uparrow}\} \\
R_2 \\
\Delta_6 \big\| \{\mathsf{w\uparrow}\} \\
R_1 \\
\Delta_7 \big\| \{\mathsf{b\downarrow}\} \\
R
\end{array}\quad.
$$

   Apart from a decomposition into seven subsystems, the first decomposition theorem can also be read as a decomposition into three subsystems that can be called *creation*, *merging*, and *destruction*. In the creation subsystem, each rule increases the size of the structure; in the merging system, each rule does some rearranging of substructures, without changing

the size of the structures; and in the destruction system, each rule decreases the size of the structure. In a decomposed derivation, the merging part is in the middle of the derivation, and (depending on your preferred reading of a derivation) the creation and destruction are at the top and at the bottom:

$$
\begin{array}{c}
T \\
\text{destruction } \big(\!\big\Updownarrow\!\big) \text{ creation} \\
T' \\
\Updownarrow \text{ merging} \qquad . \\
R' \\
\text{creation } \big(\!\big\Updownarrow\!\big) \text{ destruction} \\
R
\end{array}
$$

In system SELS the merging part contains the rules s, p$\downarrow$ and p$\uparrow$. In the top-down reading of a derivation, the creation part (where the size of the structure is increased) contains the rules b$\uparrow$, w$\downarrow$ and ai$\downarrow$, and the destruction part (where the size of the structure is decreased) consists of b$\downarrow$, w$\uparrow$ and ai$\uparrow$. In the bottom-up reading, creation and destruction are exchanged.

The second decomposition is almost the same statement, with the only difference that the rules w$\downarrow$ and w$\uparrow$ are exchanged.

4.3.2 Theorem (Second Decomposition) *For every derivation* $\begin{array}{c} T \\ \Delta\|\text{SELS} \\ R \end{array}$ *there are structures* $T_1, T_2, T_3, R_1, R_2, R_3$, *and derivations* $\Delta_1, \ldots, \Delta_7$, *such that*

$$
\begin{array}{c}
T \\
\Delta_1 \big\| \{\text{b}\uparrow\} \\
T_1 \\
\Delta_2 \big\| \{\text{w}\uparrow\} \\
T_2 \\
\Delta_3 \big\| \{\text{ai}\downarrow\} \\
T_3 \\
\Delta_4 \big\| \{\text{s},\text{p}\downarrow,\text{p}\uparrow\} \qquad . \\
R_3 \\
\Delta_5 \big\| \{\text{ai}\uparrow\} \\
R_2 \\
\Delta_6 \big\| \{\text{w}\downarrow\} \\
R_1 \\
\Delta_7 \big\| \{\text{b}\downarrow\} \\
R
\end{array}
$$

The second decomposition theorem allows a separation between core and noncore of the system, such that the up fragment and the down fragment of the noncore are not merged, as it is the case in the first decomposition. More precisely, we can separate the following

five subsystems:

$$
\begin{array}{l}
T \\
\;\|\text{noncore (up)} \\
T' \\
\;\|\text{interaction (down)} \\
T'' \\
\;\|\text{core (up and down)} \qquad . \\
R'' \\
\;\|\text{interaction (up)} \\
R' \\
\;\|\text{noncore (down)} \\
R
\end{array}
$$

This decomposition will also considerably simplify the cut elimination argument because we no longer have to deal with the noncore rules (up and down) which usually make the cut elimination proof hard.

In the following, I will first give only a sketch of the proofs of the decomposition theorems, and then the detailed proofs will follow in Sections 4.3.1 and 4.3.2.

The first decomposition is obtained in three steps as follows:

$$
\begin{array}{ccc}
\begin{array}{l}
T \\
\Delta\|\text{SELS} \\
R
\end{array}
\;\overset{1}{\leadsto}\;
&
\begin{array}{l}
T \\
\Delta_1\|\{\text{b}\uparrow\} \\
T_1 \\
\Delta'\|\text{SELS}\setminus\{\text{b}\downarrow,\text{b}\uparrow\} \\
R_1 \\
\Delta_7\|\{\text{b}\downarrow\} \\
R
\end{array}
\;\overset{2}{\leadsto}\;
&
\begin{array}{l}
T \\
\Delta_1\|\{\text{b}\uparrow\} \\
T_1 \\
\Delta_2\|\{\text{w}\downarrow\} \\
T_2 \\
\Delta''\|\{\text{ai}\downarrow,\text{ai}\uparrow,\text{s},\text{p}\downarrow,\text{p}\uparrow\} \\
R_2 \\
\Delta_6\|\{\text{w}\uparrow\} \\
R_1 \\
\Delta_7\|\{\text{b}\downarrow\} \\
R
\end{array}
\;\overset{3}{\leadsto}\;
\begin{array}{l}
T \\
\Delta_1\|\{\text{b}\uparrow\} \\
T_1 \\
\Delta_2\|\{\text{w}\downarrow\} \\
T_2 \\
\Delta_3\|\{\text{ai}\downarrow\} \\
T_3 \\
\Delta_4\|\{\text{s},\text{p}\downarrow,\text{p}\uparrow\} \\
R_3 \\
\Delta_5\|\{\text{ai}\downarrow\} \\
R_2 \\
\Delta_6\|\{\text{w}\uparrow\} \\
R_1 \\
\Delta_7\|\{\text{b}\downarrow\} \\
R
\end{array}
\quad .
\end{array}
$$

The second and third step are realised by the results of Section 4.2: we can use Proposition 4.2.10 together with Remark 4.2.11. The first step is more difficult. The basic idea is to permute the instances of b↑ over all the other rules. The most problematic case is when

$$
\text{p}\downarrow\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \text{is replaced by} \qquad
\begin{array}{l}
\text{b}\uparrow\dfrac{S\{![R,T]\}}{S(![R,T],[R,T])} \\[4pt]
\text{p}\downarrow\dfrac{}{S([!R,?T],[R,T])} \\[4pt]
\text{s}\dfrac{}{S[([!R,?T],R),T]} \\[4pt]
\text{s}\dfrac{}{S[(!R,R),?T,T]} \\[4pt]
\text{b}\downarrow\dfrac{}{S[(!R,R),?T]}
\end{array} \quad ,
$$

where the left-hand replaced derivation is

$$
\text{b}\uparrow\frac{\text{p}\downarrow\dfrac{S\{![R,T]\}}{S[!R,?T]}}{S[(!R,R),?T]}
$$

$$
\begin{array}{ccccccc}
&&T&T&T&&T\\
&T&\big\|{\scriptstyle\{b\uparrow\}}&\big\|{\scriptstyle\{b\uparrow\}}&\big\|{\scriptstyle\{b\uparrow\}}&&\big\|{\scriptstyle\{b\uparrow\}}\\
T&\big\|{\scriptstyle\{b\uparrow\}}&T_1&T_2&T_2&&T_k\\
\big\|{\scriptstyle\mathsf{SELS}}\ \rightsquigarrow\ T_1&&\big\|{\scriptstyle\mathsf{SELS}\backslash\{b\downarrow\}}\ \rightsquigarrow&\big\|{\scriptstyle\mathsf{SELS}\backslash\{b\uparrow\}}\ \rightsquigarrow&\big\|{\scriptstyle\mathsf{SELS}\backslash\{b\downarrow\}}\ \rightsquigarrow\dots\rightsquigarrow&&\big\|{\scriptstyle\mathsf{SELS}\backslash\{b\downarrow,b\uparrow\}}\\
R&\big\|{\scriptstyle\mathsf{SELS}\backslash\{b\uparrow\}}&R_1&R_1&R_2&&R_h\\
&R&\big\|{\scriptstyle\{b\downarrow\}}&\big\|{\scriptstyle\{b\downarrow\}}&\big\|{\scriptstyle\{b\downarrow\}}&&\big\|{\scriptstyle\{b\downarrow\}}\\
&&R&R&R&&R
\end{array}
$$
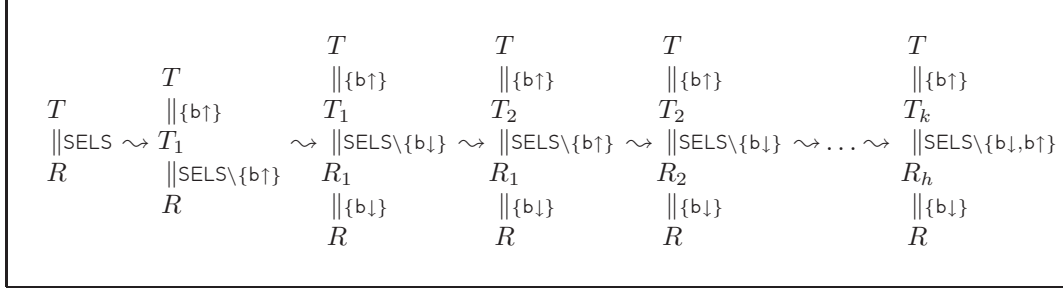
Figure 4.7: Permuting $b\uparrow$ up and $b\downarrow$ down

because a new instance of $b\downarrow$ is introduced. After all $b\uparrow$ have reached the top of the derivation, the instances of $b\downarrow$ are permuted down by the dual procedure, where new instances of $b\uparrow$ might be introduced; and so on.

The problem is to show that this process, visualised in Figure 4.7, does terminate eventually. This is done as follows: I will introduce the notion of a certain type of cycle inside a derivation. Then, I will show that the assumption of a nontermination can be reduced to the existence of such a cycle. From a derivation that contains such a cycle, I will extract a derivation

$$
\begin{array}{c}
([!R_1,?T_1],[!R_2,?T_2],\dots[!R_n,?T_n])\\
\Delta\big\|{\scriptstyle\{s\}}\\
[(!R_2,?T_1),(!R_3,?T_2),\dots(!R_1,?T_n)]
\end{array}\quad,
$$

for some $n \geqslant 1$ and structures $R_1,\dots,R_n$ and $T_1,\dots,T_n$. Then, I will show that such a derivation cannot exist. This will prove the termination. In fact, the nonexistence of such a derivation is the deep reason behind the decomposition theorems, and the only purpose of all the effort made in Sections 4.3.1 and 4.3.2 is to carry out the technical details. (Here we have another example where syntax seems to be the enemy of proof theory.)

The proof of the second decomposition is also done in three steps:

$$
\begin{array}{ccccccc}
&&&&&&T\\
&&&&&&\Delta_1\big\|{\scriptstyle\{b\uparrow\}}\\
&&&&T&&T_1\\
&&&&\Delta_1\big\|{\scriptstyle\{b\uparrow\}}&&\Delta_2\big\|{\scriptstyle\{w\uparrow\}}\\
&&T&&T_1&&T_2\\
&&\Delta_1\big\|{\scriptstyle\{b\uparrow\}}&&\Delta_2\big\|{\scriptstyle\{w\uparrow\}}&&\Delta_3\big\|{\scriptstyle\{ai\downarrow\}}\\
T&&T_1&&T_2&&T_3\\
\Delta\big\|{\scriptstyle\mathsf{SELS}}\ \ \stackrel{1}{\rightsquigarrow}&&\Delta'\big\|{\scriptstyle\mathsf{SELS}\backslash\{b\downarrow,b\uparrow\}}\ \ \stackrel{2}{\rightsquigarrow}&&\Delta''\big\|{\scriptstyle\{ai\downarrow,ai\uparrow,s,p\downarrow,p\uparrow\}}\ \ \stackrel{3}{\rightsquigarrow}&&\Delta_4\big\|{\scriptstyle\{s,p\downarrow,p\uparrow\}}\quad.\\
R&&R_1&&R_2&&R_3\\
&&\Delta_7\big\|{\scriptstyle\{b\downarrow\}}&&\Delta_6\big\|{\scriptstyle\{w\downarrow\}}&&\Delta_5\big\|{\scriptstyle\{ai\downarrow\}}\\
&&R&&R_1&&R_2\\
&&&&\Delta_7\big\|{\scriptstyle\{b\downarrow\}}&&\Delta_6\big\|{\scriptstyle\{w\downarrow\}}\\
&&&&R&&R_1\\
&&&&&&\Delta_7\big\|{\scriptstyle\{b\downarrow\}}\\
&&&&&&R
\end{array}
$$

The first and the last step are exactly the same as for the first decomposition. In the second step, we have to permute the rule $w\uparrow$ up and the rule $w\downarrow$ down. When permuting $w\uparrow$ up, the problematic case is when

$$
p\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \\
w\uparrow \frac{}{S[1,?T]}
\qquad \text{is replaced by} \qquad
w\uparrow \frac{S\{![R,T]\}}{S\{1\}} \\
= \frac{}{S[1,\bot]} \\
w\downarrow \frac{}{S[1,?T]}
\quad ,
$$

because a new instance of $w\downarrow$ is introduced. After all $w\uparrow$ have reached the top of the derivation, the instances of $w\downarrow$ are permuted down by the dual procedure, where new instances of $w\uparrow$ might be introduced; and so on. This means that there is the same termination problem as for the separation of absorption in the first step. However, since the weakening rule deletes material instead of duplicating it, termination will be much easier to show than for the separation of absorption.

4.3.1   Chains and Cycles in Derivations

In this section, I will provide a tool for dealing with the rules $b\downarrow$ and $b\uparrow$ in the decomposition theorems. The goal is to permute in any derivation all instances of $b\uparrow$ up to the top and all instances of $b\downarrow$ down to the bottom. If we try to permute the rule $b\uparrow$ over the other rules in system SELS applying the schema in 4.2.3, we encounter (among others) the following case:

$$
p\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \\
b\uparrow \frac{}{S[(!R,R),?T]}
\quad ,
$$

where it is unclear, how the instance of $b\uparrow$ could be permuted over $p\downarrow$. But the derivation can be replaced by

$$
b\uparrow \frac{S\{![R,T]\}}{S(![R,T],[R,T])} \\
p\downarrow \frac{}{S([!R,?T],[R,T])} \\
s \frac{}{S[([!R,?T],R),T]} \\
s \frac{}{S[(!R,R),?T,T]} \\
b\downarrow \frac{}{S[(!R,R),?T]}
\quad .
$$

This seems to solve the problem because now the instance of $b\uparrow$ is above the instance of $p\downarrow$. However, there is now a new instance of $b\downarrow$ which needs to be permuted down to the bottom of the derivation. Applying the schema in 4.2.3 again, we encounter the dual case:

$$
b\downarrow \frac{S(!R,[?T,T])}{S(!R,?T)} \\
p\uparrow \frac{}{S\{?(R,T)\}}
\quad .
$$

This has now to be replaced by

$$
\mathsf{b}\!\uparrow \frac{
\mathsf{s} \dfrac{
\mathsf{s} \dfrac{
\mathsf{p}\!\uparrow \dfrac{
\mathsf{b}\!\downarrow \dfrac{S(!R, [?T, T])}{S(!R, R, [?T, T])}}{S([(!R, ?T), T], R)}}{S[(!R, ?T), (R, T)]}}{S[?(R, T), (R, T)]}}{S\{?(R, T)\}} \quad ,
$$

which introduces a new instance of b↑. And so on.

The problem is to show that this cannot run forever, but must terminate eventually (see Figure 4.7). In order to do so, we have to inspect the "path" that is taken by an instance of b↑ while it moves up to the top and the "path" taken by a b↓ while it moves down. This is the motivation for the definition of !-*chains* and ?-*chains*. These chains can be composed to give complex chains. I will show that in the process described above, the instances of b↑ and b↓ travel up and down along such chains. Further, the process will not terminate if such a chain has the form of a cycle. The purpose of this section is to show that there is no such cycle.

In Definition 3.1.7, I introduced the concept of a context as a structure with a hole, and then used $S\{R\}$ for denoting a structure with a substructure R. In this section, I also need to denote structures with more than one distinctive substructure. For example, I will write $S\{R_1\}\{R_2\}$ to denote a structure with two independent substructures R_1 and R_2. More generally, $S\{R_1\}\ldots\{R_n\}$ has $R_1, \ldots, R_n$ as substructures. Removing those substructures and replacing them by { }, yields an *n*-ary context $S\{\ \}\ldots\{\ \}$. For example, $[!\{\ \}, (a, \{\ \}, b)]$ is a binary context.

4.3.3   Definition   Let Δ be a derivation. A !-*link* in Δ is any !-structure $!R$ that occurs as substructure of a structure S inside a derivation.

The purpose of the definition of a !-link is to define !-chains, which consist of !-links. In general, in a given derivation Δ, most of the !-links in Δ do not belong to a !-chain. Later, I will mark those !-links that belong to a !-chain or that are for other reason under discussion with a **!▲**.

4.3.4   Example   The derivation

$$
\mathsf{p}\!\uparrow \frac{
\mathsf{s} \dfrac{
\mathsf{p}\!\downarrow \dfrac{(!^{\blacktriangle}[(b, !a), \bar{a}], !c)}{([!^{\blacktriangle}(b, !a), ?\bar{a}], !c)}}{[!(b, !^{\blacktriangle}a), (?\bar{a}, !^{\blacktriangle}c)]}}{[!(b, !a), ?(\bar{a}, c)]}
$$

contains many !-links, but only four of them are marked. (So far, there is no relation between markings and chains.)

$$
\begin{array}{cll}
(1) & \rho\,\dfrac{S\{!^{\blacktriangle}R\}\{W\}}{S\{!^{\blacktriangle}R\}\{\boxed{Z}\}} & \rho\,\dfrac{S\{?^{\blacktriangledown}T\}\{W\}}{S\{?^{\blacktriangledown}T\}\{\boxed{Z}\}} \\[3ex]
(2) & \rho\,\dfrac{S\{W\{!^{\blacktriangle}R\}\}}{S\{\boxed{Z\{!^{\blacktriangle}R\}}\}} & \rho\,\dfrac{S\{W\{?^{\blacktriangledown}T\}\}}{S\{\boxed{Z\{?^{\blacktriangledown}T\}}\}} \\[3ex]
(3) & \rho\,\dfrac{S\{!^{\blacktriangle}R\{W\}\}}{S\{!^{\blacktriangle}R\{\boxed{Z}\}\}} & \rho\,\dfrac{S\{?^{\blacktriangledown}T\{W\}\}}{S\{?^{\blacktriangledown}T\{\boxed{Z}\}\}} \\[3ex]
(4.\mathrm{i}) & \mathsf{p}{\downarrow}\,\dfrac{S\{!^{\blacktriangle}[R,T]\}}{S\boxed{[!^{\blacktriangle}R,?T]}} & \mathsf{p}{\uparrow}\,\dfrac{S(!R,?^{\blacktriangledown}T)}{S\{\boxed{?^{\blacktriangledown}(R,T)}\}} \\[3ex]
(4.\mathrm{ii}) & \mathsf{b}{\uparrow}\,\dfrac{S\{!^{\blacktriangle}R\}}{S\boxed{(!^{\blacktriangle}R,R)}} & \mathsf{b}{\uparrow}\,\dfrac{S(?^{\blacktriangledown}T,T)}{S\{\boxed{?^{\blacktriangledown}T}\}} \\[3ex]
(4.\mathrm{iii}) & \mathsf{b}{\uparrow}\,\dfrac{S\{!V\{!^{\blacktriangle}R\}\}}{S\boxed{(!V\{!^{\blacktriangle}R\},V\{!R\})}} & \mathsf{b}{\downarrow}\,\dfrac{S[?U\{?^{\blacktriangledown}T\},U\{?T\}]}{S\{\boxed{?U\{?^{\blacktriangledown}T\}}\}} \\[3ex]
(4.\mathrm{iv}) & \mathsf{b}{\uparrow}\,\dfrac{S\{!V\{!^{\blacktriangle}R\}\}}{S\boxed{(!V\{!R\},V\{!^{\blacktriangle}R\})}} & \mathsf{b}{\downarrow}\,\dfrac{S[?U\{?T\},U\{?^{\blacktriangledown}T\}]}{S\{\boxed{?U\{?^{\blacktriangledown}T\}}\}} \\[3ex]
(4.\mathrm{v}) & \mathsf{b}{\downarrow}\,\dfrac{S[?U\{!^{\blacktriangle}R\},U\{!R\}]}{S\{\boxed{?U\{!^{\blacktriangle}R\}}\}} & \mathsf{b}{\uparrow}\,\dfrac{S\{!V\{?^{\blacktriangledown}T\}\}}{S\boxed{(!V\{?^{\blacktriangledown}T\},V\{?T\})}} \\[3ex]
(4.\mathrm{vi}) & \mathsf{b}{\downarrow}\,\dfrac{S[?U\{!R\},U\{!^{\blacktriangle}R\}]}{S\{\boxed{?U\{!^{\blacktriangle}R\}}\}} & \mathsf{b}{\uparrow}\,\dfrac{S\{!V\{?^{\blacktriangledown}T\}\}}{S\boxed{(!V\{?T\},V\{?^{\blacktriangledown}T\})}}
\end{array}
$$

Figure 4.8: Connection of !-links and ?-links

4.3.5   Definition   Two !-links $!^{\blacktriangle}R$ and $!^{\blacktriangle}R'$ inside a derivation Δ are *connected* if they occur in two consecutive structures, i.e. Δ is of the shape

$$
\begin{array}{c}
P \\
\parallel \\
\rho\,\dfrac{S'\{!^{\blacktriangle}R'\}}{S\{!^{\blacktriangle}R\}} \\
\parallel \\
Q
\end{array}\quad,
$$

and one of the following cases holds:

(1) The link $!^{\blacktriangle}R$ is inside the context of ρ, i.e. $R = R'$ and $S\{!^{\blacktriangle}R\} = S''\{!^{\blacktriangle}R\}\{Z\}$ and

$S'\{!^{\blacktriangle}R'\} = S''\{!^{\blacktriangle}R\}\{W\}$ for some context $S''\{\ \}\{\ \}$, where Z and W are redex and contractum of ρ:

$$\rho\,\frac{S''\{!^{\blacktriangle}R\}\{W\}}{S''\{!^{\blacktriangle}R\}\{Z\}}\quad.$$

(2) The link $!^{\blacktriangle}R$ is inside a passive structure of the redex of ρ, i.e. $R = R'$ and there are contexts $S''\{\ \}$, $Z'\{\ \}$ and $W'\{\ \}$ such that $S\{!^{\blacktriangle}R\} = S''\{Z\{!^{\blacktriangle}R\}\}$ and $S'\{!^{\blacktriangle}R'\} = S''\{W\{!^{\blacktriangle}R\}\}$, where $Z\{!^{\blacktriangle}R\}$ and $W\{!^{\blacktriangle}R\}$ are redex and contractum of ρ:

$$\rho\,\frac{S''\{W\{!^{\blacktriangle}R\}\}}{S''\{Z\{!^{\blacktriangle}R\}\}}\quad.$$

(3) The redex of ρ is inside R, i.e. $S\{\ \} = S'\{\ \}$ and there is a context $R''\{\ \}$ such that $S\{!^{\blacktriangle}R\} = S\{!^{\blacktriangle}R''\{Z\}\}$ and $S'\{!^{\blacktriangle}R'\} = S\{!^{\blacktriangle}R''\{W\}\}$, where Z and W are redex and contractum of ρ:

$$\rho\,\frac{S\{!^{\blacktriangle}R''\{W\}\}}{S\{!^{\blacktriangle}R''\{Z\}\}}\quad.$$

(4) The link $!^{\blacktriangle}R$ is inside an active structure of the redex of ρ, but not inside a passive one. Then six subcases are possible:

(i) $\rho = \mathsf{p}{\downarrow}$ and there is a structure T such that we have $S\{!^{\blacktriangle}R\} = S'[!^{\blacktriangle}R, ?T]$ and $S'\{!^{\blacktriangle}R'\} = S'\{!^{\blacktriangle}[R,T]\}$, i.e. $R' = [R,T]$:

$$\mathsf{p}{\downarrow}\,\frac{S'\{!^{\blacktriangle}[R,T]\}}{S'[!^{\blacktriangle}R, ?T]}\quad.$$

(ii) $\rho = \mathsf{b}{\uparrow}$, $R = R'$, $S\{!^{\blacktriangle}R\} = S'(!^{\blacktriangle}R, R)$ and $S'\{!^{\blacktriangle}R'\} = S'\{!^{\blacktriangle}R\}$:

$$\mathsf{b}{\uparrow}\,\frac{S'\{!^{\blacktriangle}R\}}{S'(!^{\blacktriangle}R, R)}\quad.$$

(iii) $\rho = \mathsf{b}{\uparrow}$, $R = R'$ and there are contexts $S''\{\ \}$ and $V\{\ \}$ such that $S\{!^{\blacktriangle}R\} = S''(!V\{!^{\blacktriangle}R\}, V\{!R\})$ and $S'\{!^{\blacktriangle}R'\} = S''\{!V\{!^{\blacktriangle}R\}\}$:

$$\mathsf{b}{\uparrow}\,\frac{S''\{!V\{!^{\blacktriangle}R\}\}}{S''(!V\{!^{\blacktriangle}R\}, V\{!R\})}\quad.$$

(iv) $\rho = \mathsf{b}{\uparrow}$, $R = R'$ and there are contexts $S''\{\ \}$ and $V\{\ \}$ such that $S\{!^{\blacktriangle}R\} = S''(!V\{!R\}, V\{!^{\blacktriangle}R\})$ and $S'\{!^{\blacktriangle}R'\} = S''\{!V\{!^{\blacktriangle}R\}\}$:

$$\mathsf{b}{\uparrow}\,\frac{S''\{!V\{!^{\blacktriangle}R\}\}}{S''(!V\{!R\}, V\{!^{\blacktriangle}R\})}\quad.$$

(v) $\rho = \mathsf{b}{\downarrow}$, $R = R'$ and there are contexts $S''\{\ \}$ and $U\{\ \}$ such that $S\{!^{\blacktriangle}R\} = S''\{?U\{!^{\blacktriangle}R\}\}$ and $S'\{!^{\blacktriangle}R'\} = S''[?U\{!^{\blacktriangle}R\}, U\{!R\}]$:

$$\mathsf{b}{\downarrow}\,\frac{S''[?U\{!^{\blacktriangle}R\}, U\{!R\}]}{S''\{?U\{!^{\blacktriangle}R\}\}}\quad.$$

(vi) $\rho = \mathsf{b}{\downarrow}$, $R = R'$ and there are contexts $S''\{\ \}$ and $U\{\ \}$ such that
$S\{!^{\blacktriangle}R\} = S''\{?U\{!^{\blacktriangle}R\}\}$ and $S'\{!^{\blacktriangle}R'\} = S''[?U\{!R\}, U\{!^{\blacktriangle}R\}]$:

$$\mathsf{b}{\downarrow}\,\frac{S''[?U\{!R\}, U\{!^{\blacktriangle}R\}]}{S''\{?U\{!^{\blacktriangle}R\}\}}\quad.$$

On the left-hand side of Figure 4.8, I have listed all possibilities, how !-links can be connected.

4.3.6   Example   In the derivation shown in Example 4.3.4, the two !-links $!^{\blacktriangle}[(b, !a), \bar{a}]$ and $!^{\blacktriangle}(b, !a)$ are connected (by case (4.i)), whereas the !-link $!^{\blacktriangle}a$ is neither connected to $!^{\blacktriangle}(b, !a)$ nor to $!^{\blacktriangle}c$.

4.3.7   Definition   A !-*chain* χ inside a derivation Δ is a sequence of connected !-links. The bottommost !-link of χ is called its *tail* and the topmost !-link of χ is called its *head*.

Throughout this work, I will visualise !-chains by giving the derivation and marking all !-links of the chain by $!^{\blacktriangle}$.

4.3.8   Example   The following derivation shows a !-chain with tail $!^{\blacktriangle}(b, ?a)$ and head $!^{\blacktriangle}b$:

$$
\mathsf{b}{\uparrow}\,\frac{(!^{\blacktriangle}b, !c)}{
= \dfrac{(!^{\blacktriangle}b, b, !c)}{\dfrac{(!^{\blacktriangle}b, !c, b)}{
\mathsf{ai}{\downarrow}\,\dfrac{}{
\mathsf{w}{\downarrow}\,\dfrac{(!^{\blacktriangle}(b, [a, \bar{a}]), !c, b)}{
\mathsf{s}\,\dfrac{(!^{\blacktriangle}(b, [?a, a, \bar{a}]), !c, b)}{
\mathsf{b}{\downarrow}\,\dfrac{(!^{\blacktriangle}[(b, [?a, a]), \bar{a}], !c, b)}{
\mathsf{p}{\downarrow}\,\dfrac{(!^{\blacktriangle}[(b, ?a), \bar{a}], !c, b)}{
\mathsf{s}\,\dfrac{([!^{\blacktriangle}(b, ?a), ?\bar{a}], !c, b)}{
\mathsf{p}{\uparrow}\,\dfrac{([!^{\blacktriangle}(b, ?a), (?\bar{a}, !c)], b)}{([!^{\blacktriangle}(b, ?a), ?(\bar{a}, c)], b)}}}}}}}}}\,(4.\text{ii})}
$$

(On the right-hand side labels: (4.ii), (3), (3), (3), (3), (4.i), (2), (1))

On the right-hand side of the derivation I indicated by which case (of Figure 4.8) the links are connected. (Observe that every *subchain* is also a !-chain. For example the bottommost four !-links are also a !-chain.)

4.3.9   Definition   The notion of ?-*link* is defined in the same way as the one of !-link. The notion of ?-*chain* is defined dually to !-chain, in particular, the *tail* of a ?-chain is its topmost ?-link and its *head* is its bottommost ?-link.

Similar to !-links, I will mark ?-links that are under discussion with $?^{\blacktriangledown}$. In particular, ?-chains are marked by marking every link of the ?-chain by $?^{\blacktriangledown}$. For convenience, I have listed on the right-hand side of Figure 4.8 all possibilities how ?-links can be connected inside a ?-chain. Observe that cases (4.i) and (4.ii) are the only cases in which both sides differ. The cases (1) to (3) are exactly the same on both sides. The cases (4.iii) and (4.v) as well as the cases (4.iv) and (4.vi) are exchanged in order to maintain the duality.

4.3.10 **Example** The following derivation shows an example for a ?-chain with tail $?^\blacktriangledown a$
and head $?^\blacktriangledown(a,c)$:

$$
\mathsf{p}{\downarrow}\,\cfrac{\cfrac{![!c,(?^\blacktriangledown a,!c),1]}{[?!c,![(?^\blacktriangledown a,!c),1]]}\;(2)}{\cfrac{\cfrac{[?!([\bot,1],!c),![(?^\blacktriangledown a,!c),1]]}{\mathsf{w}{\downarrow}\,\cfrac{[?!([?a,1],!c),![(?^\blacktriangledown a,!c),1]]}{\mathsf{s}\,\cfrac{[?![(?a,!c),1],![(?^\blacktriangledown a,!c),1]]}{\mathsf{b}{\downarrow}\,\cfrac{?![(?^\blacktriangledown a,!c),1]}{\mathsf{ai}{\downarrow}\,\cfrac{?![(?^\blacktriangledown a,!c),b,\bar{b}]}{\mathsf{p}{\downarrow}\,\cfrac{?[![(?^\blacktriangledown a,!c),b],?\bar{b}]}{\mathsf{b}{\uparrow}\,\cfrac{?[(![(?^\blacktriangledown a,!c),b],[(?a,!c),b]),?\bar{b}]}{\mathsf{p}{\uparrow}\,\cfrac{?[(![?^\blacktriangledown(a,c),b],[(?a,!c),b]),?\bar{b}]}{}}}}}}}}\;}{}
$$

Again, I indicated on the right-hand side of the derivation the cases by which the links are
connected.

4.3.11 **Definition** Let Δ be a derivation. An *upper link* in Δ is any structure $[!R,?T]$
that occurs as substructure of a structure S inside Δ. Dually, a *lower link* is any structure
$(?T,!R)$ that occurs as substructure of a structure S inside Δ.

Similar to the marking of !-links and ?-links, I will mark upper links as $[!^\blacktriangle R, ?^\blacktriangledown T]$ and
lower links as $(?^\blacktriangledown T, !^\blacktriangle R)$.

4.3.12 **Definition** Let Δ be a derivation. The set $X(\Delta)$ of *chains* in Δ is defined
inductively as follows:

(1) For every !-chain χ in Δ, we have $\chi \in X(\Delta)$.

(2) For every ?-chain χ in Δ, we have $\chi \in X(\Delta)$.

(3) If Δ contains two chains χ_1 and χ_2 and an upper link $[!^\blacktriangle R, ?^\blacktriangledown T]$ such that $!^\blacktriangle R$ is
the head of χ_1 and $?^\blacktriangledown T$ is the tail of χ_2, then the concatenation of χ_1 and χ_2 forms a
chain $\chi_3 \in X(\Delta)$. The tail of χ_3 is the tail of χ_1 and the head of χ_3 is the head of χ_2.

(4) If Δ contains two chains χ_1 and χ_2 and a lower link $(?^\blacktriangledown T, !^\blacktriangle R)$ such that $?^\blacktriangledown T$ is the
head of χ_1 and $!^\blacktriangle R$ is the tail of χ_2, then the concatenation of χ_1 and χ_2 forms a
chain $\chi_3 \in X(\Delta)$. The tail of χ_3 is the tail of χ_1 and the head of χ_3 is the head of χ_2.

(5) There are no other chains in $X(\Delta)$.

4.3.13 **Definition** The *length* of a chain χ is the number of !-chains and ?-chains it is
composed of.
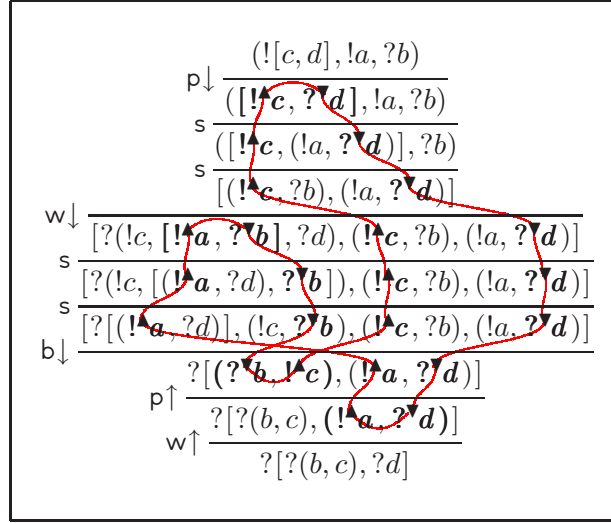
Figure 4.9: A cycle χ with $n(\chi) = 2$

4.3.14   Example   Here are two examples of chains in derivations (for TEXnical reasons the redices of the rule applications are no longer marked):

and
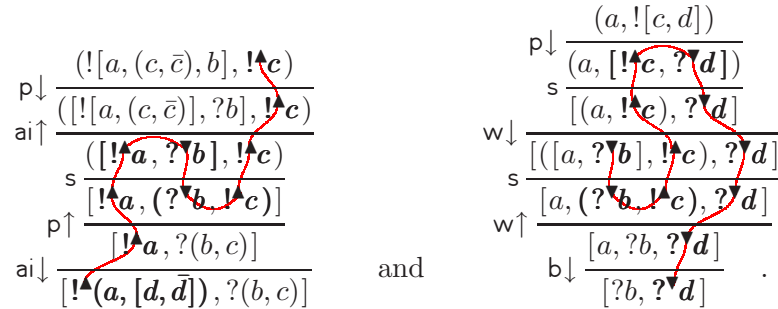
In the first chain, the tail is $!^{\blacktriangle}(a, [d, \bar{d}])$ and the head is $!^{\blacktriangle}c$. In the second example the tail is $?^{\blacktriangledown}b$ and the head is $?^{\blacktriangledown}d$. Both have length $l = 3$.

4.3.15   Definition   Let Δ be a derivation. A chain $\chi \in X(\Delta)$ is called a *cycle* if Δ contains an upper link $[!^{\blacktriangle}R, ?^{\blacktriangledown}T]$ such that $!^{\blacktriangle}R$ is the head of χ and $?^{\blacktriangledown}T$ is the tail of χ, or Δ contains a lower link $(?^{\blacktriangledown}T, !^{\blacktriangle}R)$ such that $?^{\blacktriangledown}T$ is the head of χ and $!^{\blacktriangle}R$ its tail.

In other words, a cycle can be seen as a chain without head or tail. Figure 4.9 shows an example for a cycle. Observe that for every cycle χ there is a number $n = n(\chi) \geqslant 1$ such that χ consists of n !-chains, n ?-chains, n upper links and n lower links. I will call this $n(\chi)$ the *characteristic number* of χ. For the example in Figure 4.9, we have $n = 2$.

4.3.16   Definition   A cycle χ is called a *promotion cycle* if every upper link of χ is redex of a p↓-rule (called *link promotion*) and every lower link of χ is contractum of a p↑-rule (called *link copromotion*).

The example in Figure 4.9 is not a promotion cycle because the upper link $[!^{\blacktriangle}a, ?^{\blacktriangledown}b]$ is not redex of a p↓-rule and the lower link $(!^{\blacktriangle}a, ?^{\blacktriangledown}d)$ is not contractum of a p↑-rule.
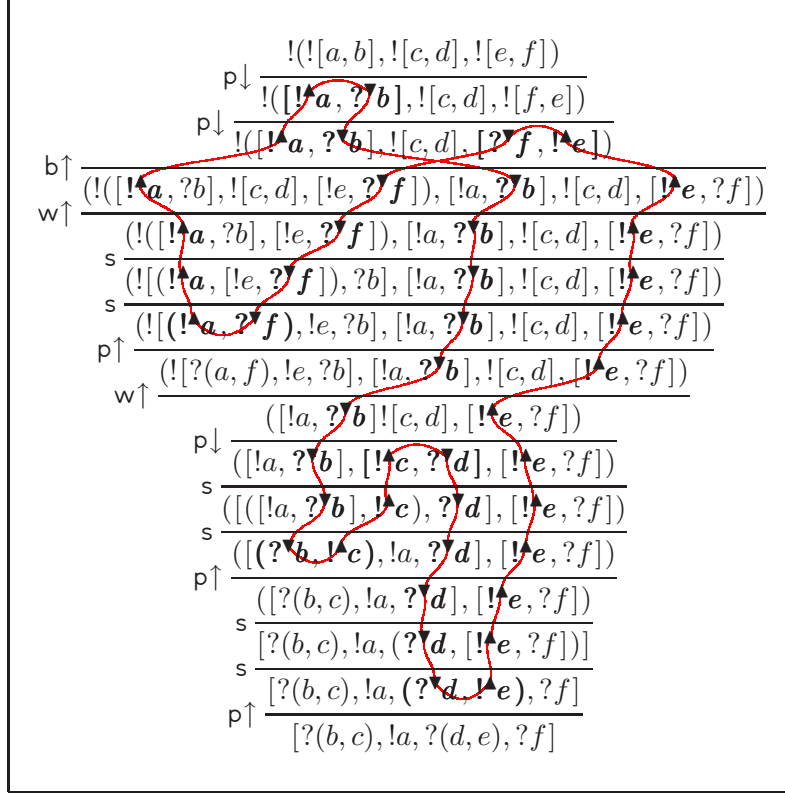
Figure 4.10: A promotion cycle χ with $n(\chi) = 3$

Figure 4.10 shows an example for a promotion cycle. Observe that it is not necessarily the case that all upper links are above all lower links in the derivation.

4.3.17 Definition Let χ be a cycle inside a derivation Δ, and let all !-links and ?-links of χ be marked with $!^{\blacktriangle}$ or $?^{\blacktriangledown}$, respectively. Then, χ is called *forked* if one of the following holds:

(i) There is an instance of $\;\mathsf{b}{\downarrow}\;\dfrac{S[?T,T]}{S\{?T\}}\;$ inside Δ, such that both substructures $?T$ and T of the contractum contain at least one substructure marked by $!^{\blacktriangle}$ or $?^{\blacktriangledown}$.

(ii) There is an instance of $\;\mathsf{b}{\uparrow}\;\dfrac{S\{!R\}}{S(!R,R)}\;$ inside Δ, such that both substructures $?R$ and R of the redex contain at least one substructure marked by $!^{\blacktriangle}$ or $?^{\blacktriangledown}$.

A cycle is called *nonforked* if it is not forked.

Both examples for cycles, that I have shown, are forked cycles. I did not show an example of a nonforked cycle because nonforked cycles cannot exist. The purpose of this section is to prove this fact.

4.3.18 Definition If a context is generated by the syntax

$$S ::= \{\ \} \mid [R,S] \mid (R,S) \quad ,$$

i.e. the hole does not occur inside an !- or ?-structure, then it is called a *basic context*.

4.3.19   Example   The contexts $[a, b, (\bar{a}, [c, d, \bar{b}, \{\ \}, a], ?c)]$ and $([!(b, ?a), \{\ \}], b)$ are basic, whereas $([!(\{\ \}, ?a), ?(\bar{a}, c)], b)$ is not basic.

4.3.20   Lemma   *Let $S\{\ \}$ be a basic context and R and T be structures. Then there is a derivation*

$$S[R, T]$$
$$\Delta \| \{s\}$$
$$[S\{R\}, T] \qquad .$$

Proof: By structural induction on $S\{\ \}$.

- $S = \{\ \}$. This case is trivial because $S[R, T] = [R, T] = [S\{R\}, T]$.

- $S = [S', S''\{\ \}]$ for some structure S' and context $S''\{\ \}$. Then by induction hypothesis we have

$$[S', S''[R, T]]$$
$$\Delta \| \{s\}$$
$$[S', S''\{R\}, T] \qquad .$$

- $S = (S', S''\{\ \})$ for some structure S' and context $S''\{\ \}$. Then let Δ be

$$(S', S''[R, T])$$
$$\Delta' \| \{s\}$$
$$(S', [S''\{R\}, T])$$
$$\mathsf{s} \, \frac{\phantom{(S', [S''\{R\}, T])}}{[(S', S''\{R\}), T]} \qquad ,$$

  where Δ' exists by induction hypothesis. $\square$

4.3.21   Definition   A cycle χ is called *pure* if

(i) for each !-chain and each ?-chain that is contained in χ, head and tail are equal, and

(ii) all upper links occur in the same structure and all lower links occur in the same structure.

For example, the two cycles in Figures 4.9 and 4.10 are not pure. Although in both cases condition (i) is fulfilled, condition (ii) is not. Figure 4.11 shows an example for a pure cycle.

If a derivation

$$P$$
$$\Delta \| \mathsf{SELS}$$
$$Q$$

contains a pure cycle then there are structures $R_1, \ldots, R_n$ and $T_1, \ldots, T_n$ (for some $n \geqslant 1$) and two n-ary contexts $S\{\ \} \ldots \{\ \}$ and $S'\{\ \} \ldots \{\ \}$, such that Δ is of the shape

$$P$$
$$\Delta_1 \| \mathsf{SELS}$$
$$S[!R_1, ?T_1][!R_2, ?T_2] \ldots [!R_n, ?T_n]$$
$$\Delta_2 \| \mathsf{SELS}$$
$$S'(?T_1, !R_2)(?T_2, !R_3) \ldots (?T_n, !R_1)$$
$$\Delta_3 \| \mathsf{SELS}$$
$$Q$$

,

$$
\begin{array}{rl}
\mathsf{p}{\downarrow} & \dfrac{[\,?(!c,[\,!a,?b\,],?d),(!\,[\,c,d\,],!a,?b)\,]}{[\,?(!c,[\,!{\blacktriangle}a,?{\blacktriangledown}b\,],?d),([\,!{\blacktriangle}c,?{\blacktriangledown}d\,],!a,?b)\,]} \\[4pt]
\mathsf{s} & \dfrac{}{[\,?(!c,[\,!{\blacktriangle}a,?{\blacktriangledown}b\,],?d),([\,!{\blacktriangle}c,(!a,?{\blacktriangledown}d)\,],?b)\,]} \\[4pt]
\mathsf{s} & \dfrac{}{[\,?(!c,[\,!{\blacktriangle}a,?{\blacktriangledown}b\,],?d),(!{\blacktriangle}c,?b),(!a,?{\blacktriangledown}d)\,]} \\[4pt]
\mathsf{s} & \dfrac{}{[\,?(!c,[(!{\blacktriangle}a,?d),?{\blacktriangledown}b\,]),(!{\blacktriangle}c,?b),(!a,?{\blacktriangledown}d)\,]} \\[4pt]
\mathsf{b}{\downarrow} & \dfrac{}{[\,?[(?d,!{\blacktriangle}a),(?{\blacktriangledown}b,!c)\,],(!{\blacktriangle}c,?b),(!a,?{\blacktriangledown}d)\,]} \\[4pt]
\mathsf{p}{\uparrow} & \dfrac{?[(?{\blacktriangledown}b,!{\blacktriangle}c),(!{\blacktriangle}a,?{\blacktriangledown}d)\,]}{?[\,?(c,b),(!a,?d)\,]}
\end{array}
$$
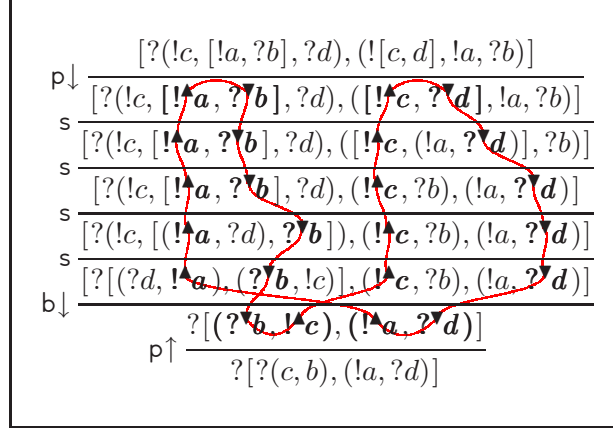
Figure 4.11: A pure cycle χ with $n(\chi)=2$

where inside Δ_1 and Δ_3 no structures are marked with $!^{\blacktriangle}$ or $?^{\blacktriangledown}$ because the structure

$$
S[\,!^{\blacktriangle}R_1\,,?^{\blacktriangledown}T_1\,][\,!^{\blacktriangle}R_2\,,?^{\blacktriangledown}T_2\,]\ldots[\,!^{\blacktriangle}R_n\,,?^{\blacktriangledown}T_n\,]
$$

contains all upper links and

$$
S'(\,?^{\blacktriangledown}T_1\,,!^{\blacktriangle}R_2\,)(\,?^{\blacktriangledown}T_2\,,!^{\blacktriangle}R_3\,)\ldots(\,?^{\blacktriangledown}T_n\,,!^{\blacktriangle}R_1\,)
$$

contains all lower links of the pure circle. In other words, the cycle is completely inside Δ_2.

It seems rather intuitive that every cycle can be transformed into a pure cycle by permuting rules up and down until all upper links are in one structure, all lower links are in one structure, and all links belonging to a !-subchain (or ?-subchain) are equal. The following proposition makes this precise for nonforked cycles. More precisely, every nonforked promotion cycle will be transformed into a nonforked pure cycle. The proof is rather long because of the technical details to be carried out. But the basic idea is quite simple: Whenever there is a situation

$$
\begin{array}{rl}
\pi & \dfrac{S\{!^{\blacktriangle}R\}}{S'\{!^{\blacktriangle}R'\}} \\[4pt]
\rho & \dfrac{}{S''\{!^{\blacktriangle}R''\}}
\end{array}\quad,
$$

where the three links are connected and ρ can be permuted over π by one of the cases (1), (2), or (3) of 4.2.3 to get

$$
\begin{array}{rl}
\rho & \dfrac{S\{!^{\blacktriangle}R\}}{V} \\[4pt]
\pi & \dfrac{}{S''\{!^{\blacktriangle}R''\}}
\end{array}\quad,
$$

for some structure V, then V must contain a link $!^{\blacktriangle}R'''$ that is connected to $!^{\blacktriangle}R$ and $!^{\blacktriangle}R''$. In other words, we have $V=S'''\{!^{\blacktriangle}R'''\}$ for some context $S'''\{\ \}$:

$$
\begin{array}{rl}
\rho & \dfrac{S\{!^{\blacktriangle}R\}}{S'''\{!^{\blacktriangle}R'''\}} \\[4pt]
\pi & \dfrac{}{S''\{!^{\blacktriangle}R''\}}
\end{array}\quad.
$$

$$
\begin{array}{ccc}
\begin{array}{c}
P \\
\| \\
\mathsf{p}\!\downarrow \dfrac{S_1\{![R_1,T_1]\}}{S_1[!^{\blacktriangle}R_1,\,?^{\blacktriangledown}T_1]} \\
\| \\
\mathsf{p}\!\downarrow \dfrac{S_2\{![R_2,T_2]\}}{S_2[!^{\blacktriangle}R_2,\,?^{\blacktriangledown}T_2]} \\
\| \\
\mathsf{p}\!\uparrow \dfrac{S_1'(!^{\blacktriangle}R_2',\,?^{\blacktriangledown}T_1')}{S_1'\{?(R_2',T_1')\}} \\
\| \\
\mathsf{p}\!\downarrow \dfrac{S_3\{![R_3,T_3]\}}{S_3[!^{\blacktriangle}R_3,\,?^{\blacktriangledown}T_3]} \\
\| \\
\mathsf{p}\!\uparrow \dfrac{S_3'(!^{\blacktriangle}R_1',\,?^{\blacktriangledown}T_3')}{S_3'\{?(R_1',T_3')\}} \\
\| \\
\mathsf{p}\!\uparrow \dfrac{S_2'(!^{\blacktriangle}R_3',\,?^{\blacktriangledown}T_2')}{S_2'\{?(R_3',T_2')\}} \\
\| \\
Q
\end{array}
&
\rightsquigarrow \quad
\begin{array}{c}
P \\
\| \\
\mathsf{p}\!\downarrow^{\pitchfork} \dfrac{S_1\{![R_1,T_1]\}}{S_1[!^{\blacktriangle}R_1,\,?^{\blacktriangledown}T_1]} \\
\| \\
\mathsf{p}\!\downarrow^{\pitchfork} \dfrac{S_2\{![R_2,T_2]\}}{S_2[!^{\blacktriangle}R_2,\,?^{\blacktriangledown}T_2]} \\
\| \\
\mathsf{p}\!\uparrow^{\mathbb{U}} \dfrac{S_1'(!^{\blacktriangle}R_2',\,?^{\blacktriangledown}T_1')}{S_1'\{?(R_2',T_1')\}} \\
\| \\
\mathsf{p}\!\downarrow^{\pitchfork} \dfrac{S_3\{![R_3,T_3]\}}{S_3[!^{\blacktriangle}R_3,\,?^{\blacktriangledown}T_3]} \\
\| \\
\mathsf{p}\!\uparrow^{\mathbb{U}} \dfrac{S_3'(!^{\blacktriangle}R_1',\,?^{\blacktriangledown}T_3')}{S_3'\{?(R_1',T_3')\}} \\
\| \\
\mathsf{p}\!\uparrow^{\mathbb{U}} \dfrac{S_2'(!^{\blacktriangle}R_3',\,?^{\blacktriangledown}T_2')}{S_2'\{?(R_3',T_2')\}} \\
\| \\
Q
\end{array}
&
\rightsquigarrow \quad
\begin{array}{c}
P \\
\| \\
\mathsf{p}\!\downarrow^{\pitchfork} \dfrac{S_1\{!^{\blacktriangle}[R_1,T_1]\}}{S_1[!^{\blacktriangle}R_1,\,?^{\blacktriangledown}T_1]} \\
\| \\
\mathsf{p}\!\downarrow^{\pitchfork} \dfrac{S_2\{!^{\blacktriangle}[R_2,T_2]\}}{S_2[!^{\blacktriangle}R_2,\,?^{\blacktriangledown}T_2]} \\
\| \\
\mathsf{p}\!\uparrow^{\mathbb{U}} \dfrac{S_1'(!^{\blacktriangle}R_2',\,?^{\blacktriangledown}T_1')}{S_1'\{?^{\blacktriangledown}(R_2',T_1')\}} \\
\| \\
\mathsf{p}\!\downarrow^{\pitchfork} \dfrac{S_3\{!^{\blacktriangle}[R_3,T_3]\}}{S_3[!^{\blacktriangle}R_3,\,?^{\blacktriangledown}T_3]} \\
\| \\
\mathsf{p}\!\uparrow^{\mathbb{U}} \dfrac{S_3'(!^{\blacktriangle}R_1',\,?^{\blacktriangledown}T_3')}{S_3'\{?^{\blacktriangledown}(R_1',T_3')\}} \\
\| \\
\mathsf{p}\!\uparrow^{\mathbb{U}} \dfrac{S_2'(!^{\blacktriangle}R_3',\,?^{\blacktriangledown}T_2')}{S_2'\{?^{\blacktriangledown}(R_3',T_2')\}} \\
\| \\
Q
\end{array}
\end{array}
$$

Figure 4.12: Example (with $n(\chi) = 3$) for the marking inside Δ

This means that a cycle cannot be destroyed by simple rule permutations.

4.3.22 Proposition *If there is a derivation* $\begin{array}{c}P\\\Delta\|\mathsf{SELS}\\Q\end{array}$ *that contains a nonforked promotion cycle, then there is a derivation* $\begin{array}{c}\tilde{P}\\\tilde{\Delta}\|\{\mathsf{ai}\downarrow,\mathsf{ai}\uparrow,\mathsf{s}\}\\\tilde{Q}\end{array}$ *that contains a pure cycle.*

Proof: Let χ be the nonforked promotion cycle inside Δ and let all !-links and ?-links of χ be marked with $!^{\blacktriangle}$ and $?^{\blacktriangledown}$, respectively (see Figure 4.12, first derivation). Further, let all instances of a link promotion (Definition 4.3.15) and all instances of a link copromotion be marked as $\mathsf{p}\!\downarrow^{\pitchfork}$ and $\mathsf{p}\!\uparrow^{\mathbb{U}}$, respectively (see Figure 4.12, second derivation). Now I will stepwise construct $\tilde{\Delta}$ from Δ by adding some further markings and by permuting, adding and removing rules until the cycle is pure. Observe that the transformations will not destroy the cycle but might change premise and conclusion of the derivation.

I. Let n be the characteristic number of χ. For each of the n marked instances of

$$
\mathsf{p}\!\downarrow^{\pitchfork} \frac{S\{![R_i,T_i]\}}{S[!^{\blacktriangle}R_i,\,?^{\blacktriangledown}T_i]}
$$

proceed as follows: Mark the contractum $![R_i,T_i]$ as $!^{\blacktriangle}[R_i,T_i]$ and continue the marking for all !-links of the (maximal) !-chain that has $!^{\blacktriangle}[R_i,T_i]$ as tail. There is

always a unique choice how to continue the marking (see Definition 4.3.5), except for one case: the marking reaches a

$$\mathsf{b}{\downarrow}\,\frac{S[?U,U]}{S\{?U\}}$$

and the last marked $!^{\blacktriangle}$-structure is inside the redex $?U$. Then there are two possibilities: either continue inside $?U$ (case (4.v) of Definition 4.3.5) or continue inside U (case (4.vi) of Definition 4.3.5). Choose that side that already contains a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure. Since the cycle χ is nonforked, it cannot happen that both sides already contain a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure. If there is no marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure inside the contractum $[?U,U]$ of the $\mathsf{b}{\downarrow}$, then choose either one.

Proceed dually for all marked

$$\mathsf{p}{\uparrow}^{\cupplus}\,\frac{S(!^{\blacktriangle}R'_i,\,?^{\blacktriangledown}T'_i)}{S\{?(R'_i,T'_i)\}}\quad,$$

i.e. mark the redex $?(R'_i,T'_i)$ as $?^{\blacktriangledown}(R'_i,T'_i)$ and mark also all links of the ?-chain that has $?^{\blacktriangledown}(R'_i,T'_i)$ as tail (see Figure 4.12, third derivation).

II. Now consider all !-substructures and all ?-substructures that occur somewhere in the derivation Δ. They can be divided into three groups:

(a) those which are marked with $!^{\blacktriangle}$ or $?^{\blacktriangledown}$,

(b) those which are a substructure of a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure, and

(c) all the others.

In this step replace all substructures $!R$ and $?T$ that fall in group (c) by R and T respectively, i.e. remove the exponential. This rather drastic step will, of course, yield a nonvalid derivation because correct rule applications might become incorrect. Observe that all instance of $\mathsf{ai}{\downarrow}$, $\mathsf{ai}{\uparrow}$ and s inside Δ do not suffer from this step, i.e. they remain valid. Let us now inspect more closely what could happen to the instances of $\mathsf{p}{\downarrow}$, $\mathsf{p}{\uparrow}$, $\mathsf{w}{\downarrow}$, $\mathsf{w}{\uparrow}$, $\mathsf{b}{\downarrow}$ and $\mathsf{b}{\uparrow}$.

- Consider any instance of $\mathsf{p}{\downarrow}\,\dfrac{S\{![R,T]\}}{S[!R,?T]}$ in Δ. Then the following cases exhaust all possibilities.

  (i) There are two contexts $S'\{\ \}$ and $S''\{\ \}$ such that $S\{\ \} = S'\{!^{\blacktriangle}S''\{\ \}\}$ or $S\{\ \} = S'\{?^{\blacktriangledown}S''\{\ \}\}$. Then redex and contractum of the $\mathsf{p}{\downarrow}$ remain unchanged and the rule remains valid.

  (ii) The $\mathsf{p}{\downarrow}$ is marked as $\mathsf{p}{\downarrow}^{\cap}\,\dfrac{S\{!^{\blacktriangle}[R,T]\}}{S[!^{\blacktriangle}R,\,?^{\blacktriangledown}T]}$. Then it also remains unchanged.

  (iii) The $\mathsf{p}{\downarrow}$ is marked as $\mathsf{p}{\downarrow}\,\dfrac{S\{!^{\blacktriangle}[R,T]\}}{S[!^{\blacktriangle}R,\,?T]}$. Then the exponentials inside $?T$ are removed, and we obtain an instance

  $$\hat{\mathsf{p}}{\downarrow}\,\frac{S\{!^{\blacktriangle}[R,T]\}}{S[!^{\blacktriangle}R,\,T']}\quad.$$

Observe that T' and T might be different because inside T all exponentials remain as they are inside $!^{\blacktriangle}[R, T]$, whereas inside T' some or all exponentials are removed.

(iv) The $\mathsf{p}{\downarrow}$ is not marked and does not occur inside a marked structure. Then it becomes

$$\mathsf{p}{\downarrow}' \,\frac{S[R', T']}{S[R', T']} \quad,$$

where R' and T' are obtained from R and T, respectively, by removing some (or all) exponentials.

There are no other cases because there are no other markings possible. Observe that the rule $\mathsf{p}{\downarrow}'$ in case (iv) is vacuous and can therefore be removed in the whole derivation. Hence, it only remains to remove all instances of the rule $\hat{\mathsf{p}}{\downarrow}$ (case (iii)). This will be done in Step V.

- The rule $\mathsf{p}{\uparrow} \,\dfrac{S(!R, ?T)}{S\{?(R, T)\}}$ is dual to the rule $\mathsf{p}{\downarrow}$. Hence the only problem lies in the new rule

$$\hat{\mathsf{p}}{\uparrow} \,\frac{S(R', ?^{\blacktriangledown}\boldsymbol{T})}{S\{?^{\blacktriangledown}(\boldsymbol{R}, \boldsymbol{T})\}} \quad,$$

where R' is obtained from R by removing the exponentials. This rule will also be removed in Step V.

- For the rule $\mathsf{w}{\downarrow} \,\dfrac{S\{\bot\}}{S\{?T\}}$ only two cases are possible.

(i) There are two contexts $S'\{\ \}$ and $S''\{\ \}$ such that $S\{\ \} = S'\{!^{\blacktriangle}S''\{\ \}\}$ or $S\{\ \} = S'\{?^{\blacktriangledown}S''\{\ \}\}$. Then redex and contractum of the $\mathsf{w}{\downarrow}$ remain unchanged and the rule remains valid.

(ii) The rule becomes

$$\hat{\mathsf{w}}{\downarrow} \,\frac{S\{\bot\}}{S\{T'\}} \quad,$$

where T' is obtained from T by removing some or all exponentials.

Observe that the marking $\mathsf{w}{\downarrow} \,\dfrac{S\{\bot\}}{S\{?^{\blacktriangledown}\boldsymbol{T}\}}$ is impossible.

- For the rule $\mathsf{w}{\uparrow} \,\dfrac{S\{!R\}}{S\{1\}}$ the situation is dual and we obtain

$$\hat{\mathsf{w}}{\uparrow} \,\frac{S\{R'\}}{S\{1\}} \quad,$$

where R' is obtained from R by removing the exponentials. The two rules $\hat{\mathsf{w}}{\downarrow}$ and $\hat{\mathsf{w}}{\uparrow}$ will be removed in Step IV.

- For $\mathsf{b}{\downarrow}\ \dfrac{S[?T,T]}{S\{?T\}}$ the situation is a little more intricate. The possible cases are:

  (i) There are two contexts $S'\{\ \}$ and $S''\{\ \}$ such that $S\{\ \} = S'\{!^{\blacktriangle}S''\{\ \}\}$ or $S\{\ \} = S'\{?^{\blacktriangledown}S''\{\ \}\}$. Then redex and contractum of the $\mathsf{b}{\downarrow}$ remain unchanged and the rule remains valid.

  (ii) The rule is marked as $\mathsf{b}{\downarrow}\ \dfrac{S[?^{\blacktriangledown}T,T]}{S\{?^{\blacktriangledown}T\}}$. Then it becomes

  $$\mathsf{b}{\downarrow}'\ \frac{S[?^{\blacktriangledown}T,T']}{S\{?^{\blacktriangledown}T\}}\quad,$$

  where T' is obtained from T by removing (some or all) exponentials.

  (iii) Neither redex nor contractum of the rule contain any marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure, nor are they contained in a marked structure. Then the rule becomes

  $$\mathsf{b}{\downarrow}''\ \frac{S[T',T']}{S\{T'\}}\quad,$$

  where T' is obtained from T by removing the exponentials.

  (iv) There are marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structures inside the structure T in the redex. Then all those markings reoccur in one of the two substructures T in the contractum whereas the other T does not contain any marking (because the cycle χ is nonforked). Hence the rule becomes

  $$\mathsf{b}{\downarrow}'''\ \frac{S[T'',T']}{S\{T''\}}\quad,$$

  where in T' all exponentials are removed and in T'' some exponentials are removed and some remain.

  Observe that all instances of $\mathsf{b}{\downarrow}'$, $\mathsf{b}{\downarrow}''$ and $\mathsf{b}{\downarrow}'''$ are instances of

  $$\hat{\mathsf{b}}{\downarrow}\ \frac{S[T,T']}{S\{T\}}\quad,$$

  where $S\{\ \}$ is a basic context, and T and T' are arbitrary structures.

- Dually, for $\mathsf{b}{\uparrow}\ \dfrac{S\{!R\}}{S(!R,R)}$, we obtain

  $$\hat{\mathsf{b}}{\uparrow}\ \frac{S\{R\}}{S(R,R')}\quad,$$

  where $S\{\ \}$ is a basic context. The new instances of $\hat{\mathsf{b}}{\downarrow}$ and $\hat{\mathsf{b}}{\uparrow}$ will be removed in the next step.

Let me summarise what is achieved after this step. The original derivation

$$
\begin{array}{c}
P \\
\Delta \| \mathsf{SELS} \\
Q
\end{array}
$$

has been transformed into

$$
\begin{array}{c}
P \\
\hat{\Delta} \| \mathsf{SELS} \cup \{\mathsf{p}{\downarrow}^{\text{\tiny \textcircled{m}}}, \mathsf{p}{\uparrow}^{\text{\tiny \textcircled{w}}}, \hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}, \hat{\mathsf{w}}{\downarrow}, \hat{\mathsf{w}}{\uparrow}, \hat{\mathsf{b}}{\downarrow}, \hat{\mathsf{b}}{\uparrow}\} \\
Q
\end{array} \quad ,
$$

where the cycle together with the extentions of its !- and ?-chains is marked. In the following steps, I will remove all rules (including $\hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}, \hat{\mathsf{w}}{\downarrow}, \hat{\mathsf{w}}{\uparrow}, \hat{\mathsf{b}}{\downarrow}, \hat{\mathsf{b}}{\uparrow}$) that prevent the cycle from being pure.

III. First, I will remove all instances of $\hat{\mathsf{b}}{\downarrow}$ and $\hat{\mathsf{b}}{\uparrow}$. Consider the bottommost occurrence of $\hat{\mathsf{b}}{\downarrow}$ inside $\hat{\Delta}$. Replace

$$
\hat{\Delta} \quad = \quad \hat{\mathsf{b}}{\downarrow}\,\cfrac{\begin{array}{c}P \\ \Delta_1 \| \\ S[T,T']\end{array}}{\begin{array}{c}S\{T\} \\ \Delta_2 \| \\ Q\end{array}} \qquad \text{by} \qquad
\begin{array}{c}
P \\
\Delta_1 \| \\
S[T,T'] \\
\Delta_3 \| \{\mathsf{s}\} \\
{[S\{T\},T']} \\
\Delta_2 \| \\
{[Q,T']}
\end{array} \quad ,
$$

where Δ_2 does not contain any $\hat{\mathsf{b}}{\downarrow}$ and Δ_3 exists by Lemma 4.3.20. Repeat this until there are no more $\hat{\mathsf{b}}{\downarrow}$ in the derivation. Then proceed dually to remove all $\hat{\mathsf{b}}{\uparrow}$, i.e. start with the topmost $\hat{\mathsf{b}}{\uparrow}$. This gives us a derivation

$$
\begin{array}{c}
P' \\
\hat{\Delta}' \| \mathsf{SELS} \cup \{\mathsf{p}{\downarrow}^{\text{\tiny \textcircled{m}}}, \mathsf{p}{\uparrow}^{\text{\tiny \textcircled{w}}}, \hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}, \hat{\mathsf{w}}{\downarrow}, \hat{\mathsf{w}}{\uparrow}\} \\
Q'
\end{array} \quad .
$$

Observe that premise and conclusion of the derivation have changed now, but the cycle is still present.

IV. In this step, I will remove all instances of $\hat{\mathsf{w}}{\downarrow}$ and $\hat{\mathsf{w}}{\uparrow}$. For this, observe that the proofs of Lemmata 4.2.4 and 4.2.9 (a) do also work for $\hat{\mathsf{w}}{\downarrow}$. Further observe that it can never happen that the contractum $\bot$ of $\hat{\mathsf{w}}{\downarrow}\,\cfrac{S\{\bot\}}{S\{T\}}$ is inside an active structure of the redex of $\mathsf{p}{\uparrow}, \hat{\mathsf{p}}{\uparrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}$ or $\mathsf{w}{\downarrow}$ because then the redex T would be inside a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure, which is impossible by the construction of $\hat{\mathsf{w}}{\downarrow}$ in Step II. Hence, the rule $\hat{\mathsf{w}}{\downarrow}$ permutes (by s) over all other rules in the derivation $\hat{\Delta}'$. Dually, $\hat{\mathsf{w}}{\uparrow}$ permutes

under all other rules in $\hat{\Delta}'$ (by s). This means that $\hat{\Delta}'$ can easily be transformed into

$$
\begin{array}{c}
P' \\
\Delta_1' \Big\| \{\hat{\mathsf{w}}{\downarrow}\} \\
P'' \\
\hat{\Delta}'' \Big\| \mathsf{SELS} \cup \{\mathsf{p}{\downarrow}^{\varowdot},\mathsf{p}{\uparrow}^{\uplus},\hat{\mathsf{p}}{\downarrow},\hat{\mathsf{p}}{\uparrow}\} \\
Q'' \\
\Delta_2' \Big\| \{\hat{\mathsf{w}}{\uparrow}\} \\
Q'
\end{array}
$$

by permuting stepwise all $\hat{\mathsf{w}}{\downarrow}$ up and all $\hat{\mathsf{w}}{\uparrow}$ down. Let us now consider only

$$
\begin{array}{c}
P'' \\
\hat{\Delta}'' \Big\| \mathsf{SELS} \cup \{\mathsf{p}{\downarrow}^{\varowdot},\mathsf{p}{\uparrow}^{\uplus},\hat{\mathsf{p}}{\downarrow},\hat{\mathsf{p}}{\uparrow}\} \\
Q''
\end{array} \qquad ,
$$

in which the cycle χ is still present.

V. Inside $\hat{\Delta}''$ mark all rules ρ whose redex is inside a marked $!^{\blacktriangle}$-structure as $\rho^{\triangle}$. Additionally, mark all instances of $\hat{\mathsf{p}}{\downarrow}$ as $\hat{\mathsf{p}}{\downarrow}^{\triangle}$. Dually, mark all rules $\hat{\mathsf{p}}{\uparrow}$ as well as all rules ρ whose contractum is inside a marked $?^{\blacktriangledown}$-structure as $\rho^{\triangledown}$. Now mark all remaining, i.e. not yet marked, rules ρ as ρ°. This means, we now have a derivation

$$
\begin{array}{c}
P'' \\
\hat{\Delta}'' \Big\| \{\mathsf{p}{\downarrow}^{\varowdot},\mathsf{p}{\uparrow}^{\uplus},\rho^{\triangle},\rho^{\triangledown},\rho^{\circ}\} \\
Q''
\end{array} \qquad ,
$$

which will in this step be decomposed into

$$
\begin{array}{c}
P'' \\
\hat{\Delta}_1'' \Big\| \{\rho^{\triangle}\} \\
P''' \\
\hat{\Delta}_2'' \Big\| \{\mathsf{p}{\downarrow}^{\varowdot}\} \\
\tilde{P} \\
\tilde{\Delta} \Big\| \{\rho^{\circ}\} \\
\tilde{Q} \\
\hat{\Delta}_3'' \Big\| \{\mathsf{p}{\uparrow}^{\uplus}\} \\
Q''' \\
\hat{\Delta}_4'' \Big\| \{\rho^{\triangledown}\} \\
Q''
\end{array}
$$

only by permutation of rules. In order to obtain this decomposition, we need to show that

(a) all rules marked as $\rho^{\triangle}$ permute over all other rules,

(b) all rules marked as $\rho^{\triangledown}$ permute under all other rules,

(c) all rules $\mathsf{p}{\downarrow}^{\varowdot}$ permute over all rules marked as ρ° or $\mathsf{p}{\uparrow}^{\uplus}$, and

(d) all rules $\mathsf{p}{\uparrow}^{\uplus}$ permute under all rules marked as ρ° or $\mathsf{p}{\downarrow}^{\varowdot}$.

All four statements are proved according to the scheme discussed in Section 4.2.

(a) Consider

$$\pi \frac{\begin{array}{c} Q \\ \hline S\{W\} \end{array}}{S\{Z\}} \rho^{\vartriangle} \quad ,$$

where π is not marked as $\pi^{\vartriangle}$ and nontrivial. According to 4.2.3 the cases to consider are:

(4) The redex of π is inside an active structure of the contractum W of $\rho^{\vartriangle}$. This is impossible because

   (i) if the redex of $\rho^{\vartriangle}$ is inside a $!^{\blacktriangle}$-structure, then the contractum of $\rho^{\vartriangle}$ is also inside a $!^{\blacktriangle}$-structure, and hence, the redex of π is inside a $!^{\blacktriangle}$-structure, and therefore π is $\pi^{\vartriangle}$;

   (ii) if $\rho^{\vartriangle} = \hat{\mathsf{p}}{\downarrow}^{\vartriangle}$, then the redex of π is also inside a $!^{\blacktriangle}$-structure, and therefore π is $\pi^{\vartriangle}$.

(5) The contractum W of $\rho^{\vartriangle}$ is inside an active structure of the redex of π but not inside a passive one. There are the following subcases:

   (i) The redex of $\rho^{\vartriangle}$ is inside a $!^{\blacktriangle}$-structure. This is impossible because then the contractum of $\rho^{\vartriangle}$ is also inside a $!^{\blacktriangle}$-structure. Since it is also inside an active structure of the redex of π, we have that either this active structure is a $!^{\blacktriangle}$-structure and therefore $\pi = \hat{\mathsf{p}}{\downarrow}^{\vartriangle}$, or the whole redex of π is inside a $!^{\blacktriangle}$-structure and therefore π must be marked as $\pi^{\vartriangle}$.

   (ii) $\rho^{\vartriangle} = \hat{\mathsf{p}}{\downarrow}^{\vartriangle}$ and $\pi = \hat{\mathsf{p}}{\downarrow}$. This is impossible because then π is marked as $\pi^{\vartriangle}$.

   (iii) $\rho^{\vartriangle} = \hat{\mathsf{p}}{\downarrow}^{\vartriangle}$ and $\pi = \mathsf{p}{\downarrow}$. Then $\pi = \mathsf{p}{\downarrow}^{\between}$ because there are no other $\mathsf{p}{\downarrow}$ that have a marked $!^{\blacktriangle}$-structure in the redex, and we can replace

$$\begin{array}{c} \mathsf{p}{\downarrow}^{\between} \dfrac{\begin{array}{c} S'\{!^{\blacktriangle}[R,T_1,T_2]\} \\ \hline S'[!^{\blacktriangle}[R,T_1],?^{\blacktriangledown}T_2] \end{array}}{\hat{\mathsf{p}}{\downarrow}^{\vartriangle} \, S'[!^{\blacktriangle}R,T_1',?^{\blacktriangledown}T_2]} \end{array} \quad \text{by} \quad \begin{array}{c} \hat{\mathsf{p}}{\downarrow}^{\vartriangle} \dfrac{\begin{array}{c} S'\{!^{\blacktriangle}[R,T_1,T_2]\} \\ \hline S'[!^{\blacktriangle}[R,T_2],T_1'] \end{array}}{\mathsf{p}{\downarrow}^{\between} \, S'[[!^{\blacktriangle}R,?^{\blacktriangledown}T_2],T_1']} \end{array} \quad .$$

(6) The contractum W of ρ° and the redex of π overlap. This is impossible.

(b) Dual to (a).

(c) Consider

$$\mathsf{p}{\downarrow}^{\between} \frac{\begin{array}{c} Q \\ \hline S\{!^{\blacktriangle}[R,T]\} \end{array}}{S[!^{\blacktriangle}R,?^{\blacktriangledown}T]} \pi \quad ,$$

where $\pi \in \{\rho^{\circ}, \mathsf{p}{\uparrow}^{\mathbb{U}}\}$ is nontrivial.

(4) The redex of π is inside an active structure of the contractum of $\mathsf{p}{\downarrow}^{\between}$. This is impossible because then the redex of π is inside a $!^{\blacktriangle}$-structure, and therefore π is $\pi^{\vartriangle}$.

(5) The contractum $!^{\blacktriangle}[R,T]$ of $\mathsf{p}{\downarrow}^{\tiny\textcircled{m}}$ is inside an active structure of the redex of π but not inside a passive one. This is impossible because then π were $\hat{\mathsf{p}}{\downarrow}^{\vartriangle}$ or $\mathsf{p}{\downarrow}^{\tiny\textcircled{m}}$.

(6) The contractum of $\mathsf{p}{\downarrow}^{\tiny\textcircled{m}}$ and the redex of π overlap. As before, this is impossible.

(d) Dual to (c).

Now it only remains to show that the subderivation

$$\tilde{P}$$
$$\tilde{\Delta}\|\{\rho^\circ\}$$
$$\tilde{Q}$$

obtained in the last step has indeed the desired properties (i.e. contains a pure cycle and consists only of the rules $\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}$ and s). Observe that all rules $\rho \in \{\mathsf{p}{\downarrow}, \mathsf{p}{\uparrow}, \mathsf{w}{\downarrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}\}$ in Δ

- either have been transformed into $\hat{\rho}$ in Step II and then been removed in the Steps III, IV and V,

- or remained unchanged in Step II (because they occurred inside a marked $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure) and have then been marked as $\rho^{\vartriangle}$ or $\rho^{\triangledown}$ and removed in Step V.

This means that only the rules $\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}$ and s are left inside $\tilde{\Delta}$. Now consider the premise $\tilde{P}$ of $\tilde{\Delta}$. Since it is also the conclusion of $\hat{\Delta}_2''$ which consists only of $\mathsf{p}{\downarrow}^{\tiny\textcircled{m}}$, it is of the shape

$$S[!^{\blacktriangle}R_1, ?^{\blacktriangledown}T_1][!^{\blacktriangle}R_2, ?^{\blacktriangledown}T_2]\ldots[!^{\blacktriangle}R_n, ?^{\blacktriangledown}T_n]$$

for some structures $R_1, \ldots, R_n, T_1, \ldots, T_n$ and some n-ary context $S\{\ \}\{\ \}\ldots\{\ \}$. Similarly, we have that

$$\tilde{Q} = S'(!^{\blacktriangle}R_1', ?^{\blacktriangledown}T_1')(!^{\blacktriangle}R_2', ?^{\blacktriangledown}T_2')\ldots(!^{\blacktriangle}R_n', ?^{\blacktriangledown}T_n')$$

for some structures $R_1', \ldots, R_n', T_1', \ldots, T_n'$ and some n-ary context $S'\{\ \}\{\ \}\ldots\{\ \}$. Since no transformation in Steps II to V destroyed the cycle, it must still be present in $\tilde{\Delta}$. Since $\tilde{\Delta}$ contains no rule that operates inside a $!^{\blacktriangle}$- or $?^{\blacktriangledown}$-structure, we have that $R_1' = R_2$, $R_2' = R_3$, $\ldots$, $R_n' = R_1$ and $T_1' = T_1$, $T_2' = T_2$, $\ldots$, $T_n' = T_n$. This means that $\tilde{\Delta}$ does indeed contain a pure cycle. $\qquad\square$

4.3.23 Definition Let S be a structure and R and T be substructures of S. Then the structures R and T are in *par-relation* in S if there are contexts $S'\{\ \}$, $S''\{\ \}$ and $S'''\{\ \}$ such that $S = S'[S''\{R\}, S'''\{T\}]$. Similarly, R and T are in *times-relation* in S if $S = S'(S''\{R\}, S'''\{T\})$ for some contexts $S'\{\ \}$, $S''\{\ \}$ and $S'''\{\ \}$.

4.3.24 Lemma *If there is a derivation* $\begin{array}{c}P\\\Delta\|\{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow},\mathsf{s}\}\\Q\end{array}$ *that contains a pure cycle* χ, *then there is a derivation*

$$\begin{array}{c}([!R_1, ?T_1], [!R_2, ?T_2], \ldots, [!R_n, ?T_n])\\\tilde{\Delta}\|\{\mathsf{s}\}\\{}[(!R_2, ?T_1), (!R_3, ?T_2), \ldots, (!R_1, ?T_n)]\end{array}\ ,$$

for some structures $R_1, \ldots, R_n, T_1, \ldots, T_n$, where n is the characteristic number of χ.

Proof: By Lemma 4.2.6 and Lemma 4.2.7, the derivation Δ can be decomposed into

$$
\begin{array}{c}
P \\
\Delta_1 \big\| \{\mathsf{ai}\downarrow\} \\
P' \\
\Delta_2 \big\| \{\mathsf{s}\} \\
Q' \\
\Delta_3 \big\| \{\mathsf{ai}\uparrow\} \\
Q
\end{array} \quad .
$$

As before, this transformation does not destroy the cycle. Hence, the pure cycle is contained in Δ_2. In other words, Δ_2 has a subderivation

$$
\begin{array}{c}
S[\,!^{\blacktriangle}R_1\,,\, ?^{\blacktriangledown}T_1\,][\,!^{\blacktriangle}R_2\,,\, ?^{\blacktriangledown}T_2\,]\ldots[\,!^{\blacktriangle}R_n\,,\, ?^{\blacktriangledown}T_n\,] \\
\Delta' \big\| \{\mathsf{s}\} \\
S'(\,!^{\blacktriangle}R_2\,,\, ?^{\blacktriangledown}T_1\,)(\,!^{\blacktriangle}R_3\,,\, ?^{\blacktriangledown}T_2\,)\ldots(\,!^{\blacktriangle}R_1\,,\, ?^{\blacktriangledown}T_n\,)
\end{array}
$$

for some structures $R_1, \ldots, R_n$ and $T_1, \ldots, T_n$, and for two n-ary contexts $S\{\ \}\ldots\{\ \}$ and $S'\{\ \}\ldots\{\ \}$. In the premise of Δ', for every $i = 1, \ldots, n$, the substructures $!^{\blacktriangle}R_i$ and $?^{\blacktriangledown}T_i$ are in par-relation. The switch rule is not able (and also no other rule in system SELS) to transform a par-relation into a times-relation while going down in a derivation. Hence, for every $i = 1, \ldots, n$, the substructures $!^{\blacktriangle}R_i$ and $?^{\blacktriangledown}T_i$ are also in par-relation in the conclusion of Δ'. This means that the context $S'\{\ \}\ldots\{\ \} = S'_0[S'_1\{\ \},\ldots,S'_n\{\ \}]$ for some contexts $S'_0\{\ \}, S'_1\{\ \}, \ldots, S'_n\{\ \}$. Dually, we have that $S\{\ \}\ldots\{\ \} = S_0(S_1\{\ \},\ldots,S_n\{\ \})$ for some contexts $S_0\{\ \}, S_1\{\ \}, \ldots, S_n\{\ \}$. Hence, the derivation Δ' has the shape

$$
\begin{array}{c}
S_0(S_1[\,!^{\blacktriangle}R_1\,,\, ?^{\blacktriangledown}T_1\,], S_2[\,!^{\blacktriangle}R_2\,,\, ?^{\blacktriangledown}T_2\,],\ldots,S_n[\,!^{\blacktriangle}R_n\,,\, ?^{\blacktriangledown}T_n\,]) \\
\Delta' \big\| \{\mathsf{s}\} \\
S'_0[S'_1(\,!^{\blacktriangle}R_2\,,\, ?^{\blacktriangledown}T_1\,), S'_2(\,!^{\blacktriangle}R_3\,,\, ?^{\blacktriangledown}T_2\,),\ldots,S'_n(\,!^{\blacktriangle}R_1\,,\, ?^{\blacktriangledown}T_n\,)]
\end{array} \quad .
$$

Observe that the two contexts $S_0(S_1\{\ \},\ldots,S_n\{\ \})$ and $S'_0[S'_1\{\ \},\ldots,S'_n\{\ \}]$ must contain the same atoms because Δ' contains no rules that could create or destroy any atoms. Hence, the derivation Δ' remains valid if those atoms are removed from the derivation, which gives us the derivation

$$
\begin{array}{c}
(\,[\,!^{\blacktriangle}R_1\,,\, ?^{\blacktriangledown}T_1\,], [\,!^{\blacktriangle}R_2\,,\, ?^{\blacktriangledown}T_2\,],\ldots, [\,!^{\blacktriangle}R_n\,,\, ?^{\blacktriangledown}T_n\,]) \\
\tilde{\Delta} \big\| \{\mathsf{s}\} \\
[\,(\,!^{\blacktriangle}R_2\,,\, ?^{\blacktriangledown}T_1\,), (\,!^{\blacktriangle}R_3\,,\, ?^{\blacktriangledown}T_2\,),\ldots, (\,!^{\blacktriangle}R_1\,,\, ?^{\blacktriangledown}T_n\,)]
\end{array} \quad . \qquad \square
$$

4.3.25   Lemma   *Let $n \geqslant 1$ and $R_1, \ldots, R_n, T_1, \ldots, T_n$ be any structures. Then there is no derivation*

$$
\begin{array}{c}
(\,[!R_1, ?T_1], [!R_2, ?T_2],\ldots, [!R_n, ?T_n]) \\
\Delta \big\| \{\mathsf{s}\} \\
[\,(!R_2, ?T_1), (!R_3, ?T_2),\ldots, (!R_1, ?T_n)]
\end{array} \quad .
$$

Proof: By induction on n.

Base case: Let $n = 1$. Then it is easy to see that there is no derivation

$$[!R_1, ?T_1]$$
$$\Delta \Big\|_{\{s\}}$$
$$(!R_1, ?T_1)$$

because a times-relation can never become a par-relation while going up in a derivation.

Inductive case: By way of contradiction suppose there is a derivation

$$([!R_1, ?T_1], [!R_2, ?T_2], \ldots, [!R_n, ?T_n])$$
$$\Delta \Big\|_{\{s\}}$$
$$[(!R_2, ?T_1), (!R_3, ?T_2), \ldots, (!R_1, ?T_n)]$$
.

Now consider the bottommost instance of

$$s \frac{S([\tilde{R}, \tilde{U}], \tilde{T})}{S[(\tilde{R}, \tilde{T}), \tilde{U}]}$$

in Δ. Without loss of generality we can assume that $\tilde{R} = !R_2$ and $\tilde{T} = ?T_1$. (The case where $\tilde{R} = ?T_1$ and $\tilde{T} = !R_2$ is similar.) For matching $\tilde{U}$, we have

$$\tilde{U} \quad = \quad [(!R_{k_1+1}, ?T_{k_1}), (!R_{k_2+1}, ?T_{k_2}), \ldots, (!R_{k_m+1}, ?T_{k_m})] \quad ,$$

for some m and $k_1, \ldots, k_m$. Hence, we get:

$$([!R_1, ?T_1], [!R_2, ?T_2], \ldots [!R_n, ?T_n])$$
$$\Delta' \Big\|_{\{s\}}$$
$$s \frac{[(([!R_2, (!R_{k_1+1}, ?T_{k_1}), \ldots, (!R_{k_m+1}, ?T_{k_m})], ?T_1), \ldots, (!R_1, ?T_n)]}{[(!R_2, ?T_1), (!R_3, ?T_2), (!R_4, ?T_3), \ldots (!R_1, ?T_n)]} \quad .$$

Inside Δ' the structures $!R_1, \ldots, !R_n, ?T_1, \ldots, ?T_n$ occur only inside passive structures of instances of s. Therefore, if we replace inside Δ' any structure $!R_j$ or $?T_j$ by some other structure V, then the derivation Δ' must remain valid. Without loss of generality, assume that $k_1 < k_2 < \ldots < k_m$ and for every $i = 2, \ldots, k_m$ replace inside Δ' the structures $!R_i$ by $\bot$ and the structures $?T_i$ by 1, which yields a derivation

$$= \frac{([!R_1, ?T_1], [!R_{k_m+1}, ?T_{k_m+1}], \ldots, [!R_n, ?T_n])}{([!R_1, ?T_1], [\bot, 1], \ldots, [\bot, 1], [!R_{k_m+1}, ?T_{k_m+1}], \ldots, [!R_n, ?T_n])}$$
$$\Delta'' \Big\|_{\{s\}}$$
$$= \frac{[(([\bot, (\bot, 1), \ldots, (\bot, 1), (!R_{k_m+1}, 1)], ?T_1), \ldots, (!R_1, ?T_n)]}{[(!R_{k_m+1}, ?T_1), (!R_{k_m+2}, ?T_{k_m+1}), \ldots, (!R_1, ?T_n)]} \quad ,$$

which cannot exist by induction hypothesis. $\qquad \square$

As mentioned before, the property of the switch rule described by Lemma 4.3.25 seems to be at the heart of the decomposition theorems. I will come back to this property of the switch in Section 6.2.

4.3.26 Theorem *There exists no derivation containing a nonforked promotion cycle.*

Proof: By way of contradiction, suppose there is a derivation Δ containing a nonforked promotion cycle χ. By Proposition 4.3.22, there is a derivation containing a pure cycle χ'. Let n be its characteristic number. By Lemma 4.3.24 there is a derivation

$$
\begin{array}{c}
([!R_1, ?T_1], [!R_2, ?T_2], \ldots, [!R_n, ?T_n]) \\
\Delta \| \{\mathsf{s}\} \\
[(!R_2, ?T_1), (!R_3, ?T_2), \ldots, (!R_1, ?T_n)]
\end{array} \quad .
$$

for some structures $R_1, \ldots, R_n, T_1, \ldots, T_n$. This is a contradiction to Lemma 4.3.25. $\square$

4.3.27 Corollary *There exists no derivation containing a nonforked cycle.*

Proof: Any (nonforked) cycle can easily be transformed into a (nonforked) promotion cycle by adding instances of $\mathsf{p}\!\downarrow$ and $\mathsf{p}\!\uparrow$. $\square$

4.3.2 Separation of Absorption and Weakening

The process of permuting up all instances of $\mathsf{b}\!\uparrow$ in a given derivation $\begin{array}{c}T\\\Delta\|\mathsf{SELS}\\R\end{array}$ is realised by the procedure $\mathsf{b}\!\uparrow\mathsf{up}$:

4.3.28 Algorithm $\mathsf{b}\!\uparrow\mathsf{up}$ for permuting coabsorption up Consider the topmost occurrence of a subderivation

$$
\mathsf{b}\!\uparrow \frac{\pi\dfrac{Q}{S\{!R\}}}{S(!R, R)} \quad ,
$$

where $\pi \neq \mathsf{b}\!\uparrow$. According to 4.2.3 there are the following cases (cases (3) and (6) are impossible):

(1) If the redex of π is inside $S\{\ \}$, or

(2) if the contractum $!R$ of $\mathsf{b}\!\uparrow$ is inside a passive structure of the redex of π, then replace

$$
\mathsf{b}\!\uparrow \frac{\pi \dfrac{S'\{!R\}}{S\{!R\}}}{S(!R, R)} \qquad \text{by} \qquad \pi \frac{\mathsf{b}\!\uparrow \dfrac{S'\{!R\}}{S'(!R, R)}}{S(!R, R)} \quad .
$$

(4) If the redex of π is inside the contractum $!R$ of $\mathsf{b}\!\uparrow$, then replace

$$
\mathsf{b}\!\uparrow \frac{\pi \dfrac{S\{!R'\}}{S\{!R\}}}{S(!R, R)} \qquad \text{by} \qquad \pi \frac{\pi \dfrac{\mathsf{b}\!\uparrow \dfrac{S\{!R'\}}{S(!R', R')}}{S(!R', R)}}{S(!R, R)} \quad .
$$

(5) If the contractum $!R$ of $\mathsf{b}\!\uparrow$ is inside an active structure of the redex of π but not inside a passive one, then there are three subcases:

(i) If $\pi = \mathsf{p}{\downarrow}$ and $S\{!R\} = S'[!R, ?T]$, then replace

$$
\mathsf{b}{\uparrow}\,\frac{\mathsf{p}{\downarrow}\,\dfrac{S'\{![R,T]\}}{S'[!R,?T]}}{S'[(!R,R),?T]} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\,\frac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'\{![R,T]\}}{S'(![R,T],[R,T])}}{S'([!R,?T],[R,T])}}{S'[([!R,?T],R),T]}}{S'[(!R,R),?T,T]}}{S'[(!R,R),?T]} \quad .
$$

(ii) If $\pi = \mathsf{w}{\downarrow}$ and $S\{!R\} = S'\{?S''\{!R\}\}$, then replace

$$
\mathsf{b}{\uparrow}\,\frac{\mathsf{w}{\downarrow}\,\dfrac{S'\{\bot\}}{S'\{?S''\{!R\}\}}}{S'\{?S''(!R,R)\}} \qquad \text{by} \qquad \mathsf{w}{\downarrow}\,\frac{S'\{\bot\}}{S'\{?S''(!R,R)\}} \quad .
$$

(iii) If $\pi = \mathsf{b}{\downarrow}$ and $S\{!R\} = S'\{?S''\{!R\}\}$, then replace

$$
\mathsf{b}{\uparrow}\,\frac{\mathsf{b}{\downarrow}\,\dfrac{S'[?S''\{!R\},S''\{!R\}]}{S'\{?S''\{!R\}\}}}{S'\{?S''(!R,R)\}} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\,\frac{\mathsf{b}{\uparrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S'[?S''\{!R\},S''\{!R\}]}{S'[?S''(!R,R),S''\{!R\}]}}{S'[?S''(!R,R),S''(!R,R)]}}{S'\{?S''(!R,R)\}} \quad .
$$

Repeat until all instances of $\mathsf{b}{\uparrow}$ are at the top of the derivation.

It is easy to see that if the Algorithm 4.3.28 terminates, the resulting derivation has the shape

$$
\begin{array}{c}
T \\
\|{\scriptstyle\{\mathsf{b}{\uparrow}\}} \\
T' \\
\|{\scriptstyle\mathsf{SELS}\setminus\{\mathsf{b}{\uparrow}\}} \\
R
\end{array} \quad .
$$

Although it is rather intuitive that $\mathsf{b}{\uparrow}\mathsf{up}$ should terminate, it is not easy to prove because while permuting the rule $\mathsf{b}{\uparrow}$ up, it might happen that new instances of $\mathsf{b}{\uparrow}$ as well as new instances of $\mathsf{b}{\downarrow}$ are introduced.

4.3.29 Lemma *For an input derivation*

$$
\mathsf{b}{\uparrow}\,\frac{\Delta\|{\scriptstyle\mathsf{SELS}\setminus\{\mathsf{b}{\uparrow}\}}\atop S\{!R\}}{S(!R,R)} \quad ,
$$

the $\mathsf{b}{\uparrow}\mathsf{up}$ *procedure terminates.*

Proof: The problem of showing termination is that the number of instances of $\mathsf{b}{\uparrow}$ might increase during the process of permuting up $\mathsf{b}{\uparrow}$. This always happens when an upwards

moving $b{\uparrow}$ meets a $b{\downarrow}$ as in case (5.iii) in 4.3.28. Further, the number of instances of $b{\downarrow}$ inside Δ is not fixed. The number of $b{\downarrow}$ might increase when an upwards moving $b{\uparrow}$ meets an instance of $p{\downarrow}$ as in case (5.i) in 4.3.28 or an instance of $b{\downarrow}$ as in case (4). Therefore, the difficulty lies in finding the right induction measure. For this, I will mark inside Δ all !-chains that have the contractum $!R$ of the $b{\uparrow}$ instance as tail. But I will mark the links not with $!^{\blacktriangle}$, but with $!^{n}$ for some $n \geqslant 1$. Start with the contractum $!R$ of the $b{\uparrow}$ by marking it with $!^{1}R$. Now continue the marking as indicated in Figure 4.8 (on page 102) by propagating the number n from conclusion to premise in each rule, with one exception: If in case (2) of Definition 4.3.5 the rule $\rho = p{\downarrow}$ and the situation is

$$
\begin{array}{c}
T \\
\big\Vert \text{\scriptsize SELS}\backslash\{b{\uparrow}\} \\
p{\downarrow}\,\dfrac{S'\{![U, V\{!R'\}]\}}{S'[!U, ?V\{!^{n}R'\}]} \\
\big\Vert \text{\scriptsize SELS}\backslash\{b{\uparrow}\} \\
b{\uparrow}\,\dfrac{S\{!^{1}R\}}{S(!R, R)} \qquad ,
\end{array}
$$

then continue the marking as follows:

$$
\begin{array}{c}
T \\
\big\Vert \text{\scriptsize SELS}\backslash\{b{\uparrow}\} \\
p{\downarrow}\,\dfrac{S'\{![U, V\{!^{2n}R'\}]\}}{S'[!U, ?V\{!^{n}R'\}]} \\
\big\Vert \text{\scriptsize SELS}\backslash\{b{\uparrow}\} \\
b{\uparrow}\,\dfrac{S\{!^{1}R\}}{S(!R, R)} \qquad ,
\end{array}
$$

i.e. the marking number is multiplied by 2. For example, the derivation

$$
\begin{array}{c}
p{\downarrow}\,\dfrac{![b, (\bar{b}, ![([!a, ?b], [c, \bar{c}]), a])]}{![b, (\bar{b}, [?([!a, ?b], [c, \bar{c}]), !a])]} \\
ai{\downarrow}\,\dfrac{}{![b, (\bar{b}, [?([!a, ?b], [c, \bar{c}]), (!a, [c, \bar{c}])])]} \\
w{\downarrow}\,\dfrac{}{![b, (\bar{b}, [?([!a, ?b], [c, \bar{c}]), ([!a, ?b], [c, \bar{c}])])]} \\
p{\downarrow}\,\dfrac{}{[!b, ?(\bar{b}, [?([!a, ?b], [c, \bar{c}]), ([!a, ?b], [c, \bar{c}])])]} \\
b{\downarrow}\,\dfrac{}{[!b, ?(\bar{b}, ?([!a, ?b], [c, \bar{c}]))]} \\
s\,\dfrac{}{[!b, ?(\bar{b}, ?[!a, (?b, [c, \bar{c}])])]} \\
b{\uparrow}\,\dfrac{}{[!b, ?(\bar{b}, ?[(!a, a), (?b, [c, \bar{c}])])]}
\end{array}
$$

is marked as

$$
\mathsf{b}{\uparrow}\frac{\mathsf{s}\frac{\mathsf{b}{\downarrow}\frac{\mathsf{p}{\downarrow}\frac{\mathsf{w}{\downarrow}\frac{\mathsf{ai}{\downarrow}\frac{\mathsf{p}{\downarrow}\frac{![b,(\bar{b},!^{2}[(([!^{4}\boldsymbol{a},?\boldsymbol{b}],[c,\bar{c}]),a])]}{![b,(\bar{b},[?([!^{2}\boldsymbol{a},?b],[c,\bar{c}]),!^{2}\boldsymbol{a}])]}}{![b,(\bar{b},[?([!^{2}\boldsymbol{a},?b],[c,\bar{c}]),(!^{2}\boldsymbol{a},[c,\bar{c}])])]}}{![b,(\bar{b},[?([!^{2}\boldsymbol{a},?b],[c,\bar{c}]),([!^{2}\boldsymbol{a},?b],[c,\bar{c}])])]}}{[!b,?(\bar{b},[?([!^{1}\boldsymbol{a},?b],[c,\bar{c}]),([!^{1}\boldsymbol{a},?b],[c,\bar{c}])])]}}{[!b,?(\bar{b},?([!^{1}\boldsymbol{a},?b],[c,\bar{c}]))]}}{[!b,?(\bar{b},?[!^{1}\boldsymbol{a},(?b,[c,\bar{c}])])]}}{[!b,?(\bar{b},?[(!a,a),(?b,[c,\bar{c}])])]}\quad.
$$

Observe that it might happen that one marking is inside another. This happens if one marking "is pulled inside" another by an instance of $\mathsf{p}{\downarrow}$ (as in the example above). In this case the marking that is "pulled inside" is doubled. As a consequence we have that whenever there is a marked structure $!^{n}R$ which has other markings inside, then those markings are even. For notational convenience, let in the following R_{*} denote the structure R where all markings inside R are divided by two. (I will use this only for structures R, where all markings are even.) Further, let $R_{\oplus}$ denote the structure R where all markings inside R are multiplied by two. During the run of the Algorithm 4.3.28, the markings are now removed as follows:

- In cases (1) and (2) replace

$$
\mathsf{b}{\uparrow}\frac{\pi\frac{S'\{!^{n}R\}}{S\{!^{n}R\}}}{S(!R_{*},R_{*})} \qquad \text{by} \qquad \pi\frac{\mathsf{b}{\uparrow}\frac{S'\{!^{n}R\}}{S'(!R_{*},R_{*})}}{S(!R_{*},R_{*})}\quad.
$$

- In case (4) replace

$$
\mathsf{b}{\uparrow}\frac{\pi\frac{S\{!^{n}R'\}}{S\{!^{n}R\}}}{S(!R_{*},R_{*})} \qquad \text{by} \qquad \pi\frac{\pi\frac{\mathsf{b}{\uparrow}\frac{S\{!^{n}R'\}}{S(!R'_{*},R'_{*})}}{S(!R'_{*},R_{*})}}{S(!R_{*},R_{*})}\quad.
$$

- In case (5.i) replace

$$
\mathsf{b}{\uparrow}\frac{\mathsf{p}{\downarrow}\frac{S'\{!^{n}[R,T_{\oplus}]\}}{S'[!^{n}R,?T]}}{S'[(!R_{*},R_{*}),?T]} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\frac{\mathsf{s}\frac{\mathsf{s}\frac{\mathsf{p}{\downarrow}\frac{\mathsf{b}{\uparrow}\frac{S'\{!^{n}[R,T_{\oplus}]\}}{S'(![R_{*},T],[R_{*},T])}}{S'([!R_{*},?T],[R_{*},T])}}{S'[([!R_{*},?T],R_{*}),T]}}{S'[(!R_{*},R_{*}),?T,T]}}{S'[(!R_{*},R_{*}),?T]}\quad.
$$

- In case (5.ii) replace

$$
\mathsf{b}{\uparrow}\frac{\mathsf{w}{\downarrow}\frac{S'\{\bot\}}{S'\{?S''\{!^{n}R\}\}}}{S'\{?S''(!R_{*},R_{*})\}} \qquad \text{by} \qquad \mathsf{w}{\downarrow}\frac{S'\{\bot\}}{S'\{?S''(!R_{*},R_{*})\}}\quad.
$$

- In case (5.iii) replace

$$\mathsf{b}{\downarrow}\cfrac{\cfrac{S'[?S''\{!^{m}R\}, S''\{!^{m}R\}]}{S'\{?S''\{!^{m}R\}\}}}{\mathsf{b}{\uparrow}\,S'\{?S''(!R_*, R_*)\}} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\cfrac{\mathsf{b}{\uparrow}\cfrac{\mathsf{b}{\uparrow}\cfrac{S'[?S''\{!^{m}R\}, S''\{!^{m}R\}]}{S'[?S''(!R_*, R_*), S''\{!^{m}R\}]}}{S'[?S''(!R_*, R_*), S''(!R_*, R_*)]}}{S'\{?S''(!R_*, R_*)\}} \quad .$$

All instances of $\mathsf{b}{\uparrow}$ travel up along a !-chain that has been marked in the beginning. Since in the beginning there was only one instance of $\mathsf{b}{\uparrow}$, each marked !-chain can be used only once (and is used exactly once) by an instance of $\mathsf{b}{\uparrow}$, and then the marking is removed. But it might happen that new markings are introduced during the process because the length of the derivation can increase. For a given structure S let $\sigma(S)$ denote the sum of the markings inside S. (For example for $S = ![b, (\bar{b}, !^{2}[([!^{4}a, ?b], [c, \bar{c}]), a])]$ we have $\sigma(S) = 6$.) Then, for any two structures S and S' occurring in Δ, such that S' occurs above S, we have $\sigma(S') \geqslant \sigma(S)$. Further, during the process of permuting up $\mathsf{b}{\uparrow}$, the value $\sigma(S)$ never increases for a structure S occurring in Δ. When a new structure S' is inserted (in cases (4), (5.i) and (5.iii)), we have $\sigma(S') = \sigma(S)$ for some structure S occurring below S'. With this observation we can show termination by assigning to Δ a pair $\langle n_\Delta, m_\Delta \rangle \in \mathbf{N} \times \mathbf{N}$, where $\mathbf{N} \times \mathbf{N}$ is endowed with the lexicographic ordering

$$\langle n, m \rangle < \langle n', m' \rangle \quad \Longleftrightarrow \quad \begin{aligned} & n < n' \quad \text{or} \\ & n = n' \quad \text{and} \quad m < m' \quad . \end{aligned}$$

and the values of n_Δ and m_Δ are defined as follows: During the process of permuting up $\mathsf{b}{\uparrow}$, the derivation has always the shape

$$\begin{array}{c} T \\ \Delta_1 \big\| \{\mathsf{b}{\uparrow}\} \\ T' \\ \Delta_2 \big\| \mathsf{SELS}\setminus\{\mathsf{b}{\uparrow}\} \\ \mathsf{b}{\uparrow}\,\dfrac{S\{!^{m}R\}}{S(!R, R)} \\ \Delta_3 \big\| \mathsf{SELS} \\ U \end{array} \quad ,$$

where Δ_1 contains the instances of $\mathsf{b}{\uparrow}$ that already have reached the top and Δ_2 is nontrivial and the instance of $\mathsf{b}{\uparrow}$ between Δ_2 and Δ_3 is the topmost to be permuted up. Now let $n_\Delta = \sigma(T')$ and $m_\Delta = \text{length}(\Delta_2)$. Then we have that $\langle n_\Delta, m_\Delta \rangle$ strictly decreases in each permutation step and we have $\langle n_\Delta, m_\Delta \rangle = \langle 0, 0 \rangle$ when all instances of $\mathsf{b}{\uparrow}$ have reached the top. $\qquad \square$

4.3.30 Lemma   *The* $\mathsf{b}{\uparrow}\mathsf{up}$ *algorithm terminates for any input derivation* $\begin{array}{c} T \\ \Delta \big\| \mathsf{SELS} \\ R \end{array}$ *.*

Proof: Apply Lemma 4.3.29 to every instance of $\mathsf{b}{\uparrow}$ in Δ. $\qquad \square$

The dual procedure to $\mathsf{b}{\uparrow}\mathsf{up}$ is $\mathsf{b}{\downarrow}\mathsf{down}$, in which all instances of $\mathsf{b}{\downarrow}$ are moved down in the derivation.

4.3.31 **Algorithm** b↓down **for permuting absorption down** Repeat the following until all instances of b↓ are at the bottom of the derivation. Consider the bottommost occurrence of a subderivation

$$\rho \frac{\mathsf{b}{\downarrow} \dfrac{S[?T,T]}{S\{?T\}}}{P} \quad,$$

where $\rho \neq \mathsf{b}{\downarrow}$. The possible cases are (dual to 4.2.3):

(1) The contractum of ρ is inside $S\{\ \}$, or

(2) the redex $?T$ of b↓ is inside a passive structure of the contractum of ρ. Then replace

$$\rho \frac{\mathsf{b}{\downarrow}\dfrac{S[?T,T]}{S\{?T\}}}{S'\{?T\}} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\frac{\rho\dfrac{S[?T,T]}{S'[?T,T]}}{S'\{?T\}}\quad.$$

(4) The contractum of ρ is inside the redex $?T$ of b↓ Then replace

$$\rho \frac{\mathsf{b}{\downarrow}\dfrac{S[?T,T]}{S\{?T\}}}{S\{?T'\}} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\frac{\rho\dfrac{\rho\dfrac{S[?T,T]}{S[?T,T']}}{S[?T',T']}}{S\{?T'\}}\quad.$$

(5) The redex $?T$ of b↓ is inside an active structure of the contractum of ρ but not inside a passive one. Then there are three cases:

(i) If $\rho = \mathsf{p}{\uparrow}$ and $S\{?T\} = S'(!R,?T)$, then replace

$$\mathsf{p}{\uparrow}\frac{\mathsf{b}{\downarrow}\dfrac{S'(!R,[?T,T])}{S'(!R,?T)}}{S'\{?(R,T)\}} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\frac{\mathsf{p}{\uparrow}\dfrac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{\mathsf{b}{\uparrow}\dfrac{S'(!R,[?T,T])}{S'(!R,R,[?T,T])}}{S'([(!R,?T),T],R)}}{S'[(!R,?T),(R,T)]}}{S'[?(R,T),(R,T)]}}{S'\{?(R,T)\}}\quad.$$

(ii) If $\rho = \mathsf{w}{\uparrow}$ and $S\{?T\} = S'\{!S''\{?T\}\}$, then replace

$$\mathsf{w}{\uparrow}\frac{\mathsf{b}{\downarrow}\dfrac{S'\{!S''[?T,T]\}}{S'\{!S''\{?T\}\}}}{S'\{1\}} \qquad \text{by} \qquad \mathsf{w}{\uparrow}\frac{S'\{!S''[?T,T]\}}{S'\{1\}}\quad.$$

(iii) If $\rho = \mathsf{b}{\uparrow}$ and $S\{?T\} = S'\{!S''\{?T\}\}$, then replace

$$\mathsf{b}{\uparrow}\frac{\mathsf{b}{\downarrow}\dfrac{S'\{!S''[?T,T]\}}{S'\{!S''\{?T\}\}}}{S'(!S''\{?T\},S''\{?T\})} \qquad \text{by} \qquad \mathsf{b}{\downarrow}\frac{\mathsf{b}{\downarrow}\dfrac{\mathsf{b}{\uparrow}\dfrac{S'\{!S''[?T,T]\}}{S'(!S''[?T,T],S''[?T,T])}}{S'(!S''[?T,T],S''\{?T\})}}{S'(!S''\{?T\},S''\{?T\})}\quad.$$

Remark: Cases (3) and (6) are impossible.

4.3.32   Lemma   *The* b↓down *procedure terminates for every input derivation* $\Delta\|\overset{T}{\underset{R}{}}\mathsf{SELS}$
and yields a derivation

$$
\begin{array}{c}
T \\
\Delta'\|\mathsf{SELS}\backslash\{b\downarrow\} \\
R' \\
\Delta''\|\{b\downarrow\} \\
R
\end{array} \qquad .
$$

Proof: Dual to Lemma 4.3.30.   □

4.3.33   Algorithm b↑↓split **for separating absorption and coabsorption**

  I. If there are no subderivations of the shape

$$
\pi\,\dfrac{Q}{\underset{b\uparrow}{\dfrac{U}{P}}} \quad,
$$

   where $\pi \neq$ b↑, or of the shape

$$
b\downarrow\,\dfrac{Q}{\underset{\rho}{\dfrac{V}{P}}} \quad,
$$

   where $\rho \neq$ b↓, then terminate.

 II. Permute all instances of b↑ up by applying b↑up.

III. Permute all instances of b↓ down by applying b↓down.

 IV. Go to Step I.

  Lemma 4.3.30 and Lemma 4.3.32 ensure that each step of Algorithm 4.3.33 (depicted in Figure 4.7, page 99) does terminate. It remains to show that the whole algorithm b↑↓split does terminate eventually.

4.3.34   Lemma   *Let* Δ *be a derivation that does not contain a promotion cycle. Then the algorithm* b↑↓split *does terminate for* Δ.

Proof: Without loss of generality, let Δ be the outcome of a run of b↓down, i.e. there are no instances of b↓ to consider. Since Δ is finite, it contains only finitely many instances of b↑. Hence, there are only finitely many chains, say $\chi_1, \ldots, \chi_n$, that have the contractum $!R$ of a b↑ as tail. Mark all those chains with $!^{\blacktriangle}$ and $?^{\blacktriangledown}$ as in the previous section, and let $l_i = l(\chi_i)$ be the length of χ_i (see Definition 4.3.13) for each $i = 1, \ldots, n$. Now run b↑up and remove the markings $!^{\blacktriangle}$ as in the proof of Lemma 4.3.29 while the instances of b↑ are

permuted up. In case (5.i) replace

$$
\mathsf{p}\downarrow \frac{S'\{!^{\blacktriangle}[R,T]\,\}}{\mathsf{b}\uparrow \frac{S'[!^{\blacktriangle}R, ?^{\blacktriangledown}T\,]}{S'[(!R,R), ?^{\blacktriangledown}T\,]}}
\qquad \text{by} \qquad
\mathsf{b}\uparrow \frac{S'\{!^{\blacktriangle}[R,T]\,\}}{\mathsf{p}\downarrow \frac{S'(![R,T],[R,T])}{\mathsf{s}\frac{S'([!R,?T],[R,T])}{\mathsf{s}\frac{S'[((!R,?T),R),T]}{\mathsf{b}\downarrow \frac{S'[(!R,R),?T,T]}{S'[(!R,R),?^{\blacktriangledown}T\,]}}}}}
\quad .
$$

After this, all chains χ_i with length $l_i = 1$ are no longer marked. If $l(\chi_i) \geqslant 1$, then after the run of $\mathsf{b}\uparrow\mathsf{up}$ only a subchain χ_i' of χ_i with length $l_i' = l(\chi_i') = l(\chi_i) - 1$ remains marked because it is impossible to add links to a chain at the head. (Each of the chains $\chi_1, \ldots, \chi_n$ has a head since there is no promotion cycle inside Δ.) It is only possible to duplicate chains (case (4) of in 4.3.28). The situation for $\mathsf{b}\downarrow\mathsf{down}$ is dual. Hence the number $l_{max} = \max\{l(\chi_i)\}$ is reduced at each run of $\mathsf{b}\uparrow\mathsf{up}$ and $\mathsf{b}\downarrow\mathsf{down}$. This ensures the termination. $\qquad\square$

4.3.35   Lemma *Let Δ be a derivation that is obtained by a consecutive run of $\mathsf{b}\uparrow\mathsf{up}$ and $\mathsf{b}\downarrow\mathsf{down}$. Then Δ does not contain a promotion cycle.*

Proof: By way of contradiction, assume Δ contains a promotion cycle χ. Since Δ is the outcome of a run of $\mathsf{b}\downarrow\mathsf{down}$, all instances of $\mathsf{b}\downarrow$ are at the bottom of Δ. Hence, the cycle χ can only be forked by instances of $\mathsf{b}\uparrow$, more precisely, χ is forked by k_χ different instances of $\mathsf{b}\uparrow$. If Δ contains more than one promotion cycle, we can, without loss of generality, assume that χ is the one for which k_χ is minimal. I will now proceed by induction on k_χ to show a contradiction.

Base case: If $k_\chi = 0$, then we have an immediate contradiction to Theorem 4.3.26.

Inductive case: Now let $k_\chi \geqslant 1$ and consider the bottommost instance of $\mathsf{b}\uparrow$ that forks χ and mark it as $\mathsf{b}\uparrow^\wedge$. It has been introduced during the run of $\mathsf{b}\downarrow\mathsf{down}$, when

$$
\mathsf{b}\downarrow \frac{S'(!R,[?T,T])}{\mathsf{p}\uparrow \frac{S'(!R,?T)}{S'\{?(R,T)\}}}
\qquad \text{was replaced by} \qquad
\mathsf{b}\uparrow^\wedge \frac{S'(!R,[?T,T])}{\mathsf{s}\frac{S'(!R^1,R^2,[?T,T])}{\mathsf{s}\frac{S'([(!R^1,?T),T],R^2)}{\mathsf{p}\uparrow \frac{S'[(!R^1,?T),(R^2,T)]}{\mathsf{b}\downarrow^\vee \frac{S'[?(R^1,T),(R^2,T)]}{S'\{?(R,T)\}}}}}}
$$

(see case (5.i) in Algorithm 4.3.31). Here, I have marked the down moving $\mathsf{b}\downarrow$ as $\mathsf{b}\downarrow^\vee$ and the two copies of R as R^1 and R^2. By inspecting the cases of $\mathsf{b}\downarrow\mathsf{down}$ (see Algorithm 4.3.31), it is easy to see that while $\mathsf{b}\downarrow^\vee$ travels further down, the two copies R^1 and R^2 are treated equally, i.e. whenever a rule ρ modifies R^1, then there is another instance of ρ that modifies R^2 in same way. Further, if another $\mathsf{b}\downarrow$ is moved down in the same run of $\mathsf{b}\downarrow\mathsf{down}$ and meets the new $\mathsf{b}\uparrow^\wedge$ as in case (5.iii) in Algorithm 4.3.31, then it is duplicated into both copies R^1 and R^2. Hence, after finishing the run of

b↓down, every !-chain with head in R^1 has a counterpart !-chain with head in R^2, and vice versa. Similarly, all ?-chains with tail in R^1 and R^2 correspond to each other. This means that we can construct from χ a new promotion cycle χ' by replacing each subchain of χ with head or tail inside R^1 by the corresponding chain with head or tail in R^2. Then the new cycle is not forked by b↑$^\wedge$ since there are no more links inside R^1. Hence, the cycle χ' is forked by $k_{\chi'} = k_\chi - 1$ instances of b↑, which is a contradiction to the induction hypothesis. □

4.3.36 Proposition *For every derivation* $\begin{array}{c} T \\ \Delta \| \mathsf{SELS} \\ R \end{array}$ *there is a derivation*

$$
\begin{array}{c}
T \\
\Delta_1 \| \{\mathsf{b}\uparrow\} \\
T' \\
\Delta' \| \mathsf{SELS}\backslash\{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\
R' \\
\Delta_2 \| \{\mathsf{b}\downarrow\} \\
R
\end{array} \qquad .
$$

Proof: Apply the algorithm b↑↓split, which terminates by Lemmata 4.3.35 and 4.3.34. □

This completes the proof of the first decomposition theorem. For the second decomposition, we have to separate weakening and coweakening.

4.3.37 Proposition *For every derivation* $\begin{array}{c} T \\ \Delta \| \mathsf{SELS}\backslash\{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\ R \end{array}$ *there is a derivation*

$$
\begin{array}{c}
T \\
\Delta_1 \| \{\mathsf{w}\uparrow\} \\
T' \\
\Delta' \| \mathsf{SELS}\backslash\{\mathsf{b}\downarrow,\mathsf{b}\uparrow,\mathsf{w}\downarrow,\mathsf{w}\uparrow\} \\
R' \\
\Delta_2 \| \{\mathsf{w}\downarrow\} \\
R
\end{array} \qquad .
$$

Proof: First, all instances of w↑ inside Δ are permuted up to the top of the derivation. For this, consider the topmost subderivation

$$
\pi \frac{\begin{array}{c} Q \\ S\{!R\} \end{array}}{\mathsf{w}\uparrow \; S\{1\}} \qquad ,
$$

where $\pi \in \mathsf{SELS}\backslash\{\mathsf{b}\downarrow,\mathsf{b}\uparrow,\mathsf{w}\uparrow\}$ is nontrivial. According to 4.2.3 there are the following cases:

(4) The redex of π is inside an active structure of the contractum $!R$ of w↑. Then replace

$$
\pi \frac{\dfrac{S\{!R'\}}{S\{!R\}}}{\mathsf{w}\uparrow \; S\{1\}} \qquad \text{by} \qquad \mathsf{w}\uparrow \frac{S\{!R'\}}{S\{1\}} \qquad .
$$

(5) The contractum $!R$ of $\mathsf{w}\uparrow$ is inside an active structure of the redex of π but not inside a passive one. There are two possibilities:

(i) $\pi = \mathsf{p}\downarrow$ and $S\{!R\} = S'[!R, ?T]$ for some context $S'\{\ \}$ and some structure T. Then replace

$$
\mathsf{p}\downarrow \cfrac{S'\{![R,T]\}}{\mathsf{w}\uparrow \cfrac{S'[!R, ?T]}{S'[1, ?T]}} \qquad \text{by} \qquad \mathsf{w}\downarrow \cfrac{\mathsf{w}\uparrow \cfrac{S'\{![R,T]\}}{S'\{1\}} = \cfrac{S'[1, \bot]}{S'[1, ?T]}}{} \quad .
$$

(ii) $\pi = \mathsf{w}\downarrow$ and $S\{!R\} = S'\{?S''\{!R\}\}$ for some contexts $S'\{\ \}$ and $S''\{\ \}$. Then replace

$$
\mathsf{w}\uparrow \cfrac{\mathsf{w}\downarrow \cfrac{S'\{\bot\}}{S'\{?S''\{!R\}\}}}{S'\{?S''\{1\}\}} \qquad \text{by} \qquad \mathsf{w}\downarrow \cfrac{S'\{\bot\}}{S'\{?S''\{1\}\}} \quad .
$$

(6) The contractum $!R$ of $\mathsf{w}\uparrow$ and the redex of π (properly) overlap. This is impossible.

This terminates because the number of instances of $\mathsf{w}\uparrow$ does not increase and all reach the top eventually. Then, proceed dually, to permute all instances of $\mathsf{w}\downarrow$ down to the bottom of the derivation. Repeat the permuting up of $\mathsf{w}\uparrow$ and the permuting down of $\mathsf{w}\downarrow$ until the derivation has the desired shape. It remains to show that this does indeed terminate, because during the permuting up of $\mathsf{w}\uparrow$ new instances of $\mathsf{w}\downarrow$ are introduced and during the permuting down of $\mathsf{w}\downarrow$ new instances of $\mathsf{w}\uparrow$ are introduced. The only possibility of introducing a new $\mathsf{w}\downarrow$ while a $\mathsf{w}\uparrow$ is permuted up, is in case (5.i), when the $\mathsf{w}\uparrow$ meets a $\mathsf{p}\downarrow$. But then this $\mathsf{p}\downarrow$ disappears. Since the number of instances of $\mathsf{p}\downarrow$ in the initial derivation is finite and this number is not increased during the process of permuting $\mathsf{w}\uparrow$ up and $\mathsf{w}\downarrow$ down, the whole process must terminate eventually. □

4.4 Cut Elimination

In this section, I will use the second decomposition theorem together with the technique of permuting rules to show that the up fragment of system SELS is admissible.

More precisely, the second decomposition theorem is used to show that the rules $\mathsf{b}\uparrow$ and $\mathsf{w}\uparrow$ (i.e. the noncore up rules) are admissible. Then, the rules $\mathsf{p}\uparrow$ and $\mathsf{ai}\uparrow$ are eliminated by using the technique that has already been employed by G. Gentzen [Gen34]: For both rules, I will give a super rule that is more general and that helps in the book-keeping of the context. The super rules are permuted over all other rules until they reach the top of the proof where they disappear.

This permutability is distributed over several lemmata. If new rules are added to the system, then those lemmata remain valid: If rule ρ permutes over rule π, then the introduction of a rule σ does not change this fact. This modularity in the cut elimination argument is not available in the sequent calculus.

Let me now start with the first step, which is a corollary of the second decomposition theorem (Theorem 4.3.2).

4.4.1 Corollary *For every proof* $\Pi\|\mathsf{SELSU\{1\downarrow\}}\atop R$ *, there is a proof*

$$
\begin{array}{c}
1\downarrow\,\dfrac{}{1} \\[4pt]
\|\{\mathsf{ai}\downarrow\} \\
T_3 \\
\|\{\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow\} \\
R_3 \\
\|\{\mathsf{ai}\uparrow\} \\
R_2 \\
\|\{\mathsf{w}\downarrow\} \\
R_1 \\
\|\{\mathsf{b}\downarrow\} \\
R
\end{array}
$$

for some structures R_1, R_2, R_3 *and* T_3.

Proof: Apply Theorem 4.3.2 to the derivation $\begin{array}{c}1\\\|\mathsf{SELS}\\R\end{array}$ which is obtained from Π by removing the axiom. This yields a derivation

$$
\begin{array}{c}
1 \\
\Delta_1\|\{\mathsf{b}\uparrow\} \\
T_1 \\
\Delta_2\|\{\mathsf{w}\uparrow\} \\
T_2 \\
\Delta_3\|\{\mathsf{ai}\downarrow\} \\
T_3 \\
\Delta_4\|\{\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow\} \\
R_3 \\
\Delta_5\|\{\mathsf{ai}\uparrow\} \\
R_2 \\
\Delta_6\|\{\mathsf{w}\downarrow\} \\
R_1 \\
\Delta_7\|\{\mathsf{b}\downarrow\} \\
R
\end{array}
\quad .
$$

Now every application of $\mathsf{b}\uparrow$ in Δ_1 must be trivial, i.e. of the shape $\mathsf{b}\uparrow\dfrac{(!1,1)}{!1}$. Hence $T_1 = 1$. The same is true for the applications of $\mathsf{w}\uparrow$ in Δ_2. Hence $T_2 = 1$.   □

This shows that the noncore rules $\mathsf{w}\uparrow$ and $\mathsf{b}\uparrow$ are admissible. Consider now the two rules

$$
\mathsf{r}\downarrow\,\frac{S\{?[R,T]\}}{S[?R,?T]} \qquad \text{and} \qquad \mathsf{r}\uparrow\,\frac{S(!R,!T)}{S\{!(R,T)\}} \quad .
$$

The following derivation (together with Proposition 3.2.16) shows that both rules are sound,

i.e. derivable in system SELS:

$$
i{\downarrow} \cfrac{= \cfrac{S\{?[R,T]\}}{S(!1,?[R,T])}}{
p{\downarrow} \cfrac{S(![R,T,(\bar{R},\bar{T})],?[R,T])}{
p{\downarrow} \cfrac{S([?R,![T,(\bar{R},\bar{T})]],?[R,T])}{
s \cfrac{S([?R,?T,!(\bar{R},\bar{T})],?[R,T])}{
i{\uparrow} \cfrac{S[?R,?T,(!(\bar{R},\bar{T}),?[R,T])]}{
= \cfrac{S[?R,?T,\bot]}{S[?R,?T]}}}}}} \quad .
$$

For technical reasons, I was not able to simply eliminate the rule $\mathsf{p}{\uparrow}$. Instead, I will eliminate the rules $\mathsf{p}{\uparrow}$ and $\mathsf{r}{\uparrow}$ simultaneously, with the result that instances of $\mathsf{r}{\downarrow}$ might be introduced. Those instances will be eliminated afterwards. For this reason, let me define system ELSr to be system ELS extended by $\mathsf{r}{\downarrow}$, i.e.

$$\mathsf{ELSr} = \mathsf{ELS} \cup \{\mathsf{r}{\downarrow}\} \quad .$$

All three rules $\mathsf{p}{\uparrow}$, $\mathsf{r}{\uparrow}$ and $\mathsf{ai}{\uparrow}$ are removed by a method that has already been used in [Gug99] for proving the cut elimination for system BV, namely, by removing the super rules $\mathsf{sp}{\uparrow}$, $\mathsf{sr}{\uparrow}$ and $\mathsf{sai}{\uparrow}$, respectively:

$$
\mathsf{sp}{\uparrow} \frac{S([?R,U],[!T,V])}{S[?(R,T),U,V]} \quad , \quad \mathsf{sr}{\uparrow} \frac{S([!R,U],[!T,V])}{S[!(R,T),U,V]} \quad \text{and} \quad \mathsf{sai}{\uparrow} \frac{S([a,U],[\bar{a},V])}{S[U,V]} \quad .
$$

Observe that the three rules $\mathsf{p}{\uparrow}$, $\mathsf{r}{\uparrow}$ and $\mathsf{ai}{\uparrow}$ are instances of their super rules $\mathsf{sp}{\uparrow}$, $\mathsf{sr}{\uparrow}$ and $\mathsf{sai}{\uparrow}$, respectively. I will now show that every super rule can be permuted up in the proof until it disappears or its application becomes trivial.

Before we can start, a few more definitions are necessary.

4.4.2 Definition A structure R is called a *proper par*, if there are two structures R' and R'' with $R = [R',R'']$ and $R' \neq \bot \neq R''$. Similarly, a structure R is a *proper times*, if there are two structures R' and R'' with $R = (R',R'')$ and $R' \neq 1 \neq R''$.
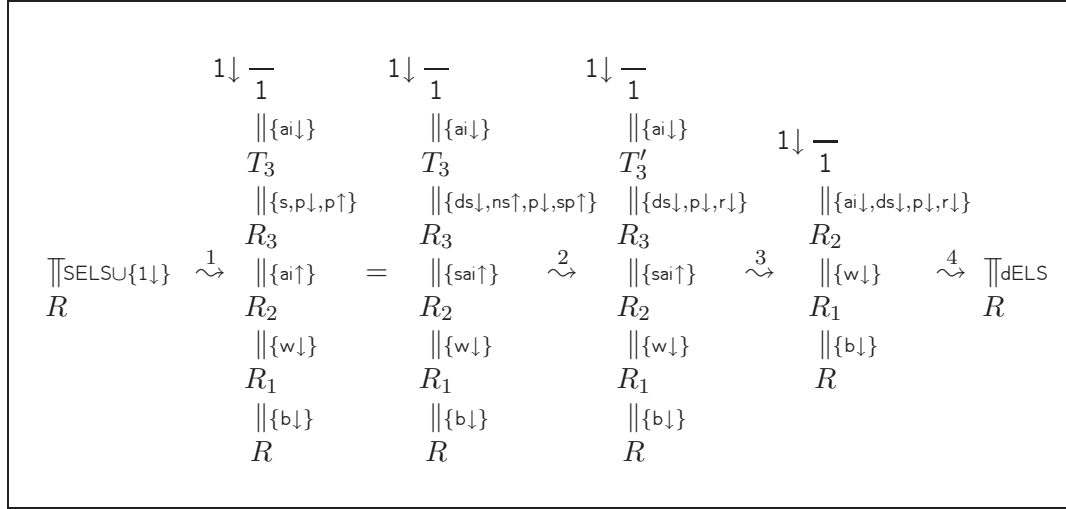
4.4.3 Definition Let *deep switch* be the rule

$$\mathsf{ds}{\downarrow} \frac{S([R,T],U)}{S[(R,U),T]} \quad ,$$

where the structure R is not a proper times. The rule

$$\mathsf{ns}{\uparrow} \frac{S([(R,R'),T],U)}{S[(R,R',U),T]} \quad ,$$

where $R \neq 1 \neq R'$, will be called *nondeep switch*. Let *deep system* ELS (or *system* dELS) be the system that is obtained from system ELS by replacing the switch rule by deep switch:

$$\mathsf{dELS} = \{1{\downarrow}, \mathsf{ai}{\downarrow}, \mathsf{ds}{\downarrow}, \mathsf{p}{\downarrow}, \mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}\} \quad ,$$

$$
\begin{array}{cccccccc}
& 1\downarrow\dfrac{}{1} & & 1\downarrow\dfrac{}{1} & & 1\downarrow\dfrac{}{1} & & \\[4pt]
& \|\{\mathsf{ai}\downarrow\} & & \|\{\mathsf{ai}\downarrow\} & & \|\{\mathsf{ai}\downarrow\} & 1\downarrow\dfrac{}{1} & \\
& T_3 & & T_3 & & T_3' & & \\
& \|\{\mathsf{s},\mathsf{p}\downarrow,\mathsf{p}\uparrow\} & & \|\{\mathsf{ds}\downarrow,\mathsf{ns}\uparrow,\mathsf{p}\downarrow,\mathsf{sp}\uparrow\} & & \|\{\mathsf{ds}\downarrow,\mathsf{p}\downarrow,\mathsf{r}\downarrow\} & \|\{\mathsf{ai}\downarrow,\mathsf{ds}\downarrow,\mathsf{p}\downarrow,\mathsf{r}\downarrow\} & \\
& R_3 & & R_3 & & R_3 & R_2 & \\
\|\mathsf{SELS}\cup\{1\downarrow\} \overset{1}{\leadsto} & \|\{\mathsf{ai}\uparrow\} & = & \|\{\mathsf{sai}\uparrow\} & \overset{2}{\leadsto} & \|\{\mathsf{sai}\uparrow\} \overset{3}{\leadsto} & \|\{\mathsf{w}\downarrow\} \overset{4}{\leadsto} & \|\mathsf{dELS} \\
R & R_2 & & R_2 & & R_2 & R_1 & R \\
& \|\{\mathsf{w}\downarrow\} & & \|\{\mathsf{w}\downarrow\} & & \|\{\mathsf{w}\downarrow\} & \|\{\mathsf{b}\downarrow\} & \\
& R_1 & & R_1 & & R_1 & R & \\
& \|\{\mathsf{b}\downarrow\} & & \|\{\mathsf{b}\downarrow\} & & \|\{\mathsf{b}\downarrow\} & & \\
& R & & R & & R & &
\end{array}
$$

Figure 4.13: Cut elimination for system $\mathsf{SELS}\cup\{1\downarrow\}$

and accordingly

$$\mathsf{dELSr}=\mathsf{dELS}\cup\{\mathsf{r}\downarrow\}=\{1\downarrow,\mathsf{ai}\downarrow,\mathsf{ds}\downarrow,\mathsf{p}\downarrow,\mathsf{r}\downarrow,\mathsf{w}\downarrow,\mathsf{b}\downarrow\}\quad,$$

Both rules, the deep switch and the nondeep switch, are instances of the switch rule, and every instance of the switch rule is either an instance of deep switch or an instance of nondeep switch.

Now we have all the sufficient material to outline the scheme of the full cut elimination proof (shown in Figure 4.13). We start by applying Corollary 4.4.1. Then, all instances of the rule s are replaced by $\mathsf{ds}\downarrow$ or $\mathsf{ns}\uparrow$, and all instances of $\mathsf{p}\uparrow$ and $\mathsf{ai}\uparrow$ are replaced by their super rules. Then, all instances of the rules $\mathsf{ns}\uparrow$, $\mathsf{sp}\uparrow$ and $\mathsf{sr}\uparrow$ are permuted over $\mathsf{ai}\downarrow$, $\mathsf{ds}\downarrow$, $\mathsf{p}\downarrow$ and $\mathsf{r}\downarrow$ in Step 2 (in this step the rules $\mathsf{sr}\uparrow$ and $\mathsf{r}\downarrow$ are introduced). In Step 3, the rule $\mathsf{sai}\uparrow$ is permuted up. Finally, the rule $\mathsf{r}\downarrow$ is eliminated. Observe, that by this procedure, we obtain a result which is stronger than Theorem 4.1.8, because we obtain a proof in system dELS instead of ELS.

4.4.4   Lemma   *The rule* $\mathsf{ns}\uparrow$ *permutes over the rules* $\mathsf{ai}\downarrow$, $\mathsf{ds}\downarrow$, $\mathsf{p}\downarrow$ *and* $\mathsf{r}\downarrow$ *by the rule* $\mathsf{ds}\downarrow$.

Proof: Consider the derivation

$$
\mathsf{ns}\uparrow\dfrac{\pi\dfrac{Q}{S([(R,R'),T],U)}}{S[(R,R',U),T]}\quad,
$$

where the application of $\pi\in\{\mathsf{ai}\downarrow,\mathsf{ds}\downarrow,\mathsf{p}\downarrow,\mathsf{r}\downarrow\}$ is nontrivial. Without loss of generality, we can assume that R is not a proper times. According to 4.2.3, the cases to consider are:

(4) The redex of π is inside the contractum $([(R,R'),T],U)$ of $\mathsf{ns}\uparrow$, but not inside one of

the passive structures R, R', T or U. Only one case is possible ($\pi = \mathsf{ds}{\downarrow}$):

$$\mathsf{ns}{\uparrow}\frac{\mathsf{ds}{\downarrow}\dfrac{S([R,T],R',U)}{S([(R,R'),T],U)}}{S[(R,R',U),T]} \quad \text{yields} \quad \mathsf{ds}{\downarrow}\frac{S([R,T],R',U)}{S[(R,R',U),T]} \quad .$$

(5) The contractum $([(R,R'),T],U)$ of $\mathsf{ns}{\uparrow}$ is inside an active structure of the redex of π but not inside a passive one. Then $\pi = \mathsf{ds}{\downarrow}$ and

$$S([(R,R'),T],U) = S'[([(R,R'),T],U,V),W] \quad .$$

There are two possibilities:

$$\mathsf{ns}{\uparrow}\frac{\mathsf{ds}{\downarrow}\dfrac{S'([(R,R'),T,W],U,V)}{S'[([(R,R'),T],U,V),W]}}{S'[([(R,R',U),T],V),W]} \quad \text{yields} \quad \mathsf{ds}{\downarrow}\frac{\mathsf{ns}{\uparrow}\dfrac{S'([(R,R'),T,W],U,V)}{S'([(R,R',U),T,W],V)}}{S'[([(R,R',U),T],V),W]} \quad ,$$

and

$$\mathsf{ns}{\uparrow}\frac{\mathsf{ds}{\downarrow}\dfrac{S'([(R,R'),T],[U,W],V)}{S'[([(R,R'),T],U,V),W]}}{S'[([(R,R',U),T],V),W]} \quad \text{yields} \quad \mathsf{ds}{\downarrow}\frac{\mathsf{ds}{\downarrow}\dfrac{\mathsf{ns}{\uparrow}\dfrac{S'([(R,R'),T],[U,W],V)}{S'([(R,R',[U,W]),T],V)}}{S'([(R,R',U),T,W],V)}}{S'[([(R,R',U),T],V),W]} \quad .$$

Note: The second case is only possible if U is not a proper times.

(6) The redex of π and the contractum $([(R,R'),T],U)$ of $\mathsf{ns}{\uparrow}$ overlap. This is impossible, because the redex of π is always a par structure which cannot properly overlap with a times structure. $\square$

4.4.5 Proposition *For every proof* $\underset{R}{\overset{\;}{\parallel}}\mathsf{ELS}\backslash\{\mathsf{b}{\downarrow},\mathsf{w}{\downarrow}\}$ *there is a proof* $\underset{R}{\overset{\;}{\parallel}}\mathsf{dELS}\backslash\{\mathsf{b}{\downarrow},\mathsf{w}{\downarrow}\}$, *and*

for every proof $\underset{R}{\overset{\;}{\parallel}}\mathsf{ELSr}\backslash\{\mathsf{b}{\downarrow},\mathsf{w}{\downarrow}\}$ *there is a proof* $\underset{R}{\overset{\;}{\parallel}}\mathsf{dELSr}\backslash\{\mathsf{b}{\downarrow},\mathsf{w}{\downarrow}\}$.

Proof: All instances of nondeep switch are permuted up applying Lemma 4.4.4. They either disappear or reach the top of the proof. In this case they must be trivial because the premise is 1. $\square$

4.4.6 Lemma *The rules* $\mathsf{sai}{\uparrow}$, $\mathsf{sp}{\uparrow}$ *and* $\mathsf{sr}{\uparrow}$ *permute over the rule* $\mathsf{ds}{\downarrow}$.

Proof: All three rules are of the shape

$$\mathsf{sx}{\uparrow}\frac{S([P,U],[P',V])}{S[P'',U,V]} \quad ,$$

where neither P nor P' is a proper par or a proper times. Now consider the derivation

$$\mathsf{sx}{\uparrow}\frac{\mathsf{ds}{\downarrow}\dfrac{Q}{S([P,U],[P',V])}}{S[P'',U,V]} \quad ,$$

where the application of $\mathsf{ds}{\downarrow}$ is nontrivial. According to 4.2.3, the cases to consider are:

(4) The redex of ds↓ is inside the contractum $([P,U],[P',V])$ of sx↑, but not inside a passive structure. (Remark: If sx↑ is sai↑, then the passive structures are U and V. If sx↑ is sp↑ or sr↑, then U, V, R and T are passive structures.) Observe that the redex of ds↓ cannot be inside P or P' because they are neither a proper par nor a proper times. Therefore, there are only two remaining cases.

   (i) $U = (U',U'')$. Without loss of generality assume that U' is not a proper times. Then

$$\text{sx↑}\cfrac{\text{ds↓}\cfrac{S([P,U'],[P',V],U'')}{S([P,(U',U'')],[P',V])}}{S[P'',(U',U''),V]} \quad \text{yields} \quad \text{ds↓}\cfrac{\text{sx↑}\cfrac{S([P,U'],[P',V],U'')}{S([P'',U',V],U'')}}{S[P'',(U',U''),V]} \quad .$$

   (ii) $V = (V',V'')$. Similar to (i).

(5) The contractum $([P,U],[P',V])$ of sx↑ is inside an active structure of the redex of ds↓ but not inside a passive one. In the most general case we have that

$$S([P,U],[P',V]) = S'[([P,U],[P',V],W),Z] \quad ,$$

for some context $S'\{\ \}$ and some structures W and Z. Then

$$\text{sx↑}\cfrac{\text{ds↓}\cfrac{S([P,U,Z],[P',V],W)}{S'[([P,U],[P',V],W),Z]}}{S'[([P'',U,V],W),Z]} \quad \text{yields} \quad \text{ds↓}\cfrac{\text{sx↑}\cfrac{S([P,U,Z],[P',V],W)}{S'([P'',U,V,Z],W)}}{S'[([P'',U,V],W),Z]} \quad ,$$

and

$$\text{sx↑}\cfrac{\text{ds↓}\cfrac{S([P,U],[P',V,Z],W)}{S'[([P,U],[P',V],W),Z]}}{S'[([P'',U,V],W),Z]} \quad \text{yields} \quad \text{ds↓}\cfrac{\text{sx↑}\cfrac{S([P,U],[P',V,Z],W)}{S'([P'',U,V,Z],W)}}{S'[([P'',U,V],W),Z]} \quad .$$

(6) The redex of ds↓ and the contractum $([P,U],[P',V])$ of sx↑ overlap. As in the previous lemma, this case is impossible. □

Observe that the rules sai↑, sp↑ and sr↑ do not permute over the rule s. For example in the derivation

$$\text{sai↑}\cfrac{\text{s}\cfrac{S([a,U],[([\bar{a},V],W),Z])}{S[([a,U],[\bar{a},V],W),Z]}}{S[([U,V],W),Z]}$$

it is unclear how how the rule sai↑ could be permuted over the switch. This is the reason why the deep switch has been introduced in [Gug99] in the first place.

4.4.7 Lemma *For every derivation* $\rho\dfrac{\pi\dfrac{Q}{Z}}{P}$ *with* $\rho \in \{\mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow\}$ *and* $\pi \in \{\mathsf{p}\!\downarrow, \mathsf{r}\!\downarrow\}$, *there is either a derivation* $\pi\dfrac{\rho\dfrac{Q}{Z'}}{P}$ *for some structure* Z' *or a derivation* $\pi'\dfrac{\mathsf{s}\dfrac{\rho'\dfrac{Q}{Z'}}{Z''}}{P}$ *for some structures*

Z' *and* Z'' *and rules* $\rho' \in \{\mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow\}$ *and* $\pi' \in \{\mathsf{p}\!\downarrow, \mathsf{r}\!\downarrow\}$.

Proof: Consider the derivation

$$\rho\dfrac{\pi\dfrac{Q}{S([*R,U],[!T,V])}}{S[*(R,T),U,V]}\quad,$$

where $\rho \in \{\mathsf{sp}\!\uparrow, \mathsf{sr}\!\uparrow\}$, $* \in \{?,!\}$ and the application of $\pi \in \{\mathsf{p}\!\downarrow, \mathsf{r}\!\downarrow\}$ is nontrivial. According to 4.2.3, the cases to consider are:

(4) The redex of π is inside the contractum $([*R,U],[!T,V])$ of ρ but not inside R, U, T or V. There are the following five subcases:

(i) $\rho = \mathsf{sp}\!\uparrow, * = \ ?, \pi = \mathsf{p}\!\downarrow$ and $U = [!U', U'']$. Then

$$\mathsf{sp}\!\uparrow\dfrac{\mathsf{p}\!\downarrow\dfrac{S([![R,U'],U''],[!T,V])}{S([?R,!U',U''],[!T,V])}}{S[?(R,T),!U',U'',V]}\qquad\text{yields}\qquad\mathsf{p}\!\downarrow\dfrac{\mathsf{s}\dfrac{\mathsf{sr}\!\uparrow\dfrac{S([![R,U'],U''],[!T,V])}{S[!([R,U'],T),U'',V]}}{S[![(R,T),U'],U'',V]}}{S[?(R,T),!U',U'',V]}\quad.$$

(ii) $\rho = \mathsf{sp}\!\uparrow, * = \ ?, \pi = \mathsf{p}\!\downarrow$ and $V = [?V', V'']$. Then

$$\mathsf{sp}\!\uparrow\dfrac{\mathsf{p}\!\downarrow\dfrac{S([?R,U],[![T,V'],V''])}{S([?R,U],[!T,?V',V''])}}{S[?(R,T),U,?V',V'']}\qquad\text{yields}\qquad\mathsf{r}\!\downarrow\dfrac{\mathsf{s}\dfrac{\mathsf{sp}\!\uparrow\dfrac{S([?R,U],[![T,V'],V''])}{S[?(R,[T,V']),U,V'']}}{S[?[(R,T),V'],U,V'']}}{S[?(R,T),U,?V',V'']}\quad.$$

(iii) $\rho = \mathsf{sp}\!\uparrow, * = \ ?, \pi = \mathsf{r}\!\downarrow$ and $U = [?U', U'']$. Then

$$\mathsf{sp}\!\uparrow\dfrac{\mathsf{r}\!\downarrow\dfrac{S([?[R,U'],U''],[!T,V])}{S([?R,?U',U''],[!T,V])}}{S[?(R,T),?U',U'',V]}\qquad\text{yields}\qquad\mathsf{r}\!\downarrow\dfrac{\mathsf{s}\dfrac{\mathsf{sp}\!\uparrow\dfrac{S([?[R,U'],U''],[!T,V])}{S[?([R,U'],T),U'',V]}}{S[?[(R,T),U'],U'',V]}}{S[?(R,T),?U',U'',V]}\quad.$$

(iv) $\rho = \mathsf{sr}\!\uparrow, * = \ !, \pi = \mathsf{p}\!\downarrow$ and $U = [?U', U'']$. Then

$$\mathsf{sr}\!\uparrow\dfrac{\mathsf{p}\!\downarrow\dfrac{S([![R,U'],U''],[!T,V])}{S([!R,?U',U''],[!T,V])}}{S[!(R,T),?U',U'',V]}\qquad\text{yields}\qquad\mathsf{p}\!\downarrow\dfrac{\mathsf{s}\dfrac{\mathsf{sr}\!\uparrow\dfrac{S([![R,U'],U''],[!T,V])}{S[!([R,U'],T),U'',V]}}{S[![(R,T),U'],U'',V]}}{S[!(R,T),?U',U'',V]}\quad.$$

(v) $\rho = \mathsf{sr}{\uparrow}, * = \;!, \pi = \mathsf{p}{\downarrow}$ and $V = [?V', V'']$. Then

$$
\mathsf{p}{\downarrow}\frac{S([!R,U],[![T,V'],V''])}{\underset{\mathsf{sr}{\uparrow}}{S([!R,U],[!T,?V',V''])}}\frac{}{S[!(R,T),U,?V',V'']} \qquad \text{yields} \qquad \mathsf{sr}{\uparrow}\frac{S([!R,U],[![T,V'],V''])}{\underset{\mathsf{s}}{\frac{S[!(R,[T,V']),U,V'']}{\underset{\mathsf{p}{\downarrow}}{\frac{S[![(R,T),V'],U,V'']}{S[!(R,T),U,?V',V'']}}}}} \quad .
$$

(5) The contractum $([*R,U],[!T,V])$ of ρ is inside an active structure of the redex of π, but not inside a passive one. This is impossible for $\pi \in \{\mathsf{p}{\downarrow}, \mathsf{r}{\downarrow}\}$.

(6) The redex of π and the contractum $([*R,U],[!T,V])$ of ρ overlap. This case is impossible because a par structure cannot overlap with a times structure. $\qquad\square$

4.4.8   **Lemma**   *The rules* $\mathsf{sp}{\uparrow}$ *and* $\mathsf{sr}{\uparrow}$ *permute over the rule* $\mathsf{ai}{\downarrow}$.

Proof: Consider the derivation

$$
\rho\frac{\mathsf{ai}{\downarrow}\dfrac{Q}{S\{W\}}}{S\{Z\}} \quad ,
$$

where the application of $\rho \in \{\mathsf{sr}{\uparrow}, \mathsf{sp}{\uparrow}\}$ is nontrivial. According to 4.2.3, the cases to consider are:

(4) The redex of $\mathsf{ai}{\downarrow}$ is inside an active structure of the contractum W of ρ but not inside a passive one. This is impossible because an atom cannot be matched with a why-not or with an of-course structure.

(5) The contractum W of ρ is inside an active structure of the redex of $\mathsf{ai}{\downarrow}$. This is impossible because the application of ρ is nontrivial.

(6) The contractum W of ρ and the redex of π overlap. As before, this is impossible. $\quad\square$

4.4.9   **Observation**   If an instance of a rule $\rho \in \{\mathsf{sai}{\uparrow}, \mathsf{sp}{\uparrow}, \mathsf{sr}{\uparrow}\}$ is topmost in a derivation with premise 1, then the application of ρ must be trivial. This is easy to see because its premise is 1, and hence its conclusion is also 1. (In case of $\mathsf{sai}{\uparrow}$, this is only possible if one of a and $\bar{a}$ is 1 and the other is $\bot$.) This means that whenever a rule $\mathsf{sai}{\uparrow}, \mathsf{sp}{\uparrow}$, or $\mathsf{sr}{\uparrow}$ occurs at the top of a proof, it can be removed.

4.4.10   **Lemma (Atomic Cut Elimination)**   *The rule* $\mathsf{sai}{\uparrow}$ *permutes over the rules* $\mathsf{ai}{\downarrow}, \mathsf{ds}{\downarrow}, \mathsf{p}{\downarrow}$ *and* $\mathsf{r}{\downarrow}$ *by the rule* $\mathsf{ds}{\downarrow}$.

Proof: Consider the derivation

$$
\mathsf{sai}{\uparrow}\frac{\pi\dfrac{Q}{S([a,U],[\bar{a},V])}}{S[U,V]} \quad .
$$

If $\pi = \mathsf{ds}{\downarrow}$, then Lemma 4.4.6 applies. Now let $\pi \in \{\mathsf{w}{\downarrow}, \mathsf{ai}{\downarrow}, \mathsf{p}{\downarrow}, \mathsf{r}{\downarrow}\}$ be nontrivial. According to 4.2.3, the cases to consider are:

(4) The redex of π is inside the contractum $([a,U],[\bar{a},V])$ of $\mathsf{sai}{\uparrow}$ but not inside U or V.

(i) $\pi = \mathsf{ai}\downarrow$ and $U = [\bar{a}, U']$. Then

$$
\mathsf{sai}\uparrow \frac{\mathsf{ai}\downarrow \dfrac{S([1, U'], [\bar{a}, V])}{S([a, \bar{a}, U'], [\bar{a}, V])}}{S[\bar{a}, U', V]} \qquad \text{yields} \qquad \mathsf{ds}\downarrow \frac{S([1, U'], [\bar{a}, V])}{S[(1, [\bar{a}, V]), U']} \qquad .
$$

(ii) $\pi = \mathsf{ai}\downarrow$ and $V = [a, V']$. This is similar to (i).

(5) The contractum $([a, U], [\bar{a}, V])$ of $\mathsf{sai}\uparrow$ is inside an active structure of the redex of π but not inside a passive one. This is impossible because an atom cannot match with a why-not structure, an of-course structure or a proper times structure.

(6) The redex of π and the contractum of $\mathsf{sai}\uparrow$ overlap. This is impossible because no rule $\pi \in \{\mathsf{w}\downarrow, \mathsf{ai}\downarrow, \mathsf{p}\downarrow, \mathsf{r}\downarrow\}$ can have a proper times structure as redex. $\qquad\square$

4.4.11 Proposition *For every proof*

$$
\rho \frac{\overset{\Pi \Big\| \mathsf{ELSr} \backslash \{\mathsf{b}\downarrow, \mathsf{w}\downarrow\}}{T}}{R} \qquad , \quad \text{where } \rho \in \{\mathsf{sai}\uparrow, \mathsf{sp}\uparrow, \mathsf{sr}\uparrow\} \quad ,
$$

there is a proof $\begin{array}{c} \Pi' \Big\| \mathsf{dELSr} \backslash \{\mathsf{b}\downarrow, \mathsf{w}\downarrow\} \\ R \end{array}$.

Proof: First, the proof Π is transformed into a proof Π_d in system dELSr by applying Proposition 4.4.5. Then, if $\rho \in \{\mathsf{sp}\uparrow, \mathsf{sr}\uparrow\}$, apply Lemmata 4.4.6, 4.4.7, and 4.4.8, to permute the instance of ρ up. By Observation 4.4.9, it disappears at the top. If $\rho = \mathsf{sai}\uparrow$, then apply Lemma 4.4.10 to permute it up. $\qquad\square$

4.4.12 Lemma *For every proof* $\begin{array}{c} \Big\| \mathsf{dELSr} \backslash \{\mathsf{b}\downarrow\} \\ R \end{array}$ *there is a proof* $\begin{array}{c} \Big\| \mathsf{dELS} \backslash \{\mathsf{b}\downarrow\} \\ R \end{array}$.

Proof: Instead of eliminating the rule $\mathsf{r}\downarrow$, I will eliminate the rule

$$
\mathsf{sr}\downarrow \frac{S\{?U\}}{S[?R, ?T]} \quad ,
$$

where U is any structure such that there is a derivation

$$
\begin{array}{c} U \\ \Delta \Big\| \{\mathsf{w}\downarrow, \mathsf{ai}\downarrow, \mathsf{ds}\downarrow, \mathsf{p}\downarrow\} \\ [R, T] \end{array} \qquad .
$$

Note that $\mathsf{r}\downarrow$ is an instance of $\mathsf{sr}\downarrow$. Now consider the topmost instance of $\mathsf{sr}\downarrow$ and permute it up by applying the scheme in 4.2.3, i.e. consider a derivation

$$
\mathsf{sr}\downarrow \frac{\pi \dfrac{Q}{S\{?U\}}}{S[?R, ?T]} \quad ,
$$

where $\pi \in \mathsf{dELS} \setminus \{\mathsf{b}\downarrow\} = \{1\downarrow, \mathsf{w}\downarrow, \mathsf{ai}\downarrow, \mathsf{ds}\downarrow, \mathsf{p}\downarrow\}$ is nontrivial. According to 4.2.3, the cases to consider are:

(4) The redex of π is inside the contractum $?U$ of $\mathsf{sr}{\downarrow}$. Then replace

$$\mathsf{sr}{\downarrow}\frac{\pi\dfrac{S\{?U'\}}{S\{?U\}}}{S[?R,?T]} \qquad \text{by} \qquad \mathsf{sr}{\downarrow}\frac{S\{?U'\}}{S[?R,?T]} \quad .$$

(5) The contractum $?U$ of $\mathsf{sr}{\downarrow}$ is inside an active structure of the redex of π but not inside a passive one. Then the following subcases are possible:

(i) $\pi = \mathsf{p}{\downarrow}$ and $S\{?U\} = S'[!V,?U]$. Then replace

$$\mathsf{sr}{\downarrow}\frac{\mathsf{p}{\downarrow}\dfrac{S\{![V,U]\}}{S[!V,?U]}}{S[!V,?R,?T]} \qquad \text{by} \qquad \mathsf{p}{\downarrow}\frac{\mathsf{p}{\downarrow}\dfrac{\Delta\|\dfrac{S\{![V,U]\}}{S\{![V,R,T]\}}}{S[![V,R],?T]}}{S[!V,?R,?T]} \quad ,$$

where Δ is the derivation that exists by definition of $\mathsf{sr}{\downarrow}$.

(ii) $\pi = \mathsf{w}{\downarrow}$ and $?U$ is the redex. Then replace

$$\mathsf{sr}{\downarrow}\frac{\mathsf{w}{\downarrow}\dfrac{S\{\bot\}}{S\{?U\}}}{S[?R,?T]} \qquad \text{by} \qquad \mathsf{w}{\downarrow}\frac{\mathsf{w}{\downarrow}\dfrac{S\{\bot\}}{S\{?R\}}}{S[?R,?T]} \quad .$$

(iii) $\pi = \mathsf{w}{\downarrow}$ and $S\{?U\} = S'\{?S''\{?U\}\}$. Then replace

$$\mathsf{sr}{\downarrow}\frac{\mathsf{w}{\downarrow}\dfrac{S'\{\bot\}}{S'\{?S''\{?U\}\}}}{S'\{?S''[?R,?T]\}} \qquad \text{by} \qquad \mathsf{w}{\downarrow}\frac{S'\{\bot\}}{S'\{?S''[?R,?T]\}} \quad .$$

(iv) $\pi = 1{\downarrow}$ and $S\{?U\} = 1$. This is only possible if $U = \bot$, i.e. $S\{?U\} = S\{\bot\}$. Then

$$\mathsf{sr}{\downarrow}\frac{S\{\bot\}}{S[?R,?T]} \qquad \text{can be replaced by} \qquad \mathsf{w}{\downarrow}\frac{\mathsf{w}{\downarrow}\dfrac{S\{\bot\}}{S\{?R\}}}{S[?R,?T]} \quad .$$

(6) The redex of π and the contractum $?U$ of $\mathsf{sr}{\downarrow}$ overlap. This is impossible.

By this procedure, the topmost instance of $\mathsf{sr}{\downarrow}$ must disappear eventually. Repeat this for all instances of $\mathsf{sr}{\downarrow}$ in the derivation.   □

Now we can give the complete proof of the cut elimination theorem.

4.4.13   Theorem (Cut Elimination)   *System* dELS *is equivalent to every subsystem of the system* SELS $\cup\,\{1{\downarrow}\}$ *containing* ELS.

Proof: The proof follows the steps shown in Figure 4.13. For the first step apply Corollary 4.4.1, for the second and third step apply repeatedly Proposition 4.4.11, and for the last step use Lemma 4.4.12.   □

Finally, there is also a decomposition result for proofs in system dELS.

4.4.14 Proposition *For every proof* $\Pi \| \mathsf{dELS}$ *, there is a proof*
$$R$$

$$
1\downarrow \frac{}{1} \\
\quad \| \{\mathsf{ai}\downarrow\} \\
R_3 \\
\quad \| \{\mathsf{ds}\downarrow,\mathsf{p}\downarrow\} \\
R_2 \\
\quad \| \{\mathsf{w}\downarrow\} \\
R_1 \\
\quad \| \{\mathsf{b}\downarrow\} \\
R
$$

for some structures R_1, R_2, R_3.

Proof: We can easily permute $\mathsf{b}\downarrow$ and $\mathsf{w}\downarrow$ down. (There are no problematic cases, see 4.3.31 and the proof of Proposition 4.3.37.) Further, $\mathsf{ai}\downarrow$ can be permuted up by the results of Section 4.2. The details are left to the reader. $\qquad\square$

4.5 Interpolation

The technique of permuting rules, as it has been used in the previous sections, allows to obtain a top-down symmetric version of cut elimination: Instead of starting from a proof an permuting the rules of the up fragment to the top, where they disappear, we can start from a derivation and permute the rules of the up fragment to the top, where they remain. Dually, one can permute the rules of the down fragment to the bottom. As a result, one can obtain the following decomposition:

$$
T \\
\quad \| \text{up fragment} \\
I \\
\quad \| \text{down fragment} \\
R
$$
,

for some structure I. Observe that the structure I can only contain atoms that occur in T as well as in R. This resembles Craig's interpolation theorem for first-order classical logic [Cra57, Bus98]. This is the motivation for calling the structure I *interpolant*. In [Brü03a], K. Brünnler shows that for classical logic, in its presentation in the calculus of structures, the interpolation theorem can be stated in terms of derivations, as shown above.

Here, I will show such a theorem for multiplicative exponential linear logic. More precisely, with the second decomposition theorem (Theorem 4.3.2) and the lemmata of the previous section, we can get such an interpolation result for system SELS. Because of Proposition 4.4.14, the result can then be further decomposed. The only drawback is that it is unclear how the rules $\mathsf{r}\downarrow$ and $\mathsf{r}\uparrow$ could be eliminated.

4.5.1   Theorem (Interpolation)   *For every derivation* $\overset{T}{\underset{R}{\Delta}}\|\text{SELS}$ *there are structures*

$I, T_1, T_2, T_3, R_1, R_2, R_3$, *and derivations* $\Delta_1, \ldots, \Delta_8$, *such that*

$$
\begin{array}{c}
T \\
\Delta_1 \| \{\mathsf{b\uparrow}\} \\
T_1 \\
\Delta_2 \| \{\mathsf{w\uparrow}\} \\
T_2 \\
\Delta_3 \| \{\mathsf{s,p\uparrow,r\uparrow}\} \\
T_3 \\
\Delta_4 \| \{\mathsf{ai\uparrow}\} \\
I \\
\Delta_5 \| \{\mathsf{ai\downarrow}\} \\
R_3 \\
\Delta_6 \| \{\mathsf{s,p\downarrow,r\downarrow}\} \\
R_2 \\
\Delta_7 \| \{\mathsf{w\downarrow}\} \\
R_1 \\
\Delta_8 \| \{\mathsf{b\downarrow}\} \\
R
\end{array} \quad .
$$

Proof: The decomposition is obtained in four steps as indicated in Figure 4.14. They are quite similar to the four steps of the cut elimination proof (see Figure 4.13).

(1) The first step is an application of the second decomposition theorem (Theorem 4.3.2). Then, the instances of $\mathsf{p\uparrow}$ are replaced by $\mathsf{sp\uparrow}$, the instances of $\mathsf{ai\uparrow}$ by $\mathsf{sai\uparrow}$, and the instances of s by $\mathsf{ds\downarrow}$ and $\mathsf{ns\uparrow}$.

(2) In the second step the instances of $\mathsf{ns\uparrow}$ and $\mathsf{sp\uparrow}$ (and $\mathsf{sr\uparrow}$) are permuted up as in the cut elimination proof (which causes the introduction of $\mathsf{r\downarrow}$). The only difference is, that the rules do not disappear.

(3) In third step, the instances of $\mathsf{sai\uparrow}$ are permuted up (by Lemma 4.4.10).

(4) The last step is different from the cut elimination proof. First, the instances of $\mathsf{ai\downarrow}$ are permuted up (they have to permute only over the switch, which can be done by Lemma 4.2.6). Then, the instances of $\mathsf{sp\uparrow}$ are replaced by

$$
\mathsf{p\uparrow} \frac{\mathsf{s} \frac{\mathsf{s} \frac{S([?R,U],[!T,V])}{S[([?R,U],!T),V]}}{S[(?R,!T),U,V]}}{S[?(R,T),U,V]} \quad .
$$

Similarly, each instance of $\mathsf{sr\uparrow}$ is replaced by two instances of s and one instance of $\mathsf{r\uparrow}$, and each instance of $\mathsf{sai\uparrow}$ is replaced by two instances of s and one instance of $\mathsf{ai\uparrow}$. Finally the instances of $\mathsf{ai\uparrow}$ are permuted under the instances of s (by Lemma 4.2.7).

$\square$

$$
\begin{array}{ccccccccc}
 & & & & T & & & & T \\
 & & T & & \|\{b\uparrow\} & & T & & \|\{b\uparrow\} \\
 & & \|\{b\uparrow\} & & T_1 & & \|\{b\uparrow\} & & T_1 \\
 & & T_1 & & \|\{w\uparrow\} & & T_1 & & \|\{w\uparrow\} \\
 & & \|\{w\uparrow\} & & T_2 & & \|\{w\uparrow\} & & T_2 \\
 & & T_2 & & \|\{ns\uparrow,sp\uparrow,sr\uparrow\} & & T_2 & & \|\{s,p\uparrow,r\uparrow\} \\
 & & \|\{ai\downarrow\} & & T_3'' & & \|\{ns\uparrow,sp\uparrow,sr\uparrow\} & & T_3 \\
 T & & T_3' & & \|\{ai\downarrow\} & & T_3'' & & \|\{ai\uparrow\} \\
 \|SELS & \overset{1}{\rightsquigarrow} & \|\{ds\downarrow,ns\uparrow,p\downarrow,sp\uparrow\} & \overset{2}{\rightsquigarrow} & T_3''' & \overset{3}{\rightsquigarrow} & \|\{sai\uparrow\} & \overset{4}{\rightsquigarrow} & I \\
 R & & R_3' & & \|\{ds\downarrow,p\downarrow,r\downarrow\} & & I & & \|\{ai\downarrow\} \\
 & & \|\{sai\uparrow\} & & R_3' & & \|\{ai\downarrow,ds\downarrow,p\downarrow,r\downarrow\} & & R_3 \\
 & & R_2 & & \|\{sai\uparrow\} & & R_2 & & \|\{s,p\downarrow,r\downarrow\} \\
 & & \|\{w\downarrow\} & & R_2 & & \|\{w\downarrow\} & & R_2 \\
 & & R_1 & & \|\{w\downarrow\} & & R_1 & & \|\{w\downarrow\} \\
 & & \|\{b\downarrow\} & & R_1 & & \|\{b\downarrow\} & & R_1 \\
 & & R & & \|\{b\downarrow\} & & R & & \|\{b\downarrow\} \\
 & & & & R & & & & R \\
\end{array}
$$
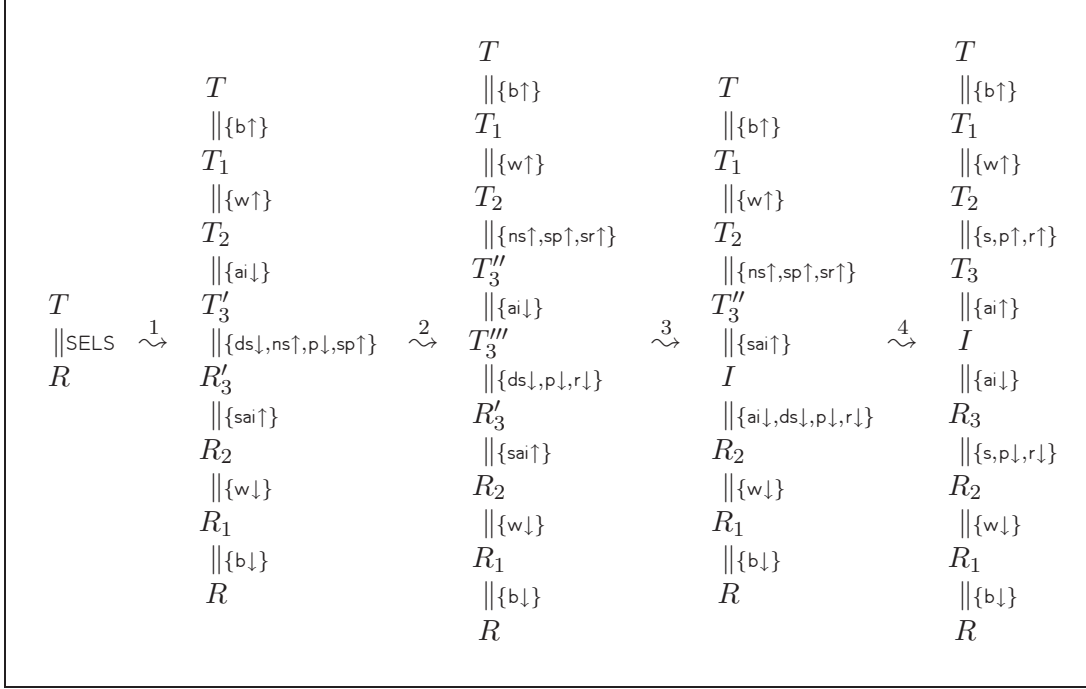
Figure 4.14: Proof of the interpolation theorem for system SELS

4.5.2 Remark Observe that the interpolation theorem could also be proved in the dual way, i.e. by introducing "super down rules" and permuting them down. In the general case the resulting derivation would be different from the one obtained via the previous proof.

Via the translation between structures and formulae as discussed in Chapter 3, it is also possible to obtain an interpolation result for MELL.

4.5.3 Corollary *Let A and B be two MELL formulae such that $A \multimap B$ is provable in LL. Then there is a MELL formula C such that C contains only atoms that occur in A as well as in B and such that $A \multimap C$ and $C \multimap B$ are provable in LL.*

Proof: Apply Theorem 4.5.1 to the derivation

$$
\underset{\underline{B}_{\mathsf{S}}}{\overset{\underline{A}_{\mathsf{S}}}{\|\mathsf{SELS}}} \quad ,
$$

which can be obtained from Theorem 4.1.7 (and Corollary 4.1.10). Then let $C = \underline{I}_{\mathsf{L}}$, where I is (a representation of) the interpolant obtained via Theorem 4.5.1. The formulae $A \multimap C$ and $C \multimap B$ are provable in LL because of Theorem 4.1.6. □

4.6 Discussion

The results presented in this chapter were the first that I found during my work on the calculus of structures. The main results of this chapter are the two decomposition theorems,

the cut elimination proof via the permutation of rules, and the interpolation theorem.

There is an interesting observation connected to the cut elimination argument. The separation between core and noncore allows us to split the cut elimination proof into two very different phases. First, the noncore up fragment is eliminated and at the same time the noncore down fragment is moved down, below all other rules. The surprising fact is now, that this step is responsible for the hyperexponential blow-up that is caused by cut elimination. Furthermore, all the difficulty connected to cut elimination went to the proof of the decomposition theorems, more precisely, to the separation of absorption. After this has been done, the cut elimination becomes rather simple. The elimination of the core up fragment, which is the second phase of the whole procedure, causes then only an exponential blow-up.

The use of permutation for eliminating the core up fragment, as it has been done in this chapter, will probably become obsolete after the discovery of splitting in [Gug02e] (see also Section 7.3). I included the proof in this thesis for the following reasons:

- It was the first correct purely syntactical cut elimination proof inside the calculus of structures.

- The permutation lemmata are needed to prove the interpolation theorem. At the current state of the art, I cannot see, how the technique of splitting can be employed to get interpolation.

- In Section 6.2, I will use some of the permutation results again.

- The various permutation results exhibit the wide variety of possibilities how derivations in the calculus of structures can be manipulated.

5

A Local System for Linear Logic

In Chapter 3, I introduced the calculus of structures and used linear logic as an example to show the the general principles. In this chapter, I will go one step further and explore the new possibilities and the new freedom of the calculus of structures. More precisely, I will show a system for linear logic, in which all rules are *local*, in the sense that their application requires only a bounded amount of computational resources. In particular, the contraction rule will be reduced to an atomic version, which is impossible in the sequent calculus [Brü02b].

In Section 5.1, I will give a more detailed discussion on locality. Then, in Section 5.2, I will present the local system, which is called SLLS. Although this system is quite big, it bears various regularities. In particular, the top-down symmetry of the calculus of structures is present also in system SLLS. I will show that SLLS it is strongly equivalent to system SLS, which implies that it is indeed a system for linear logic. Further, I will show cut elimination, by showing that the up fragment is admissible. Besides being local, system SLLS unveils regularities of the connectives of linear logic and their mutual relationship, that are hidden in the sequent calculus system. Then, in Section 5.3, I will show several decomposition theorems for the new system.

5.1 Locality via Atomicity

Formal proofs (which are the object of research in proof theory), consist of applications of *inference rules*. Proof theory can provide tools to modify proofs, for example, permuting inference rules, or replacing inference rules by other inference rules. But proof theory cannot look *into* the inference rules. An inference rule is like a black box, of which we can see only the input and the output, i.e. premise and conclusion. Proof theory can say more about the computational aspect of proof construction if less computation is hidden inside the inference rules. This means in particular, that when it comes to the implementation of a logical system, the application of each inference rule should consume only a bounded amount of computational resources (i.e. time and space). However, most deductive systems contain inference rules that do not have this property. Let me use as an example the sequent calculus system for linear logic (introduced in Chapter 2, see Figure 2.1). In particular,

consider the following three rules,

$$?\mathsf{c}\,\frac{\vdash\,?A,?A,\varPhi}{\vdash\,?A,\varPhi}\quad,\qquad \&\,\frac{\vdash A,\varPhi\quad\vdash B,\varPhi}{\vdash A\,\&\,B,\varPhi}\quad\text{and}\quad !\,\frac{\vdash A,?B_1,\ldots,?B_n}{\vdash\,!A,?B_1,\ldots,?B_n}\quad,$$

which are called *contraction*, *with* and *promotion*, respectively. If the contraction rule is applied in a proof search, going from bottom to top, the formula $?A$ has to be duplicated. Whatever mechanism is used for this duplication, it needs to inspect all of $?A$. In other words, the contraction rule needs a global view on $?A$. Further, the computational resources needed for applying contraction are not bounded, but depend on the size of $?A$. A similar situation occurs when the with rule is applied because the whole context $\varPhi$ of the formula $A\,\&\,B$ has to be copied. Another rule which involves a global knowledge of the context is the promotion rule, where for each formula in the context of $!A$ it has to be checked whether it has the form $?B$. Inference rules, like contraction, with and promotion, that require such a global view on formulae or sequents of unbounded size, are called *nonlocal*, and all other rules are called *local* [BT01]. For example, the two rules

$$\otimes\,\frac{\vdash A,\varPhi\quad\vdash B,\varPsi}{\vdash A\otimes B,\varPhi,\varPsi}\qquad\text{and}\qquad \otimes\,\frac{\vdash A,B,\varPhi}{\vdash A\,\otimes\,B,\varPhi}\quad,$$

which are called *times* and *par*, respectively, are local because they do not need to look into the formulae A and B or their contexts. They require only a bounded amount of computational resources because it would suffice to operate on pointers to A and B, which depend not on the size of the formulae.

Observe that sharing cannot be used for implementing a nonlocal rule because after copying a formula A in an application of the contraction or with rule, both copies of A might be used and modified in a very different way.

In [BT01] it has been shown that it is possible to design a local system (i.e. a deductive system in which all rules are local) for classical logic. Here, I will show a local system for linear logic.

The basic idea for making a system local is replacing each nonlocal inference rule by a local version. I have shown in Chapter 3, how the promotion rule can be made local in the calculus of structures. For the other rules, locality can be achieved by restricting their application to atoms, which are formulae of bounded size. This idea is not new: the (nonlocal) identity rule in the sequent calculus can be replaced by its atomic counterpart (which is considered to be local) without affecting provability. To make the contraction rule admissible for its atomic version, it is necessary to add new inference rules to the system in order to maintain completeness. As already observed in [BT01], these new rules cannot be given in the sequent calculus which is too rigid. But they can be presented in the calculus of structures which provides the necessary freedom for designing inference rules.

The atomicity of contraction is achieved by copying formulae stepwise, i.e. atom by atom, and by using the new rules to restore the original formula to be copied. Operationally this itself is not very interesting. The surprising fact is that this can be done *inside* a logical system without losing important properties like cut elimination, soundness and completeness.

Apart from the proof theoretical interest in local rules, there is also interest from the viewpoint of applications:

- Proof construction, in particular the notion of uniform provability, is mainly concerned with controlling the nondeterminism that is intrinsically connected to proof search. For linear logic, this is based on permutability properties, which yield optimised presentations of cut free proofs [And92, HM94]. Although in the calculus of structures (and in particular in the local systems) the nondeterminism in the proof search is much higher than in the sequent calculus, the local system for linear logic provides much more possibilities for permuting rules. This might yield new normal forms of cut free proofs. From this, new methods for controlling the nondeterminism could emerge.

- Since distributed computing moves more and more into the focus of research in computer science, it is natural to ask whether it is conceivable to implement proof search in a distributed way. For this it is essential that each application of an inference rule consumes only a bounded amount time and space. This is provided by a local system.

5.2 Rules and Cut Elimination

In this section, I will start from system SLS and produce a strongly equivalent system, in which all rules are local, in the sense as discussed in the previous section.

Before discussing the new system, I will first list the nonlocal rules of system SLS (Figure 3.2 on page 35) and give some informal reason why they are not local. Obviously the rules c↓ and b↓ are nonlocal because they involve the duplication of a structure of unbounded size. Also their corules c↑ and b↑ are nonlocal because they involve the comparison and the deletion of structures of unbounded size (or, again a duplication if one reasons top-down). Similarly, I consider the rules t↓ and w↓, as well as their corules t↑ and w↑ to be nonlocal because they involve the deletion or introduction of structures of unbounded size. One could argue that the deletion of a structure of unbounded size can be considered local because it might suffice to delete just the pointer to the structure. But in a real implementation, garbage collection would then become a problem. Another more important reason is that the symmetry exhibited in system SLS should be carried through to the local system, and therefore, the locality of a rule should be invariant under forming the contrapositive, i.e. the corule.

Observe that all other rules of system SLS are already local. In particular, the two rules ai↓ and ai↑ only involve atoms. In the switch rule

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

the structures R, T and U all occur exactly once in redex and contractum, which means that no structure of unbounded size has to be copied or deleted. For this reason, the switch rule is considerd to be local. Informally, this can be motivated as follows: The rule could be implemented by changing the marking of two nodes and exchanging two pointers, as already observed in [BT01]:
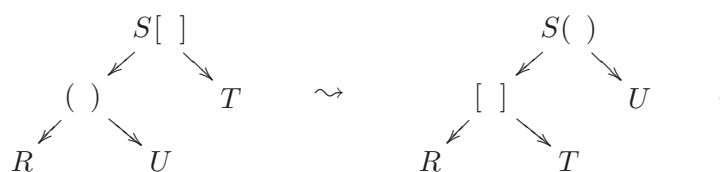
$$\mathsf{ai}{\downarrow}\,\frac{S\{1\}}{S[a,\bar a]} \qquad\qquad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar a)}{S\{\bot\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}{\downarrow}\,\frac{S\langle[R,U],[T,V]\rangle}{S[\langle R,T\rangle,\langle U,V\rangle]} \qquad \mathsf{d}{\uparrow}\,\frac{S(\langle R,U\rangle,\langle T,V\rangle)}{S\langle(R,T),(U,V)\rangle}$$

$$\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad\qquad \mathsf{p}{\uparrow}\,\frac{S(?R,!T)}{S\{?(R,T)\}}$$

$$\mathsf{at}{\downarrow}\,\frac{S\{0\}}{S\{a\}} \qquad \mathsf{ac}{\downarrow}\,\frac{S\langle a,a\rangle}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\,\frac{S\{a\}}{S\langle a,a\rangle} \qquad \mathsf{at}{\uparrow}\,\frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{nm}{\downarrow}\,\frac{S\{0\}}{S\langle 0,0\rangle} \qquad \mathsf{m}\,\frac{S\langle\langle R,U\rangle,\langle T,V\rangle\rangle}{S\langle\langle R,T\rangle,\langle U,V\rangle\rangle} \qquad \mathsf{nm}{\uparrow}\,\frac{S\langle\top,\top\rangle}{S\{\top\}}$$

$$\mathsf{nm_1}{\downarrow}\,\frac{S\{0\}}{S[0,0]} \qquad \mathsf{m_1}{\downarrow}\,\frac{S\langle[R,U],[T,V]\rangle}{S[\langle R,T\rangle,\langle U,V\rangle]} \qquad \mathsf{m_1}{\uparrow}\,\frac{S(\langle R,U\rangle,\langle T,V\rangle)}{S\langle(R,T),(U,V)\rangle} \qquad \mathsf{nm_1}{\uparrow}\,\frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{nm_2}{\downarrow}\,\frac{S\{0\}}{S(0,0)} \qquad \mathsf{m_2}{\downarrow}\,\frac{S\langle(R,U),(T,V)\rangle}{S(\langle R,T\rangle,\langle U,V\rangle)} \qquad \mathsf{m_2}{\uparrow}\,\frac{S[\langle R,U\rangle,\langle T,V\rangle]}{S\langle[R,T],[U,V]\rangle} \qquad \mathsf{nm_2}{\uparrow}\,\frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{nl_1}{\downarrow}\,\frac{S\{0\}}{S\{?0\}} \qquad \mathsf{l_1}{\downarrow}\,\frac{S\langle?R,?T\rangle}{S\{?\langle R,T\rangle\}} \qquad \mathsf{l_1}{\uparrow}\,\frac{S\{!\langle R,T\rangle\}}{S\langle!R,!T\rangle} \qquad \mathsf{nl_1}{\uparrow}\,\frac{S\{!\top\}}{S\{\top\}}$$

$$\mathsf{nl_2}{\downarrow}\,\frac{S\{0\}}{S\{!0\}} \qquad \mathsf{l_2}{\downarrow}\,\frac{S\langle!R,!T\rangle}{S\{!\langle R,T\rangle\}} \qquad \mathsf{l_2}{\uparrow}\,\frac{S\{?\langle R,T\rangle\}}{S\langle?R,?T\rangle} \qquad \mathsf{nl_2}{\uparrow}\,\frac{S\{?\top\}}{S\{\top\}}$$

$$\mathsf{nz}{\downarrow}\,\frac{S\{\bot\}}{S\{?0\}} \qquad \mathsf{z}{\downarrow}\,\frac{S[?R,T]}{S\{?\langle R,T\rangle\}} \qquad \mathsf{z}{\uparrow}\,\frac{S\{!\langle R,T\rangle\}}{S(!R,T)} \qquad \mathsf{nz}{\uparrow}\,\frac{S\{!\top\}}{S\{1\}}$$

Figure 5.1: System SLLS

For the rules d↓, d↑, p↓ and p↑, the situation is similar.

  Let me summarise this discussion by explaining when I call a rule in the calculus of structures local. For this, let us interpret the inference rules in the calculus of structures, as I have shown them in the previous chapters, as term rewriting rules (see e.g. [BN98]). We then have to distinguish between two sorts of variables, namely, those which stand for

arbitrary structures and those which stand for atoms. I will consider a rule to be *local*, if each variable of the first kind, which occurs in the rule, occurs exactly once in the redex and exactly once in the contractum. Further, there should be no restriction on the context. In the terminology of term rewriting, locality would correspond to linearity. To make this clear, let me use as example the following two rules:

$$\mathsf{b}{\uparrow}\ \frac{S\{!R\}}{S(!R,R)} \qquad \text{and} \qquad \mathsf{sai}{\uparrow}\ \frac{S([a,U],[\bar{a},V])}{S[U,V]}\quad.$$

Whereas the first is nonlocal because the R occurs twice in the redex, the second rule is local because U and V occur exactly once in redex and contractum and a and $\bar{a}$ are atoms.

Let us now have a look at the new system, which is called *system* SLLS (the second L stands for "local") and which is shown in Figure 5.1. The reader should not be frightened by the size of the system. I will discuss the rules later on. At this point I will draw the attention to the fact that all rules in system SLLS are local. Either they handle only atoms, or their implementation can be realised by exchanging pointers in a similar way as for the switch rule. The equations shown in Figure 3.1 can be made local by implementing them in the same way as the inference rules. This means that system SLLS is indeed a local system. It remains to show that it is linear logic.

In order to do so, I will show that it is strongly equivalent to system SLS. Further, I will define system LLS by adding the axiom $1{\downarrow}$ to the down fragment of system SLLS. System LLS will be strongly equivalent to system LS. As a corollary we get a cut elimination result for the local system.

Consider now the rules

$$\mathsf{ac}{\downarrow}\ \frac{S\{a,a\}}{S\{a\}} \qquad \text{and} \qquad \mathsf{ac}{\uparrow}\ \frac{S\{a\}}{S(a,a)}\quad,$$

which are called *atomic contraction* and *atomic cocontraction*, respectively. They replace their general nonlocal counterparts $\mathsf{c}{\downarrow}$ and $\mathsf{c}{\uparrow}$. But they are not powerful enough to ensure completeness. For this reason, the *medial* rule

$$\mathsf{m}\ \frac{S\{(R,U),(T,V)\}}{S((R,T),(U,V))}\quad,$$

and its variations

$$\mathsf{m}_1{\downarrow}\ \frac{S\{[R,U],[T,V]\}}{S[(R,T),(U,V)]} \quad,\qquad \mathsf{m}_1{\uparrow}\ \frac{S((R,U),(T,V))}{S((R,T),(U,V))}\quad,$$

$$\mathsf{m}_2{\downarrow}\ \frac{S\{(R,U),(T,V)\}}{S((R,T),(U,V))} \quad,\qquad \mathsf{m}_2{\uparrow}\ \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}\quad,$$

$$\mathsf{l}_1{\downarrow}\ \frac{S\{?R,?T\}}{S\{?(R,T)\}} \quad,\qquad \mathsf{l}_1{\uparrow}\ \frac{S\{!(R,T)\}}{S(!R,!T)}\quad,$$

$$\mathsf{l}_2{\downarrow}\ \frac{S\{!R,!T\}}{S\{!(R,T)\}} \quad,\qquad \mathsf{l}_2{\uparrow}\ \frac{S\{?(R,T)\}}{S(?R,?T)}\quad,$$

are introduced. These rules have the same purpose as the medial rule in [BT01]. There are more medial rules in system SLLS because there are more connectives in linear logic than in classical logic.

5.2.1   Proposition   *The rule* $\mathsf{c}{\downarrow}$ *is derivable in* $\{\mathsf{ac}{\downarrow}, \mathsf{m}, \mathsf{m}_1{\downarrow}, \mathsf{m}_2{\downarrow}, \mathsf{l}_1{\downarrow}, \mathsf{l}_2{\downarrow}\}$. *Dually, the rule* $\mathsf{c}{\uparrow}$ *is derivable in* $\{\mathsf{ac}{\uparrow}, \mathsf{m}, \mathsf{m}_1{\uparrow}, \mathsf{m}_2{\uparrow}, \mathsf{l}_1{\uparrow}, \mathsf{l}_2{\uparrow}\}$.

Proof: For a given instance $\mathsf{c}{\downarrow}\dfrac{S\{R, R\}}{S\{R\}}$, I will construct a derivation

$$
\begin{array}{c}
S\{R, R\} \\
\Delta \left\| {\scriptstyle \{\mathsf{ac}{\downarrow},\mathsf{m},\mathsf{m}_1{\downarrow},\mathsf{m}_2{\downarrow},\mathsf{l}_1{\downarrow},\mathsf{l}_2{\downarrow}\}} \right. \\
S\{R\}
\end{array}
$$

by structural induction on R.

- If R is an atom then the given instance of $\mathsf{c}{\downarrow}$ is an instance of $\mathsf{ac}{\downarrow}$.

- If $R = \{P, Q\}$, where $P \neq 0 \neq Q$, then apply the induction hypothesis to

$$
\mathsf{c}{\downarrow}\dfrac{\mathsf{c}{\downarrow}\dfrac{S\{P, Q, P, Q\}}{S\{P, P, Q\}}}{S\{P, Q\}} \quad .
$$

- If $R = [P, Q]$, where $P \neq \bot \neq Q$, then apply the induction hypothesis to

$$
\mathsf{c}{\downarrow}\dfrac{\mathsf{c}{\downarrow}\dfrac{\mathsf{m}_1{\downarrow}\dfrac{S\{[P, Q], [P, Q]\}}{S[\{P, P\}, \{Q, Q\}]}}{S[\{P, P\}, Q]}}{S[P, Q]} \quad .
$$

- If $R = (P, Q)$, where $P \neq 1 \neq Q$, then apply the induction hypothesis to

$$
\mathsf{c}{\downarrow}\dfrac{\mathsf{c}{\downarrow}\dfrac{\mathsf{m}_2{\downarrow}\dfrac{S\{(P, Q), (P, Q)\}}{S(\{P, P\}, \{Q, Q\})}}{S(\{P, P\}, Q)}}{S(P, Q)} \quad .
$$

- If $R = \langle P, Q\rangle$, where $P \neq \top \neq Q$, then apply the induction hypothesis to

$$
\mathsf{c}{\downarrow}\dfrac{\mathsf{c}{\downarrow}\dfrac{\mathsf{m}\dfrac{S\{\langle P, Q\rangle, \langle P, Q\rangle\}}{S\langle\{P, P\}, \{Q, Q\}\rangle}}{S\langle\{P, P\}, Q\rangle}}{S\langle P, Q\rangle} \quad .
$$

- If $R = {?}P$, where $P \neq \bot$, then apply the induction hypothesis to

$$
\mathsf{c}{\downarrow}\dfrac{\mathsf{l}_1{\downarrow}\dfrac{S\{{?}P, {?}P\}}{S\{{?}\{P, P\}\}}}{S\{{?}P\}} \quad .
$$

- If $R = !P$, where $P \neq 1$, then apply the induction hypothesis to

$$
\mathsf{c}{\downarrow} \frac{\mathsf{l}_2{\downarrow} \dfrac{S\{!P, !P\}}{S\{!\{P, P\}\}}}{S\{!P\}} \quad .
$$

The proof for $\mathsf{c}{\uparrow}$ is dual. □

Let us now consider the rules

$$
\mathsf{at}{\downarrow} \frac{S\{0\}}{S\{a\}} \qquad \text{and} \qquad \mathsf{at}{\uparrow} \frac{S\{a\}}{S\{\top\}} \quad ,
$$

called *atomic thinning* and *atomic cothinning*, respectively. Again, they are the replacement for the general thinning and cothinning rules, and in order to keep completeness, we need to add the rules

$$
\mathsf{nm}{\downarrow} \frac{S\{0\}}{S(0,0)} \quad , \qquad \mathsf{nm}_1{\downarrow} \frac{S\{0\}}{S[0,0]} \quad , \qquad \mathsf{nm}_2{\downarrow} \frac{S\{0\}}{S(0,0)} \quad , \qquad \mathsf{nl}_1{\downarrow} \frac{S\{0\}}{S\{?0\}} \quad , \qquad \mathsf{nl}_2{\downarrow} \frac{S\{0\}}{S\{!0\}} \quad ,
$$

$$
\mathsf{nm}{\uparrow} \frac{S\{\top, \top\}}{S\{\top\}} \quad , \qquad \mathsf{nm}_1{\uparrow} \frac{S(\top, \top)}{S\{\top\}} \quad , \qquad \mathsf{nm}_2{\uparrow} \frac{S[\top, \top]}{S\{\top\}} \quad , \qquad \mathsf{nl}_1{\uparrow} \frac{S\{!\top\}}{S\{\top\}} \quad , \qquad \mathsf{nl}_2{\uparrow} \frac{S\{?\top\}}{S\{\top\}} \quad ,
$$

which are the nullary versions of the medial rules. In the local system for classical logic [BT01] these rules are hidden in the equational theory for structures. It might be argued about doing the same for linear logic. In this presentation, I chose not to do so because of the following reasons: First, not all of them are equivalences in linear logic, e.g. we have $0 \multimap ?0$ but not $?0 \multimap 0$, and second, for obvious reasons I want to use the same equational theory for both systems, SLS and SLLS. But the new equations, e.g. $0 = !0$, would be redundant for system SLS.

5.2.2 Definition For notational convenience, let me define the following two systems:

$$
\begin{aligned}
\mathsf{SLLSt}{\downarrow} &= \{\mathsf{at}{\downarrow}, \mathsf{nm}{\downarrow}, \mathsf{nm}_1{\downarrow}, \mathsf{nm}_2{\downarrow}, \mathsf{nl}_1{\downarrow}, \mathsf{nl}_2{\downarrow}\} \quad \text{and} \\
\mathsf{SLLSt}{\uparrow} &= \{\mathsf{at}{\uparrow}, \mathsf{nm}{\uparrow}, \mathsf{nm}_1{\uparrow}, \mathsf{nm}_2{\uparrow}, \mathsf{nl}_1{\uparrow}, \mathsf{nl}_2{\uparrow}\} \quad .
\end{aligned}
$$

5.2.3 Proposition *The rule* $\mathsf{t}{\downarrow}$ *is derivable in* $\mathsf{SLLSt}{\downarrow}$*. Dually,* $\mathsf{t}{\uparrow}$ *is derivable in* $\mathsf{SLLSt}{\uparrow}$*.*

Proof: Similar to Proposition 5.2.1, I will for a given instance $\mathsf{t}{\downarrow} \dfrac{S\{0\}}{S\{R\}}$, construct a derivation

$$
\begin{array}{c}
S\{0\} \\
\Delta \Big\| \mathsf{SLLSt}{\downarrow} \\
S\{R\}
\end{array} \quad ,
$$

by structural induction on R.

- If R is an atom then the given instance of $\mathsf{t}{\downarrow}$ is an instance of $\mathsf{at}{\downarrow}$.

- If $R = \langle P, Q \rangle$, where $P \neq \top \neq Q$, then apply the induction hypothesis to

$$
\begin{array}{c}
\mathsf{nm}{\downarrow} \dfrac{S\{0\}}{S\langle 0, 0 \rangle} \\
\mathsf{t}{\downarrow} \dfrac{}{S\langle P, 0 \rangle} \\
\mathsf{t}{\downarrow} \dfrac{}{S\langle P, Q \rangle}
\end{array} \quad .
$$

The other cases are similar. The proof for $\mathsf{t}{\uparrow}$ is dual. □

Now arises the problem that the rules $\mathsf{w}{\downarrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}$ cannot be reduced to their atomic versions in the same way as this has been done for the rules $\mathsf{t}{\downarrow}, \mathsf{t}{\uparrow}, \mathsf{c}{\downarrow}, \mathsf{c}{\uparrow}$. This is not really a surprise since the basic idea of the exponentials is to guard whole subformulas such that no arbitrary weakening and contraction is possible. For this reason, I will reduce the rules $\mathsf{w}{\downarrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}$ to the rules $\mathsf{t}{\downarrow}, \mathsf{t}{\uparrow}, \mathsf{c}{\downarrow}, \mathsf{c}{\uparrow}$, respectively, by using the well-known equivalence

$$
!(A \,\&\, B) \;\equiv\; !A \otimes !B \quad ,
$$

which is encoded in the rules

$$
\mathsf{z}{\downarrow} \dfrac{S[?R, T]}{S\{?\langle R, T \rangle\}} \qquad \text{and} \qquad \mathsf{z}{\uparrow} \dfrac{S\{!\langle R, T \rangle\}}{S(!R, T)} \quad ,
$$

and their nullary versions

$$
\mathsf{nz}{\downarrow} \dfrac{S\{\bot\}}{S\{?0\}} \qquad \text{and} \qquad \mathsf{nz}{\uparrow} \dfrac{S\{!\top\}}{S\{1\}} \quad .
$$

5.2.4   Proposition   *The rule* $\mathsf{w}{\downarrow}$ *is derivable in* $\{\mathsf{nz}{\downarrow}, \mathsf{t}{\downarrow}\}$, *and the rule* $\mathsf{b}{\downarrow}$ *is derivable in* $\{\mathsf{z}{\downarrow}, \mathsf{c}{\downarrow}\}$. *Dually, the rule* $\mathsf{w}{\uparrow}$ *is derivable in* $\{\mathsf{nz}{\uparrow}, \mathsf{t}{\uparrow}\}$, *and the rule* $\mathsf{b}{\uparrow}$ *is derivable in* $\{\mathsf{z}{\uparrow}, \mathsf{c}{\uparrow}\}$.

Proof: Use the derivations

$$
\begin{array}{c}
\mathsf{nz}{\downarrow} \dfrac{S\{\bot\}}{S\{?0\}} \\
\mathsf{t}{\downarrow} \dfrac{}{S\{?R\}}
\end{array} \qquad \text{and} \qquad
\begin{array}{c}
\mathsf{z}{\downarrow} \dfrac{S[?R, R]}{S\{?\langle R, R \rangle\}} \\
\mathsf{c}{\downarrow} \dfrac{}{S\{?R\}}
\end{array} \quad ,
$$

for $\mathsf{w}{\downarrow}$ and $\mathsf{b}{\downarrow}$, respectively. □

All new rules, introduced in this section, are sound, i.e. correspond to linear implications. More precisely, we have:

5.2.5   Proposition   *The rules* $\mathsf{ac}{\downarrow}$, m, $\mathsf{m}_1{\downarrow}$, $\mathsf{m}_2{\downarrow}$, $\mathsf{l}_1{\downarrow}$, $\mathsf{l}_2{\downarrow}$ *are derivable in* $\{\mathsf{t}{\downarrow}, \mathsf{c}{\downarrow}\}$, *the rule* $\mathsf{z}{\downarrow}$ *is derivable in* $\{\mathsf{t}{\downarrow}, \mathsf{b}{\downarrow}\}$, *the rules* $\mathsf{at}{\downarrow}$, $\mathsf{nm}{\downarrow}$, $\mathsf{nm}_1{\downarrow}$, $\mathsf{nm}_2{\downarrow}$, $\mathsf{nl}_1{\downarrow}$, $\mathsf{nl}_2{\downarrow}$ *are derivable in* $\{\mathsf{t}{\downarrow}\}$, *and the rule* $\mathsf{nz}{\downarrow}$ *is derivable in* $\{\mathsf{w}{\downarrow}\}$. *Dually, the rules* $\mathsf{ac}{\uparrow}$, m, $\mathsf{m}_1{\uparrow}$, $\mathsf{m}_2{\uparrow}$, $\mathsf{l}_1{\uparrow}$, $\mathsf{l}_2{\uparrow}$ *are derivable in* $\{\mathsf{t}{\uparrow}, \mathsf{c}{\uparrow}\}$, *the rule* $\mathsf{z}{\uparrow}$ *is derivable in* $\{\mathsf{t}{\uparrow}, \mathsf{b}{\uparrow}\}$, *the rules* $\mathsf{at}{\uparrow}$, $\mathsf{nm}{\uparrow}$, $\mathsf{nm}_1{\uparrow}$, $\mathsf{nm}_2{\uparrow}$, $\mathsf{nl}_1{\uparrow}$, $\mathsf{nl}_2{\uparrow}$ *are derivable in* $\{\mathsf{t}{\uparrow}\}$, *and the rule* $\mathsf{nz}{\uparrow}$ *is derivable in* $\{\mathsf{w}{\uparrow}\}$.

Proof: The rule $\mathsf{ac}{\downarrow}$ is an instance of $\mathsf{c}{\downarrow}$. For the rule $\mathsf{m}_1{\downarrow}$, use the derivation

$$
\mathsf{c}{\downarrow}\frac{\mathsf{t}{\downarrow}\dfrac{\mathsf{t}{\downarrow}\dfrac{\mathsf{t}{\downarrow}\dfrac{\mathsf{t}{\downarrow}\dfrac{S\{[R,U],[T,V]\}}{S\{[R,U],[T,\{U,V\}]\}}}{S\{[R,U],[\{R,T\},\{U,V\}]\}}}{S\{[R,\{U,V\}],[\{R,T\},\{U,V\}]\}}}{S\{[\{R,T\},\{U,V\}],[\{R,T\},\{U,V\}]\}}}{S[\{R,T\},\{U,V\}]}\quad .
$$

The cases for the rules m, $\mathsf{m}_2{\downarrow}$, $\mathsf{l}_1{\downarrow}$, $\mathsf{l}_2{\downarrow}$ and $\mathsf{z}{\downarrow}$ are similar. The rules $\mathsf{at}{\downarrow}, \mathsf{nm}{\downarrow}, \mathsf{nm}_1{\downarrow}, \mathsf{nm}_2{\downarrow}$, $\mathsf{nl}_1{\downarrow}$ and $\mathsf{nl}_2{\downarrow}$ are instances of $\mathsf{t}{\downarrow}$, and $\mathsf{nz}{\downarrow}$ is an instance of $\mathsf{w}{\downarrow}$. $\qquad\square$

5.2.6   Theorem   *Systems* SLLS *and* SLS *are strongly equivalent.*

Proof: Immediate consequence of Propositions 5.2.1–5.2.5. $\qquad\square$

Observe that Propositions 5.2.1–5.2.5 show that there is a certain modularity involved in the equivalence of the two systems. For instance, the user can choose to have either the rules $\mathsf{at}{\downarrow}, \mathsf{nm}{\downarrow}, \mathsf{nm}_1{\downarrow}, \mathsf{nm}_2{\downarrow}, \mathsf{nl}_1{\downarrow}, \mathsf{nl}_2{\downarrow}$ or the rule $\mathsf{t}{\downarrow}$ in the system without affecting the other rules.

5.2.7   Definition   *System* LLS, shown in Figure 5.2, is obtained from the down fragment of system SLLS by adding the axiom $1{\downarrow}$.

5.2.8   Theorem   *The systems* LLS *and* LS *are strongly equivalent.*

Proof: Again, use Propositions 5.2.1–5.2.5. $\qquad\square$

As an immediate consequence of Theorem 5.2.8 we have that splitting (Lemma 3.4.5) and context reduction (Lemma 3.4.22) do also hold for system LLS. In particular, we also have that the up fragment is admissible.

5.2.9   Corollary (Cut Elimination)   *System* LLS *is equivalent to every subsystem of* $\mathsf{SLLS} \cup \{1{\downarrow}\}$ *containing* LLS.

5.2.10   Corollary   *The rule* $\mathsf{i}{\uparrow}$ *is admissible for system* LLS.

System SLLS bears many regularities. For example, all medial rules follow the same pattern. In Section 8.2, I will come back to this, and discuss the possibility of formulating a general "recipe" of designing inference rules in the calculus of structures.

5.3   Decomposition

In this section, I will show several decomposition theorems for system SLLS. They are of a different nature than the decompositions shown before, because the atomicity of contraction allows manipulations that are impossible in system SLS. But compared to decomposition results of system SELS in the previous chapter, in Section 4.3, the results presented here

$$1\downarrow \frac{}{1}$$

$$\mathsf{ai}\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \mathsf{nm_1}\downarrow \frac{S\{0\}}{S[0,0]} \qquad \mathsf{m_1}\downarrow \frac{S\{[R,U],[T,V]\}}{S[\{R,T\},\{U,V\}]}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]} \qquad \mathsf{nm_2}\downarrow \frac{S\{0\}}{S(0,0)} \qquad \mathsf{m_2}\downarrow \frac{S\{(R,U),(T,V)\}}{S(\{R,T\},\{U,V\})}$$

$$\mathsf{d}\downarrow \frac{S\langle[R,U],[T,V]\rangle}{S[\langle R,T\rangle,\langle U,V\rangle]} \qquad \mathsf{nm}\downarrow \frac{S\{0\}}{S\langle 0,0\rangle} \qquad \mathsf{m}\,\frac{S\langle(R,U),(T,V)\rangle}{S\langle\{R,T\},\{U,V\}\rangle}$$

$$\mathsf{p}\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{nl_1}\downarrow \frac{S\{0\}}{S\{?0\}} \qquad \mathsf{l_1}\downarrow \frac{S\{?R,?T\}}{S\{?\{R,T\}\}}$$

$$\mathsf{at}\downarrow \frac{S\{0\}}{S\{a\}} \qquad \mathsf{nl_2}\downarrow \frac{S\{0\}}{S\{!0\}} \qquad \mathsf{l_2}\downarrow \frac{S\{!R,!T\}}{S\{!\{R,T\}\}}$$

$$\mathsf{ac}\downarrow \frac{S\{a,a\}}{S\{a\}} \qquad \mathsf{nz}\downarrow \frac{S\{\bot\}}{S\{?0\}} \qquad \mathsf{z}\downarrow \frac{S[?R,T]}{S\{?\{R,T\}\}}$$

Figure 5.2: System LLS

are rather simple. They should be considered as preliminary work for future research in the topic of decomposition.

All proofs are based on simple permutation lemmata, as I have shown them before. However, now there not only two, but four types of contexts. Furthermore, system SLLS is considerably bigger than system SELS, which means that there are more rules to consider in a case analysis. Nevertheless, the basic idea is always the same. Since this is a dissertation thesis, I put some effort in listing all the cases, although they are in principle all the same. This has the effect that the proofs in this section look rather involved. But the reader should keep in mind that this is not the case. We have here just another example, where syntax is an obstacle to short and concise proofs. This means that what has been said in the previous chapter does also apply here: the impatient reader is invited to read only the statements of the theorems at the beginning of each subsection and skip the proofs.

5.3.1   Separation of Atomic Contraction

The first theorem deals with atomic contraction. The intuition behind contraction is to copy parts of a structure to make them usable more than once in a proof or derivation. This intuition tells us that it should be possible to copy first everything as much as it is

needed and then go on with the proof without copying anything anymore. This is exactly, what the following theorem says.

5.3.1 **Theorem** *For every proof* $\overset{\displaystyle}{\underset{R}{\|}}\mathsf{LLS}$ *there is a proof*

$$\overset{\displaystyle}{\underset{R'}{\|}}\mathsf{(LLS\cup\{nm\uparrow\})\backslash\{ac\downarrow\}}$$
$$\overset{\displaystyle R'}{\underset{R}{\|}}\mathsf{\{ac\downarrow\}}$$

for some structure R'.

The top-down symmetry allows us, to extend this result to derivations:

5.3.2 **Theorem** *For every derivation* $\Delta\overset{\displaystyle T}{\underset{R}{\|}}\mathsf{SLLS}$ *there is a derivation*

$$T$$
$$\overset{\displaystyle}{\underset{T'}{\|}}\mathsf{\{ac\uparrow\}}$$
$$\overset{\displaystyle T'}{\underset{R'}{\|}}\mathsf{SLLS\backslash\{ac\uparrow,ac\downarrow\}}$$
$$\overset{\displaystyle R'}{\underset{R}{\|}}\mathsf{\{ac\downarrow\}}$$

for some structures T' and R'.

Before showing the proofs of the two theorems, let me point out the difficulties that arise. In the proof of the decomposition theorems in the previous chapter, the separation of absorption rule was problematic. The same problem has to be expected for contraction. In fact, the case for contraction is even more difficult because the substructure to be contracted is not guarded by an exponential. For example, in the following two derivations it is unclear how the instances of cocontraction could be permuted up:

$$\mathsf{c\uparrow}\frac{\mathsf{s}\dfrac{S([R,U_1,U_2],T)}{S[(R,T),U_1,U_2]}}{S[\langle[(R,T),U_1],[(R,T),U_1]\rangle,U_2]}\quad\text{and}$$

$$\mathsf{c\uparrow}\frac{\mathsf{s}\dfrac{S([(R_1,R_2),U],T_1,T_2)}{S[(R_1,R_2,T_1,T_2),U]}}{S[(R_1,\langle(R_2,T_1),(R_2,T_1)\rangle,T_2),U]}\quad.$$

For permuting contraction down, the situation is dual. This is the reason why I could not give a decomposition theorem for system SLS, as it has been done in Section 4.3. The situation becomes considerably simpler in the case of atomic contraction. But there are still two major problems to solve.

(1) While permuting an instance of $\mathsf{ac}{\uparrow}$ up, it might happen that it meets an instance of the interaction rule as follows:

$$
\mathsf{ac}{\uparrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{S\{1\}}{S[a,\bar{a}]}}{S[\langle a,a\rangle,\bar{a}]}
\quad,\quad\text{which can be replaced by}\quad
\mathsf{ac}{\downarrow}\cfrac{\mathsf{d}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{=\cfrac{S\{1\}}{S\langle 1,1\rangle}}{S\langle[a,\bar{a}],1\rangle}}{S\langle[a,\bar{a}],[a,\bar{a}]\rangle}}{S[\langle a,a\rangle,\langle\bar{a},\bar{a}\rangle]}}{S[\langle a,a\rangle,\bar{a}]}\quad.
$$

Here a new instance of $\mathsf{ac}{\downarrow}$ is introduced, which has to be permuted down afterwards. The problem is to show that this does indeed terminate eventually.

(2) The atom to be contracted in an instance of $\mathsf{ac}{\uparrow}$ or $\mathsf{ac}{\downarrow}$ can also be one of the constants. Then we might encounter cases like

$$
\mathsf{ac}{\uparrow}\cfrac{=\cfrac{\mathsf{p}{\uparrow}\cfrac{S(?R,!T)}{S\{?(R,T)\}}}{S\{?[(R,T),0]\}}}{S\{?[(R,T),\langle 0,0\rangle]\}}\quad,
$$

where it is not obvious how the instance of $\mathsf{ac}{\uparrow}$ can be permuted up.

The first problem is of the same nature as the problem with coabsorption and promotion in Section 4.3. This indicates a possible solution via and analysis of chains and cycles in derivations as in Section 4.3.1 (which must be more difficult here because there are four binary connectives instead of two). However, instead of going through such a difficult termination argument again, I will employ cut elimination, as it has been done in [BT01] for the same theorem for classical logic. This can be done here because I will not use Theorem 5.3.2 in a cut elimination argument.

For solving the second problem we have to use a case analysis and then find for each problematic rule an individual solution. In the case of classical logic this rather painful task has not to be done because there are only two connectives and two constants and there are no problematic cases.

In the following, I will show how the rule $\mathsf{ac}{\uparrow}$ can be permuted up. In order to keep the situation as clear as possible, I introduce the following five rules:

$$
\mathsf{ac}{\uparrow}_{\perp}\cfrac{S\{\perp\}}{S\langle\perp,\perp\rangle}\quad,\quad
\mathsf{ac}{\uparrow}_{1}\cfrac{S\{1\}}{S\langle 1,1\rangle}\quad,\quad
\mathsf{ac}{\uparrow}_{0}\cfrac{S\{0\}}{S\langle 0,0\rangle}\quad,\quad
\mathsf{ac}{\uparrow}_{\top}\cfrac{S\{\top\}}{S\langle\top,\top\rangle}\quad,\quad\text{and}
$$

$$
\mathsf{ac}{\uparrow}_{a}\cfrac{S\{a\}}{S\langle a,a\rangle}\quad,\qquad\text{where } a \text{ is not a constant.}
$$

Obviously, each of them is a special case of $\mathsf{ac}{\uparrow}$, and each instance of $\mathsf{ac}{\uparrow}$ is an instance of one of $\mathsf{ac}{\uparrow}_{\perp}$, $\mathsf{ac}{\uparrow}_{1}$, $\mathsf{ac}{\uparrow}_{0}$, $\mathsf{ac}{\uparrow}_{\top}$, $\mathsf{ac}{\uparrow}_{a}$. The $\mathsf{nm}{\uparrow}$ in the statement of Theorem 5.3.1 is due to the fact that $\mathsf{ac}{\uparrow}_{0}$ is the same as $\mathsf{nm}{\downarrow}$, or dually, $\mathsf{ac}{\downarrow}_{\top}$ is the same as $\mathsf{nm}{\uparrow}$. Furthermore, all instances of the two rules $\mathsf{ac}{\uparrow}_{1}$ and $\mathsf{ac}{\uparrow}_{\top}$ are trivial because $1 = \langle 1,1\rangle$ and $\top = \langle\top,\top\rangle$. Hence, the only cases to deal with are $\mathsf{ac}{\uparrow}_{a}$ and $\mathsf{ac}{\uparrow}_{\perp}$.

5.3.3 **Lemma** *The rule* $\mathsf{ac}\!\uparrow_a$ *permutes over every rule* $\pi \in \mathsf{SLLS} \setminus \{\mathsf{ai}\!\downarrow, \mathsf{at}\!\downarrow, \mathsf{ac}\!\downarrow, \mathsf{ac}\!\uparrow\}$.

Proof: Consider a derivation

$$\mathsf{ac}\!\uparrow_a \frac{\pi \dfrac{Q}{S\{a\}}}{S\langle a, a\rangle} \quad,$$

where $\pi \in \mathsf{SLLS} \setminus \{\mathsf{ai}\!\downarrow, \mathsf{at}\!\downarrow, \mathsf{ac}\!\downarrow, \mathsf{ac}\!\uparrow\}$. Without loss of generality, assume that the application of π is nontrivial. According to the case analysis in 4.2.3, the following cases are to consider:

(4) The redex of π is inside the contractum a. This is impossible because nothing can be inside an atom.

(5) The contractum a of $\mathsf{ac}\!\uparrow_a$ is inside an active structure of the redex of π, but not inside a passive one. The rules of system SLLS are designed such that this is impossible.

(6) The contractum a of $\mathsf{ac}\!\uparrow_a$ and the redex of π overlap. This is impossible because an atom cannot properly ovelap with anything. □

5.3.4 **Definition** A rule ρ *permutes over* a rule π *by a system* $\mathscr{S}$, if for every derivation $\rho \dfrac{\pi \dfrac{Q}{U}}{P}$ there is a derivation

$$\frac{\pi \dfrac{\rho \dfrac{Q}{Q'}}{P'}}{\underset{P}{\|\mathscr{S}}} \quad.$$

For notational convenience, let me define the following two systems:

$$\begin{aligned}
\mathsf{SLLS}\!\downarrow &= \mathsf{SLLSt}\!\downarrow \cup \{\mathsf{nz}\!\downarrow, \mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{d}\!\downarrow, \mathsf{p}\!\downarrow, \mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow, \mathsf{l}_1\!\downarrow, \mathsf{l}_2\!\downarrow, \mathsf{z}\!\downarrow\} \\
\mathsf{SLLS}\!\uparrow &= \mathsf{SLLSt}\!\uparrow \cup \{\mathsf{nz}\!\uparrow, \mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{d}\!\uparrow, \mathsf{p}\!\uparrow, \mathsf{ac}\!\uparrow, \mathsf{m}, \mathsf{m}_1\!\uparrow, \mathsf{m}_2\!\uparrow, \mathsf{l}_1\!\uparrow, \mathsf{l}_2\!\uparrow, \mathsf{z}\!\uparrow\}
\end{aligned}$$

In other words, the system $\mathsf{SLLS}\!\downarrow$ contains the down fragment of system SLLS and system $\mathsf{SLLS}\!\uparrow$ contains the up fragment.

5.3.5 **Lemma** *The rule* $\mathsf{ac}\!\uparrow_\perp$ *permutes over every rule* $\pi \in (\mathsf{SLLS}\!\uparrow \cup \{\mathsf{nm}\!\downarrow\}) \setminus \{\mathsf{ac}\!\uparrow\}$ *by the system* $\{\mathsf{s}, \mathsf{d}\!\uparrow, \mathsf{m}_1\!\uparrow\}$.

Proof: Consider a derivation

$$\mathsf{ac}\!\uparrow_\perp \frac{\pi \dfrac{Q}{S\{\perp\}}}{S\langle \perp, \perp\rangle} \quad,$$

where $\pi \in \mathsf{SLLS}\!\uparrow \setminus \{\mathsf{ac}\!\uparrow\}$. Without loss of generality, assume that the application of π is nontrivial. Now, follow again the case analysis in 4.2.3.

(4) The redex of π is inside the contractum $\perp$. This is impossible because the application of π is nontrivial.

(5) The contractum $\perp$ of $\mathsf{ac}{\uparrow}_{\perp}$ is inside an active structure of the redex of π, but not inside a passive one. This case is the problematic one. The situation is similar to Section 4.2, where the weakening rule has been permuted up. Since here the additives are also involved, there are more possibilities of interference. Since there are more rules π, there are more cases to consider. But in principle, they are all the same. Each rule π will be treated separately.

(i) If $\pi \in \mathsf{SLLSt}{\uparrow} \cup \{\mathsf{ai}{\uparrow}, \mathsf{nz}{\uparrow}\}$, then this case cannot occur.

(ii) If $\pi = \mathsf{m}_2{\uparrow}$, we can reduce this case to an instance of case (2) by the associativity of the par context.

(iii) Let us now consider the cases $\pi \in \{\mathsf{s}, \mathsf{d}{\uparrow}, \mathsf{p}{\uparrow}, \mathsf{m}_1{\uparrow}\}$. Let me use the rule $\mathsf{m}_1{\uparrow}$ to explain the situation. We have to check all possibilities, how the contractum $\perp$ can appear inside the redex $\langle(R,T),(U,V)\rangle$. An example is $\langle[(R,T),\perp],(U,V)\rangle$. The following derivation shows the most general case:

$$
\mathsf{ac}{\uparrow}_{\perp} \frac{\mathsf{m}_1{\uparrow} \dfrac{S'(\langle(R,R'),U\rangle,\langle(T,T'),V\rangle)}{S'\langle(R,R',T,T'),(U,V)\rangle} = \dfrac{}{S'\langle(R,[(R',T),\perp],T'),(U,V)\rangle}}{S'\langle(R,[(R',T),\langle\perp,\perp\rangle],T'),(U,V)\rangle} \quad .
$$

(If we set in this derivation $R = T' = 1$, we get the example mentioned before. The case where the $\perp$ occurs inside (U,V) is does not need to be considered because of commutativity.) This is in fact very similar to the cases (5.i) and (5.ii) of the proof of part (a) of Lemma 4.2.9 (page 93), where the switch rule has been used to move the disturbing structure inside a passive structure of the redex of π. Here, we do exactly the same:

$$
\mathsf{s} \frac{\mathsf{m}_1{\uparrow} \dfrac{\mathsf{ac}{\uparrow}_{\perp} \dfrac{= \dfrac{S'(\langle(R,R'),U\rangle,\langle(T,T'),V\rangle)}{S'(\langle([R,\perp],R'),U\rangle,\langle(T,T'),V\rangle)}}{S'(\langle([R,\langle\perp,\perp\rangle],R'),U\rangle,\langle(T,T'),V\rangle)}}{S'\langle(R,[R',\langle\perp,\perp\rangle],T,T'),(U,V)\rangle}}{S'\langle(R,[(R',T),\langle\perp,\perp\rangle],T'),(U,V)\rangle} \quad .
$$

For the rules s, $\mathsf{d}{\uparrow}$ and $\mathsf{p}{\uparrow}$, the situation is similar.

(iv) If $\pi = \mathsf{z}{\uparrow}$, we also use the switch, in a different way, namely, to move the disturbing structure out of the redex of π into the context.

$$
\mathsf{ac}{\uparrow}_{\perp} \frac{\mathsf{z}{\uparrow} \dfrac{= \dfrac{S'\{!\langle R,T\rangle\}}{S'(!R,T)}}{S'([!R,\perp],T)}}{S'([!R,\langle\perp,\perp\rangle],T)} \quad \text{yields} \quad \mathsf{s} \frac{\mathsf{z}{\uparrow} \dfrac{\mathsf{ac}{\uparrow}_{\perp} \dfrac{= \dfrac{S'\{!\langle R,T\rangle\}}{S'(!\langle R,T\rangle,[1,\perp])}}{S'(!\langle R,T\rangle,[1,\langle\perp,\perp\rangle])}}{S'(!R,[1,\langle\perp,\perp\rangle],T)}}{= \dfrac{S'([(!R,1),\langle\perp,\perp\rangle],T)}{S'([!R,\langle\perp,\perp\rangle],T)}} \quad .
$$

(v) If $\pi \in \{l_1\uparrow, l_2\uparrow, nm\downarrow\}$, we use the switch together with the rule $m_1\uparrow$:

$$
l_1\uparrow \frac{S'\{!\langle R,T\rangle\}}{\frac{S'\langle !R,!T\rangle}{ac\uparrow_\perp \frac{S'\langle [!R,\perp],!T\rangle}{S'\langle [!R,\langle \perp,\perp\rangle],!T\rangle}}}
\qquad \text{yields} \qquad
\begin{array}{c}
= \dfrac{S'\{!\langle R,T\rangle\}}{ac\uparrow_\perp \dfrac{S'(!\langle R,T\rangle,\langle 1,[1,\perp]\rangle)}{l_1\uparrow \dfrac{S'(!\langle R,T\rangle,\langle 1,[1,\langle \perp,\perp\rangle]\rangle)}{m_1\uparrow \dfrac{S'(\langle !R,!T\rangle,\langle 1,[1,\langle \perp,\perp\rangle]\rangle)}{= \dfrac{S'\langle(!R,[1,\langle \perp,\perp\rangle]),(!T,1)\rangle}{s \dfrac{S'\langle(!R,[1,\langle \perp,\perp\rangle],!T)\rangle}{= \dfrac{S'\langle[(!R,1),\langle \perp,\perp\rangle],!T\rangle}{S'([!R,\langle \perp,\perp\rangle],!T)}}}}}}
\end{array}
\;.
$$

For $\pi = l_2\uparrow$ and $\pi = nm\downarrow$, the situation is similar.

(vi) If $\pi = m$, the situation is similar as before:

$$
m \frac{S'\{(\langle R,R'\rangle,U),(\langle T,T'\rangle,V)\}}{\frac{S'\langle\langle R,R',T,T'\rangle,\langle U,V\rangle\rangle}{ac\uparrow_\perp \frac{S'\langle\langle R,[\langle R',T\rangle,\perp],T'\rangle,\langle U,V\rangle\rangle}{S'\langle\langle R,[\langle R',T\rangle,\langle \perp,\perp\rangle],T'\rangle,\langle U,V\rangle\rangle}}} \;,
$$

but we have to insert an instance of $d\uparrow$:

$$
\begin{array}{c}
= \dfrac{S'\{(\langle R,R'\rangle,U),(\langle T,T'\rangle,V)\}}{ac\uparrow_\perp \dfrac{S'(\{(\langle R,R'\rangle,U),(\langle T,T'\rangle,V)\},\langle 1,1,[1,\perp]\rangle)}{m \dfrac{S'(\{(\langle R,R'\rangle,U),(\langle T,T'\rangle,V)\},\langle 1,1,[1,\langle \perp,\perp\rangle]\rangle)}{m_1\uparrow \dfrac{S'(\langle\langle R,R',T,T'\rangle,\langle U,V\rangle\rangle,\langle 1,1,[1,\langle \perp,\perp\rangle]\rangle)}{= \dfrac{S'\langle(\langle R,R',T,T'\rangle,\langle 1,[1,\langle \perp,\perp\rangle]\rangle),(\langle U,V\rangle,1)\rangle}{d\uparrow \dfrac{S'\langle(\langle R,R',T,T'\rangle,\langle 1,[1,\langle \perp,\perp\rangle]\rangle),\langle U,V\rangle\rangle}{= \dfrac{S'\langle\langle(1,\langle R,T'\rangle),(\langle R',T\rangle,[1,\langle \perp,\perp\rangle])\rangle,\langle U,V\rangle\rangle}{= \dfrac{S'\langle\langle R,(\langle R',T\rangle,[1,\langle \perp,\perp\rangle]),T'\rangle,\langle U,V\rangle\rangle}{s \dfrac{S'\langle\langle R,[(\langle R',T\rangle,1),\langle \perp,\perp\rangle],T'\rangle,\langle U,V\rangle\rangle}{= S'\langle\langle R,[\langle R',T\rangle,\langle \perp,\perp\rangle],T'\rangle,\langle U,V\rangle\rangle}}}}}}}}
\end{array}
\;.
$$

(6) The contractum $\perp$ of $ac\uparrow_\perp$ and the redex of π overlap. This is impossible. $\square$

5.3.6 Remark The interested reader might observe, that it is also possible, to permute the rule $ac\uparrow_\perp$ over most rules of the down fragment of system SLLS. For example, for the rules $ai\downarrow$, $p\downarrow$ and $nm_1\downarrow$, the situation is the same as for $m_2\uparrow$. For the rules $m_1\downarrow$ and $m_2\downarrow$, we could construct similar derivations as for m. But I do not want to trouble the reader with more details than necessary.

5.3.7 Lemma *For every derivation* $\Delta\|$SLLS↑ *there is a derivation*

$$
\begin{array}{l}
T \\
\|\{\text{ac}↑\} \\
T' \\
\|(\text{SLLS}↑\cup\{\text{nm}↓\})\backslash\{\text{ac}↑\} \\
R
\end{array}
$$

for some structure T'.

Proof: Each instance of ac↑ inside Δ is an instance of $\text{ac}↑_\bot$, $\text{ac}↑_1$, $\text{ac}↑_0$, $\text{ac}↑_\top$ or $\text{ac}↑_a$. The instances of $\text{ac}↑_1$ and $\text{ac}↑_\top$ can be removed because they are trivial, and the instances of $\text{ac}↑_0$ are replaced by nm↓. In order to permute $\text{ac}↑_a$ and $\text{ac}↑_\bot$ to the top of Δ, repeatedly apply Lemmata 5.3.3 and 5.3.5. □

5.3.8 Lemma *For every derivation* $\Delta\|$SLLS↓ *there is a derivation*

$$
\begin{array}{l}
T \\
\|(\text{SLLS}↓\cup\{\text{nm}↑\})\backslash\{\text{ac}↓\} \\
R' \\
\|\{\text{ac}↓\} \\
R
\end{array}
$$

for some structure R'.

Proof: Dual to Lemma 5.3.7. □

Proof of Theorem 5.3.1: Immediate consequence of Lemma 5.3.8. □

Now we have done all necessary preparations for the proof of Theorem 5.3.2, which is similar to the proof of the same theorem for classical logic, presented in [BT01].

Proof of Theorem 5.3.2: Let $\Delta\|$SLLS be given. We perform the following transformations:

$$
\begin{array}{ccccccc}
 & & & & \text{1}↓\,\dfrac{}{1} & & \\
 & & & \text{i}↓\,\dfrac{}{[T,\bar{T}]} & & \\
T & & [T,\bar{T}] & & & & \\
\Delta\|\text{SLLS} & \overset{1}{\rightsquigarrow} & \Delta\|\text{SLLS} & \overset{2}{\rightsquigarrow} & \Delta\|\text{SLLS} & \overset{3}{\rightsquigarrow} & \Pi\|\text{LLS} \quad . \\
R & & [R,\bar{T}] & & [R,\bar{T}] & & [R,\bar{T}]
\end{array}
$$

In the first step we replace each structure S that occurs inside Δ by $[S,\bar{T}]$, which is in Step 2 transformed into a proof by adding an instance of i↓ and 1↓. Then, in Step 3, we

apply Proposition 3.2.16 and cut elimination (Theorem 5.2.8) to transform this proof into a proof in system LLS.

$$
\overset{4}{\rightsquigarrow}\quad
\begin{array}{c}
1 \\
\Pi'\big\|(\mathsf{LLS}\cup\{\mathsf{nm}\!\uparrow\})\backslash\{\mathsf{ac}\!\downarrow\} \\
U \\
\tilde{\Delta}\big\|\{\mathsf{ac}\!\downarrow\} \\
[R,\bar{T}]
\end{array}
\qquad
\overset{5}{\rightsquigarrow}\quad
\begin{array}{c}
1 \\
\Pi'\big\|\mathsf{SLLS}\backslash\{\mathsf{ac}\!\downarrow,\mathsf{ac}\!\uparrow\} \\
U \\
\tilde{\Delta}\big\|\{\mathsf{ac}\!\downarrow\} \\
[R,\bar{T}]
\end{array}
\qquad
\overset{6}{\rightsquigarrow}\quad
\begin{array}{c}
1 \\
\Pi'\big\|\mathsf{SLLS}\backslash\{\mathsf{ac}\!\downarrow,\mathsf{ac}\!\uparrow\} \\
[R',\bar{T}'] \\
\tilde{\Delta}\big\|\{\mathsf{ac}\!\downarrow\} \\
[R,\bar{T}]
\end{array}
\quad.
$$

In Step 4, we apply Theorem 5.3.1. Step 5 is a triviality because $(\mathsf{LLS}\cup\{\mathsf{nm}\!\uparrow\})\setminus\{\mathsf{ac}\!\downarrow\}\subseteq\mathsf{SLLS}\backslash\{\mathsf{ac}\!\downarrow,\mathsf{ac}\!\uparrow\}$. Since the rule $\mathsf{ac}\!\downarrow$ cannot modify a par structure, we have that $U=[R',\bar{T}']$ for some structures R' and $\bar{T}'$ such that there are derivations

$$
\begin{array}{c}
R' \\
\tilde{\Delta}_R\big\|\{\mathsf{ac}\!\downarrow\} \\
R
\end{array}
\qquad\text{and}\qquad
\begin{array}{c}
\bar{T}' \\
\tilde{\Delta}_{\bar{T}}\big\|\{\mathsf{ac}\!\downarrow\} \\
\bar{T}
\end{array}
\quad.
$$

Those two derivations are now separated in Step 7:

$$
\overset{7}{\rightsquigarrow}\quad
\begin{array}{c}
1 \\
\Pi'\big\|\mathsf{SLLS}\backslash\{\mathsf{ac}\!\downarrow,\mathsf{ac}\!\uparrow\} \\
[R',\bar{T}'] \\
\tilde{\Delta}_{\bar{T}}\big\|\{\mathsf{ac}\!\downarrow\} \\
[R',\bar{T}] \\
\tilde{\Delta}_R\big\|\{\mathsf{ac}\!\downarrow\} \\
[R,\bar{T}]
\end{array}
\qquad
\overset{8}{\rightsquigarrow}\quad
\begin{array}{c}
(1,T) \\
\Pi'\big\|\mathsf{SLLS}\backslash\{\mathsf{ac}\!\downarrow,\mathsf{ac}\!\uparrow\} \\
([R',\bar{T}'],T) \\
\tilde{\Delta}_{\bar{T}}\big\|\{\mathsf{ac}\!\downarrow\} \\
([R',\bar{T}],T) \\
\tilde{\Delta}_R\big\|\{\mathsf{ac}\!\downarrow\} \\
([R,\bar{T}],T)
\end{array}
\quad.
$$

Step 8 is very similar to the first step: we set every structure in the derivation in a times relation with T.

$$
\overset{9}{\rightsquigarrow}\quad
\begin{array}{c}
T \\
\Pi'\big\|\mathsf{SLLS}\backslash\{\mathsf{ac}\!\downarrow,\mathsf{ac}\!\uparrow\} \\
([R',\bar{T}'],T) \\
\tilde{\Delta}_{\bar{T}}\big\|\{\mathsf{ac}\!\downarrow\} \\
([R',\bar{T}],T) \\
\tilde{\Delta}_R\big\|\{\mathsf{ac}\!\downarrow\} \\
([R,\bar{T}],T) \\
\mathsf{s}\,\dfrac{}{[R,(\bar{T},T)]} \\
\mathsf{i}\!\uparrow\,\dfrac{}{R}
\end{array}
\qquad
\overset{10}{\rightsquigarrow}\quad
\begin{array}{c}
T \\
\Pi'\big\|\mathsf{SLLS}\backslash\{\mathsf{ac}\!\downarrow,\mathsf{ac}\!\uparrow\} \\
([R',\bar{T}'],T) \\
\tilde{\Delta}_{\bar{T}}\big\|\{\mathsf{ac}\!\downarrow\} \\
\mathsf{s}\,\dfrac{([R',\bar{T}],T)}{[R',(\bar{T},T)]} \\
\mathsf{i}\!\uparrow\,\dfrac{}{R'} \\
\tilde{\Delta}_R\big\|\{\mathsf{ac}\!\downarrow\} \\
R
\end{array}
\quad.
$$

In Step 9, we add an instance of switch and cut to produce a derivation with conclusion R. Since no instance of $\mathsf{ac}\!\downarrow$ in $\tilde{\Delta}_R$ does modify T or $\bar{T}$ we can in Step 10 move $\tilde{\Delta}_R$ below the

instances of switch and cut.

$$
\overset{11}{\rightsquigarrow} \quad
\begin{array}{c}
T \\
\Pi' \big\| \mathsf{SLLS} \backslash \{\mathsf{ac}\downarrow, \mathsf{ac}\uparrow\} \\
([R', \bar{T}'], T) \\
\tilde{\Delta}_T \big\| \{\mathsf{ac}\uparrow\} \\
\mathsf{s} \dfrac{([R', \bar{T}'], T')}{[R', (\bar{T}', T')]} \\
\mathsf{i}\uparrow \dfrac{}{R'} \\
\tilde{\Delta}_R \big\| \{\mathsf{ac}\downarrow\} \\
R
\end{array}
\qquad
\overset{12}{\rightsquigarrow} \quad
\begin{array}{c}
T \\
\tilde{\Delta}_T \big\| \{\mathsf{ac}\uparrow\} \\
T' \\
\Pi' \big\| \mathsf{SLLS} \backslash \{\mathsf{ac}\downarrow, \mathsf{ac}\uparrow\} \\
\mathsf{s} \dfrac{([R', \bar{T}'], T')}{[R', (\bar{T}', T')]} \\
\mathsf{i}\uparrow \dfrac{}{R'} \\
\tilde{\Delta}_R \big\| \{\mathsf{ac}\downarrow\} \\
R
\end{array}
\qquad
\overset{13}{\rightsquigarrow} \quad
\begin{array}{c}
T \\
\tilde{\Delta}_T \big\| \{\mathsf{ac}\uparrow\} \\
T' \\
\Delta' \big\| \mathsf{SLLS} \backslash \{\mathsf{ac}\downarrow, \mathsf{ac}\uparrow\} \\
R' \\
\tilde{\Delta}_R \big\| \{\mathsf{ac}\downarrow\} \\
R
\end{array} \quad .
$$

In Step 11, the derivation $\begin{array}{c} \bar{T}' \\ \tilde{\Delta}_{\bar{T}} \big\| \{\mathsf{ac}\downarrow\} \\ \bar{T} \end{array}$ is replace by the dual derivation $\begin{array}{c} T \\ \tilde{\Delta}_T \big\| \{\mathsf{ac}\uparrow\} \\ T' \end{array}$. In order to keep everything sound, T and $\bar{T}$ have to be replaced by T' and $\bar{T}'$ in the instance of $\mathsf{i}\uparrow$. Since the derivation $\begin{array}{c} 1 \\ \Pi' \big\| \mathsf{SLLS} \backslash \{\mathsf{ac}\downarrow, \mathsf{ac}\uparrow\} \\ [R', \bar{T}'] \end{array}$ does not modify the substructure T, the derivation $\tilde{\Delta}_T$ can be moved to the top in Step 12. Step 13 is then another application of Proposition 3.2.16 to get the desired derivation. $\qquad\qquad\square$

5.3.2 Separation of Atomic Interaction

The next theorem deals with interaction and cut. As already shown in Section 4.2, it is possible to permute the identity up and the cut down. But here the situation is more complex since the additive contexts have also be considered in the permutation lemmata. However, since contraction is reduced to an atomic version, we can get the following result, which was impossible for system SELS.

5.3.9 Theorem *For every derivation* $\begin{array}{c} T \\ \Delta \big\| \mathsf{SLLS} \\ R \end{array}$ *there is a derivation*

$$
\begin{array}{c}
T \\
\big\| \{\mathsf{ai}\downarrow\} \\
T' \\
\big\| \mathsf{SLLS} \backslash \{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow\} \\
R' \\
\big\| \{\mathsf{ai}\uparrow\} \\
R
\end{array}
$$

for some structures T' *and* R'.

Before I go into the proof, which will again use a permutation argument, let me explain why this result is impossible without the atomicity of the contraction rule. More precisely,

it does not hold for system SLS, which has been discussed in Chapter 3, because the rule
ai↓ cannot be permuted over the rule c↑. The problematic case is the following:

$$
\mathsf{ai}{\downarrow}\frac{\mathsf{c}{\uparrow}\dfrac{S\{R\{1\}\}}{S\langle R\{1\}, R\{1\}\rangle}}{S\langle R[a, \bar{a}], R\{1\}\rangle} \quad .
$$

This is also the reason why in both decomposition theorems in Section 4.3, the rule b↑
appears at the top and the rule b↓ at the bottom.

Although this problem cannot occur in the case of atomic contraction, there is another
problem which is caused by atomic contraction. More precisely, it is caused by the presence
of the medial rules. It is not (trivially) possible, to permute ai↓ over the rule $\mathsf{l}_2{\downarrow}$. The
problematic case is the following:

$$
\mathsf{ai}{\downarrow}\frac{=\dfrac{\mathsf{l}_2{\downarrow}\dfrac{S\{!R, !T\}}{S\{!\langle R, T\rangle\}}}{S\{!(\langle R, T\rangle, 1)\}}}{S\{!(\langle R, T\rangle, [a, \bar{a}])\}} \quad .
$$

Fortunately, this concerns only the medial down rules and not the medial up rules. In
order to prove Theorem 5.3.9, we can therefore use the fact that all medial down rules
are derivable in $\{\mathsf{t}{\downarrow}, \mathsf{c}{\downarrow}\}$ (by Proposition 5.2.5) together with Lemma 5.3.11, which follows
below.

5.3.10 **Definition** A rule ρ *weakly permutes over* a rule π,
if for every derivation $\pi\dfrac{Q}{\rho\dfrac{U}{P}}$ there is a derivation

$$
\pi\dfrac{\begin{array}{c}Q\\ \|\{\rho\}\\ U'\end{array}}{P} \quad ,
$$

for some structure U'.

5.3.11 **Lemma** *The rule* ai↓ *weakly permutes over the rules* t↓ *and* c↓.

Proof: Consider a derivation

$$
\mathsf{ai}{\downarrow}\dfrac{\pi\dfrac{Q}{S\{1\}}}{S[a, \bar{a}]} \quad ,
$$

where the application of $\pi \in \{\mathsf{t}{\downarrow}, \mathsf{c}{\downarrow}\}$ is nontrivial. By the case analysis in 4.2.3, we have
to consider the following cases:

(4) The redex of π is inside the contractum 1. This is possible since π is nontrivial.

(5) The contractum 1 of ai↓ is inside an active structure of the redex of π, but not inside
   a passive one. There are two subcases:

(i) $\pi = \mathsf{t}\downarrow$. Then

$$\mathsf{ai}\downarrow\frac{\mathsf{t}\downarrow\dfrac{S'\{0\}}{S'\{R\{1\}\}}}{S'\{R[a,\bar{a}]\}} \qquad \text{yields} \qquad \mathsf{t}\downarrow\frac{S'\{0\}}{S'\{R[a,\bar{a}]\}} \quad .$$

(ii) $\pi = \mathsf{c}\downarrow$. Then

$$\mathsf{ai}\downarrow\frac{\mathsf{c}\downarrow\dfrac{S'\{R\{1\},R\{1\}\}}{S'\{R\{1\}\}}}{S'\{R[a,\bar{a}]\}} \qquad \text{yields} \qquad \mathsf{c}\downarrow\frac{\mathsf{ai}\downarrow\dfrac{\mathsf{ai}\downarrow\dfrac{S'\{R\{1\},R\{1\}\}}{S'\{R\{1\},R[a,\bar{a}]\}}}{S'\{R[a,\bar{a}],R[a,\bar{a}]\}}}{S'\{R[a,\bar{a}]\}} \quad .$$

(6) The contractum 1 of $\mathsf{ai}\downarrow$ and the redex of π overlap. As before, this is impossible. $\square$

Before showing how $\mathsf{ai}\downarrow$ is permuted over all other rules, let me, for notational convenience, define the following two systems:

$$\begin{aligned}
\mathsf{SLLSm}\downarrow &= \mathsf{SLLSt}\downarrow \cup \{\mathsf{ac}\downarrow, \mathsf{m}_1\downarrow, \mathsf{m}_2\downarrow, \mathsf{l}_1\downarrow, \mathsf{l}_2\downarrow\} \\
\mathsf{SLLSm}\uparrow &= \mathsf{SLLSt}\uparrow \cup \{\mathsf{ac}\uparrow, \mathsf{m}_1\uparrow, \mathsf{m}_2\uparrow, \mathsf{l}_1\uparrow, \mathsf{l}_2\uparrow\}
\end{aligned}$$

They will be needed only inside this section for the proof of Theorem 5.3.9.

5.3.12 Lemma   *The rule $\mathsf{ai}\downarrow$ permutes over all rules $\pi \in \mathsf{SLLS} \setminus \mathsf{SLLSm}\downarrow$ by the system $\{\mathsf{s}, \mathsf{d}\uparrow, \mathsf{p}\uparrow, \mathsf{m}_1\uparrow\}$.*

Proof: Consider a derivation

$$\mathsf{ai}\downarrow\frac{\pi\dfrac{Q}{S\{1\}}}{S[a,\bar{a}]} \quad ,$$

where $\pi \in \mathsf{SLLS}\setminus\mathsf{SLLSm}\downarrow$ is nontrivial. According to 4.2.3, we have to consider the following cases:

(4) The redex of π is inside the contractum 1. This is impossible.

(5) The contractum 1 of $\mathsf{ai}\downarrow$ is inside an active structure of the redex of π, but not inside a passive one. This case is essentially the same, as case (5) in Lemma 5.3.7, but this time the contractum is 1 instead of $\bot$. There are the following subcases.

   (i) If $\pi \in \mathsf{SLLSt}\uparrow \cup \{\mathsf{ai}\uparrow, \mathsf{nz}\uparrow\}$, then this case is impossible.

   (ii) If $\pi \in \{\mathsf{s}, \mathsf{d}\uparrow, \mathsf{p}\uparrow, \mathsf{m}_1\uparrow, \mathsf{z}\uparrow\}$, then because of the associativity of the times context, this case can be reduced to cases (1) and (2).

   (iii) If $\pi = \mathsf{p}\downarrow$, then we have the same case as shown in case (5.i) in the proof of part (c) of Lemma 4.2.9 (see page 93), i.e. an instance of the switch is inserted.

(iv) If $\pi = \mathsf{d}{\downarrow}$, then there are two possibilities:

$$\mathsf{ai}{\downarrow}\frac{\mathsf{d}{\downarrow}\dfrac{S'\langle[\langle R,R'\rangle,U],[\langle T,T'\rangle,V]\rangle}{S'\,[\langle R,R',T,T'\rangle,\{U,V\}]}}{\dfrac{S'\,[\langle R,(\langle R',T\rangle,1),T'\rangle,\{U,V\}]}{S'\,[\langle R,(\langle R',T\rangle,[a,\bar a]),T'\rangle,\{U,V\}]}}\quad\text{and}$$

$$\mathsf{ai}{\downarrow}\frac{\mathsf{d}{\downarrow}\dfrac{S'\langle[R,\{U,U'\}],[T,\{V,V'\}]\rangle}{S'\,[\langle R,T\rangle,\{U,U',V,V'\}]}}{\dfrac{S'\,[\langle R,T\rangle,\{U,(\{U',V\},1),V'\}]}{S'\,[\langle R,T\rangle,\{U,(\{U',V\},[a,\bar a]),V'\}]}}\quad.$$

In the first case, we use the rules s and $\mathsf{m}_1{\uparrow}$ together to move the substructure $[a,\bar a]$ out of the way of π:

$$=\frac{\mathsf{ai}{\downarrow}\dfrac{S'\langle[\langle R,R'\rangle,U],[\langle T,T'\rangle,V]\rangle}{S'(\langle[\langle R,R'\rangle,U],[\langle T,T'\rangle,V]\rangle,\langle 1,1\rangle)}}{\mathsf{d}{\downarrow}\dfrac{S'(\langle[\langle R,R'\rangle,U],[\langle T,T'\rangle,V]\rangle,\langle 1,[a,\bar a]\rangle)}{\mathsf{s}\dfrac{S'([\langle R,R',T,T'\rangle,\{U,V\}],\langle 1,[a,\bar a]\rangle)}{\mathsf{m}_1{\uparrow}\dfrac{S'[(\langle R,R',T,T'\rangle,\langle 1,[a,\bar a]\rangle),\{U,V\}]}{=\dfrac{S'[\langle(\langle R,T'\rangle,1),(\langle R',T\rangle,[a,\bar a])\rangle,\{U,V\}]}{S'[\langle R,(\langle R',T\rangle,[a,\bar a]),T'\rangle,\{U,V\}]}}}}}\quad.$$

The second case is similar, but we use $\mathsf{d}{\uparrow}$ instead of $\mathsf{m}_1{\uparrow}$:

$$=\frac{\mathsf{ai}{\downarrow}\dfrac{S'\langle[R,\{U,U'\}],[T,\{V,V'\}]\rangle}{S'(\langle[R,\{U,U'\}],[T,\{V,V'\}]\rangle,\langle 1,1\rangle)}}{\mathsf{d}{\downarrow}\dfrac{S'(\langle[R,\{U,U'\}],[T,\{V,V'\}]\rangle,\langle 1,[a,\bar a]\rangle)}{\mathsf{s}\dfrac{S'([\langle R,T\rangle,\{U,U',V,V'\}],\langle 1,[a,\bar a]\rangle)}{\mathsf{d}{\uparrow}\dfrac{S'[\langle R,T\rangle,(\{U,U',V,V'\},\langle 1,[a,\bar a]\rangle)]}{=\dfrac{S'[\langle R,T\rangle,\{(\{U,V'\},1),(\{U',V\},[a,\bar a])\}]}{S'[\langle R,T\rangle,\{U,(\{U',V\},[a,\bar a]),V'\}]}}}}}\quad.$$

(v) If $\pi \in \{\mathsf{ac}{\uparrow}, \mathsf{l}_1{\uparrow}, \mathsf{l}_2{\uparrow}\}$, then we can employ the rule $\mathsf{m}_1{\uparrow}$ in the following way. I will show only the case for $\mathsf{l}_2{\uparrow}$ because the others are similar:

$$=\frac{\mathsf{ai}{\downarrow}\dfrac{\mathsf{l}_2{\uparrow}\dfrac{S'\{?\langle R,T\rangle\}}{S'\langle ?R,?T\rangle}}{\dfrac{S'\langle(?R,1),?T\rangle}{S'\langle(?R,[a,\bar a]),?T\rangle}}\qquad\text{yields}\qquad =\frac{\mathsf{ai}{\downarrow}\dfrac{S'\{?\langle R,T\rangle\}}{\mathsf{l}_2{\uparrow}\dfrac{S'(?\langle R,T\rangle,\langle 1,1\rangle)}{\mathsf{m}_1{\uparrow}\dfrac{S'(?\langle R,T\rangle,\langle[a,\bar a],1\rangle)}{\dfrac{S'(\langle ?R,?T\rangle,\langle[a,\bar a],1\rangle)}{S'\langle(?R,[a,\bar a]),(?T,1)\rangle}}}}}{S'\langle(?R,[a,\bar a]),?T\rangle}}\quad.$$

(vi) If $\pi = \mathsf{m}$, then we have

$$
\mathsf{ai}{\downarrow}\;\cfrac{\;\mathsf{m}\;\cfrac{S'[\langle\{R,R'\},U\rangle,\langle\{T,T'\},V\rangle]}{S'\langle\{R,R',T,T'\},\{U,V\}\rangle}\;}{\cfrac{S'\langle\{R,(\{R',T\},1),T'\},\{U,V\}\rangle}{S'\langle\{R,(\{R',T\},[a,\bar a]),T'\},\{U,V\}\rangle}}\quad,
$$

where we can use the rules $\mathsf{d}{\uparrow}$ and $\mathsf{m_1}{\uparrow}$ together:

$$
=\cfrac{\;\mathsf{d}{\uparrow}\;\cfrac{\;=\;\cfrac{\;\mathsf{m_1}{\uparrow}\;\cfrac{\;\mathsf{m}\;\cfrac{\;\mathsf{ai}{\downarrow}\;\cfrac{S'[\langle\{R,R'\},U\rangle,\langle\{T,T'\},V\rangle]}{S'(\langle\langle\{R,R'\},U\rangle,\langle\{T,T'\},V\rangle\rangle,\langle 1,1,1\rangle)}}{S'(\langle\langle\{R,R'\},U\rangle,\langle\{T,T'\},V\rangle\rangle,\langle 1,1,[a,\bar a]\rangle)}}{S'(\langle\{R,R',T,T'\},\{U,V\}\rangle,\langle 1,1,[a,\bar a]\rangle)}}{S'\langle(\{R,R',T,T'\},\langle 1,[a,\bar a]\rangle),(\{U,V\},1)\rangle}}{S'\langle(\{R,R',T,T'\},\langle 1,[a,\bar a]\rangle),\{U,V\}\rangle}}{S'\langle\{(\{R,T'\},1),(\{R',T\},[a,\bar a])\},\{U,V\}\rangle}}{S'\langle\{R,(\{R',T\},[a,\bar a]),T'\},\{U,V\}\rangle}\quad.
$$

Observe the similarity to case (5.vi) in the proof of Lemma 5.3.5.

(vii) If $\pi = \mathsf{m_2}{\uparrow}$, then we have

$$
\mathsf{ai}{\downarrow}\;\cfrac{\;\mathsf{m_2}{\uparrow}\;\cfrac{S'[\langle[R,R'],U\rangle,\langle[T,T'],V\rangle]}{S'\langle[R,R',T,T'],[U,V]\rangle}\;}{\cfrac{S'\langle[R,([R',T],1),T'],[U,V]\rangle}{S'\langle[R,([R',T],[a,\bar a]),T'],[U,V]\rangle}}\quad,
$$

where we can use the rules s and $\mathsf{m_1}{\uparrow}$ together:

$$
=\cfrac{\;\mathsf{s}\;\cfrac{\;=\;\cfrac{\;\mathsf{m_1}{\uparrow}\;\cfrac{\;\mathsf{m_2}{\uparrow}\;\cfrac{\;\mathsf{ai}{\downarrow}\;\cfrac{S'[\langle[R,R'],U\rangle,\langle[T,T'],V\rangle]}{S'([\langle[R,R'],U\rangle,\langle[T,T'],V\rangle],\langle 1,1\rangle)}}{S'([\langle[R,R'],U\rangle,\langle[T,T'],V\rangle],\langle 1,[a,\bar a]\rangle)}}{S'(\langle[R,R',T,T'],[U,V]\rangle,\langle 1,[a,\bar a]\rangle)}}{S'\langle([R,R',T,T'],[a,\bar a]),([U,V],1)\rangle}}{S'\langle([R,R',T,T'],[a,\bar a]),[U,V]\rangle}}{S'\langle[R,([R',T],[a,\bar a]),T'],[U,V]\rangle}\quad.
$$

(viii) If $\pi = \mathsf{z}{\downarrow}$, then we have

$$
\mathsf{ai}{\downarrow}\;\cfrac{\;\mathsf{z}{\downarrow}\;\cfrac{S'[?\{R,R'\},\{T,T'\}]}{S'\{?\{R,R',T,T'\}\}}\;}{\cfrac{S'\{?\{R,(\{R',T\},1),T'\}\}}{S'\{?\{R,(\{R',T\},[a,\bar a]),T'\}\}}}\quad,
$$

where we can use the rules d↑ and p↑ together:

$$
= \cfrac{S'[?\{R,R'\},\{T,T'\}]}{\cfrac{S([?\{R,R'\},\{T,T'\}],!\langle 1,1\rangle)}{\cfrac{\text{ai}\downarrow\ \ \ S([?\{R,R'\},\{T,T'\}],!\langle 1,[a,\bar a]\rangle)}{\cfrac{\text{z}\downarrow\ \ \ S'(?\{R,R',T,T'\},!\langle 1,[a,\bar a]\rangle)}{\cfrac{\text{p}\uparrow\ \ \ S'\{?(\{R,R',T,T'\},\langle 1,[a,\bar a]\rangle)\}}{\cfrac{\text{d}\uparrow\ \ \ S'\{?\{(\{R,T'\},1),(\{R',T\},[a,\bar a])\}\}}{= \ \ S'\{?\{R,(\{R',T\},[a,\bar a]),T'\}\}}}}}}} \quad .
$$

(ix) If $\pi = \text{nz}\downarrow$, then the situation is similar as in the previous case, with the only difference that the rule d↑ is not needed.

(6) The contractum 1 of ai↓ and the redex of π overlap. As before, this is impossible. □

Now we can show the complete proof of Theorem 5.3.9.

Proof of Theorem 5.3.9:   We can apply the following transformations to the given derivation:

$$
\begin{array}{c}T\\ \|\text{SLLS}\\ R\end{array}
\quad \overset{1}{\rightsquigarrow} \quad
\begin{array}{c}T\\ \|(\text{SLLS}\cup\{\text{t}\downarrow,\text{c}\downarrow\})\backslash\text{SLLSm}\downarrow\\ R\end{array}
\quad \overset{2}{\rightsquigarrow} \quad
\begin{array}{c}T\\ \|\{\text{ai}\downarrow\}\\ T'\\ \|(\text{SLLS}\cup\{\text{t}\downarrow,\text{c}\downarrow\})\backslash(\text{SLLSm}\downarrow\cup\{\text{ai}\downarrow\})\\ R\end{array}
\quad .
$$

In the first step, we apply Proposition 5.2.5 to replace all instances of rules in SLLSm↓ by derivation consisting of t↓ and c↓. Then, Lemmata 5.3.11 and 5.3.12 are applied in Step 2, to permute all instances of ai↓ to the top of the derivation.

$$
\overset{3}{\rightsquigarrow}
\begin{array}{c}T\\ \|\{\text{ai}\downarrow\}\\ T'\\ \|\text{SLLS}\backslash\{\text{ai}\downarrow\}\\ R\end{array}
\quad \overset{4}{\rightsquigarrow} \quad
\begin{array}{c}T\\ \|\{\text{ai}\downarrow\}\\ T'\\ \|(\text{SLLS}\cup\{\text{t}\uparrow,\text{c}\uparrow\})\backslash(\text{SLLSm}\uparrow\cup\{\text{ai}\downarrow\})\\ R\end{array}
\quad .
$$

In Step 3, we apply Propositions 5.2.1 and 5.2.3 to get again a derivation in SLLS: Observe, that the number of instances of ai↑ in the whole derivation did not change during that process. Step 4 is dual to Step 1: we replace all instances of rules in SLLSm↑ by instances of t↑ and c↑.

$$
\overset{5}{\rightsquigarrow}
\begin{array}{c}T\\ \|\{\text{ai}\downarrow\}\\ T'\\ \|(\text{SLLS}\cup\{\text{t}\uparrow,\text{c}\uparrow\})\backslash(\text{SLLSm}\uparrow\cup\{\text{ai}\downarrow,\text{ai}\uparrow\})\\ R'\\ \|\{\text{ai}\uparrow\}\\ R\end{array}
\quad \overset{6}{\rightsquigarrow} \quad
\begin{array}{c}T\\ \|\{\text{ai}\downarrow\}\\ T'\\ \|\text{SLLS}\backslash\{\text{ai}\downarrow,\text{ai}\uparrow\}\\ R'\\ \|\{\text{ai}\uparrow\}\\ R\end{array}
\quad .
$$

In Step 5, we continue by permuting all instances of ai↑ down (dual to Step 2). This yields the desired derivation because no new instances of ai↓ are introduced while ai↑ is permuted down. In Step 6 we apply again Propositions 5.2.1 and 5.2.3 (dual to Step 3). □

5.3.13   Remark   The way how the rules in SLLSm↓ are handled while ai↓ is permuted up is a compromise for simplifying the presentation. There are the following alternatives: For the rule nm↓, we can use the m_1↑ as in case (5.v) of Lemma 5.3.12. For the rule nm_1↓ we can use the switch as in case (5.iii), and for m_1↓ we use d↑ and s as in case (5.iv). The rules nm_2↓ and m_2↓ fall under case (5.ii), and the rules nl_1↓ and l_1↓, are similar to nz↓ and z↓, respectively. Only for the rules nl_2↓ and l_2↓ there is no such solution. For them we need Lemma 5.3.11. An alternative would be to introduce the rule r↑, that has been used in the cut elimination argument in Section 4.4. Then we can replace for example

$$
\mathsf{ai}{\downarrow}\,\frac{\mathsf{l}_2{\downarrow}\,\dfrac{S\{!R,!T\}}{S\{!\{R,T\}\}}}{S\{!(\{R,T\},[a,\bar{a}])\}}
\qquad \text{by} \qquad
\mathsf{r}{\uparrow}\,\frac{\mathsf{l}_2{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{S\{!R,!T\}}{S(\{!R,!T\},![a,\bar{a}])}}{S(!\{R,T\},![a,\bar{a}])}}{S\{!(\{R,T\},[a,\bar{a}])\}}
\quad .
$$

The rule r↑ is derivable in system SLLS, but we need cut and interaction for doing so. This means that employing the rule r↑ (and r↓) in the proof of Theorem 5.3.9 would cause a difficult termination problem.

5.3.14   Definition   We can define the following two rules:

$$
\mathsf{ai}{\downarrow}_{0\top}\,\frac{S\{1\}}{S[0,\top]} \quad \text{and} \quad \mathsf{ai}{\downarrow}_a\,\frac{S\{1\}}{S[a,\bar{a}]} \quad , \text{ where } a \text{ is not a constant.}
$$

Obviously, both are special cases of ai↓, and each instance of ai↓ is either an instance of $\mathsf{ai}{\downarrow}_{0\top}$, an instance of $\mathsf{ai}{\downarrow}_a$, or trivial. The rules $\mathsf{ai}{\uparrow}_{0\top}$ and $\mathsf{ai}{\uparrow}_a$ are defined dually:

$$
\mathsf{ai}{\uparrow}_{0\top}\,\frac{S(0,\top)}{S\{\bot\}} \quad \text{and} \quad \mathsf{ai}{\uparrow}_a\,\frac{S(a,\bar{a})}{S\{\bot\}} \quad , \text{ where } a \text{ is not a constant.}
$$

Now we can strengthen Theorem 5.3.9.

5.3.15   Corollary   *For every derivation* $\begin{array}{c} T \\ \Delta \| \mathsf{SLLS} \\ R \end{array}$ *there is a derivation*

$$
\begin{array}{c}
T \\
\| \{\mathsf{ai}{\downarrow}_{0\top}\} \\
T' \\
\| \{\mathsf{ai}{\downarrow}_a\} \\
T'' \\
\| \mathsf{SLLS}\backslash\{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow}\} \\
R'' \\
\| \{\mathsf{ai}{\uparrow}_a\} \\
R' \\
\| \{\mathsf{ai}{\uparrow}_{0\top}\} \\
R
\end{array}
$$

for some structures T', T'', R'' and R'.

Proof: It is easy to see, that the two rules $\mathsf{ai}{\downarrow}_{0\top}$ and $\mathsf{ai}{\downarrow}_a$ permute over each other by the switch rule. The critical case looks as follows:

$$\mathsf{ai}{\downarrow}_{0\top} \frac{\mathsf{ai}{\downarrow}_a \dfrac{S\{1\}}{S[a,\bar{a}]}}{S[(a,[0,\top]),\bar{a}]} \qquad \text{yields} \qquad \mathsf{s} \frac{\mathsf{ai}{\downarrow}_a \dfrac{\mathsf{ai}{\downarrow}_{0\top} \dfrac{S\{1\}}{S[0,\top]}}{S([0,\top],[a,\bar{a}])}}{S[(a,[0,\top]),\bar{a}]} \quad .$$

Now we can get our result by first applying Theorem 5.3.9, and then permute all instances of $\mathsf{ai}{\downarrow}_{0\top}$ to the top. Then the remaining instances of $\mathsf{ai}{\downarrow}_a$ are permuted over the newly introduced instances of s. Proceed dually to separate $\mathsf{ai}{\uparrow}_{0\top}$ and $\mathsf{ai}{\uparrow}_a$ at the bottom. □

5.3.3 Lazy Separation of Thinning

In the sequent calculus system for linear logic (see Chapter 2), the weakening rule can be applied in a lazy way. This means that in a proof search, the application of a weakening is delayed until the very last moment. This is also the case for classical logic [TS00, BT01].

In system SLLS, the thinning rule can also be applied in such a lazy way: In a derivation all instances of rules $\rho \in \mathsf{SLLSt}{\downarrow}$ (which by Propositions 5.2.3 and 5.2.5 is strongly equivalent to the rule $\mathsf{t}{\downarrow}$) can be permuted to the top and all instances of rules in $\mathsf{SLLSt}{\uparrow}$ (i.e. the rule $\mathsf{t}{\uparrow}$) can be permuted to the bottom of the derivation. The rule $\mathsf{t}{\downarrow}$ can even be permuted over the rule $\mathsf{ai}{\downarrow}_a$ (see Definition 5.3.14), but not over $\mathsf{ai}{\downarrow}_{0\top}$. This can be seen from the following derivation:

$$\mathsf{t}{\downarrow} \frac{\mathsf{ai}{\downarrow}_{0\top} \dfrac{S\{1\}}{S[0,\top]}}{S[R,\top]} \quad .$$

5.3.16 Theorem *For every derivation* $\begin{array}{c} T \\ \Delta\|\mathsf{SLLS} \\ R \end{array}$ *there is a derivation*

$$\begin{array}{c} T \\ \|\{\mathsf{ai}{\downarrow}\} \\ T' \\ \|\mathsf{SLLSt}{\downarrow} \\ T'' \\ \|\mathsf{SLLS}\backslash(\mathsf{SLLSt}{\downarrow}\cup\mathsf{SLLSt}{\uparrow}\cup\{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow}\}) \\ R'' \\ \|\mathsf{SLLSt}{\uparrow} \\ R' \\ \|\{\mathsf{ai}{\uparrow}\} \\ R \end{array}$$

for some structures T', T'', R'' and R.

5.3.17   Theorem *For every derivation* $\Delta\|{\sf SLLS}\begin{smallmatrix}T\\\\R\end{smallmatrix}$ *there is a derivation*

$$
\begin{array}{l}
T\\
\|\{\mathsf{ai}\!\downarrow_{0\top}\}\\
T'\\
\|\mathsf{SLLSt}\!\downarrow\\
T''\\
\|\{\mathsf{ai}\!\downarrow_a\}\\
T'''\\
\|\mathsf{SLLS}\backslash(\mathsf{SLLSt}\!\downarrow\cup\mathsf{SLLSt}\!\uparrow\cup\{\mathsf{ai}\!\downarrow,\mathsf{ai}\!\uparrow\})\\
R'''\\
\|\{\mathsf{ai}\!\uparrow_a\}\\
R''\\
\|\mathsf{SLLSt}\!\uparrow\\
R'\\
\|\{\mathsf{ai}\!\uparrow_{0\top}\}\\
R
\end{array}
$$

for some structures T',T'',T''',R''',R'' and R'.

Before I can show the proofs of these two results, I have to investigate the permutation of the rules $\rho \in \mathsf{SLLSt}\!\downarrow$ over the other rules of system SLLS. In order to simplify the situation, I will permute the rule $\mathsf{t}\!\downarrow$ over the rules in $\mathsf{SLLS} \setminus \mathsf{SLLSt}\!\downarrow$.

5.3.18   Lemma *The rule $\mathsf{t}\!\downarrow$ permutes over every rule $\pi \in (\mathsf{SLLS} \cup \{\mathsf{ai}\!\downarrow_a\}) \setminus (\mathsf{SLLSt}\!\downarrow \cup \{\mathsf{ai}\!\downarrow\})$ by $\{\mathsf{m},\mathsf{m}_1\!\downarrow,\mathsf{m}_2\!\downarrow,\mathsf{l}_1\!\downarrow\}.$*

Proof: Consider a derivation

$$
\mathsf{t}\!\downarrow\dfrac{\pi\dfrac{Q}{S\{0\}}}{S\{Z\}}\quad,
$$

where $\pi \in (\mathsf{SLLS} \cup \{\mathsf{ai}\!\downarrow_a\}) \setminus (\mathsf{SLLSt}\!\downarrow \cup \{\mathsf{ai}\!\downarrow\})$. Without loss of generality, assume that the application of π is nontrivial. Now, follow again the case analysis in 4.2.3.

(4) The redex of π is inside the contractum 0 of $\mathsf{t}\!\downarrow$. This is not possible because the application of π is nontrivial.

(5) The contractum 0 of $\mathsf{t}\!\downarrow$ is inside an active structure of the redex of π, but not inside a passive one. As before, this case is the problematic one and as before, for each rule π, I will show the most general case how the contractum 0 can interfere.

 (i) If $\pi \in \mathsf{SLLSt}\!\uparrow \cup \{\mathsf{ac}\!\downarrow,\mathsf{ai}\!\uparrow,\mathsf{nz}\!\uparrow\}$, then this case is impossible.

 (ii) If $\pi \in \{\mathsf{m},\mathsf{m}_1\!\downarrow,\mathsf{m}_2\!\downarrow,\mathsf{l}_1\!\downarrow,\mathsf{l}_2\!\downarrow,\mathsf{z}\!\downarrow\}$, we can reduce this case to case (2) by the associativity of the plus context.

(iii) If $\pi \in \{\mathsf{ai}\downarrow_a, \mathsf{p}\downarrow\}$, then we can use the rule $\mathsf{m}_1\downarrow$:

$$
\mathsf{p}\downarrow \frac{S'\{![R,T]\}}{S'[!R,?T]} \\
= \frac{}{S'[\{!R,0\},?T]} \\
\mathsf{t}\downarrow \frac{}{S'[\{!R,Z\},?T]}
\qquad \text{yields} \qquad
\begin{array}{c}
= \dfrac{S'\{![R,T]\}}{S'\{![R,T],0\}} \\
\mathsf{t}\downarrow \dfrac{}{S'\{![R,T],[Z,0]\}} \\
\mathsf{p}\downarrow \dfrac{}{S'\{[!R,?T],[Z,0]\}} \\
\mathsf{m}_1\downarrow \dfrac{}{S'[\{!R,Z\},\{?T,0\}]} \\
= \dfrac{}{S'[\{!R,Z\},?T]}
\end{array}
\quad .
$$

For $\pi = \mathsf{ai}\downarrow_a$, the situation is similar.

(iv) If $\pi = \mathsf{d}\downarrow$, then we have

$$
\mathsf{d}\downarrow \frac{S'\langle[\langle R,R'\rangle,U],[\langle T,T'\rangle,V]\rangle}{S'[\langle R,R',T,T'\rangle,\{U,V\}]} \\
= \frac{}{S'[\langle R,\{\langle R',T\rangle,0\},T'\rangle,\{U,V\}]} \\
\mathsf{t}\downarrow \frac{}{S'[\langle R,\{\langle R',T\rangle,Z\},T'\rangle,\{U,V\}]}
\quad ,
$$

where we use the rules m and $\mathsf{m}_1\downarrow$ together:

$$
\begin{array}{c}
= \dfrac{S'\langle[\langle R,R'\rangle,U],[\langle T,T'\rangle,V]\rangle}{S'\{\langle[\langle R,R'\rangle,U],[\langle T,T'\rangle,V]\rangle,0\}} \\
\mathsf{t}\downarrow \dfrac{}{S'\{\langle[\langle R,R'\rangle,U],[\langle T,T'\rangle,V]\rangle,[0,\langle 0,Z\rangle]\}} \\
\mathsf{d}\downarrow \dfrac{}{S'\{[\langle R,R',T,T'\rangle,\{U,V\}],[0,\langle 0,Z\rangle]\}} \\
\mathsf{m}_1\downarrow \dfrac{}{S'[\{\langle R,R',T,T'\rangle,\langle 0,Z\rangle\},\{U,V,0\}]} \\
= \dfrac{}{S'[\{\langle R,R',T,T'\rangle,\langle 0,Z\rangle\},\{U,V\}]} \\
\mathsf{m} \dfrac{}{S'[\langle\{\langle R,T'\rangle,0\},\{\langle R',T\rangle,Z\}\rangle,\{U,V\}]} \\
= \dfrac{}{S'[\langle R,\{\langle R',T\rangle,Z\},T'\rangle,\{U,V\}]}
\end{array}
\quad .
$$

(v) If $\pi = \mathsf{d}\uparrow$, then we have

$$
\mathsf{d}\uparrow \frac{S'(\{(R,R'),U\},\langle(T,T'),V\rangle)}{S'\{(R,R',T,T'),(U,V)\}} \\
= \frac{}{S'\{(R,\{(R',T),0\},T'),(U,V)\}} \\
\mathsf{t}\downarrow \frac{}{S'\{(R,\{(R',T),Z\},T'),(U,V)\}}
\quad ,
$$

where we use the rule $\mathsf{m}_2\downarrow$:

$$
\begin{array}{c}
= \dfrac{S'(\{(R,R'),U\},\langle(T,T'),V\rangle)}{S'\{(\{(R,R'),U\},\langle(T,T'),V\rangle),0\}} \\
\mathsf{t}\downarrow \dfrac{}{S'\{(\{(R,R'),U\},\langle(T,T'),V\rangle),(0,Z)\}} \\
\mathsf{d}\uparrow \dfrac{}{S'\{(R,R',T,T'),(0,Z),(U,V)\}} \\
\mathsf{m}_2\downarrow \dfrac{}{S'\{(\{(R,T'),0\},\{(R',T),Z\}),(U,V)\}} \\
= \dfrac{}{S'\{(R,\{(R',T),Z\},T'),(U,V)\}}
\end{array}
\quad .
$$

(vi) If $\pi = \mathsf{p}{\uparrow}$, then we have

$$
=\cfrac{\mathsf{t}{\downarrow}\,\cfrac{\mathsf{p}{\uparrow}\,\cfrac{S'(?(R,R'),!(T,T'))}{S'\{?(R,R',T,T')\}}}{S'\{?(R,[(R',T),0],T')\}}}{S'\{?(R,[(R',T),Z],T')\}}\quad,
$$

where we can use the rules $\mathsf{m}_2{\downarrow}$ and $\mathsf{l}_1{\downarrow}$:

$$
=\cfrac{\mathsf{m}_2{\downarrow}\,\cfrac{\mathsf{l}_1{\downarrow}\,\cfrac{\mathsf{p}{\uparrow}\,\cfrac{\mathsf{t}{\downarrow}\,\cfrac{S'(?(R,R'),!(T,T'))}{S'[(?(R,R'),!(T,T')),0]}}{S'[(?(R,R'),!(T,T')),?(0,Z)]}}{S'[?(R,R',T,T'),?(0,Z)]}}{S'\{?[(R,R',T,T'),(0,Z)]\}}}{S'\{?([(R,T'),0],[(R',T),Z],T')\}}{S'\{?(R,[(R',T),Z],T')\}}\quad.
$$

(vii) If $\pi \in \{\mathsf{ac}{\uparrow}, \mathsf{l}_1{\uparrow}, \mathsf{l}_2{\uparrow}\}$, then use the rule m as follows:

$$
=\cfrac{\mathsf{t}{\downarrow}\,\cfrac{\mathsf{l}_1{\uparrow}\,\cfrac{S'\{![R,T]\}}{S'([R,!T)}}{S'([!R,0],!T)}}{S'([!R,Z],!T)}
\qquad\text{yields}\qquad
=\cfrac{\mathsf{m}\,\cfrac{\mathsf{l}_1{\uparrow}\,\cfrac{\mathsf{t}{\downarrow}\,\cfrac{S'\{![R,T]\}}{S'[![R,T],0]}}{S'[![R,T],(Z,0)]}}{S'[(!R,!T),(Z,0)]}}{S'([!R,Z],[!T,0])}{S'([!R,Z],!T)}\quad.
$$

(viii) If $\pi = \mathsf{m}_1{\uparrow}$, then we have

$$
=\cfrac{\mathsf{t}{\downarrow}\,\cfrac{\mathsf{m}_1{\uparrow}\,\cfrac{S'((R,R'),U),((T,T'),V))}{S'((R,R',T,T'),(U,V))}}{S'((R,[(R',T),0],T'),(U,V))}}{S'((R,[(R',T),Z],T'),(U,V))}\quad,
$$

where we can use the rules $\mathsf{m}_2{\downarrow}$ and m:

$$
=\cfrac{\mathsf{m}_2{\downarrow}\,\cfrac{\mathsf{m}\,\cfrac{\mathsf{m}_1{\uparrow}\,\cfrac{\mathsf{t}{\downarrow}\,\cfrac{S'((R,R'),U),((T,T'),V))}{S'[((R,R'),U),((T,T'),V)),0]}}{S'[((R,R'),U),((T,T'),V)),((Z,0),0)]}}{S'[(R,R',T,T'),(U,V)),((Z,0),0)]}}{S'([(R,R',T,T'),(Z,0)],[(U,V),0])}}{S'([(R,R',T,T'),(Z,0)],(U,V))}}{S'(([(R,T'),0],[(R',T),Z]),(U,V))}{S'((R,[(R',T),Z],T'),(U,V))}\quad.
$$

(ix) If $\pi = \mathsf{m_2}\!\uparrow$, then the situation is similar to the previous case, but we use the rule $\mathsf{m_1}\!\downarrow$ instead of $\mathsf{m_2}\!\downarrow$.

(x) If $\pi = \mathsf{s}$, then the situation is also similar to the case of $\mathsf{m_1}\!\uparrow$, but we use the rules $\mathsf{m_2}\!\downarrow$ and $\mathsf{m_1}\!\downarrow$ together.

(xi) If $\pi = \mathsf{z}\!\uparrow$, then we can use the rule $\mathsf{m_2}\!\downarrow$:

$$
\mathsf{z}\!\uparrow\frac{\mathsf{z}\!\uparrow\dfrac{S'\{!\langle R,T\rangle\}}{S'(!R,T)}}{\mathsf{t}\!\downarrow\dfrac{S'(\{!R,0\},T)}{S'(\{!R,Z\},T)}}
\qquad \text{yields} \qquad
\mathsf{m_2}\!\downarrow\frac{\mathsf{z}\!\uparrow\dfrac{\mathsf{t}\!\downarrow\dfrac{=\dfrac{S'\{!\langle R,T\rangle\}}{S'(\{!\langle R,T\rangle,0\}}}{S'\{!\langle R,T\rangle,(Z,0)\}}}{S'\{(!R,T),(Z,0)\}}}{=\dfrac{S'(\{!R,Z\},\{T,0\})}{S'(\{!R,Z\},T)}} \quad .
$$

(6) The contractum 0 of $\mathsf{t}\!\downarrow$ and the redex of π overlap. impossible. $\qquad\square$

Proof of Theorem 5.3.16: Apply the following transformations to the given derivation:

$$
\begin{array}{ccccc}
 & & T & & T \\
 & & \|\{\mathsf{ai}\!\downarrow\} & & \|\{\mathsf{ai}\!\downarrow\} \\
T & & T' & & T' \\
\|\mathsf{SLLS} & \overset{1}{\rightsquigarrow} & \|\mathsf{SLLS}\backslash\{\mathsf{ai}\!\downarrow,\mathsf{ai}\!\uparrow\} & \overset{2}{\rightsquigarrow} & \|(\mathsf{SLLS}\cup\{\mathsf{t}\!\downarrow\})\backslash(\mathsf{SLLSt}\!\downarrow\cup\{\mathsf{ai}\!\downarrow,\mathsf{ai}\!\uparrow\}) \\
R & & R' & & R' \\
 & & \|\{\mathsf{ai}\!\uparrow\} & & \|\{\mathsf{ai}\!\uparrow\} \\
 & & R & & R
\end{array} \quad .
$$

In Step 1 we apply Theorem 5.3.9 and in Step 2 we replace all instances of rules in $\mathsf{SLLSt}\!\downarrow$ by an instance of $\mathsf{t}\!\downarrow$.

$$
\begin{array}{ccccc}
 & & T & & T \\
 & & \|\{\mathsf{ai}\!\downarrow\} & & \|\{\mathsf{ai}\!\downarrow\} \\
 & & T' & & T' \\
 & & \|\{\mathsf{t}\!\downarrow\} & & \|\{\mathsf{t}\!\downarrow\} \\
\overset{3}{\rightsquigarrow} & & T'' & \overset{4}{\rightsquigarrow} & T'' \\
 & & \|\mathsf{SLLS}\backslash(\mathsf{SLLSt}\!\downarrow\cup\{\mathsf{ai}\!\downarrow,\mathsf{ai}\!\uparrow\}) & & \|(\mathsf{SLLS}\cup\{\mathsf{t}\!\uparrow\})\backslash(\mathsf{SLLSt}\!\downarrow\cup\mathsf{SLLSt}\!\uparrow\cup\{\mathsf{ai}\!\downarrow,\mathsf{ai}\!\uparrow\}) \\
 & & R' & & R' \\
 & & \|\{\mathsf{ai}\!\uparrow\} & & \|\{\mathsf{ai}\!\uparrow\} \\
 & & R & & R
\end{array} \quad .
$$

In Step 3, we continue by permuting up all instances of $\mathsf{t}\!\downarrow$ (by applying Lemma 5.3.18). Then, dually, all instances of rules in $\mathsf{SLLSt}\!\uparrow$ are replaced by $\mathsf{t}\!\uparrow$ (in Step 4) and are permuted

down (in Step 5).

$$
\overset{5}{\rightsquigarrow}
\quad
\begin{array}{l}
T \\
\Big\|{}_{\{\mathsf{ai}\downarrow\}} \\
T' \\
\Big\|{}_{\{\mathsf{t}\downarrow\}} \\
T'' \\
\Big\|{}_{\mathsf{SLLS}\backslash(\mathsf{SLLSt}\downarrow\cup\mathsf{SLLSt}\uparrow\cup\{\mathsf{ai}\downarrow,\mathsf{ai}\uparrow\})} \\
R'' \\
\Big\|{}_{\{\mathsf{t}\uparrow\}} \\
R' \\
\Big\|{}_{\{\mathsf{ai}\uparrow\}} \\
R
\end{array}
\qquad
\overset{6}{\rightsquigarrow}
\quad
\begin{array}{l}
T \\
\Big\|{}_{\{\mathsf{ai}\downarrow\}} \\
T' \\
\Big\|{}_{\mathsf{SLLSt}\downarrow} \\
T'' \\
\Big\|{}_{\mathsf{SLLS}\backslash(\mathsf{SLLSt}\downarrow\cup\mathsf{SLLSt}\uparrow\cup\{\mathsf{ai}\downarrow,\mathsf{ai}\uparrow\})} \\
R'' \\
\Big\|{}_{\mathsf{SLLSt}\uparrow} \\
R' \\
\Big\|{}_{\{\mathsf{ai}\uparrow\}} \\
R
\end{array}
\qquad .
$$

In the end, Proposition 5.2.3 is applied, to get the desired derivation (Step 6).   □

Proof of Theorem 5.3.17:   Apply the following transformations to the given derivation:

$$
\begin{array}{l}
T \\
\Big\|{}_{\mathsf{SLLS}} \\
R
\end{array}
\overset{1}{\rightsquigarrow}
\begin{array}{l}
T \\
\Big\|{}_{\{\mathsf{ai}\downarrow_{0\top}\}} \\
T' \\
\Big\|{}_{\{\mathsf{ai}\downarrow_{a}\}} \\
Q \\
\Big\|{}_{\mathsf{SLLS}\backslash\{\mathsf{ai}\downarrow,\mathsf{ai}\uparrow\}} \\
P \\
\Big\|{}_{\{\mathsf{ai}\uparrow_{a}\}} \\
R' \\
\Big\|{}_{\{\mathsf{ai}\uparrow_{0\top}\}} \\
R
\end{array}
\overset{2}{\rightsquigarrow}
\begin{array}{l}
T \\
\Big\|{}_{\{\mathsf{ai}\downarrow_{0\top}\}} \\
T' \\
\Big\|{}_{\{\mathsf{ai}\downarrow_{a}\}} \\
Q \\
\Big\|{}_{\{\mathsf{t}\downarrow\}} \\
Q' \\
\Big\|{}_{\mathsf{SLLS}\backslash(\mathsf{SLLSt}\downarrow\cup\mathsf{SLLSt}\uparrow\cup\{\mathsf{ai}\downarrow,\mathsf{ai}\uparrow\})} \\
P' \\
\Big\|{}_{\{\mathsf{t}\uparrow\}} \\
P \\
\Big\|{}_{\{\mathsf{ai}\uparrow_{a}\}} \\
R' \\
\Big\|{}_{\{\mathsf{ai}\uparrow_{0\top}\}} \\
R
\end{array}
\qquad .
$$

First, we apply Corollary 5.3.15. The second step is the same as Steps 2 to 5 of the previous proof. Now, all instances of $\mathsf{t}\downarrow$ are permuted over $\mathsf{ai}\downarrow_{a}$ and all instances of $\mathsf{t}\uparrow$ are permuted

under $\mathsf{ai}{\uparrow}_a$ (by Lemma 5.3.18 and the dual statement).

$$
\overset{3}{\rightsquigarrow}
\quad
\begin{array}{c}
T \\
\|\{\mathsf{ai}{\downarrow}_{0\top}\} \\
T' \\
\|\{\mathsf{t}{\downarrow}\} \\
T'' \\
\|\{\mathsf{ai}{\downarrow}_a,\mathsf{m}_1{\downarrow}\} \\
Q' \\
\|\mathsf{SLLS}\backslash(\mathsf{SLLSt}{\downarrow}\cup\mathsf{SLLSt}{\uparrow}\cup\{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow}\}) \\
P' \\
\|\{\mathsf{ai}{\uparrow}_a,\mathsf{m}_1{\uparrow}\} \\
R'' \\
\|\{\mathsf{t}{\uparrow}\} \\
R' \\
\|\{\mathsf{ai}{\uparrow}_{0\top}\} \\
R
\end{array}
\qquad
\overset{4}{\rightsquigarrow}
\quad
\begin{array}{c}
T \\
\|\{\mathsf{ai}{\downarrow}_{0\top}\} \\
T' \\
\|\{\mathsf{t}{\downarrow}\} \\
T'' \\
\|\{\mathsf{ai}{\downarrow}_a\} \\
T''' \\
\|\mathsf{SLLS}\backslash(\mathsf{SLLSt}{\downarrow}\cup\mathsf{SLLSt}{\uparrow}\cup\{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow}\}) \\
R''' \\
\|\{\mathsf{ai}{\uparrow}_a\} \\
R'' \\
\|\{\mathsf{t}{\uparrow}\} \\
R' \\
\|\{\mathsf{ai}{\uparrow}_{0\top}\} \\
R
\end{array}
\qquad .
$$

The proof of Lemma 5.3.18 shows that new instances of $\mathsf{m}_1{\downarrow}$ might be introduced when $\mathsf{t}{\downarrow}$ is permuted over $\mathsf{ai}{\downarrow}_a$. By Remark 5.3.13, we can permute $\mathsf{ai}{\downarrow}_a$ over $\mathsf{m}_1{\downarrow}$. This is done in Step 4. For $\mathsf{ai}{\uparrow}_a$ we proceed dually. $\qquad\square$

5.3.19 Remark It should be mentioned here that I conjecture that it is also possible to obtain a derivation

$$
\begin{array}{c}
T \\
\|\{\mathsf{t}{\downarrow}\} \\
T' \\
\|\mathsf{SLLS}\backslash(\mathsf{SLLSt}{\downarrow}\cup\mathsf{SLLSt}{\uparrow}) \\
R' \\
\|\{\mathsf{t}{\uparrow}\} \\
R
\end{array}
\qquad ,
$$

by applying the following method: The rule $\mathsf{t}{\downarrow}$ is permuted up as described above, and if it encounters an instance of $\mathsf{ai}{\downarrow}_{0\top}$ (which is the only problematic case), then replace

$$
\mathsf{t}{\downarrow}\ \frac{\mathsf{ai}{\downarrow}_{0\top}\ \dfrac{S\{1\}}{S[0,\top]}}{S[R,\top]}
\qquad \text{by} \qquad
\mathsf{t}{\uparrow}\ \frac{\mathsf{i}{\downarrow}\ \dfrac{S\{1\}}{S[R,\bar{R}]}}{S[R,\top]}
\qquad ,
$$

and apply Proposition 3.2.16. Then proceed dually while the rule $\mathsf{t}{\uparrow}$ is permuted down. In order to show termination, a similar argument as for the decomposition theorems for system SELS in Section 4.3 has to be used.

5.3.4 Eager Separation of Atomic Thinning

The ability of applying inferences deep inside structures, allows us to apply thinning not only in a lazy, but also in an eager way. This means that in a proof, we can first forget all the material that is not needed by applying thinning and then go on with the proof without applying thinning (or weakening) again.

5.3.20 Theorem *For every proof* $\overline{\parallel}$LLS *there is a proof*
$$R$$

$$\overline{\parallel}\text{LLS}\backslash\{\text{at}\downarrow\}$$
$$R'$$
$$\parallel\{\text{at}\downarrow\}$$
$$R$$

for some structure R'.

A similar result is impossible in the sequent calculus because it does not allow deep inference. Again, the top-down symmetry allows us, to extend this result to derivations:

5.3.21 Theorem *For every derivation* $\Delta\parallel$SLLS *there is a derivation*

where the derivation starts from T at top and R at bottom.

$$T$$
$$\parallel\{\text{at}\uparrow\}$$
$$T'$$
$$\parallel\text{SLLS}\backslash\{\text{at}\uparrow,\text{at}\downarrow\}$$
$$R'$$
$$\parallel\{\text{at}\downarrow\}$$
$$R$$

for some structures T' *and* R'.

The proofs of the two theorems will be very similar to the previous decomposition results. I will show that at$\uparrow$ can be permuted up and at$\downarrow$ can be permuted down.

Before I start, I will again show the main difficulties that occur. First observe, that it is again atomicity that makes everything work. For example, in the following two derivations it is not clear how the instances of cothinning could be permuted up:

$$\text{t}\uparrow\frac{\text{s}\dfrac{S([R,U_1,U_2],T)}{S[(R,T),U_1,U_2]}}{S[\top,U_2]} \qquad \text{and} \qquad \text{t}\uparrow\frac{\text{s}\dfrac{S([(R_1,R_2),U],T_1,T_2)}{S[(R_1,R_2,T_1,T_2),U]}}{S[(R_1,\top,T_2),U]} \quad .$$

The reason why this problem did not occur in the proof of Theorem 4.3.2 for the rule w$\uparrow$, is that in the case of w$\uparrow$ the structure that is inserted is guarded by an exponential, which is not the case for t$\uparrow$.

But also in the case of atomic thinning, the result is not completely trivial. We encounter in fact the same problems as for atomic contraction in Section 5.3.1:

(1) While permuting an instance of at$\uparrow$ up, it might happen that it meets an instance of the interaction rule as follows:

$$\text{at}\uparrow\frac{\text{ai}\downarrow\dfrac{S\{1\}}{S[a,\bar{a}]}}{S[\top,\bar{a}]} \quad , \qquad \text{which can be replaced by} \qquad \text{at}\downarrow\frac{\text{ai}\downarrow\dfrac{S\{1\}}{S[\top,0]}}{S[\top,\bar{a}]} \quad .$$

Here a new instance of $\mathsf{at}{\downarrow}$ is introduced, which has to be permuted down afterwards. The problem is to show that this does indeed terminate eventually.

(2) The atom to be inserted in an instance of $\mathsf{at}{\uparrow}$ or $\mathsf{at}{\downarrow}$ can be one of the constants. Then we might encounter cases like

$$\mathsf{at}{\uparrow}\,\frac{\mathsf{p}{\uparrow}\,\dfrac{S(?R,!T)}{S\{?(R,T)\}}}{=\dfrac{S\{?\{(R,T),0\}\}}{S\{?\{(R,T),\boxed{\top}\}\}}}\quad,$$

where it is not obvious how the instance of $\mathsf{at}{\uparrow}$ can be permuted up.

Both problems will be solved in the same way as it has been done for atomic contraction in Section 5.3.1. In other words, I will use cut elimination instead of permuting $\mathsf{at}{\uparrow}$ up and $\mathsf{at}{\downarrow}$ down repeatedly until termination is reached. In [Brü03b], K. Brünnler actually uses such a termination argument to show the same theorem for classical logic. However, the reader familiar with that work should observe that in classical logic the situation is much simpler because the instance of $\mathsf{ai}{\downarrow}$ is destroyed when "hit" by an atomic cothinning, which is not the case in linear logic.

In the following, I will show how the rule $\mathsf{at}{\uparrow}$ can be permuted up. In order to keep the situation as clear as possible, I introduce the following five rules:

$$\mathsf{at}{\uparrow}_\perp\,\frac{S\{\perp\}}{S\{\top\}}\quad,\quad \mathsf{at}{\uparrow}_1\,\frac{S\{1\}}{S\{\top\}}\quad,\quad \mathsf{at}{\uparrow}_0\,\frac{S\{0\}}{S\{\top\}}\quad,\quad \mathsf{at}{\uparrow}_\top\,\frac{S\{\top\}}{S\{\top\}}\quad\text{and}$$

$$\mathsf{ac}{\uparrow}_a\,\frac{S\{a\}}{S\{\top\}}\quad,\qquad\text{where } a \text{ is not a constant.}$$

Obviously, each of them is a special case of $\mathsf{at}{\uparrow}$, and each instance of $\mathsf{at}{\uparrow}$ is an instance of one of $\mathsf{at}{\uparrow}_\perp$, $\mathsf{at}{\uparrow}_1$, $\mathsf{at}{\uparrow}_0$, $\mathsf{at}{\uparrow}_\top$, $\mathsf{at}{\uparrow}_a$. Further, all instances of the rule $\mathsf{ac}{\uparrow}_\top$ are trivial.

5.3.22 **Lemma** *The rule* $\mathsf{at}{\uparrow}_a$ *permutes over every rule* $\pi \in \mathsf{SLLS} \setminus \{\mathsf{ai}{\downarrow}, \mathsf{at}{\downarrow}, \mathsf{ac}{\downarrow}, \mathsf{at}{\uparrow}\}$.

Proof: Consider a derivation

$$\mathsf{at}{\uparrow}_a\,\frac{\pi\,\dfrac{Q}{S\{a\}}}{S\{\top\}}\quad,$$

where $\pi \in \mathsf{SLLS} \setminus \{\mathsf{ai}{\downarrow}, \mathsf{at}{\downarrow}, \mathsf{ac}{\downarrow}, \mathsf{at}{\uparrow}\}$. Without loss of generality, assume that the application of π is nontrivial. Now, follow the case analysis in 4.2.3.

(4) The redex of π is inside the contractum a. impossible.

(5) The contractum a of $\mathsf{at}{\uparrow}_a$ is inside an active structure of the redex of π, but not inside a passive one. This is only possible, if $\pi = \mathsf{ac}{\uparrow}$. Then we have

$$\mathsf{at}{\uparrow}\,\frac{\mathsf{ac}{\uparrow}\,\dfrac{S'\{a\}}{S'\langle a, a\rangle}}{S'\langle a, \boxed{\top}\rangle}\quad,\qquad\text{which yields}\qquad =\frac{S'\{a\}}{S'\langle a, \top\rangle}\quad.$$

(6) The contractum a of $\mathsf{at}{\uparrow}_a$ and the redex of π overlap. This is impossible. □

It remains to show, how the rules $\mathsf{at}{\uparrow}_\perp$, $\mathsf{at}{\uparrow}_1$ and $\mathsf{at}{\uparrow}_0$ can be permuted up. For this, the following method would be possible: For permuting $\mathsf{at}{\uparrow}_\perp$ up, one can use the same method as for $\mathsf{ac}{\uparrow}_\perp$ in Section 5.3.1; for $\mathsf{at}{\uparrow}_1$, proceed similar as for $\mathsf{ai}{\downarrow}$ in Section 5.3.2; and for $\mathsf{at}{\uparrow}_0$ proceed similar as for $\mathsf{t}{\downarrow}$ in Section 5.3.3.

However, for the sake of simplification, I will here use the following observation. Every instance of $\mathsf{at}{\uparrow}_0$ can be replaced by an instance of the rule m:

$$\mathsf{at}{\uparrow}_0 \frac{S\{0\}}{S\{\top\}} \quad = \quad \mathsf{at}{\uparrow}_0 \frac{S[0,0]}{S\langle\top,\top\rangle} \quad = \quad \mathsf{m}\,\frac{S[\langle 0,\top\rangle,\langle\top,0\rangle]}{S\langle[0,\top],[\top,0]\rangle} \quad .$$

Similarly, every instance of $\mathsf{at}{\uparrow}_1$ is an instance of $\mathsf{m}_1{\uparrow}$, and every instance of $\mathsf{at}{\uparrow}_\perp$ is an instance of $\mathsf{m}_2{\uparrow}$. This is enough to prove the following lemma:

5.3.23   Lemma   *For every derivation* $\begin{array}{c}T\\ \Delta\|\mathsf{SLLS}{\uparrow}\\ R\end{array}$ *there is a derivation*

$$\begin{array}{c}T\\ \|\{\mathsf{at}{\uparrow}\}\\ T'\\ \|\mathsf{SLLS}{\uparrow}\backslash\{\mathsf{at}{\uparrow}\}\\ R\end{array}$$

for some structure T'.

Proof: Each instance of $\mathsf{at}{\uparrow}$ inside Δ is an instance of $\mathsf{at}{\uparrow}_\perp$, $\mathsf{at}{\uparrow}_1$, $\mathsf{at}{\uparrow}_0$, $\mathsf{at}{\uparrow}_\top$ or $\mathsf{at}{\uparrow}_a$. The instances of $\mathsf{ac}{\uparrow}_\top$ can be removed because they are trivial, and the instances of $\mathsf{ac}{\uparrow}_0$, $\mathsf{at}{\uparrow}_1$ and $\mathsf{ac}{\uparrow}_\perp$ are replaced by m, $\mathsf{m}_1{\uparrow}$ and $\mathsf{m}_2{\uparrow}$, respectively. It remains to permute all instances of $\mathsf{at}{\uparrow}_a$ to the top of Δ. For this, repeatedly apply Lemmata 5.3.22. □

5.3.24   Lemma   *For every derivation* $\begin{array}{c}T\\ \Delta\|\mathsf{SLLS}{\downarrow}\\ R\end{array}$ *there is a derivation*

$$\begin{array}{c}T\\ \|\mathsf{SLLS}{\downarrow}\backslash\{\mathsf{at}{\downarrow}\}\\ R'\\ \|\{\mathsf{at}{\downarrow}\}\\ R\end{array}$$

for some structure R'.

Proof: Dual to Lemma 5.3.23. □

Proof of Theorem 5.3.20:   Immediate consequence of Lemma 5.3.24. □

Proof of Theorem 5.3.21:   Similar to the proof of Theorem 5.3.2. □

The inspection of the proofs of Theorems 5.3.2 and 5.3.21 show that the rules at↑ and ac↑ can both be permuted up independently, i.e. without introducing new instances of at↑ or ac↑. They can also be permuted over each other without introducing new rule instances. Consequently, the results of Theorems 5.3.2 and 5.3.21 can be put together in the following way:

5.3.25 Theorem *For every derivation* $\begin{matrix} T \\ \Delta \| \text{SLLS} \\ R \end{matrix}$ *there is are derivations*

$$
\begin{matrix}
T \\
\| \{\text{ac}\uparrow\} \\
T'' \\
\| \{\text{at}\uparrow\} \\
T' \\
\| \text{SLLS}\backslash\{\text{at}\uparrow,\text{ac}\uparrow,\text{at}\downarrow,\text{ac}\downarrow\} \\
R' \\
\| \{\text{at}\downarrow\} \\
R'' \\
\| \{\text{ac}\downarrow\} \\
R
\end{matrix}
\qquad and \qquad
\begin{matrix}
T \\
\| \{\text{at}\uparrow\} \\
T''' \\
\| \{\text{ac}\uparrow\} \\
T' \\
\| \text{SLLS}\backslash\{\text{at}\uparrow,\text{ac}\uparrow,\text{at}\downarrow,\text{ac}\downarrow\} \\
R' \\
\| \{\text{ac}\downarrow\} \\
R''' \\
\| \{\text{at}\downarrow\} \\
R
\end{matrix}
\quad ,
$$

for some structures T', T'', T''' *and* R', R'', R'''.

Proof: Again, the proof is similar to the proof of Theorem 5.3.2. $\square$

5.4 Discussion

In this chapter, I have shown that it is possible to present in the calculus of structures a local system for linear logic, in particular, it is possible to reduce the application of contraction to atoms. This is due to the ability of applying inferences deep inside structures and therefore impossible in the sequent calculus.

There is one drawback in system LLS that has to be mentioned. Since the additives are needed for reducing absorption and weakening to local rules, the separation property is lost in system LLS. However, it is possible to consider the multiplicative additive fragment independently from the exponentials.

The restriction of contraction to atoms may be useful in proof search if it is done in a distributed way. However, the huge amount of nondeterminism induced by the freedom of the applicability of the rules and the number of rules makes the system LLS, as it stands now, unsuitable for practical proof search. This means that an important problem of future research should be to control this nondeterminism. This might happen via focusing proofs based on the relations between the connectives as it has been done in [And92, Mil96]. But it might also be possible to find a way to focus proofs based on the depth of structures.

In Section 5.3, I have shown several decomposition results. All have been proved by the use of permutation results, which in turn are based on tedious case analyses. The question that arises is whether there is a general underlying principle that can be used to simplify those proofs. Another open question is whether it is possible to find for system SLLS (or equivalently system SLS) a similar decomposition as for system SELS in Theorem 4.3.2,

which allows a separation of core and noncore:

$$
\begin{array}{l}
T \\
\quad \|\text{noncore (up)} \\
T' \\
\quad \|\text{interaction (down)} \\
T'' \\
\quad \|\text{core (up and down)} \qquad . \\
R'' \\
\quad \|\text{interaction (up)} \\
R' \\
\quad \|\text{noncore (down)} \\
R
\end{array}
$$

I conjecture that this is possible and that there is a relation to cut elimination (see Section 8.3 for further explanation). However, I think that the methods presented in this thesis are not sufficient to bring light into this matter.

The maybe most interesting result of this chapter is system SLLS itself, together with the fact that it *is* linear logic. To me, this was rather surprising because I found the system on the basis of A. Guglielmi's *relation webs* [Gug99], when I tried to reduce the absorbtion rule of system SELS to an atomic version. I realised that $\langle\ldots\rangle$ and $[\ldots]$ are the additives of linear logic only after A. Tiu found the medial rule for classical logic [BT01] for reducing contraction to an atomic version. It might amuse the reader to learn that system SLLS was actually discovered before system SLS.

6

Mix and Switch

This chapter contains only two small sections. In Section 6.1, I will investigate how the mix rule and its nullary version, which can be added to the sequent calculus system for linear logic, are incorporated in the calculus of structures. The two rules are of importance for this thesis because they are necessary for the noncommutative connective that will be discussed in Chapter 7. Therefore, this section can be seen as a link between the previous chapters (in particular Chapter 4) and the next chapter.

Then, in Section 6.2, I will restrict myself to the multiplicative fragment of linear logic, extended by mix and nullary mix. In particular, I will study the derivability in the system that contains only the switch rule. The purpose is to give an algebraic characterisation for derivability, which can be seen as a symmetric version of the correctness criterion for multiplicative proof nets [Ret99a].

Whereas Section 6.1 is crucial for the following chapter, Section 6.2 contains supplementary results that are not needed later. However, from a mathematical viewpoint, the results of Section 6.2 are of importance for the whole thesis. They can be seen as a first step towards unveiling the algebraic principles behind the decomposition theorems of Chapter 4. The ideas discussed here do also reappear in Chapter 7.

6.1 Adding the Rules Mix and Nullary Mix

The sequent calculus system for linear logic, as it has been presented in Chapter 2, is sometimes extended by the the mix rule and its nullary version [AJ94, FR94, Ret93]:

$$\mathsf{mix}\,\frac{\vdash \Phi \quad \vdash \Psi}{\vdash \Phi, \Psi} \qquad \text{and} \qquad \mathsf{mix0}\,\frac{}{\vdash} \quad .$$

One of the reasons is that these rules are valid in many semantics for linear logic, for example coherence spaces [Gir87a] and game semantics [AJ94]. Both rules can be easily translated into rules in the calculus of structures:

$$\times\,\frac{S(R,T)}{S[R,T]} \qquad \text{and} \qquad \mathsf{nx}\,\frac{S\{1\}}{S\{\bot\}} \quad .$$

Both rules are self-dual, and therefore part of both, the down fragment and the up fragment. Here, I will consider only multiplicative exponential linear logic, as it has been discussed in Chapter 4. If we add the two rules, we obtain the following systems:

$$
\begin{aligned}
\mathsf{MELL'} &= \mathsf{MELL} \cup \{\mathsf{mix}, \mathsf{mix0}\} \quad, \\
\mathsf{SELS'} &= \mathsf{SELS} \cup \{\mathsf{x}, \mathsf{nx}\} \quad, \\
\mathsf{ELS'} &= \mathsf{ELS} \cup \{\mathsf{x}, \mathsf{nx}\} \quad.
\end{aligned}
$$

Theorems 4.1.5, 4.1.6 and 4.1.7 do also hold for the system $\mathsf{SELS'}$, $\mathsf{ELS'}$ and $\mathsf{MELL'}$. In particular, we also have cut elimination (Theorem 4.1.8). However, it is not obvious what happens with decomposition. How should the rules x and nx, which belong to the noncore, be incorporated in the decomposition theorems?

Instead of simply adding the rules x and nx, the calculus of structures offers a more elegant solution, which employs the well-known fact that the mix and $\mathsf{mix0}$ rules together say that $1 = \bot$. This can be seen as follows. First, the rule nx presents the linear implication $1 \multimap \bot$, and second, the rule

$$
\mathsf{x'} \frac{S\{\bot\}}{S\{1\}} \quad,
$$

which stands for the linear implication $\bot \multimap 1$, is derivable in the system $\{\mathsf{x}\}$:

$$
= \frac{S\{\bot\}}{\mathsf{x}\dfrac{S(1,\bot)}{\dfrac{S[1,\bot]}{S\{1\}}}} \quad.
$$

Hence, the presence of the rule x implies that $\bot \multimap 1$. Conversely, the rule x is derivable in the system $\{\mathsf{x'}, \mathsf{s}\}$:

$$
= \frac{S(R,T)}{\mathsf{x'}\dfrac{S(R,[\bot,T])}{\dfrac{S(R,[1,T])}{\mathsf{s}\dfrac{S[(R,1),T]}{S[R,T]}}}} \quad.
$$

This means that the presence of the rules x and nx is equivalent to the presence of the equation $1 = \bot$.

For this reason, I will follow [Gug99] and introduce a single symbol $\circ$ to denote the *unit* for both, par and times, instead of introducing two different symbols together with an equation that identifies them. This unit $\circ$ will not be considered as an atom. Since the additives are not considered here, the set of atoms does now no longer contain any constants.

6.1.1   Definition   Let $\mathcal{A}$ be a countable set of *atoms*. Atoms are denoted by a, b, c, …, and the set $\mathcal{A}$ is equipped with a bijective function $\bar{\cdot} : \mathcal{A} \to \mathcal{A}$, such that $\bar{\bar{a}} = a$ and $\bar{a} \neq a$ for every $a \in \mathcal{A}$. Let $\mathcal{R}$ be the set of expressions generated by the following syntax:

$$
R ::= \circ \mid a \mid [R, R] \mid (R, R) \mid \,!R \mid \,?R \mid \bar{R} \quad,
$$

Associativity	Exponentials	Negation
$[R,[T,U]] \ = \ [[R,T],U]$	$??R \ = \ ?R$	$\bar{\circ} \ = \ \circ$
$(R,(T,U)) \ = \ ((R,T),U)$	$!!R \ = \ !R$	$\overline{[R,T]} \ = \ (\bar{R},\bar{T})$
Commutativity	**Unit**	$\overline{(R,T)} \ = \ [\bar{R},\bar{T}]$
		$\overline{?R} \ = \ !\bar{R}$
$[R,T] \ = \ [T,R]$	$[\circ,R] \ = \ R$	$\overline{!R} \ = \ ?\bar{R}$
$(R,T) \ = \ (T,R)$	$(\circ,R) \ = \ R$	$\bar{\bar{R}} \ = \ R$
	$?\circ \ = \ \circ$	
	$!\circ \ = \ \circ$	

Figure 6.1: Basic equations for the syntactic congruence of ELS° structures

where $\circ$, the *unit*, is not an atom, and a stands for any atom. On the set $\mathcal{R}$, the relation $=$ is defined to be the smallest congruence relation induced by the equations shown in Figure 6.1. The ELS° *structures* (denoted with P, Q, R, S, ...) are the elements of $\mathcal{R}/=$, i.e. equivalence classes of expressions.

6.1.2   Remark   What has been said in Chapter 3 (see 3.1.3–3.1.8) does also apply here. In particular, I will omit superfluous parentheses when referring to structures.

The rules s, p↓, p↑, b↓ and b↑ are not influenced by the new syntax. But the interaction and weakening rules have to be modified in such a way that the two constants $\perp$ and 1 are replaced by the unit $\circ$:

$$\mathsf{i}{\downarrow}\,\frac{S\{\circ\}}{S[R,\bar{R}]} \quad , \quad \mathsf{ai}{\downarrow}\,\frac{S\{\circ\}}{S[a,\bar{a}]} \quad , \quad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar{a})}{S\{\circ\}} \quad , \quad \mathsf{i}{\uparrow}\,\frac{S(R,\bar{R})}{S\{\circ\}} \quad ,$$

$$\mathsf{w}{\downarrow}\,\frac{S\{\circ\}}{S\{?R\}} \quad , \quad \mathsf{w}{\uparrow}\,\frac{S\{!R\}}{S\{\circ\}} \quad .$$

Although those rules are formally different rules than the ones defined in Section 3.2, I will use the same names. However, in order to avoid ambiguities, I will use the name SELS° for the new system, which is shown in Figure 6.2. As a consequence of the new syntax, we also have to change the axiom. The rule

$$\circ{\downarrow}\,\frac{}{\circ}$$

is called *unit*. System ELS°, which is obtained from the down fragment of system SELS° together with the axiom is shown in Figure 6.3.

All results concerning systems SELS and ELS presented in Chapter 4 do also hold for systems SELS° and ELS°. In particular, we have the two decomposition theorems and cut

$$\mathsf{ai}{\downarrow}\ \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad\qquad \mathsf{ai}{\uparrow}\ \frac{S(a,\bar{a})}{S\{\circ\}}$$

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{p}{\downarrow}\ \frac{S\{![R,T]\}}{S[!R,?T]} \qquad\qquad \mathsf{p}{\uparrow}\ \frac{S(?R,!T)}{S\{?(R,T)\}}$$

$$\mathsf{w}{\downarrow}\ \frac{S\{\circ\}}{S\{?R\}} \qquad\qquad \mathsf{w}{\uparrow}\ \frac{S\{!R\}}{S\{\circ\}}$$

$$\mathsf{b}{\downarrow}\ \frac{S[?R,R]}{S\{?R\}} \qquad\qquad \mathsf{b}{\uparrow}\ \frac{S\{!R\}}{S(!R,R)}$$

Figure 6.2: System SELS°

$$\circ{\downarrow}\ \frac{}{\circ} \qquad \mathsf{ai}{\downarrow}\ \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{p}{\downarrow}\ \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{w}{\downarrow}\ \frac{S\{\circ\}}{S\{?R\}} \qquad \mathsf{b}{\downarrow}\ \frac{S[?R,R]}{S\{?R\}}$$

Figure 6.3: System ELS°

elimination, also in its stronger version (Theorem 4.4.13), where system dELS° is obtained from system ELS° by replacing the switch rule by the deep switch. Consequently, the interpolation theorem (Theorem 4.5.1) does also hold for system SELS°.

All proofs are literally the same, except for the permutation results in Section 4.2. Because of the collapsing of the units, some of the results presented there are no longer true. For example the rule ai↓ does no longer permute over p↑. This can be seen from the derivation

$$= \ \frac{\mathsf{p}{\uparrow}\ \dfrac{S(?U,!V)}{S\{?(U,V)\}}}{\mathsf{ai}{\downarrow}\ \dfrac{S\{?[(U,V),\circ]\}}{S\{?[(U,V),[a,\bar{a}]]\}}}\ \ .$$

But we can use the switch rule in the same way as it has been done for the rule w↓ in Lemma 4.2.9. I will not go into further details here, because in Chapter 7, I will discuss

$$\mathsf{ai}{\downarrow}\ \frac{S\{\circ\}}{S[a,\bar{a}]}\qquad\qquad\mathsf{ai}{\uparrow}\ \frac{S(a,\bar{a})}{S\{\circ\}}$$

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

Figure 6.4: System SS°

$$\circ{\downarrow}\ \frac{}{\circ}\qquad\qquad\mathsf{ai}{\downarrow}\ \frac{S\{\circ\}}{S[a,\bar{a}]}\qquad\qquad\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

Figure 6.5: System S°

a conservative extension of ELS°. In particular, in Section 7.2.1, I will show all relevant permutation results.

Let us now consider only the multiplicative fragment.

6.1.3 Definition An S° *structure* is an ELS° structure that does not contain any exponentials, i.e. that does not have any substructure $!R$ or $?R$ for some R.

6.1.4 Definition *System* SS° is built from the rules ai↓, ai↑, and s, and is shown in Figure 6.4. *System* S° is obtained from the down fragment of SS° together with the axiom ○↓ and is shown in Figure 6.5.

From the results of Section 3.3 and the discussion at the beginning of this section, it follows immediately that System S°, is equivalent to multiplicative linear logic (MLL) plus mix and mix0. (A direct proof can also be found in [Gug02e], where system S° is called system FBV.)

All the results mentioned for the systems SELS° and ELS° before, do obviously also hold for the systems SS° and S°. Since there are no noncore rules, the proofs become even simpler. This is in particular the case for the decomposition theorems, which collapse to a single decomposition theorem:

6.1.5   Theorem *For every derivation* $\overset{T}{\underset{R}{\Delta\|\mathsf{SS}^\circ}}$ *there are structures T_1 and R_1, and derivations $\Delta_1, \dots, \Delta_3$, such that*

$$
\begin{array}{c}
T \\
\Delta_1\|\{\mathsf{ai}{\downarrow}\} \\
T_1 \\
\Delta_2\|\{\mathsf{s}\} \\
R_1 \\
\Delta_3\|\{\mathsf{ai}{\uparrow}\} \\
R
\end{array}
\quad.
$$

Let me also mention the interpolation theorem, which will be needed in the next section.

6.1.6   Theorem (Interpolation) *For every derivation* $\overset{T}{\underset{R}{\Delta\|\mathsf{SS}^\circ}}$ *there are structures I, T_1, R_1 and derivations $\Delta_1, \dots, \Delta_4$, such that*

$$
\begin{array}{c}
T \\
\Delta_1\|\{\mathsf{s}\} \\
T_1 \\
\Delta_2\|\{\mathsf{ai}{\uparrow}\} \\
I \\
\Delta_3\|\{\mathsf{ai}{\downarrow}\} \\
R_1 \\
\Delta_4\|\{\mathsf{s}\} \\
R
\end{array}
\quad.
$$

Proof: This is in fact an immediate consequence of the interpolation theorem for SELS°, whose proof is literally the same as for Theorem 4.5.1 (as mentioned before, the relevant permutation results are refined in Section 7.2.1). However, because for system SS°, the situation is much simpler, let me give here the direct proof. Observe that in the case of system SS°, it is not necessary to start from the decomposition theorem in order to obtain the interpolation. It is only necessary to replace inside Δ all instances of $\mathsf{ai}{\uparrow}$ by $\mathsf{sai}{\uparrow}$ (see page 131) and all instances of s by $\mathsf{ds}{\downarrow}$ or $\mathsf{ns}{\uparrow}$ (Definition 4.4.3 on page 131). The derivation $\overset{T}{\underset{R}{\Delta'\|\{\mathsf{ai}{\downarrow},\mathsf{ds}{\downarrow},\mathsf{ns}{\uparrow},\mathsf{sai}{\uparrow}\}}}$ can now decomposed into

$$
\begin{array}{c}
T \\
\Delta'_1\|\{\mathsf{sai}{\uparrow},\mathsf{ns}{\uparrow}\} \\
I \\
\Delta'_2\|\{\mathsf{ai}{\downarrow},\mathsf{ds}{\downarrow}\} \\
R
\end{array}
$$

by applying Lemma 4.4.4 and Lemma 4.4.10. (For these two Lemmata the collapsing of the constants is not problematic since the the rule to be permuted up is always considered to be nontrivial.) The rules $\mathsf{ds}{\downarrow}$ and $\mathsf{ns}{\uparrow}$ are instances of s, and the instances of $\mathsf{sai}{\uparrow}$ inside

Δ_1 can be replaced by

$$
\mathsf{s}\,\frac{S([a,U],[\bar{a},V])}{\mathsf{s}\,\frac{S\big[([a,U],\bar{a}),V\big]}{\mathsf{ai}{\uparrow}\,\frac{S\big[(a,\bar{a}),U,V\big]}{=\,\frac{S[\circ,U,V]}{S[U,V]}}}}\quad.
$$

Finally the instances of $\mathsf{ai}{\downarrow}$ are permuted up and the instances of $\mathsf{ai}{\uparrow}$ are permuted down until the derivation has the desired shape. □

The cut elimination result is an immediate consequence of interpolation.

6.1.7 Theorem (Cut Elimination) *Systems* S° *and* $\mathsf{SS}^{\circ} \cup \{\circ{\downarrow}\}$ *are equivalent.*

The following corollary will also be needed in the next section.

6.1.8 Corollary *Let R and T be two S° structures. Then we have*

$$
\begin{array}{c} T \\ \|\mathsf{SS}^{\circ} \\ R \end{array} \qquad \textit{if and only if} \qquad \begin{array}{c} \|\mathsf{S}^{\circ} \\ [\bar{T},R] \end{array} \quad .
$$

6.2 The Switch Rule

In this section, I will inspect the systems S° and SS° more closely. The main result will be an algebraic criterion for the derivability of the switch rule. More precisely, I will show what kind of criterion two S° structures R and T have to fulfill in order to have a derivation

$$
\begin{array}{c} T \\ \|\{\mathsf{s}\} \\ R \end{array} \quad .
$$

The reader familiar with *proof nets* [Gir87a, DR89, Ret99a] might observe that this can be seen as a top-down symmetric version of a correctness criterion for proof nets. Let me make the relation to proof nets more precise. (The reader who is not familiar with proof nets does not need to worry about this paragraph.) In [Ret99a], Ch. Retoré introduces a way of presenting proof nets as series-parallel R&B-graphs. For these so called *series-parallel R&B proof structures*, a correctness criterion is shown: A series-parallel R&B proof structure is a proof net (i.e. corresponds to a sequent calculus proof in MLL plus mix plus mix0) if and only if it does not contain a cordless Æ-*circuit*. In [Gug99], A. Guglielmi introduces *relation webs* and shows their one-to-one correspondence to structures as they are presented in this thesis. Furthermore, there is an obvious one-to-one correspondence between A. Guglielmi's relation webs and Ch. Retoré's series-parallel R&B-graphs. (Below I will give the basic ideas behind these objects.)

It is therefore possible to obtain from the correctness criterion for proof nets a criterion for the provability of S° structures. In other words, for a given structure R it is possible to judge from its algebraic properties whether there is a proof

$$
\begin{array}{c} \|\mathsf{S}^{\circ} \\ R \end{array} \quad .
$$

Since I will explain this criterion in terms of structures, there is no need for the reader to be familiar with proof nets. I will also show directly how the relation between the criterion and provability is established, without going the detour over proof nets.

6.2.1   Definition   The multiset of *occurrences* of atoms in a $\mathsf{S}°$ structure R, denoted by $\mathsf{occ}\, R$ is defined inductively as follows:

$$
\begin{aligned}
\mathsf{occ}\, \circ &= \varnothing \quad , \\
\mathsf{occ}\, a &= \{a\} \quad , \\
\mathsf{occ}\,[R,T] = \mathsf{occ}(R,T) &= \mathsf{occ}\, R \cup \mathsf{occ}\, T \quad .
\end{aligned}
$$

6.2.2   Definition   A structure R is called *balanced* if each atom in R occurs exactly once, i.e. $\mathsf{occ}\, R$ is a set.

6.2.3   Example   If $R = [(a,[b,c],d),\bar{d},([\bar{c},c],b,a,a)]$ and $T = [(a,b),(\bar{b};\bar{a})]$, then

$$
\begin{aligned}
\mathsf{occ}\, R &= \{a,a,a,b,b,c,c,\bar{c},d,\bar{d}\} \quad \text{and} \\
\mathsf{occ}\, T &= \{a,b,\bar{a},\bar{b}\} \quad .
\end{aligned}
$$

T is balanced, R is not.

6.2.4   Definition   For a given balanced structure R, let me define the two binary relations $\downarrow_R$ and $\uparrow_R$ on the set $\mathsf{occ}\, R$ as follows. For all atoms $a,b \in \mathsf{occ}\, R$, let $a \downarrow_R b$ if and only if there exist contexts $S\{\ \}$, $S_a\{\ \}$ and $S_b\{\ \}$ such that

$$
R = S[S_a\{a\},S_b\{b\}] \quad ,
$$

and let $a \uparrow_R b$ if and only if there exist contexts $S\{\ \}$, $S_a\{\ \}$ and $S_b\{\ \}$ such that

$$
R = S(S_a\{a\},S_b\{b\}) \quad .
$$

In other words, I will write $a \downarrow_R b$ if a and b are in par-relation inside R, and $a \uparrow_R b$, if a and b are in times-relation (see Definition 4.3.23 on page 117). Immediately from Definition 6.2.4, we can get the following two observations:

- For every structure R, the relations $\downarrow_R$ and $\uparrow_R$ are both symmetric.

- For any two atoms $a,b \in \mathsf{occ}\, R$ (with $a \neq b$), we have either $a \downarrow_R b$ or $a \uparrow_R b$. In other words,

$$
\begin{aligned}
\downarrow_R \cup \uparrow_R &= \big(\mathsf{occ}\, R \times \mathsf{occ}\, R\big) \setminus \big\{\, (a,a) \mid a \in \mathsf{occ}\, R \,\big\} \quad \text{and} \\
\downarrow_R \cap \uparrow_R &= \varnothing \quad .
\end{aligned}
$$

The definition of the relations $\downarrow_R$ and $\uparrow_R$ can easily be extended from balanced structures to arbitrary $\mathsf{S}°$ structures. This is essentially the idea behind A. Guglielmi's *relation webs* [Gug02e], which are coloured complete undirected graphs where the vertices are the atoms of the structure and an edge (a,b) is colored green if $a \downarrow_R b$ and red if $a \uparrow_R b$. Ch. Retoré's *series-parallel R&B proof structures* [Ret99a] are essentially the same: there is a red edge between a and b if $a \uparrow_R b$ and no edge when $a \downarrow_R b$. But there is a blue edge when a and b are a matching pair of atoms. This is the reason, why R&B proof structures can only be used for characterizing proofs and not for derivations.

6.2.5   Definition   Let R and T be two S° structures. The structure T is a *quasi-substructure* of R, if T can be obtained from R by replacing some atoms (possibly none) by the unit.

6.2.6   Remark   Every substructure of R is also a quasi-substructure of R.

6.2.7   Example   The structure $[(a,b),(\bar{c},a)]$ is a quasi-substructure (but not a substructure) of $[(a,[b,c],d),\bar{d},([\bar{c},c],b,a,a)]$, because

$$[(a,[b,\circ],\circ),\circ,([\bar{c},\circ],\circ,\circ,a)] = [(a,b),(\bar{c},a)] \quad .$$

6.2.8   Remark   In the notion of relation web, which I explained above, a quasi-substructure would correspond to a (relation) subweb.

6.2.9   Definition   Let $m \geqslant 1$ be even and let $a_1, a_2, \ldots, a_m$ be m different atoms. Then the structure $[(a_2,a_3),(a_4,a_5),\ldots,(a_m,a_1)]$ is called a $\langle a_1, a_2, \ldots, a_m \rangle$-*circuit*, and the structure $([a_1,a_2],[a_3,a_4],\ldots,[a_{m-1},a_m])$ is called a $\langle a_1, a_2, \ldots, a_m \rangle$-*cocircuit*. A circuit is called a *negation circuit* if we have that $a_2 = \bar{a}_1$, $a_4 = \bar{a}_3$, $\ldots$, $a_m = \bar{a}_{m-1}$.

6.2.10   Remark   In terms of the relation webs and the relations $\downarrow_R$ and $\uparrow_R$, the notions of circuit and cocircuit can be explained as follows. Consider the graph with vertices $a_1, \ldots, a_m$, where the edges (a_1, a_2), (a_3, a_4), $\ldots$, (a_{m-1}, a_m) are green, and the edges (a_2, a_3), (a_4, a_5), $\ldots$, (a_m, a_1) are red. This structure becomes a circuit if all other edges in the complete graph are green and a cocircuit of all other edges are red. In this setting, a negation circuit corresponds exactly to Ch. Retoré's definition of a cordless Æ-circuit.

6.2.11   Definition   Let R be an S° structure and let $a_1, \ldots, a_m$ be an even number of atoms. The structure R *contains a* $\langle a_1, \ldots, a_m \rangle$-*circuit*, if R has a quasi-substructure R', which is a $\langle a_1, \ldots, a_m \rangle$-circuit. Similarly, R *contains a* $\langle a_1, \ldots, a_m \rangle$-*cocircuit*, if R has a quasi-substructure R', which is a $\langle a_1, \ldots, a_m \rangle$-cocircuit. Further, the structure R *contains a negation circuit*, if R has a quasi-substructure R', which is a negation circuit, i.e. there are atoms $c_1, \ldots, c_n$ for some $n \geqslant 1$, such that R contains a $\langle c_1, \bar{c}_1, \ldots, c_n, \bar{c}_n \rangle$-circuit.

6.2.12   Example   The structure $([(a,b),(\bar{b},\bar{c}),d],c)$ contains (among others) a $\langle \bar{c}, a, b, \bar{b} \rangle$-circuit because
$$([(a,b),(\bar{b},\bar{c}),\circ],\circ) = [(a,b),(\bar{b},\bar{c})] \quad ,$$
and a $\langle b, \bar{b} \rangle$-cocircuit because

$$([(\circ,b),(\bar{b},\circ),\circ],\circ) = [b,\bar{b}] \quad .$$

It also contains the negation circuit $(\bar{c}, c)$.

6.2.13   Proposition   *Let P be a balanced S° structure. If P contains a negation circuit, then P is not provable in S°.*

Proof: The proof of this statement is very similar to the proof of Lemma 4.3.25. But because of the collapsing of the units, there is a greater applicability of the switch rule, which has to be taken into account. Let

$$Q = [(\bar{a}_1,a_2),(\bar{a}_2,a_3),\ldots,(\bar{a}_n,a_1)] \quad ,$$

for some $n > 0$, be the negation circuit contained in P. By way of contradiction, assume that P is provable in S°. Since P is balanced, we must have that Q is also provable in S°: The proof of Q can be obtained from the proof of P by replacing everywhere all atoms not occurring in Q by $\circ$. Now let me proceed by induction on n to show a contradiction.

Base case: For $n = 1$ there is no proof

$$\begin{array}{c} \| \mathsf{S}^\circ \\ (\bar{a}_1, a_1) \end{array} \quad .$$

Inductive case: For $n > 1$ consider the bottommost rule instance ρ in the proof of Q:

$$\begin{array}{c} \Pi \| \mathsf{S}^\circ \\ [(\bar{a}_1, a_2), (\bar{a}_2, a_3), \dots, (\bar{a}_n, a_1)] \end{array} \quad .$$

Since inside Q, there is no substructure $[a, \bar{a}]$ for some a, we have that $\rho = \mathsf{s}$. There are three possibilities to apply $\mathsf{s} \dfrac{S([R,U],T)}{S[(R,T),U]}$ to

$$Q = [(\bar{a}_1, a_2), (\bar{a}_2, a_3), \dots, (\bar{a}_n, a_1)] \quad .$$

For notational convenience, I will set $Z_j = (\bar{a}_j, a_{j+1})$ for every $j \in \{1, \dots, n\}$ (with the convention that $j + 1 = 1$ for $j = n$). This means that

$$Q = [Z_1, Z_2, \dots, Z_n] \quad .$$

(i) $R = \bar{a}_j, T = a_{j+1}$ for some $j = 1, \dots, n$. Then, without loss of generality, we can assume that $j = 1$. Further, $U = [Z_{k_1}, \dots, Z_{k_u}]$ for some $u > 0$ and $k_1, \dots, k_u \in \{2, \dots, n\}$. Without loss of generality, assume that $k_1 < k_2 < \dots < k_u$. Let $\{2, \dots, n\} \setminus \{k_1, \dots, k_u\} = \{h_1, \dots, h_s\}$, where $s = n - u - 1$. Then there is a proof Π' such that

$$\mathsf{s} \dfrac{\begin{array}{c} \Pi' \| \mathsf{S}^\circ \\ [([\bar{a}_1, Z_{k_1}, \dots, Z_{k_u}], a_2), Z_{h_1}, \dots, Z_{h_s}] \end{array}}{[(\bar{a}_1, a_2), Z_2, \dots, Z_n]} \quad .$$

The proof Π' remains valid if we replace a_m and $\bar{a}_m$ by $\circ$ for every m with $m = 1$ or $m > k_1$. Then we get

$$\begin{array}{c} \Pi'' \| \mathsf{S}^\circ \\ [(\bar{a}_{k_1}, a_2), (\bar{a}_2, a_3), \dots, (\bar{a}_{k_1-1}, a_{k_1})] \end{array} \quad ,$$

which is a contradiction to the induction hypothesis.

(ii) $R = a_{j+1}, T = \bar{a}_j$ for some $j = 1, \dots, n$. Then, without loss of generality, we can assume that $j = 1$. Further, $U = [Z_{k_1}, \dots, Z_{k_u}]$ for some $u > 0$ and $k_1, \dots, k_u \in \{2, \dots, n\}$. Without loss of generality, assume that $k_1 < k_2 < \dots < k_u$. Let $\{2, \dots, n\} \setminus \{k_1, \dots, k_u\} = \{h_1, \dots, h_s\}$, where $s = n - u - 1$. Then there is a proof Π' such that

$$\mathsf{s} \dfrac{\begin{array}{c} \Pi' \| \mathsf{S}^\circ \\ [(\bar{a}_1, [Z_{k_1}, \dots, Z_{k_u}, a_2]), Z_{h_1}, \dots, Z_{h_s}] \end{array}}{[(\bar{a}_1, a_2), Z_2, \dots, Z_n]} \quad .$$

The proof Π' remains valid if we replace a_m and $\bar{a}_m$ by $\circ$ for every m with $1 < m \leqslant k_u$. Then we get

$$\Pi'' \| \mathsf{S}^\circ \\ [(\bar{a}_1, a_{k_u+1}), (\bar{a}_{k_u+1}, a_{k_u+2}), \dots, (\bar{a}_n, a_1)] \qquad ,$$

which is a contradiction to the induction hypothesis.

(iii) $R = \circ$, $T = [Z_{k_1}, \dots, Z_{k_t}]$ and $U = [Z_{l_1}, \dots, Z_{l_u}]$ for some $t, u > 0$ and $k_1, \dots, k_t, l_1, \dots, l_u \in \{1, \dots, n\}$. Then there is a proof Π' such that

$$\Pi' \| \mathsf{S}^\circ \\ \mathsf{s}\,\frac{[([Z_{k_1}, \dots, Z_{k_t}], [Z_{l_1}, \dots, Z_{l_u}]), Z_{h_1}, \dots, Z_{h_s}]}{[Z_1, Z_2, \dots, Z_n]} \qquad ,$$

where $s = n - t - u$ and $\{h_1, \dots, h_s\} = \{1, \dots, n\} \backslash \{k_1, \dots, k_t, l_1, \dots, l_u\}$. Without loss of generality, we can assume that $t = u = 1$. Otherwise we could replace

$$\mathsf{s}\,\frac{[([Z_{k_1}, \dots, Z_{k_t}], [Z_{l_1}, \dots, Z_{l_u}]), Z_{h_1}, \dots, Z_{h_s}]}{[Z_1, Z_2, \dots, Z_n]} \qquad \text{by}$$

$$\mathsf{s}\,\frac{\mathsf{s}\,\dfrac{[([Z_{k_1}, \dots, Z_{k_t}], [Z_{l_1}, \dots, Z_{l_u}]), Z_{h_1}, \dots, Z_{h_s}]}{\mathsf{s}\,\dfrac{[([Z_{k_1}, \dots, Z_{k_t}], Z_{l_1}), Z_{l_2}, \dots, Z_{l_u}, Z_{h_1}, \dots, Z_{h_s}]}{[(Z_{k_1}, Z_{l_1}), Z_{k_2}, \dots, Z_{k_t}, Z_{l_2}, \dots, Z_{l_u}, Z_{h_1}, \dots, Z_{h_s}]}}{[Z_1, Z_2, \dots, Z_n]} \qquad .$$

Now let $k = k_1$ and $l = l_1$. Then we have

$$\Pi' \| \mathsf{S}^\circ \\ \mathsf{s}\,\frac{[(Z_k, Z_l), Z_{h_1}, \dots, Z_{h_s}]}{[Z_1, Z_2, \dots, Z_n]} \qquad ,$$

where $s = n - 2$. Without loss of generality, we can assume that $k < l$. Now replace inside Π' for every m with $m \leqslant k$ or $l < m$ the atoms a_m and $\bar{a}_m$ by $\circ$. The proof Π' then becomes

$$\Pi'' \| \mathsf{S}^\circ \\ [(\bar{a}_l, a_{k+1}), (\bar{a}_{k+1}, a_{k+2}), \dots, (\bar{a}_{l-1}, a_l)] \qquad ,$$

which is a contradiction to the induction hypothesis. $\qquad\square$

6.2.14 **Definition** Let R be a balanced S° structure. Then the relation $\sim_R \, \subseteq \, \mathsf{occ}\,R \times \mathsf{occ}\,R$ is defined as follows. For any atoms $a, b \in \mathsf{occ}\,R$ (with $a \neq b$), we have $a \sim_R b$ if and only if there are atoms $c_1, \dots, c_n \in \mathsf{occ}\,R$ (for some $n \geqslant 0$), such that both,

$$[a, (\bar{a}, c_1), (\bar{c}_1, c_2), \dots, (\bar{c}_{n-1}, c_n), (\bar{c}_n, \bar{b})] \qquad \text{and} \\ [(\bar{a}, c_1), (\bar{c}_1, c_2), \dots, (\bar{c}_{n-1}, c_n), (\bar{c}_n, \bar{b}), b]$$

are quasi-substructures of R.

The reason for the definition of the relation $\sim_R$ is the following observation. For every balanced S° structure R, we have that R contains a negation circuit if and only if there are two atoms $a, b \in \mathsf{occ}\, R$ such that $a \sim_R b$ and $a \uparrow_R b$. This can be seen as follows. If R contains a negation circuit $[(\bar{a}_1, a_2), (\bar{a}_2, a_3), \ldots, (\bar{a}_n, a_1)]$ then we have $\bar{a}_1 \uparrow_R a_2$ and $\bar{a}_1 \sim_R a_2$. Conversely, if there are two atoms a and b with $a \sim_R b$ and $a \uparrow_R b$, then there are atoms $c_1, \ldots, c_n \in \mathsf{occ}\, R$ (for some $n \geqslant 0$) such that $[(b, a), (\bar{a}, c_1), (\bar{c}_1, c_2), \ldots, (\bar{c}_{n-1}, c_n), (\bar{c}_n, \bar{b})]$ is a quasi-substructure of R. This observation is useful for the following lemma.

6.2.15 Lemma *Let $P = S[(R\{a\}, T), (U\{\bar{a}\}, V)]$ be a balanced structure that does not contain a negation circuit. Then at least one of*

$$P' = S([R\{a\}, (U\{\bar{a}\}, V)], T) \quad and \quad P'' = S([U\{\bar{a}\}, (R\{a\}, T)], V)$$

does not contain a negation circuit.

Proof: By way of contradiction, assume that both, P' and P'' contain a negation circuit, and P does not contain a negation circuit. Consider now the two structures P and P'. We have that $\mathsf{occ}\, P = \mathsf{occ}\, P'$, and for all atoms $a, b \in \mathsf{occ}\, P$ with $a \notin \mathsf{occ}\, T$ or $b \notin \mathsf{occ}(U\{\bar{a}\}, V)$, that

$$a \downarrow_P b \quad \text{iff} \quad a \downarrow_{P'} b \qquad \text{and} \qquad a \uparrow_P b \quad \text{iff} \quad a \uparrow_{P'} b \quad.$$

If $a \in \mathsf{occ}\, T$ and $b \in \mathsf{occ}(U\{\bar{a}\}, V)$, then we have $a \downarrow_P b$ and $a \uparrow_{P'} b$. In other words, only the relation between the atoms in T and $(U\{\bar{a}\}, V)$ is different in P and P'. Since P' contains a negation circuit and P not, we must have and atom $t_1 \in \mathsf{occ}\, T$ and and atom $b \in \mathsf{occ}(U\{\bar{a}\}, V)$, such that $t_1 \sim_{P'} b$. We then also have $t_1 \sim_P b$ (by the definition of $\sim_P$, and the construction of P'). Observe that b cannot be $\bar{a}$ because $t_1 \uparrow_P a$ and $t_1 \uparrow_{P'} a$. We have therefore two cases:

- there is a $u_1 \in \mathsf{occ}\, U\{\ \}$ such that $t_1 \sim_P u_1$, or
- there is a $v_1 \in \mathsf{occ}\, V$ such that $t_1 \sim_P v_1$.

By the same argumentation about P'', we get a $v_2 \in \mathsf{occ}\, V$ such that either

- there is an $r_2 \in \mathsf{occ}\, R\{\ \}$ such that $v_2 \sim_P r_2$, or
- there is a $t_2 \in \mathsf{occ}\, T$ such that $v_2 \sim_P t_2$.

All together there are four cases:

- $t_1 \sim_P u_1$ and $v_2 \sim_P r_2$. By definition of $\sim_P$, we have atoms $c_1, \ldots, c_n \in \mathsf{occ}\, P$ and $d_1, \ldots, d_m \in \mathsf{occ}\, P$ for some $n, m \geqslant 0$, such that the four structures

$$[t_1, (\bar{t}_1, c_1), (\bar{c}_1, c_2), \ldots, (\bar{c}_{n-1}, c_n), (\bar{c}_n, \bar{u}_1)] \quad,$$
$$[(\bar{t}_1, c_1), (\bar{c}_1, c_2), \ldots, (\bar{c}_{n-1}, c_n), (\bar{c}_n, \bar{u}_1), u_1] \quad,$$
$$[v_2, (\bar{v}_2, d_1), (\bar{d}_1, d_2), \ldots, (\bar{d}_{m-1}, d_m), (\bar{d}_m, \bar{r}_2)] \quad,$$
$$[(\bar{v}_2, d_1), (\bar{d}_1, d_2), \ldots, (\bar{d}_{m-1}, d_m), (\bar{d}_m, \bar{r}_2), r_2]$$

  are quasi-substructures of P. This means that P contains a negation circuit:

$$[(r_2, t_1), (\bar{t}_1, c_1), \ldots, (\bar{c}_n, \bar{u}_1), (u_1, v_2), (\bar{v}_2, d_1), \ldots, (\bar{d}_m, \bar{r}_2)] \quad,$$

  which is a contradiction to the assumption.

- $t_1 \sim_P u_1$ and $v_2 \sim_P t_2$. In this case only $v_2 \sim_P t_2$ is needed to show the negation circuit inside P. We have $d_1, \ldots, d_m \in \mathsf{occ}\, P$ for some $m \geqslant 0$, such that

$$[v_2, (\bar{v}_2, d_1), (\bar{d}_1, d_2), \ldots, (\bar{d}_{m-1}, d_m), (\bar{d}_m, \bar{t}_2)] \quad \text{and}$$
$$[(\bar{v}_2, d_1), (\bar{d}_1, d_2), \ldots, (\bar{d}_{m-1}, d_m), (\bar{d}_m, \bar{t}_2), t_2]$$

are quasi-substructures of P. Then P contains the negation circuit

$$[(\bar{a}, v_2), (\bar{v}_2, d_1), \ldots, (\bar{d}_m, \bar{t}_2), (t_2, a)] \quad .$$

- $t_1 \sim_P v_1$ and $v_2 \sim_P r_2$. This case is similar to the previous case. (Use $t_1 \sim_P v_1$).

- $t_1 \sim_P v_1$ and $v_2 \sim_P t_2$. As before.

Hence, one of P' and P'' must be negation circuit free. □

6.2.16 Proposition *Let P be a balanced structure such that for every atom $a \in \mathsf{occ}\, P$, its dual $\bar{a}$ does also occur in P. If P does not contain a negation circuit, then P is provable in S°.*

Proof: By induction on the cardinality $|{\downarrow_P}|$ of the relation $\downarrow_P \subseteq \mathsf{occ}\, P \times \mathsf{occ}\, P$, I will show that there is a proof $\Pi \| \mathsf{S}^\circ$.
$$\quad P$$

Base case: If $|{\downarrow_P}| = 0$, then $P = \circ$, which is trivially provable. (If $P \neq \circ$, then there is an atom $a \in \mathsf{occ}\, P$, and therefore also $\bar{a} \in \mathsf{occ}\, P$. Since $|{\downarrow_P}| = 0$, we have $a \uparrow_P \bar{a}$ and therefore $(a, \bar{a})$ is a quasi-substructure of P).

Inductive case: Now let $|{\downarrow_P}| > 0$. Hence $P \neq \circ$. Pick inside P any atom a. Since P contains no negation circuit, we have $a \downarrow_P \bar{a}$. Hence, $P = S[S_a\{a\}, S_{\bar{a}}\{\bar{a}\}]$ for some contexts $S\{\ \}$, $S_a\{\ \}$ and $S_{\bar{a}}\{\ \}$. Without loss of generality, we can assume $S\{\ \}$ is chosen such that neither $S_a\{\ \}$ nor $S_{\bar{a}}\{\ \}$ is a par context. In other words, either $S_a\{\ \} = \{\ \}$ or $S_a\{\ \} = (R\{\ \}, T)$ for some context $R\{\ \}$ and $T \neq \circ$, and similarly for $S_{\bar{a}}\{\ \}$. There are four possibilities

- $S_a\{\ \} = \{\ \}$ and $S_{\bar{a}}\{\ \} = \{\ \}$. Then $P = S[a, \bar{a}]$, and we can obtain

$$\mathsf{ai}{\downarrow} \frac{\overset{\Pi' \| \mathsf{S}^\circ}{S\{\circ\}}}{S[a, \bar{a}]} \quad ,$$

where Π' exists by induction hypothesis.

- $S_a\{\ \} = (R\{\ \}, T)$ and $S_{\bar{a}}\{\ \} = \{\ \}$. Then $P = S[(R\{a\}, T), \bar{a}]$. Let Π be the proof

$$\mathsf{ai}{\downarrow} \frac{\overset{\Pi' \| \mathsf{S}^\circ}{S(R\{\circ\}, T)}}{\underset{\Delta \| \{\mathsf{s}\}}{\frac{S(R[a, \bar{a}], T)}{S[(R\{a\}, T), \bar{a}]}}} \quad ,$$

where Π' exists by induction hypothesis. and Δ exists by Lemma 4.3.20.

- $S_a\{\ \} = \{\ \}$ and $S_{\bar{a}}\{\ \} = (U\{\ \}, V)$. Then $P = S[a, (U\{\bar{a}\}, V)]$. This is similar to the previous case.

- $S_a\{\ \} = (R\{\ \}, T)$ and $S_{\bar{a}}\{\ \} = (U\{\ \}, V)$. Then $P = S[(R\{a\}, T), (U\{\bar{a}\}, V)]$. Let Π be

$$
\begin{array}{ccc}
\Pi'\|\mathsf{S}^\circ & & \Pi''\|\mathsf{S}^\circ \\
\mathsf{s}\dfrac{S([R\{a\}, (U\{\bar{a}\}, V)], T)}{S[(R\{a\}, T), (U\{\bar{a}\}, V)]} & \text{or} & \mathsf{s}\dfrac{S([U\{\bar{a}\}, (R\{a\}, T)], V)}{S[(R\{a\}, T), (U\{\bar{a}\}, V)]}
\end{array} \quad ,
$$

where one of Π' and Π'' must exist by Lemma 6.2.15 and the induction hypothesis. (Since $T \neq \circ \neq V$, we have that $|\!\downarrow_{P'}| < |\!\downarrow_P|$ and $|\!\downarrow_{P''}| < |\!\downarrow_P|$.) $\qquad\square$

This proof shows that proof search in the calculus of structures can be driven by atoms, in the following sense: Let the distance between two atoms a and b in a structure R be the number of atoms c in R such that

$$
(a \downarrow_P c \wedge b \uparrow_P c) \vee (a \uparrow_P c \wedge b \downarrow_P c) \quad ,
$$

i.e. the number of atoms c to which a and b are in different relation. Then in a proof search the atoms a and $\bar{a}$ move closer together until their distance is zero. Then $[a, \bar{a}]$ is a substructure of R, and the two atoms can kill each other in an instance of $\mathsf{ai}\!\downarrow$. Lemma 6.2.15 shows that for any pair of atoms a and $\bar{a}$ it is always possible to move closer together.

This in fact could be the basis for an alternative approach to uniform provability [MNPS91]. Instead of the main connective of the goal formula, a matching pair of atoms determines the first rule to be applied in a proof. Although this idea is quite speculative, it should be mentioned that it occurred already in a completely different context in W. Bibel's linear connection method [Bib86]. However, the further exploration of this idea and the precise relation to W. Bibel's work has to remain open for future research.

The following theorem puts the two previous results together and shows that the notion of negation circuit can be used as a criterion for provability in system S°.

6.2.17 Theorem *Let R be a balanced S° structure. Then we have that R is provable in S° if and only if R does not contain a negation circuit and for every atom $a \in \mathsf{occ}\, R$, its dual $\bar{a}$ does also occur in R.*

Proof: Let R be provable. Then, by Proposition 6.2.13, R does not contain a negation circuit. In a proof an atom a can only be removed in an instance of $\mathsf{ai}\!\downarrow$. Then, at the same time its dual $\bar{a}$ is removed. Hence, for every $a \in \mathsf{occ}\, R$, we have $\bar{a} \in \mathsf{occ}\, R$. For the other direction apply Proposition 6.2.16. $\qquad\square$

The interested reader might now ask what can be said when R is not balanced. Then the first condition to check is that each atom a occurs in R as often as its dual $\bar{a}$ does. Otherwise R is obviously not provable. Then we have to find a pairing between the atoms in $\mathsf{occ}\, R$ such that each atom a is paired up with $\bar{a}$. Then we can rename all atoms such that the pairing is respected and the resulting structure R' is balanced. If R' does not contain a negation circuit, then R' and R are provable. In other word, for checking whether an unbalanced structure is provable, all possible pairings have to be checked in the worst case.

For example, consider the structure $[(a, \bar{a}), a, \bar{a}]$. If we rename it as $[(a_1, \bar{a}_1), a_2, \bar{a}_2]$, then we get a negation circuit, but if we rename it as $[(a_1, \bar{a}_2), a_2, \bar{a}_1]$, then we obtain a provable structure.

So far, we were only considering provability. Let us now investigate derivability, which means that we explore another time the top-down symmetry of the calculus of structures.

6.2.18 Lemma *Let $m > 0$ be even and let $a_1, a_2, \ldots, a_m$ be m different atoms. Then there is no derivation*

$$([a_1, a_2], [a_3, a_4], \ldots, [a_{m-1}, a_m])$$
$$\Delta \| \{\mathsf{s}\}$$
$$[(a_2, a_3), (a_4, a_5), \ldots, (a_m, a_1)]$$
.

Proof: The proof could be carried out essentially the same way as it has been done for the proofs of Lemma 4.3.25 and Proposition 6.2.13. However, since this lemma will not be used in a cut elimination argument, we can employ cut elimination for proving it in a more elegant way. By way of contradiction, assume there is a derivation

$$([a_1, a_2], [a_3, a_4], \ldots, [a_{m-1}, a_m])$$
$$\Delta \| \{\mathsf{s}\}$$
$$[(a_2, a_3), (a_4, a_5), \ldots, (a_m, a_1)]$$
.

Without loss of generality, we can assume that we do not have $a_i = \bar{a}_j$ for some $i, j \in \{1, \ldots, m\}$ (otherwise rename one of them). By cut elimination (Corollary 6.1.8), there is a proof

$$\| \mathsf{S}^\circ$$
$$[(\bar{a}_1, \bar{a}_2), (\bar{a}_3, \bar{a}_4), \ldots, (\bar{a}_{m-1}, \bar{a}_m), (a_2, a_3), (a_4, a_5), \ldots, (a_m, a_1)]$$
.

Since par structures are commutative, this is the same as

$$\| \mathsf{S}^\circ$$
$$[(\bar{a}_1, \bar{a}_2), (a_2, a_3), (\bar{a}_3, \bar{a}_4), (a_4, a_5), \ldots, (\bar{a}_{m-1}, \bar{a}_m), (a_m, a_1)]$$
,

which is a proof of a negation circuit and therefore a contradiction to Proposition 6.2.13. □

Lemma 6.2.18 essentially says that there is no derivation from a cocircuit to a circuit. This in fact was the motivation for introducing them. The following theorem says that the converse is also true, i.e. whenever there is no attempt to transform a cocircuit into a circuit, then there is indeed a derivation. This rather informal statement is made precise as follows.

6.2.19 Definition On the set of balanced S° structures the relation $\lessdot$ is defined as follows. We have $T \lessdot R$ if and only if

- $\mathsf{occ}\, T = \mathsf{occ}\, R$ and
- for all $m > 0$, m even, and atoms $a_1, \ldots, a_m \in \mathsf{occ}\, T$, it is not the case that
  - T contains an $\langle a_1, \ldots, a_m \rangle$-cocircuit and
  - R contains an $\langle a_1, \ldots, a_m \rangle$-circuit.

6.2.20 Theorem *Let R and T be two balanced structures with $\operatorname{occ} R = \operatorname{occ} T$, such that for every atom a occurring in R, the atom $\bar{a}$ does not occur in R. Then we have,*

$$\begin{array}{ccccccc} \parallel\!\mathsf{s}^\circ & & T & & T & & \\ [\bar{T},R] & iff & \parallel\!\mathsf{ss}^\circ & iff & \parallel\!\{\mathsf{s}\} & iff & T \lessdot R \quad . \\ & & R & & R & & \end{array}$$

Proof: I will prove the equivalence of the four statements by proving

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1) \quad .$$

- $(1) \Rightarrow (2)$: Immediate from Corollary 6.1.8.

- $(2) \Rightarrow (3)$: Apply interpolation (Theorem 6.1.6). There cannot remain any instance of $\mathsf{ai}{\downarrow}$ or $\mathsf{ai}{\uparrow}$ because there are no atom pairs a and $\bar{a}$ inside R or T.

- $(3) \Rightarrow (4)$: By way of contradiction, assume that not $T \lessdot R$. This means that there are atoms $a_1, \ldots, a_m \in \operatorname{occ} R$ such that R contains an $\langle a_1, \ldots, a_m \rangle$-circuit and T contains an $\langle a_1, \ldots, a_m \rangle$-cocircuit. The derivation

$$\begin{array}{c} T \\ \parallel\!\{\mathsf{s}\} \\ R \end{array}$$

  remains valid if all atoms $b \notin \{a_1, \ldots, a_m\}$ are replaced by $\circ$ everywhere. This yields a derivation

$$\begin{array}{c} ([a_1,a_2],[a_3,a_4],\ldots,[a_{m-1},a_m]) \\ \Delta\parallel\!\{\mathsf{s}\} \\ [(a_2,a_3),(a_4,a_5),\ldots,(a_m,a_1)] \end{array} \quad ,$$

  which is a contradiction to Lemma 6.2.18.

- $(4) \Rightarrow (1)$: I will show that the structure $[\bar{T},R]$ cannot contain a negation circuit. We can then apply Proposition 6.2.16 to obtain the proof

$$\begin{array}{c} \parallel\!\mathsf{s}^\circ \\ [\bar{T},R] \end{array} \quad .$$

  By way of contradiction, assume now that $[\bar{T},R]$ contains a negation circuit, say

$$Q = [(\bar{a}_1,a_2),(\bar{a}_2,a_3),\ldots,(\bar{a}_n,a_1)] \quad ,$$

  for some $n > 0$. As in the proof of Proposition 6.2.13, I will set $Z_j = (\bar{a}_j, a_{j+1})$ for every $j \in \{1, \ldots, n\}$ (with the convention that $j + 1 = 1$ for $j = n$). This means that

$$Q = [Z_1, Z_2, \ldots, Z_n] \quad .$$

  Since neither R nor T contain a pair of dual atoms (i.e. a and $\bar{a}$), we have that n is even (because the same number of "Z's" has to come from $\bar{T}$ and from R) and either

  - $\bar{T}$ has $[Z_1, Z_3, \ldots, Z_{n-1}]$ as quasi-substructure and R has $[Z_2, Z_4, \ldots, Z_n]$ as quasi-substructure, or

 – R has $[Z_1, Z_3, \ldots, Z_{n-1}]$ as quasi-substructure and $\bar{T}$ has $[Z_2, Z_4, \ldots, Z_n]$ as quasi-substructure.

Without loss of generality, we can assume that $[Z_1, Z_3, \ldots, Z_{n-1}]$ is contained in $\bar{T}$ and $[Z_2, Z_4, \ldots, Z_n]$ in R. This means that T has

$$(\bar{Z}_1, \bar{Z}_3, \ldots, \bar{Z}_{n-1}) = ([a_1, \bar{a}_2], [a_3, \bar{a}_4], \ldots, [a_{n-1}, \bar{a}_n])$$

as quasi-substructure. This is a $\langle a_1, \bar{a}_2, a_3, \bar{a}_4, \ldots, a_{n-1}, \bar{a}_n \rangle$-cocircuit. Since

$$[Z_2, Z_4, \ldots, Z_n] = [(\bar{a}_2, a_3), (\bar{a}_4, a_5), \ldots, (\bar{a}_n, a_1)]$$

is a $\langle a_1, \bar{a}_2, a_3, \bar{a}_4, \ldots, a_{n-1}, \bar{a}_n \rangle$-circuit, that is contained in R, we have a contradiction to $T \lessdot R$. $\square$

As an immediate consequence from this theorem we get that the relation $\lessdot$ is a partial order, which is from Definition 6.2.19 not at all obvious.

6.2.21 Corollary *The relation $\lessdot$ is a partial order.*

Proof: Reflexivity is obvious since there is always the trivial derivation from R to R. Transitivity is also obvious because the derivability relation is transitive. Antisymmetry is obtained from the fact that the switch rule is not invertible. This means, whenever we have $\mathsf{s}\dfrac{T}{R}$ with $R \neq T$ (i.e. the switch rule is applied in a nontrivial way), then there is no derivation

$$\begin{array}{c} R \\ \|\{\mathsf{s}\} \\ T \end{array} \quad .$$

This is the case because $\left| \downarrow_T \right| < \left| \downarrow_R \right|$ (see also the proof of Lemma 6.2.15). $\square$

6.3 Discussion

In Section 6.1, I have shown that the rules mix and nullary mix can be incorporated naturally in the calculus of structures.

In Section 6.2, I have shown an algebraic characterisation of derivability using the switch rule. The attentive reader might already have guessed that this result is a byproduct of the investigation of the decomposition theorems in Section 4.3. The results of Section 6.2 can be considered as preliminary results towards a general characterisation for the derivability in systems, which are not restricted to multiplicative linear logic. In my opinion, the next step should be to include also A. Guglielmi's noncommutative connective seq [Gug02e] into the two characterisations (Theorem 6.2.17 and Theorem 6.2.20). I will come back to this topic in the next chapter, where the seq will be discussed in detail.

Section 6.2 can also be seen as starting point for further research in the direction suggested in [Gug02b], where a certain notion of implicational and structural completeness is suggested. Since this is quite speculative, I do not want to go into further details here. I only want to mention that the implication $(1) \Rightarrow (3)$ of Theorem 6.2.20 says that system

S° has this completeness, as conjectured by A. Guglielmi in [Gug02b], where S° is called system FBV.

The switch rule does also play an important role in the presentation of classical logic in the calculus of structures [BT01, Brü03b, Brü02a]. This means that also for classical logic the results of Section 6.2 could be of interest.

7

A Noncommutative Extension of the Multiplicative Exponential Fragment of Linear Logic

In this chapter, I will discuss an extension of multiplicative exponential linear logic by a noncommutative connective which is self-dual with respect to linear negation. More precisely, I will investigate the properties of a logical system, called system NEL, which is a conservative extension MELL + mix + mix0, or system ELS°, as discussed in Section 6.1. The self-dual noncommutative connective is the same as it has been introduced by A. Guglielmi in [Gug99, Gug02e] for system BV. Consequently, system NEL is also a conservative extension of system BV.

By bringing together the two multiplicative connectives of linear logic, the exponentials, and a self-dual noncommutative connective, system NEL can be seen from two perspectives: first, as an extension of BV by the ability of reusing information, and second, as an extension of MELL by the ability of modelling sequentiality.

In Section 7.1, I will define the structures and the rules for system NEL. In Section 7.2, I will show that the two decomposition theorems, which have been shown for system SELS in Section 4.3, do also hold for system NEL. For this, I will refine the permutation results of Section 4.2 and extend some of the results of Section 6.2.

Then, in Section 7.3, I will prove the cut elimination result for system NEL by using the techniques of decomposition and splitting, which has first been used by A. Guglielmi in [Gug02e]. The second decomposition theorem will be employed in the same way as in Chapter 4 for system ELS. Then splitting is used to eliminate the core up fragment. I will first explain the general idea of splitting, and then show the complete proof of the splitting lemma (Lemma 7.3.5) and show how it is used to get cut elimination. This will be a refinement of the cut elimination proof for system BV in [Gug02e].

Finally, in Section 7.4, I will show that system NEL is undecidable by simulating the computations of a two counter machine. The encoding and its proof of correctness will be direct, natural and simple. However, the proof of the soundness of the encoding is a little intricate because there is no sequent calculus and no phase semantics available for NEL.

Associativity	Exponentials	Negation

$$[R,[T,U]] \;=\; [[R,T],U]$$
$$(R,(T,U)) \;=\; ((R,T),U)$$
$$\langle R;\langle T;U\rangle\rangle \;=\; \langle\langle R;T\rangle;U\rangle$$

Commutativity

$$[R,T] \;=\; [T,R]$$
$$(R,T) \;=\; (T,R)$$

Exponentials

$$??R \;=\; ?R$$
$$!!R \;=\; !R$$

Unit

$$[\circ,R] \;=\; R$$
$$(\circ,R) \;=\; R$$
$$\langle\circ;R\rangle \;=\; R$$
$$\langle R;\circ\rangle \;=\; R$$
$$?\circ \;=\; \circ$$
$$!\circ \;=\; \circ$$

Negation

$$\overline{\circ} \;=\; \circ$$
$$\overline{[R,T]} \;=\; (\bar{R},\bar{T})$$
$$\overline{(R,T)} \;=\; [\bar{R},\bar{T}]$$
$$\overline{\langle R,T\rangle} \;=\; \langle\bar{R},\bar{T}\rangle$$
$$\overline{?R} \;=\; !\bar{R}$$
$$\overline{!R} \;=\; ?\bar{R}$$
$$\bar{\bar{R}} \;=\; R$$

Figure 7.1: Basic equations for the syntactic congruence of NEL structures

7.1   Structures and Rules

7.1.1   Definition   As before, the countable set $\mathcal{A}$ of *atoms*, which are denoted by a, b, c, ..., is equipped with a bijective function $\bar{\cdot} : \mathcal{A} \to \mathcal{A}$, such that $\bar{\bar{a}} = a$ and $\bar{a} \neq a$ for every $a \in \mathcal{A}$. Let $\mathcal{R}$ be the set of expressions generated by the following syntax:

$$R ::= \circ \mid a \mid [R,R] \mid (R,R) \mid \langle R;R\rangle \mid !R \mid ?R \mid \bar{R} \quad,$$

where a stands for any atom, and $\circ$, the *unit*, is not an atom. On the set $\mathcal{R}$, the relation $=$ is defined to be the smallest congruence relation induced by the equations shown in Figure 7.1. The NEL *structures* (denoted with P, Q, R, S, ...) are the elements of $\mathcal{R}/=$, i.e. equivalence classes of expressions.

As before, I will omit superfluous parentheses. NEL structures can be seen as extensions of ELS° structures (defined in Section 6.1) by structures of the shape $\langle R_1;\dots;R_h\rangle$, which are called *seq structures* [Gug99]. Let me now define the new rules, which deal with seq.

7.1.2   Definition   The two structural rules

$$\mathsf{q}{\downarrow}\,\frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]} \qquad \text{and} \qquad \mathsf{q}{\uparrow}\,\frac{S(\langle R;U\rangle,\langle T;V\rangle)}{S\langle(R,T);(U,V)\rangle}$$

are called *seq* and *coseq*, respectively.

Observe, that rules become more flexible in the presence of the unit $\circ$, which is common to par, times and seq. For example, the following derivation is valid:

$$\mathsf{q}{\uparrow}\,\frac{(a,b)}{\dfrac{\langle a;b\rangle}{[a,b]}}\;\mathsf{q}{\downarrow} \quad = \quad \mathsf{q}{\uparrow}\,\frac{(\langle a;\circ\rangle,\langle\circ;b\rangle)}{\dfrac{\langle[a,\circ];[\circ,b]\rangle}{[\langle a;\circ\rangle,\langle\circ;b\rangle]}}\;\mathsf{q}{\downarrow} \quad = \quad \mathsf{q}{\uparrow}\,\frac{(\langle a;\circ\rangle,\langle\circ;b\rangle)}{\dfrac{\langle(a,\circ);(\circ,b)\rangle}{[\langle a;\circ\rangle,\langle\circ;b\rangle]}}\;\mathsf{q}{\downarrow} \quad.$$

$$
\mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad\qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\circ\}} \qquad \left.\vphantom{\frac{S}{S}}\right\}\ \text{Interaction}
$$

$$
\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}
$$

$$
\mathsf{q}\!\downarrow \frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]} \qquad \mathsf{q}\!\uparrow \frac{S(\langle R;U\rangle,\langle T;V\rangle)}{S\langle(R,T);(U,V)\rangle}
$$

$$
\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad\qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}}
$$

$$
\mathsf{w}\!\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad\qquad \mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{\circ\}}
$$

$$
\mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \qquad\qquad \mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R,R)}
$$

Figure 7.2: System SNEL

Further, observe that the interaction rules are in the presence of seq still reducible to their atomic form (Proposition 3.2.16). The following derivation shows the corresponding inductive case for i↑:

$$
\mathsf{i}\!\uparrow \frac{\mathsf{q}\!\uparrow \dfrac{S(\langle P;Q\rangle,\langle\bar{P};\bar{Q}\rangle)}{S\langle(P,\bar{P});(Q,\bar{Q})\rangle}}{\dfrac{S\langle\circ;(Q,\bar{Q})\rangle}{=\,\mathsf{i}\!\uparrow\dfrac{S(Q,\bar{Q})}{S\{\circ\}}}}\quad.
$$

7.1.3 Definition *System* SNEL (*symmetric noncommutative exponential linear logic*), which is shown in Figure 7.2, is obtained from system SELS° by adding the rules q↓ and q↑. The *down fragment* of SNEL is {ai↓, s, q↓, p↓, w↓, b↓}, and the *up fragment* is {ai↑, s, q↑, p↑, w↑, b↑}. The *core* of system SNEL is SNELc = {s, q↓, q↑, p↓, p↑}, and the *noncore* is {w↓, w↑, b↓, b↑}.

As axiom we use the same rule as in Section 6.1. The rule

$$
\circ\!\downarrow \frac{}{\circ}
$$

is called *unit*.

$$\circ\downarrow \ \frac{}{\circ} \qquad \text{ai}\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \text{s} \frac{S([R,U],T)}{S[(R,T),U]} \qquad \text{q}\downarrow \frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]}$$

$$\text{p}\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \text{w}\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad \text{b}\downarrow \frac{S[?R,R]}{S\{?R\}}$$

Figure 7.3: System NEL

7.1.4   Definition   *System* NEL *is obtained from the down fragment of system* SNEL *together with the axiom. It is shown in Figure 7.3.*

As for all other systems discussed in this thesis, we have the following:

7.1.5   Theorem   *The systems* SNEL $\cup \{\circ\downarrow\}$ *and* NEL $\cup \{\text{i}\uparrow\}$ *are strongly equivalent.*

Of course, there is also a cut elimination result for system NEL.

7.1.6   Theorem (Cut Elimination)   *System* NEL *is equivalent to every subsystem of the system* SNEL $\cup \{\circ\downarrow\}$ *containing* NEL*.*

At this point we cannot use the sequent calculus for obtaining a proof of cut elimination. The reason is that there is no sequent system for NEL, which could be used. We have to rely on the methods developed for the calculus of structures. For NEL, I will use decomposition and splitting. The detailed proof of Theorem 7.1.6 will be carried out in Section 7.3.

7.1.7   Corollary   *The rule* i$\uparrow$ *is admissible for system* NEL*.*

7.1.8   Corollary   *Let* R *and* T *be two* NEL *structures. Then we have*

$$\begin{array}{c} T \\ \|\text{SNEL} \\ R \end{array} \qquad \text{if and only if} \qquad \begin{array}{c} \ \\ \overline{\|}\text{NEL} \\ [\bar{T},R] \end{array} \quad .$$

Let me now define system BV, which is the multiplicative fragment of NEL. It was the first system to be presented in the calculus of structures [Gug99]. I will need it for proving some of the results in this chapter.

7.1.9   Definition   A BV *structure* is a NEL structure that does not contain any exponentials, i.e. that does not have any substructure $!R$ or $?R$ for some R.

7.1.10   Remark   The equations defining the syntactic congruence for BV structures are the same as for NEL structures, except that the equations concerning the exponentials are removed.

7.1.11   Definition   *System* SBV, which shown in Figure 7.4, contains the rules ai$\downarrow$, ai$\uparrow$, s, q$\downarrow$, and q$\uparrow$ (i.e. the multiplicative part or SNEL).

The core of system SBV is SBVc = $\{\text{s}, \text{q}\downarrow, \text{q}\uparrow\}$. This means that there are no noncore rules in system SBV.

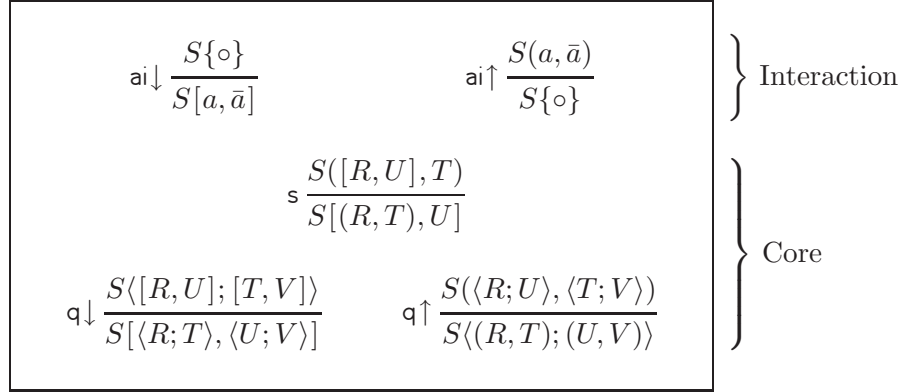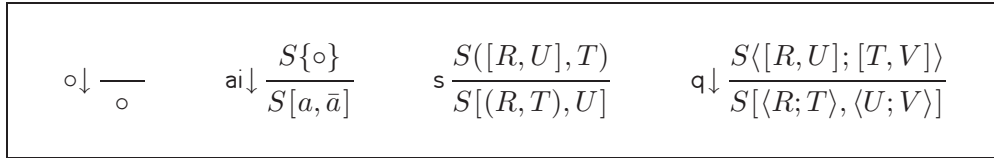$$\text{ai}\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad\qquad \text{ai}\uparrow \frac{S(a,\bar{a})}{S\{\circ\}} \qquad\Bigg\}\ \text{Interaction}$$

$$\text{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\text{q}\downarrow \frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]} \qquad \text{q}\uparrow \frac{S(\langle R;U\rangle,\langle T;V\rangle)}{S\langle(R,T);(U,V)\rangle} \qquad\Bigg\}\ \text{Core}$$

Figure 7.4: System SBV

$$\circ\downarrow \frac{}{\circ} \qquad \text{ai}\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \text{s}\,\frac{S([R,U],T)}{S[(R,T),U]} \qquad \text{q}\downarrow \frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]}$$

Figure 7.5: System BV

7.1.12   Definition   The system $\{\circ\downarrow, \text{ai}\downarrow, \text{s}, \text{q}\downarrow\}$ which is obtained from the down fragment of system SBV together with the axiom is called *system* BV (or *basic system* V). It is shown in Figure 7.5.

7.1.13   Theorem   *System* NEL *is a conservative extension of both, system* ELS° *and system* BV.

Proof: Let me first show, that NEL is a conservative extension of ELS°. Let R be an ELS° structure. Now assume, we have a proof $\Pi\|\text{NEL} \atop R$. Let Π' be the proof obtained from Π in the following way: Each seq context, occurring inside a structure S inside Π is replaced by a par context. By this each rule application remains valid, except for the seq rule, which becomes trivial:

$$\text{q}\downarrow \frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]} \qquad \rightsquigarrow \qquad \text{q}\downarrow' \frac{S[[R,U],[T,V]]}{S[[R,T],[U,V]]}\quad,$$

and can therefore be omitted. This means that Π' is a valid proof of R in system ELS°.

For showing that NEL is a conservative extension of BV, let R be a BV structure and assume, that there is a proof $\Pi\|\text{NEL} \atop R$. Observe that no rule in system NEL can introduce a ? while going up in a proof. Since there are no ?-structures present in R, there are no applications of w↓ or b↓ inside Π. This also means that all applications of the promotion

rule must be trivial (because there are also no !-structures in R):

$$\mathsf{p}{\downarrow}\ \frac{S\{![\circ,\circ]\}}{S[!\circ,?\circ]} \qquad = \qquad \mathsf{p}{\downarrow}\ \frac{S\{\circ\}}{S\{\circ\}}\ \ ,$$

and can therefore be omitted.   □

The above theorem ensures, that the cut elimination result for NEL does also hold for the systems ELS° and BV. This can be seen as follows: Let $\Pi\|\underset{R}{{}^{\mathsf{ELS°}\cup\{\mathsf{i}{\uparrow}\}}}$ be given, where R is an ELS° structure. Then we can apply Corollary 7.1.7 to Π to get a proof $\Pi'\|\underset{R}{{}^{\mathsf{NEL}}}$. Since NEL is a conservative extension of ELS°, we can obtain a proof $\Pi''\|\underset{R}{{}^{\mathsf{ELS°}}}$. Similarly we can proceed for BV. In particular, we have:

7.1.14   Corollary   *Let R and T be two* BV *structures. Then we have*

$$\begin{array}{c} T \\ \|\mathsf{SBV} \\ R \end{array} \qquad \textit{if and only if} \qquad \begin{array}{c} \ \\ \|\mathsf{BV} \\ [\bar{T},R] \end{array} \quad .$$

As shown by A. Tiu in [Tiu01], it is not possible to present system BV in the sequent calculus. Another consequence of Theorem 7.1.13 is that A. Tiu's result does also hold for NEL: there is no sequent calculus system that is equivalent to system NEL.

7.2   Decomposition

In this section, I will show that the first and the second decomposition theorem for SELS (see Section 4.3) do also hold for system SNEL.

7.2.1   Theorem (First Decomposition)   *For every derivation* $\Delta\|\underset{R}{\overset{T}{{}^{\mathsf{SNEL}}}}$ *there are struc-tures T_1, T_2, T_3, R_3, R_2 and R_1, and derivations $\Delta_1,\ldots,\Delta_7$, such that*

$$\begin{array}{c} T \\ \Delta_1\|\{\mathsf{b}{\uparrow}\} \\ T_1 \\ \Delta_2\|\{\mathsf{w}{\downarrow}\} \\ T_2 \\ \Delta_3\|\{\mathsf{ai}{\downarrow}\} \\ T_3 \\ \Delta_4\|\mathsf{SNELc} \\ R_3 \\ \Delta_5\|\{\mathsf{ai}{\uparrow}\} \\ R_2 \\ \Delta_6\|\{\mathsf{w}{\uparrow}\} \\ R_1 \\ \Delta_7\|\{\mathsf{b}{\downarrow}\} \\ R \end{array} \quad .$$

7.2.2   Theorem (Second Decomposition)   *For every derivation* $\overset{T}{\underset{R}{\Delta\|\mathsf{SNEL}}}$ *there are structures* T_1, T_2, T_3, R_3, R_2 *and* R_1, *and derivations* $\Delta_1, \ldots, \Delta_7$, *such that*

$$
\begin{array}{c}
T \\
\Delta_1 \| \{\mathsf{b\uparrow}\} \\
T_1 \\
\Delta_2 \| \{\mathsf{w\uparrow}\} \\
T_2 \\
\Delta_3 \| \{\mathsf{ai\downarrow}\} \\
T_3 \\
\Delta_4 \| \mathsf{SNELc} \\
R_3 \\
\Delta_5 \| \{\mathsf{ai\uparrow}\} \\
R_2 \\
\Delta_6 \| \{\mathsf{w\downarrow}\} \\
R_1 \\
\Delta_7 \| \{\mathsf{b\downarrow}\} \\
R
\end{array} \quad .
$$

The proof of those two theorems will be very similar (in fact, almost literally the same) to the proofs of Theorems 4.3.1 and 4.3.2. There are only two points, which are different, and which deserve attention.

- First, the collapsing of the units for the par and the times allows a more flexible application of rules. This has consequences for the permutation results. For this reason, I will in Section 7.2.1 redo all the necessary permutation proofs for separating weakening and interaction in the first decomposition.

- Second, the presence of seq has also an impact on the proof for the nonexistence of promotion cycles (for the separation of absorption): Lemma 4.3.20 (which is crucial for Proposition 4.3.22) has to be generalised. This is rather trivial and will be done below (in Lemma 7.2.3). Another consequence of the presence of seq is that instead of proving that there is no derivation

$$
\begin{array}{c}
([!R_1, ?T_1], [!R_2, ?T_2], \ldots [!R_n, ?T_n]) \\
\Delta \| \{\mathsf{s}\} \\
[(!R_2, ?T_1), (!R_3, ?T_2), \ldots (!R_1, ?T_n)]
\end{array} \quad ,
$$

for some $n \geqslant 1$ and structures $R_1, \ldots, R_n, T_1, \ldots, T_n$ (see Lemma 4.3.25), we have to show that there is no derivation

$$
\begin{array}{c}
([!R_1, ?T_1], [!R_2, ?T_2], \ldots [!R_n, ?T_n]) \\
\Delta \| \{\mathsf{s}, \mathsf{q\downarrow}, \mathsf{q\uparrow}\} \\
[(!R_2, ?T_1), (!R_3, ?T_2), \ldots (!R_1, ?T_n)]
\end{array} \quad ,
$$

for some $n \geqslant 1$ and structures $R_1, \ldots, R_n, T_1, \ldots, T_n$. This is considerably more difficult and will be done in Section 7.2.2. Fortunately, that is all that has to be done. Everything else of the proofs for the decomposition theorems is exactly the same as for system SELS.

The following lemma will be an easy extension of Lemma 4.3.20. Observe that, in order to be formally precise, it is also necessary to redefine the notion of basic context (see Definition 4.3.18) and include the seq: In the proof for the decomposition theorems for system SNEL, a *basic context* is a context $S\{\ \}$, where the hole does not occur inside the scope of an exponential.

7.2.3 Lemma *Let $S\{\ \}$ be a basic context and R and T be any structures. Then there is a derivation*

$$
\begin{array}{c}
S[R,T] \\
\Delta \big\| {\sf \{s,q\downarrow\}} \\
[S\{R\},T]
\end{array}
\quad .
$$

Proof: By structural induction on $S\{\ \}$. The cases for $S = \{\ \}$, $S = [S', S''\{\ \}]$ and $S = (S', S''\{\ \})$ have been shown in the proof of Lemma 4.3.20 on page 108. Let me now show the remaining case:

- $S = \langle S'; S''\{\ \}; S'''\rangle$ for some context $S''\{\ \}$ and structures S' and S'''. Then let Δ be

$$
\begin{array}{c}
\langle S'; S''[R,T]; S'''\rangle \\
\Delta' \big\| {\sf \{s,q\downarrow\}} \\
{\sf q\downarrow}\dfrac{\langle S'; [S''\{R\},T]; S'''\rangle}{\dfrac{\langle S'; [\langle S''\{R\}; S'''\rangle,T]\rangle}{[\langle S'; S''\{R\}; S'''\rangle,T]}} \\
\end{array}
\quad ,
$$

where Δ' exists by induction hypothesis.   □

7.2.1 Permutation of Rules

In the proof of Proposition 4.3.22 (needed for the proofs of the decomposition theorems for SELS in Chapter 4), I introduced the rules $\hat{\sf p}\downarrow$, $\hat{\sf p}\uparrow$, $\hat{\sf w}\downarrow$, and $\hat{\sf w}\uparrow$. In the proof of the decomposition theorems for SNEL, they will also be needed:

$$
\hat{\sf p}\downarrow \dfrac{S\{![R,T]\}}{S[!R,U]} \quad , \quad
\hat{\sf p}\uparrow \dfrac{S(?R,U)}{S\{?(R,T)\}} \quad , \quad
\hat{\sf w}\downarrow \dfrac{S\{\circ\}}{S\{R\}} \quad , \quad
\hat{\sf w}\uparrow \dfrac{S\{R\}}{S\{\circ\}} \quad .
$$

They have to be considered in the permutation results in this section.

7.2.4 Lemma *Every rule $\rho \in \{{\sf ai\downarrow, w\downarrow, \hat{w}\downarrow}\}$ permutes over every rule $\pi \in ({\sf SNEL} \cup \{\hat{\sf p}\downarrow, \hat{\sf p}\uparrow, \hat{\sf w}\downarrow, \hat{\sf w}\uparrow\}) \setminus \{{\sf b\downarrow, b\uparrow}\}$ by the system* SBVc.

Proof: Consider a derivation

$$
\begin{array}{c}
\pi \dfrac{Q}{S\{\circ\}} \\
\rho \dfrac{}{S\{Z\}}
\end{array}
\quad ,
$$

where $\rho \in \{{\sf ai\downarrow, w\downarrow, \hat{w}\downarrow}\}$ and $\pi \in ({\sf SNEL} \cup \{\hat{\sf p}\downarrow, \hat{\sf p}\uparrow, \hat{\sf w}\downarrow, \hat{\sf w}\uparrow\}) \setminus \{{\sf b\downarrow, b\uparrow}\}$ is nontrivial. According to 4.2.3, there are the following cases to consider:

(4) The redex of π is inside an active structure of the contractum $\circ$ of ρ. This case is impossible because no structure can be inside $\circ$.

(5) The contractum $\circ$ of ρ is inside an active structure of the redex of π but not inside a passive one. As in the permutation proofs of Chapter 5, this case is the problematic one. Since $\circ$ is the unit for par, seq, and times, we have for each rule π more cases to consider.

(i) $\pi = \mathsf{w}{\downarrow}$. Then $S\{\circ\} = S'\{?R\{\circ\}\}$ for some contexts $S'\{\ \}$ and $R\{\ \}$. We have

$$
\rho\,\cfrac{\mathsf{w}{\downarrow}\,\cfrac{S'\{\circ\}}{S'\{?R\{\circ\}\}}}{S'\{?R\{\boxed{Z}\}\}}
\qquad \text{yields} \qquad
\mathsf{w}{\downarrow}\,\cfrac{S'\{\circ\}}{S'\{?R\{\boxed{Z}\}\}}\qquad.
$$

(ii) $\pi = \hat{\mathsf{w}}{\downarrow}$. Similar to (i).

(iii) $\pi = \mathsf{q}{\uparrow}$. Then we have the following subcases:

(a) $S\{\circ\} = S'\langle(R,[(R',T),\circ],T');(U,V)\rangle$ for some context $S'\{\ \}$ and structures R,R',T,T',U and V. Then we have

$$
\rho\,\cfrac{\mathsf{q}{\uparrow}\,\cfrac{S'(\langle(R,R');U\rangle,\langle(T,T');V\rangle)}{S'\langle(R,R',T,T');(U,V)\rangle}}{S'\langle(R,[(R',T),\boxed{Z}],T');(U,V)\rangle}\qquad,
$$

which yields

$$
\mathsf{s}\,\cfrac{\mathsf{q}{\uparrow}\,\cfrac{\rho\,\cfrac{S'(\langle(R,R');U\rangle,\langle(T,T');V\rangle)}{S'(\langle(R,[R',\boxed{Z}]);U\rangle,\langle(T,T');V\rangle)}}{S'\langle(R,[R',Z],T,T');(U,V)\rangle}}{S'\langle(R,[(R',T),Z],T');(U,V)\rangle}\qquad.
$$

(b) $S\{\circ\} = S'\langle(R,T);(U,[(U',V),\circ],V')\rangle$ for some context $S'\{\ \}$ and structures R,T,U,U',V and V'. Similar to (a).

(c) $S\{\circ\} = S'\langle(R,\langle(R',T);\circ\rangle,T');(U,V)\rangle$ for some context $S'\{\ \}$ and structures R,R',T,T',U and V. Then we have

$$
\rho\,\cfrac{\mathsf{q}{\uparrow}\,\cfrac{S'(\langle(R,R');U\rangle,\langle(T,T');V\rangle)}{S'\langle(R,R',T,T');(U,V)\rangle}}{S'\langle(R,\langle(R',T);\boxed{Z}\rangle,T');(U,V)\rangle}\qquad,
$$

which yields

$$
\mathsf{q}{\uparrow}\,\cfrac{\mathsf{q}{\uparrow}\,\cfrac{\rho\,\cfrac{S'(\langle(R,R');U\rangle,\langle(T,T');V\rangle)}{S'(\langle(R,\langle R';\boxed{Z}\rangle);U\rangle,\langle(T,T');V\rangle)}}{S'\langle(R,\langle R';Z\rangle,T,T');(U,V)\rangle}}{S'\langle(R,\langle(R',T);Z\rangle,T');(U,V)\rangle}\qquad.
$$

(d) $S\{\circ\} = S'\langle(R,\langle\circ;(R',T)\rangle,T');(U,V)\rangle$ for some context $S'\{\ \}$ and structures R,R',T,T',U and V. Similar to (c).

(e) $S\{\circ\} = S'\langle(R,T);(U,\langle(U',V);\circ\rangle,V')\rangle$ for some context $S'\{\ \}$ and structures R,T,U,U',V and V'. Similar to (c).

(f) $S\{\circ\} = S'\langle(R,T);(U,\langle\circ;(U',V)\rangle,V')\rangle$ for some context $S'\{\ \}$ and structures R,T,U,U',V and V'. Similar to (c).

(g) $S\{\circ\} = S'\langle(R,T);\circ;(U,V)\rangle$ for some context $S'\{\ \}$ and structures R,T,U and V. Then

$$
\mathsf{q}\!\uparrow \frac{\mathsf{q}\!\uparrow \dfrac{S'(\langle R;U\rangle,\langle T;V\rangle)}{S'\langle(R,T);(U,V)\rangle}}{\rho\ \dfrac{S'\langle(R,T);(U,V)\rangle}{S'\langle(R,T);Z;(U,V)\rangle}}
\qquad \text{yields} \qquad
\mathsf{q}\!\uparrow \frac{\mathsf{q}\!\uparrow \dfrac{\pi\ \dfrac{S'(\langle R;U\rangle,\langle T;V\rangle)}{S'(\langle R;Z;U\rangle,\langle T;V\rangle)}}{S'\langle(\langle R;Z\rangle,T);(U,V)\rangle}}{S'\langle(R,T);Z;(U,V)\rangle}\ .
$$

(iv) $\pi = \mathsf{s}$. Then we have the following subcases:

(a) $S\{\circ\} = S'[(R,[(R',T),\circ],T'),U]$ for some context $S'\{\ \}$ and structures R,R',T,T' and U. This is similar to case (iii.a): We have

$$
\rho\ \frac{\mathsf{s}\ \dfrac{S'([(R,R'),U],T,T')}{S'[(R,R',T,T'),U]}}{S'[(R,[(R',T),Z],T'),U]}\ ,
$$

which yields

$$
\mathsf{s}\ \frac{\mathsf{s}\ \dfrac{\rho\ \dfrac{S'([(R,R'),U],T,T')}{S'([(R,[R',Z]),U],T,T')}}{S'\langle(R,[R',Z],T,T'),U\rangle}}{S'[(R,[(R',T),Z],T'),U]}\ .
$$

(b) $S\{\circ\} = S'[(R,\langle(R',T);\circ\rangle,T'),U]$ for some context $S'\{\ \}$ and structures R,R',T,T' and U. Similar to case (iii.c).

(c) $S\{\circ\} = S'[(R,\langle\circ;(R',T)\rangle,T'),U]$ for some context $S'\{\ \}$ and structures R,R',T,T' and U. Similar to case (iii.d).

(d) $S\{\circ\} = S'[([(R,T),U],\circ),U']$ for some context $S'\{\ \}$ and structures R,T,U and U'. Then we have

$$
\rho\ \frac{\mathsf{s}\ \dfrac{S'([R,U,U'],T)}{S'[(R,T),U,U']}}{S'[([(R,T),U],Z),U']}
\qquad \text{yields} \qquad
\mathsf{s}\ \frac{\mathsf{s}\ \dfrac{\rho\ \dfrac{S'([R,U,U'],T)}{S'([R,U,U'],T,Z)}}{S'([(R,T),U,U'],Z)}}{S'[([(R,T),U],Z),U']}\ .
$$

(e) $S\{\circ\} = S'[\langle[(R,T),U];\circ\rangle,U']$ for some context $S'\{\ \}$ and structures R,T,U and U'. Then we have

$$
\rho\ \frac{\mathsf{s}\ \dfrac{S'([R,U,U'],T)}{S'[(R,T),U,U']}}{S'[\langle[(R,T),U];Z\rangle,U']}
\qquad \text{yields} \qquad
\mathsf{q}\!\downarrow \frac{\mathsf{s}\ \dfrac{\rho\ \dfrac{S'([R,U,U'],T)}{S'\langle([R,U,U'],T);Z\rangle}}{S'\langle[(R,T),U,U'];Z\rangle}}{S'[\langle[(R,T),U];Z\rangle,U']}\ .
$$

(f) $S\{\circ\} = S'[(\langle\circ;(R,T),U\rangle),U']$ for some context $S'\{\ \}$ and structures R,T,U and U'. Similar to (e).

(v) $\pi = \mathsf{p}{\uparrow}$. Then we have the following subcases:

(a) $S\{\circ\} = S'\{?(R,[(R',T),\circ],T')\}$. for some context $S'\{\ \}$ and structures R,R',T and T'. Then we have

$$\rho\frac{\mathsf{p}{\uparrow}\dfrac{S'(?(R,R'),!(T,T'))}{S'\{?(R,R',T,T')\}}}{S'\{?(R,[(R',T),\boxed{Z}],T')\}}\quad,$$

which is is similar to (iii.a):

$$\mathsf{s}\frac{\mathsf{p}{\uparrow}\dfrac{\rho\dfrac{S'(?(R,R'),!(T,T'))}{S'(?(R,[R',\boxed{Z}]),!(T,T'))}}{S'\{?(R,[R',Z],T,T')\}}}{S'\{?(R,[(R',T),Z],T')\}}\quad.$$

(b) $S\{\circ\} = S'\{?(R,\langle(R',T);\circ\rangle,T')\}$. for some context $S'\{\ \}$ and structures R,R',T and T'. Similar to (iii.c).

(c) $S\{\circ\} = S'\{?(R,\langle\circ;(R',T)\rangle,T')\}$. for some context $S'\{\ \}$ and structures R,R',T and T'. Similar to (iii.d).

(vi) $\pi = \hat{\mathsf{p}}{\uparrow}$. Similar to (v).

(vii) $\pi = \mathsf{p}{\downarrow}$. Then we have the following subcases:

(a) $S\{\circ\} = S'[(!R,\circ),?T]$ for some $S'\{\ \}$ and R and T. Then the situation is similar as in (iv.d):

$$\rho\frac{\mathsf{p}{\downarrow}\dfrac{S'\{![R,T]\}}{S'[!R,?T]}}{S'[(!R,\boxed{Z}),?T]}\qquad\text{yields}\qquad \mathsf{s}\frac{\mathsf{p}{\downarrow}\dfrac{\rho\dfrac{S'\{![R,T]\}}{S'(![R,T],\boxed{Z})}}{S'([!R,?T],Z)}}{S'[(!R,Z),?T]}\quad.$$

(b) $S\{\circ\} = S'[!R,(?T,\circ)]$. Similar to (a).

(c) $S\{\circ\} = S'[\langle!R;\circ\rangle,?T]$ for some $S'\{\ \}$ and R and T. Then the situation is similar as in (iv.e):

$$\rho\frac{\mathsf{p}{\downarrow}\dfrac{S'\{![R,T]\}}{S'[!R,?T]}}{S'[\langle!R;\boxed{Z}\rangle,?T]}\qquad\text{yields}\qquad \mathsf{q}{\downarrow}\frac{\mathsf{p}{\downarrow}\dfrac{\rho\dfrac{S'\{![R,T]\}}{S'\langle![R,T];\boxed{Z}\rangle}}{S'\langle[!R,?T];Z\rangle}}{S'[\langle!R;Z\rangle,?T]}\quad.$$

(d) $S\{\circ\} = S'[\langle\circ;!R\rangle,?T]$. Similar to (c).

(e) $S\{\circ\} = S'[!R,\langle?T;\circ\rangle]$. Similar to (c).

(f) $S\{\circ\} = S'[!R, \langle \circ; ?T \rangle]$. Similar to (c).

(viii) $\pi = \hat{\mathsf{p}}\downarrow$. Similar to (vi).

(ix) $\pi = \mathsf{ai}\downarrow$. Similar to (vi).

(x) $\pi = \mathsf{q}\downarrow$. Then we have the following subcases:

(a) $S\{\circ\} = S'[\langle R; [\langle R'; T \rangle, \circ]; T' \rangle, \langle U; V \rangle]$ for some context $S'\{\ \}$ and structures R, R', T, T', U and V. Then we have

$$\rho \frac{\mathsf{q}\downarrow \dfrac{S'\langle [\langle R; R' \rangle, U]; [\langle T; T' \rangle, V] \rangle}{S'[\langle R; R'; T; T' \rangle, \langle U; V \rangle]}}{S'[\langle R; [\langle R'; T \rangle, Z]; T' \rangle, \langle U; V \rangle]} \quad ,$$

which yields

$$\mathsf{q}\downarrow \frac{\mathsf{q}\downarrow \dfrac{\rho \dfrac{S'\langle [\langle R; R' \rangle, U]; [\langle T; T' \rangle, V] \rangle}{S'\langle [\langle R; [R', Z] \rangle, U]; [\langle T; T' \rangle, V] \rangle}}{S'[\langle R; [R', Z]; T; T' \rangle, \langle U; V \rangle]}}{S'[\langle R; [\langle R'; T \rangle, Z]; T' \rangle, \langle U; V \rangle]} \quad .$$

(b) $S\{\circ\} = S'[\langle R; T \rangle, \langle U; [\langle U'; V \rangle, \circ]; V' \rangle]$ for some context $S'\{\ \}$ and structures R, T, U, U', V and V'. Similar to (a).

(c) $S\{\circ\} = S'[\langle R; (\langle R'; T \rangle, \circ); T' \rangle, \langle U; V \rangle]$ for some context $S'\{\ \}$ and structures R, R', T, T', U and V. Then we have

$$\rho \frac{\mathsf{q}\downarrow \dfrac{S'\langle [\langle R; R' \rangle, U]; [\langle T; T' \rangle, V] \rangle}{S'[\langle R; R'; T; T' \rangle, \langle U; V \rangle]}}{S'[\langle R; (\langle R'; T \rangle, Z); T' \rangle, \langle U; V \rangle]} \quad .$$

In this case we need to employ $\mathsf{q}\uparrow$ together wit s:

$$\mathsf{q}\uparrow \frac{\mathsf{q}\uparrow \dfrac{\mathsf{s} \dfrac{\mathsf{q}\downarrow \dfrac{\rho \dfrac{S'\langle [\langle R; R' \rangle, U]; [\langle T; T' \rangle, V] \rangle}{S'(\langle [\langle R; R' \rangle, U]; [\langle T; T' \rangle, V] \rangle, Z)}}{S'([\langle R; R'; T; T' \rangle, \langle U; V \rangle], Z)}}{S'[(\langle R; R'; T; T' \rangle, Z), \langle U; V \rangle]}}{S'[\langle R; (\langle R'; T; T' \rangle, Z) \rangle, \langle U; V \rangle]}}{S'[\langle R; (\langle R'; T \rangle, Z); T' \rangle, \langle U; V \rangle]} \quad .$$

(d) $S\{\circ\} = S'[\langle R; T \rangle, \langle U; (\langle U'; V \rangle, \circ); V' \rangle]$ for some context $S'\{\ \}$ and structures R, T, U, U', V and V'. Similar to (c).

(xi) If $\pi \in \{\mathsf{ai}\downarrow, \mathsf{w}\uparrow, \hat{\mathsf{w}}\uparrow\}$, then there is no problem because the redex of π is $\circ$.

(6) The contractum $\circ$ of ρ and the redex of π (properly) overlap. This is impossible because $\circ$ cannot overlap with any other structure. $\square$

7.2.5 Lemma *Every rule $\rho \in \{\mathsf{ai}{\uparrow}, \mathsf{w}{\uparrow}, \hat{\mathsf{w}}{\uparrow}\}$ permutes under every rule $\pi \in (\mathsf{SNEL} \cup \{\hat{\mathsf{p}}{\downarrow}, \hat{\mathsf{p}}{\uparrow}, \hat{\mathsf{w}}{\downarrow}, \hat{\mathsf{w}}{\uparrow}\}) \setminus \{\mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}\}$ by the system $\{\mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}\}$.*

Proof: Dual to Lemma 7.2.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

7.2.2 Cycles in Derivations

In this section, I will show that there is no derivation

$$
\begin{array}{c}
([!R_1, ?T_1], [!R_2, ?T_2], \ldots, [!R_n, ?T_n]) \\
\Delta \big\| {\scriptstyle\{\mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}\}} \\
[(!R_2, ?T_1), (!R_3, ?T_2), \ldots, (!R_1, ?T_n)]
\end{array} \quad ,
$$

for some $n \geqslant 1$ and any structures $R_1, \ldots, R_n, T_1, \ldots, T_n$. This is important for showing that there is no nonforked cycle (see Section 4.3.1) in a derivation, which in turn is crucial for the decomposition theorems. For showing this, I will show a stronger result about the core of system SBV, which will in fact be a generalisation of Lemma 6.2.18. However, this time I cannot use cut elimination for proving it because it will be used in a cut elimination argument.

7.2.6 Lemma *Let $n > 0$ and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be $2n$ different atoms. Further, let $W_1, \ldots, W_n, Z_1, \ldots, Z_n$ be structures, such that*

- $W_i = [a_i, b_i]$ *or* $W_i = \langle a_i; b_i \rangle$*, for every* $i = 1, \ldots, n$,

- $Z_j = (a_{j+1}, b_j)$ *or* $Z_j = \langle a_{j+1}; b_j \rangle$*, for every* $j = 1, \ldots, n-1$*, and*

- $Z_n = (a_1, b_n)$ *or* $Z_n = \langle a_1; b_n \rangle$.

Then there is no derivation

$$
\begin{array}{c}
(W_1, W_2, \ldots, W_n) \\
\Delta \big\| {\scriptstyle\mathsf{SBVc}} \\
[Z_1, Z_2, \ldots, Z_n]
\end{array} \quad .
$$

Proof: (In this proof, I will use the convention that $j + 1 = 1$ in the case that $j = n$.) Let m be the number of structures Z_j (for $j = 1, \ldots, n$) which are seq-structures, i.e.

$$
m = \left| \{ j \mid Z_j = \langle a_{j+1}; b_j \rangle \} \right| \quad .
$$

Proceed by induction on the pair $\langle n, m \rangle$ by using lexicographic ordering, i.e. for another pair $\langle n', m' \rangle$, we have

$$
\langle n', m' \rangle < \langle n, m \rangle \quad \Longleftrightarrow \quad n' < n \quad \text{or} \quad n' = n \text{ and } m' < m \quad .
$$

Base Case: If $n = 1$, then obviously, there is no derivation

$$
\begin{array}{ccccc}
[a_1, b_1] & & [a_1, b_1] & & \langle a_1; b_1 \rangle \\
\Delta \big\| {\scriptstyle\mathsf{SBVc}} & \text{or} & \Delta \big\| {\scriptstyle\mathsf{SBVc}} & \text{or} & \Delta \big\| {\scriptstyle\mathsf{SBVc}} \\
(a_1, b_1) & & \langle a_1; b_1 \rangle & & (a_1, b_1)
\end{array} \quad .
$$

Inductive Case: Suppose there is no derivation Δ for all $\langle n', m' \rangle < \langle n, m \rangle$, and, by way of contradiction assume there is a derivation

$$
\begin{array}{c}
(W_1, W_2, \ldots, W_n) \\
\Delta \big\| {}_{\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}} \\
[Z_1, Z_2, \ldots, Z_n]
\end{array}
\quad ,
$$

where $Z_j = (a_{j+1}, b_j)$ or $Z_j = \langle a_{j+1}; b_j \rangle$ for every $j = 1, \ldots, n$ and $W_i = [a_i, b_i]$ or $W_i = \langle a_i; b_i \rangle$, for every $i = 1, \ldots, n$. Now consider the bottommost rule instance ρ of Δ. Without loss of generality, we can assume that ρ is nontrivial.

(1) $\rho = \mathsf{q}\uparrow$. There are two possibilities to apply $\mathsf{q}\uparrow \dfrac{S(\langle R; U \rangle, \langle T; V \rangle)}{S\langle (R, T); (U, V) \rangle}$.

    (i) $R = a_{j+1}$, $T = \circ$, $U = \circ$ and $V = b_j$ for some $j = 1, \ldots, n$. Then there must be some Δ', such that

$$
\begin{array}{c}
(W_1, W_2, \ldots, W_n) \\
\Delta' \big\| {}_{\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}} \\
\mathsf{q}\uparrow \dfrac{[Z_1, \ldots, Z_{j-1}, (a_{j+1}, b_j), Z_{j+1}, \ldots, Z_n]}{[Z_1, \ldots, Z_{j-1}, \langle a_{j+1}; b_j \rangle, Z_{j+1}, \ldots, Z_n]}
\end{array}
\quad ,
$$

    which is a contradiction to the induction hypothesis.

    (ii) $R = \circ, T = a_{j+1}, U = b_j, V = \circ$ for some $j = 1, \ldots, n$. Similar to (i).

(2) $\rho = \mathsf{q}\downarrow$. There are the following possibilities to apply $\mathsf{q}\downarrow \dfrac{S\langle [R, U]; [T, V] \rangle}{S[\langle R; T \rangle, \langle U; V \rangle]}$.

    (i) $R = a_{j+1}, T = b_j$ for some $j = 1, \ldots, n$. Then, without loss of generality, we can assume that $j = 1$. We have the following subcases:

        (a) $U = a_{i+1}$ and $V = b_i$ for some $i = 2, \ldots, n$. Then we have a derivation Δ' such that

$$
\begin{array}{c}
(W_1, W_2, \ldots, W_n) \\
\Delta' \big\| {}_{\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}} \\
\mathsf{q}\downarrow \dfrac{[\langle [a_2, a_{i+1}]; [b_1, b_i] \rangle, Z_2, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_n]}{[\langle a_2; b_1 \rangle, Z_2, \ldots, Z_{i-1}, \langle a_{i+1}; b_i \rangle, Z_{i+1}, \ldots, Z_n]}
\end{array}
\quad .
$$

        The derivation Δ' remains valid if we replace a_m and b_m by $\circ$ for every m with $2 \leqslant m \leqslant i$. Then we get

$$
\begin{array}{c}
(W_1, W_{i+1}, \ldots, W_n) \\
\Delta'' \big\| {}_{\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}} \\
[\langle a_{i+1}; b_1 \rangle, Z_{i+1}, \ldots, Z_n]
\end{array}
\quad ,
$$

        which is a contradiction to the induction hypothesis.

(b) $U = \circ$ and $V = [Z_{k_1}, \ldots, Z_{k_v}]$ for some $v > 0$ and $k_1, \ldots, k_v \in \{2, \ldots, n\}$. Without loss of generality, assume that $k_1 < k_2 < \ldots < k_v$. Let $\{2, \ldots, n\} \setminus \{k_1, \ldots, k_v\} = \{h_1, \ldots, h_s\}$, where $s = n - v - 1$. Then there is a derivation Δ' such that

$$
\begin{array}{c}
(W_1, W_2, \ldots, W_n) \\
\Delta' \big\| \{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\} \\
\mathsf{q}\downarrow \dfrac{[\langle a_2; [b_1, Z_{k_1}, \ldots, Z_{k_v}]\rangle, Z_{h_1}, \ldots, Z_{h_s}]}{[\langle a_2; b_1\rangle, Z_2, \ldots, Z_n]}
\end{array}
\quad .
$$

The derivation Δ' remains valid if we replace a_m and b_m by $\circ$ for every m with $m > k_1$ and for $m = 1$. Then we get

$$
\begin{array}{c}
(W_2, \ldots, W_{k_1}) \\
\Delta'' \big\| \{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\} \\
{[\langle a_2; b_{k_1}\rangle, Z_2, \ldots, Z_{k_1-1}]}
\end{array}
$$

which is a contradiction to the induction hypothesis.

(c) $U = [Z_{k_1}, \ldots, Z_{k_u}]$ and $V = \circ$ for some $u > 0$ and $k_1, \ldots, k_u \in \{2, \ldots, n\}$. Without loss of generality, assume that $k_1 < k_2 < \ldots < k_u$. Let $\{2, \ldots, n\} \setminus \{k_1, \ldots, k_u\} = \{h_1, \ldots, h_s\}$, where $s = n - u - 1$. Then there is a derivation Δ' such that

$$
\begin{array}{c}
(W_1, W_2, \ldots, W_n) \\
\Delta' \big\| \{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\} \\
\mathsf{q}\downarrow \dfrac{[\langle [a_2, Z_{k_1}, \ldots, Z_{k_u}]; b_1\rangle, Z_{h_1}, \ldots, Z_{h_s}]}{[\langle a_2; b_1\rangle, Z_2, \ldots, Z_n]}
\end{array}
\quad .
$$

The derivation Δ' remains valid if we replace a_m and b_m by $\circ$ for every m with $1 < m \leqslant k_u$. Then we get

$$
\begin{array}{c}
(W_1, W_{k_u+1}, \ldots, W_n) \\
\Delta'' \big\| \{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\} \\
{[\langle a_{k_u+1}; b_1\rangle, Z_{k_u+1}, \ldots, Z_n]}
\end{array}
\quad ,
$$

which is a contradiction to the induction hypothesis.

(ii) $U = a_{j+1}$ and $V = b_j$ for some $j = 1, \ldots, n$. Similar to (i).

(iii) $R = [Z_{k_1}, \ldots, Z_{k_r}]$, $T = \circ$, $U = \circ$ and $V = [Z_{l_1}, \ldots, Z_{l_v}]$ for some $r, v > 0$ and $k_1, \ldots, k_r, l_1, \ldots, l_v \in \{1, \ldots, n\}$. Then there is a derivation Δ' such that

$$
\begin{array}{c}
(W_1, W_2, \ldots, W_n) \\
\Delta' \big\| \{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\} \\
\mathsf{q}\downarrow \dfrac{[\langle [Z_{k_1}, \ldots, Z_{k_r}]; [Z_{l_1}, \ldots, Z_{l_v}]\rangle, Z_{h_1}, \ldots, Z_{h_s}]}{[Z_1, Z_2, \ldots, Z_n]}
\end{array}
\quad ,
$$

where $s = n - r - v$ and $\{h_1, \ldots, h_s\} = \{1, \ldots, n\} \setminus \{k_1, \ldots, k_r, l_1, \ldots, l_v\}$. Without loss of generality, we can assume that $r = v = 1$. Otherwise we could

replace

$$\mathsf{q}{\downarrow}\frac{[\langle[Z_{k_1},\ldots,Z_{k_r}];[Z_{l_1},\ldots,Z_{l_v}]\rangle,Z_{h_1},\ldots,Z_{h_s}]}{[Z_1,Z_2,\ldots,Z_n]}\qquad \text{by}$$

$$\mathsf{q}{\downarrow}\frac{\mathsf{q}{\downarrow}\dfrac{\mathsf{q}{\downarrow}\dfrac{[\langle[Z_{k_1},\ldots,Z_{k_r}];[Z_{l_1},\ldots,Z_{l_v}]\rangle,Z_{h_1},\ldots,Z_{h_s}]}{[\langle[Z_{k_1},\ldots,Z_{k_r}];Z_{l_1}\rangle,Z_{l_2},\ldots,Z_{l_v},Z_{h_1},\ldots,Z_{h_s}]}}{[\langle Z_{k_1};Z_{l_1}\rangle,Z_{k_2},\ldots,Z_{k_r},Z_{l_2},\ldots,Z_{l_v},Z_{h_1},\ldots,Z_{h_s}]}}{[Z_1,Z_2,\ldots,Z_n]}\quad .$$

Now let $k = k_1$ and $l = l_1$. Then we have

$$\begin{array}{c}(W_1,W_2,\ldots,W_n)\\[2pt]\Delta'\Big\|{}_{\{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}}\\[2pt]\mathsf{q}{\downarrow}\dfrac{[\langle Z_k;Z_l\rangle,Z_{h_1},\ldots,Z_{h_s}]}{[Z_1,Z_2,\ldots,Z_n]}\end{array}\quad ,$$

where $s = n - 2$. There are two subcases.

(a) $k < l$. Then replace inside Δ' all atoms a_m and b_m by $\circ$ for every m with $m \leqslant k$ or $l < m$. The derivation Δ' then becomes

$$\begin{array}{c}(W_{k+1},\ldots,W_l)\\[2pt]\Delta''\Big\|{}_{\{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}}\\[2pt][\langle a_{k+1};b_l\rangle,Z_{k+1},\ldots,Z_{l-1}]\end{array}$$

which is a contradiction to the induction hypothesis.

(b) $k > l$. Then replace inside Δ' all atoms a_m and b_m by $\circ$ for every m with $k \geqslant m > l$. The derivation Δ' then becomes

$$\begin{array}{c}(W_1,\ldots,W_l,W_{k+1},\ldots,W_n)\\[2pt]\Delta''\Big\|{}_{\{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}}\\[2pt][Z_1,\ldots,Z_{l-1},\langle a_{k+1};b_l\rangle,Z_{k+1},\ldots,Z_n]\end{array}\quad ,$$

which is a contradiction to the induction hypothesis.

(iv) $R = \circ$, $T = [Z_{k_1},\ldots,Z_{k_t}]$, $U = [Z_{l_1},\ldots,Z_{l_u}]$ and $V = \circ$ for some $t, u > 0$ and $k_1,\ldots,k_t, l_1,\ldots,l_u \in \{1,\ldots,n\}$. Similar to (iii).

(3) $\rho = \mathsf{s}$. There are three possibilities to apply $\ \mathsf{s}\dfrac{S([R,U],T)}{S[(R,T),U]}\ $.

(i) $R = a_{j+1}$, $T = b_j$ for some $j = 1,\ldots,n$. Then, without loss of generality, we can assume that $j = 1$. Further, $U = [Z_{k_1},\ldots,Z_{k_u}]$ for some $u > 0$ and $k_1,\ldots,k_u \in \{2,\ldots,n\}$. Without loss of generality, assume that $k_1 < k_2 < \ldots < k_u$. Let

$\{2, \ldots, n\} \setminus \{k_1, \ldots, k_u\} = \{h_1, \ldots, h_s\}$, where $s = n - u - 1$. Then there is a derivation Δ' such that

$$
\mathsf{s} \frac{
\begin{array}{c}
(W_1, W_2, \ldots, W_n) \\
\Delta' \,\big\|\, \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\} \\
[([a_2, Z_{k_1}, \ldots, Z_{k_u}], b_1), Z_{h_1}, \ldots, Z_{h_s}]
\end{array}
}{
[(a_2, b_1), Z_2, \ldots, Z_n]
} \quad .
$$

The derivation Δ' remains valid if we replace a_m and b_m by $\circ$ for every m with $1 < m \leqslant k_u$. Then we get

$$
\begin{array}{c}
(W_1, W_{k_u+1}, \ldots, W_n) \\
\Delta'' \,\big\|\, \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\} \\
[(a_{k_u+1}, b_1), Z_{k_u+1}, \ldots, Z_n]
\end{array} \quad ,
$$

which is a contradiction to the induction hypothesis.

(ii) $R = b_j, T = a_{j+1}$ for some $j = 1, \ldots, n$. Similar to (i).

(iii) $R = \circ$, $T = [Z_{k_1}, \ldots, Z_{k_t}]$ and $U = [Z_{l_1}, \ldots, Z_{l_u}]$ for some $t, u > 0$ and $k_1, \ldots, k_t, l_1, \ldots, l_u \in \{1, \ldots, n\}$. Similar to (2.iii). (See also case (iii) in the proof of Proposition 6.2.13 on page 187.) $\qquad \square$

I believe that this property is already sufficient to completely characterise derivations in the core of SBV. In other words, I conjecture that Theorem 6.2.20 can be generalised to the case when seq is present.

In order to state this conjecture precisely, I need to generalise the definitions of Section 6.2. The Definitions 6.2.1, 6.2.2, and 6.2.5 for *occurrences*, *balanced structures*, and *quasi-substructures*, respectively, are easily extended to the case of BV structures.

7.2.7 **Example** Let $R = [(a, [b, c], d), \bar{d}, \langle [\bar{c}, c]; b; a; \bar{a} \rangle]$. Then

$$
\mathsf{occ}\, R = \{a, a, \bar{a}, b, b, c, c, \bar{c}, d, \bar{d}\} \quad .
$$

This means that R is not balanced, because a, b and c occur more than once. The structure $[(a, b), \langle \bar{c}; \bar{a} \rangle]$ is a quasi-substructure (but not a substructure) of R because

$$
[(a, [b, \circ], \circ), \circ, \langle [\bar{c}, \circ]; \circ; \circ; \bar{a} \rangle] = [(a, b), \langle \bar{c}; \bar{a} \rangle] \quad .
$$

The structure $[(a, b), \langle \bar{c}; \bar{a} \rangle]$ is balanced.

On the other hand, the definition of circuits and cocircuits needs more attention in the presence of seq. The following definition generalises Definition 6.2.9 (page 187).

7.2.8 **Definition** Let $n > 0$, let $m = 2n$, and let $a_1, a_2, \ldots, a_m$ be m different atoms. Then the structure $[Z_1, \ldots, Z_n]$ is called a $\langle a_1, a_2, \ldots, a_m \rangle$-*circuit*, if

- $Z_j = (a_{2j}, a_{2j+1})$ or $Z_j = \langle a_{2j}; a_{2j+1} \rangle$, for every $j = 1, \ldots, n-1$, and
- $Z_n = (a_m, a_1)$ or $Z_n = \langle a_m; a_1 \rangle$.

The structure $(W_1, \ldots, W_n)$ is called a $\langle a_1, a_2, \ldots, a_m \rangle$-*cocircuit*, if

- $W_i = [a_{2i-1}, a_{2i}]$ or $W_i = \langle a_{2i-1}; a_{2i} \rangle$, for every $i = 1, \ldots, n$.

A circuit is called a *negation circuit* if we have that $a_2 = \bar{a}_1$, $a_4 = \bar{a}_3$, ..., $a_m = \bar{a}_{m-1}$.

7.2.9   Example   The structure $[(a,c), \langle \bar{b}; \bar{a} \rangle]$ is a $\langle \bar{a}, a, c, \bar{b} \rangle$-circuit and $([\bar{a}, a], [c, \bar{b}])$ is a $\langle \bar{a}, a, c, \bar{b} \rangle$-cocircuit.

The following definition is a repetition of Definition 6.2.11 for BV structures.

7.2.10   Definition   Let R be a BV structure and let $a_1, \ldots, a_m$ be an even number of atoms. The structure R *contains a* $\langle a_1, \ldots, a_m \rangle$-*circuit*, if R has a quasi-substructure R', which is a $\langle a_1, \ldots, a_m \rangle$-circuit. Similarly, R *contains a* $\langle a_1, \ldots, a_m \rangle$-*cocircuit*, if R has a quasi-substructure R', which is a $\langle a_1, \ldots, a_m \rangle$-cocircuit. The structure R *contains a negation circuit*, if R has a quasi-substructure R', which is a negation circuit.

7.2.11   Example   The BV structure $R = ([(a,b), \langle \bar{b}; \bar{c} \rangle, d], c)$ contains (among others) a $\langle \bar{c}, a, b, \bar{b} \rangle$-circuit because
$$([(a,b), \langle \bar{b}; \bar{c} \rangle, \circ], \circ) = [(a,b), \langle \bar{b}; \bar{c} \rangle] \quad ,$$
and a $\langle b, \bar{b} \rangle$-cocircuit because
$$([(\circ, b), \langle \bar{b}; \circ \rangle, \circ], \circ) = [b, \bar{b}] \quad .$$

The definition of the relation $\lessdot$ for BV structures is literally the same as for S° structures. But observe that the notion of circuit has changed.

Now we are able to state the conjecture mentioned before:

7.2.12   Conjecture   *Let R and T be two balanced structures with* $\mathsf{occ}\, R = \mathsf{occ}\, T$. *Then we have,*

$$
\begin{array}{cccccc}
T & & & T & & \\
\big\| \mathsf{SBVc} & & iff & \big\| \mathsf{SBV} & iff & T \lessdot R \quad . \\
R & & & R & &
\end{array}
$$

The implication from the first to the third statement of Conjecture 7.2.12 is an immediate consequence of Lemma 7.2.6. The implication from the second to the first statement is a triviality (because SBVc is a subset of SBV). There is also the possibility of showing implication from the second to the third statement by using Proposition 7.4.30 (to be shown later in this chapter) and Corollary 7.1.14.

The difficulty of proving the other direction of Conjecture 7.2.12 consists in constructing a derivation from the given structures R and T. The method used in [BdGR97] cannot be used because of shape of the switch rule (which ensures cut elimination, see [Ret99b, Gug02e]), and the method that I used in Proposition 6.2.16 can not be used because of the behaviour of the seq (see also [Tiu01]).

Proving Conjecture 7.2.12 would not only give a good understanding of derivability in system SBV [Gug02b], but would also establish the equivalence of system BV and pomset logic [Ret93, Ret97, Ret99b].

Let me now come to the main purpose of Lemma 7.2.6 in this section.

7.2.13   Corollary   *Let $n \geqslant 1$ and $!R_1, \ldots, !R_n, ?T_1, \ldots, ?T_n$ be any NEL structures. Then there is no derivation*

$$
([!R_1, ?T_1], [!R_2, ?T_2], \ldots, [!R_n, ?T_n])
$$
$$
\Delta \big\|_{\{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\}} \qquad\qquad .
$$
$$
[(!R_2, ?T_1), (!R_3, ?T_2), \ldots, (!R_1, ?T_n)]
$$

Proof: By way of contradiction, assume there is such a derivation Δ. The rules s, q↓ and q↑ have no access to the exponentials. Hence, we can replace inside Δ, the substructures $!R_1, \ldots, !R_n, ?T_1, \ldots, ?T_n$ by new atoms $a_1, \ldots, a_n, b_1, \ldots, b_n$, respectively. This gives us a valid derivation

$$
([a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]) \\
\Delta' \big\| {\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}} \\
[(a_2, b_1), (a_3, b_2), \ldots, (a_1, b_n)]
$$

,

which is a contradiction to Lemma 7.2.6. □

7.3 Cut Elimination

In this section I will prove cut elimination for system NEL (Theorem 7.1.6). For this, I will use the second decomposition theorem in the same way as it has been done for system ELS in Section 4.4. To eliminate the core up rules, I will then use the technique of *splitting* combined with *context reduction* (as it has been done for system LS in Section 3.4). The proof of the splitting lemma presented here is much simpler than the one in Section 3.4 because the noncore rules do not need to be considered and there are no additives present.

7.3.1 Splitting

The technique of splitting has been introduced by A. Guglielmi in [Gug02e] for proving cut elimination for system BV. For explaining the idea behind splitting let me consider a purely multiplicative sequent system, for example, multiplicative linear logic. If we have a proof of a sequent $\vdash F\{A \otimes B\}, \Gamma$, where $F\{A \otimes B\}$ is a formula that contains the subformula $A \otimes B$, then we know that somewhere in the proof there is one and only one instance of the $\otimes$ rule, which splits A and B along with their context (under the assumption that identity is reduced to atomic form). In other words, the proof of $\vdash F\{A \otimes B\}, \Gamma$ must be composed of three pieces, Δ, Π_1, and Π_2, as follows:

In the calculus of structures, many different proofs correspond to the sequent calculus one. They differ because of the different possible sequencing of rules, and because rules in the calculus of structures have smaller granularity and larger applicability. But, among all

these proofs, there must also be one that fits the following scheme:

$$
\begin{array}{c}
\Pi'_1 \| \\
[A, \Phi] \\
\Pi'_2 \| \\
\cfrac{([A, \Phi], [B, \Psi])}{[([A, \Phi], B), \Psi]} \mathsf{s} \\
\cfrac{}{[(A, B), \Phi, \Psi]} \mathsf{s} \\
\Delta' \| \\
[F(A, B), \Gamma]
\end{array}
\quad ,
$$

which illustrates the idea behind the splitting technique. More precisely, the derivation Δ' above implements a *context reduction* and a proper *splitting*. These principles can be stated in general as follows:

(1) Context reduction: If $S\{R\}$ is provable, then $S\{\ \}$ can be reduced to the structure $[\{\ \}, P_R]$, such that $[R, P_R]$ is provable. In the example above, $[F\{\ \}, \Gamma]$ is reduced to $[\{\ \}, \Gamma']$, for some Γ', which is not explicitly shown in the derivation above.

(2) Splitting: If $[(R, T), P]$ is provable, then P can be reduced to $[P_R, P_T]$, such that $[R, P_R]$ and $[T, P_T]$ are provable. In the example above, the structure Γ', that was obtained by context reduction, is reduced to $[\Phi, \Psi]$.

Context reduction is proved by splitting, which is then at the core of the matter. The biggest difficulty resides in proving splitting, and this mainly requires to find the right induction measure. This particularly problematic if noncore rules like contraction need to be considered or several different units are present, as it is the case in system LS (see Section 3.4).

In the case of NEL, the induction measure will be based on the size of a structure, which is a natural number and which is, informally speaking, the number of atoms occuring in the structure. It is formally defined as follows.

7.3.1   Definition   The *size* of a NEL structure R, denoted by $\text{size}(R)$, is inductively defined as follows:

$$
\begin{aligned}
\text{size}(\circ) &= 0 \quad , \\
\text{size}(a) &= 1 \quad , \\
\text{size}([R, T]) = \text{size}((R, T)) = \text{size}(\langle R; T \rangle) &= \text{size}(R) + \text{size}(T) \quad , \\
\text{size}(!R) = \text{size}(?R) &= \text{size}(R) \quad .
\end{aligned}
$$

7.3.2   Example   The structure $R = [!(a, \bar{c}, \langle ?\bar{a}; b \rangle), c]$ has $\text{size}(R) = 5$.

7.3.3   Definition   The *size* of a proof $\begin{array}{c}\Pi \| \text{NEL} \\ R\end{array}$ is the pair

$$
\text{size}(\Pi) = \langle \text{size}(R), \text{length}(\Pi) \rangle \quad .
$$

Given two proofs $\Pi \| \mathsf{NEL}$ and $\Pi' \| \mathsf{NEL}$, then define
$R R'$

$$
\mathrm{size}(\Pi) < \mathrm{size}(\Pi') \quad \Longleftrightarrow \quad
\begin{array}{l}
\mathrm{size}(R) < \mathrm{size}(R') \quad \text{or} \\
\mathrm{size}(R) = \mathrm{size}(R') \quad \text{and} \quad \mathrm{length}(\Pi) < \mathrm{length}(\Pi')
\end{array} \quad .
$$

7.3.4 Definition For notational convenience, let me define *system* NELm to be the system obtained from NEL by removing the noncore rules:

$$
\mathsf{NELm} = \mathsf{NEL} \setminus \{\mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}\} = \{\circ{\downarrow}, \mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{p}{\downarrow}\} \quad .
$$

7.3.5 Lemma (Splitting) *Let R, T, P be any NEL structures.*

(a) *If $[(R,T),P]$ is provable in NELm, then there are structures P_R and P_T, such that*

$$
\begin{array}{lclcl}
[P_R, P_T] & & & & \\
\quad \| \mathsf{NELm} & \text{and} & \| \mathsf{NELm} & \text{and} & \| \mathsf{NELm} \\
\quad P & & [R, P_R] & & [T, P_T]
\end{array} \quad .
$$

(b) *If $[\langle R;T\rangle, P]$ is provable in NELm, then there are structures P_R and P_T, such that*

$$
\begin{array}{lclcl}
\langle P_R; P_T\rangle & & & & \\
\quad \| \mathsf{NELm} & \text{and} & \| \mathsf{NELm} & \text{and} & \| \mathsf{NELm} \\
\quad P & & [R, P_R] & & [T, P_T]
\end{array} \quad .
$$

Proof: If $R = \circ$ or $T = \circ$, then the two statements are trivially true. Let us now consider the case $R \neq \circ \neq T$. Both statements are proved simultaneously by induction on the size of the proof of $[(R,T),P]$ or $[\langle R;T\rangle, P]$, respectively. What follows is a case analysis which is conceptually similar to the cut elimination proof in the sequent calculus.

(a) Consider the bottommost rule instance ρ in the proof $\quad \Pi \| \mathsf{NELm}$. We can assume
$[(R,T),P]$
that the application of ρ is nontrivial. We can distinguish between three conceptually different cases:

   (1) The redex of ρ is inside R, T or P.

   (2) The substructure (R,T) is inside a passive structure of the redex of ρ.

   (3) The substructure (R,T) is inside an active structure of the redex of ρ.

Compared with a cut elimination proof in the sequent calculus (see Section 2.3), case (2) is similar to a commutative case and case (3) is similar to a key case. There is no counterpart to case (1) in the sequent calculus because there is no possibility of deep inference.

   (1) The redex of ρ is inside R, T or P. There are three subcases.

      (i) The redex of ρ is inside R. Then the proof Π has the shape

$$
\rho \frac{\begin{array}{c}\Pi' \| \mathsf{NELm} \\ [(R',T),P]\end{array}}{[(R,T),P]} \quad .
$$

By applying the induction hypothesis to Π', we get

$$
\begin{array}{c} [P_{R'}, P_T] \\ \Delta_P \| \mathsf{NELm} \\ P \end{array}
\qquad \text{and} \qquad
\begin{array}{c} \Pi'_R \| \mathsf{NELm} \\ [R', P_{R'}] \end{array}
\qquad \text{and} \qquad
\begin{array}{c} \Pi_T \| \mathsf{NELm} \\ [T, P_T] \end{array} \quad .
$$

Let $P_R = P_{R'}$. From Π'_R, we can get

$$
\begin{array}{c} \Pi'_R \| \mathsf{NELm} \\ [R', P_{R'}] \\ \rho \, \overline{[R, P_{R'}]} \end{array} \quad .
$$

(ii) The redex of ρ is inside T. Analogous to the previous case.

(iii) The redex of ρ is inside P. Then the proof Π has the shape

$$
\begin{array}{c} \Pi' \| \mathsf{NELm} \\ [(R, T), P'] \\ \rho \, \overline{[(R, T), \boxed{P}]} \end{array} \quad .
$$

By applying the induction hypothesis to Π', we get

$$
\begin{array}{c} [P_R, P_T] \\ \Delta'_P \| \mathsf{NELm} \\ P' \end{array}
\qquad \text{and} \qquad
\begin{array}{c} \Pi_R \| \mathsf{NELm} \\ [R, P_R] \end{array}
\qquad \text{and} \qquad
\begin{array}{c} \Pi_T \| \mathsf{NELm} \\ [T, P_T] \end{array} \quad .
$$

From Δ'_P, we can get

$$
\begin{array}{c} [P_R, P_T] \\ \Delta'_P \| \mathsf{NELm} \\ P' \\ \rho \, \overline{P} \end{array} \quad .
$$

(2) The substructure (R, T) is inside a passive structure of the redex of ρ. This is only possible if $\rho = \mathsf{s}$ or $\rho = \mathsf{q}\!\downarrow$. There are again three subcases (there are more than two because seq is noncommutative):

(i) $\rho = \mathsf{s}$ and $P = [(P_1, P_2), P_3, P_4]$ and Π is

$$
\begin{array}{c} \Pi' \| \mathsf{NELm} \\ [([(R, T), P_1, P_3], P_2), P_4] \\ \mathsf{s} \, \overline{[(R, T), (P_1, P_2), P_3, P_4]} \end{array} \quad .
$$

By applying the induction hypothesis to Π', we get

$$
\begin{array}{c} [Q_1, Q_2] \\ \Delta_1 \| \mathsf{NELm} \\ P_4 \end{array}
\qquad \text{and} \qquad
\begin{array}{c} \Pi_1 \| \mathsf{NELm} \\ [(R, T), P_1, P_3, Q_1] \end{array}
\qquad \text{and} \qquad
\begin{array}{c} \Pi_2 \| \mathsf{NELm} \\ [P_2, Q_2] \end{array} \quad .
$$

We have that $\text{size}\big([(R, T), P_1, P_3, Q_1]\big) < \text{size}\big([(R, T), (P_1, P_2), P_3, P_4]\big)$ because $\text{size}(P_2) > 0$ (because the instance of s is nontrivial) and $\text{size}(Q_1) \leqslant$

size(P_4). Therefore, we can apply the induction hypothesis again to Π_1. We get

$$
\begin{array}{ccc}
[P_R, P_T] & & \\
\Delta_2 \Big\| \mathsf{NELm} \qquad \text{and} & \Pi_R \Big\| \mathsf{NELm} \qquad \text{and} & \Pi_T \Big\| \mathsf{NELm} \qquad . \\
[P_1, P_3, Q_1] & [R, P_R] & [T, P_T]
\end{array}
$$

We can now build

$$
\begin{array}{c}
[P_R, P_T] \\
\Delta_2 \Big\| \mathsf{NELm} \\
[P_1, P_3, Q_1] \\
\Pi_2 \Big\| \mathsf{NELm} \\
\mathsf{s} \dfrac{[(P_1, [P_2, Q_2]), P_3, Q_1]}{[(P_1, P_2), P_3, Q_1, Q_2]} \\
\Delta_1 \Big\| \mathsf{NELm} \\
[(P_1, P_2), P_3, P_4]
\end{array} \quad .
$$

(ii) $\rho = \mathsf{q}{\downarrow}$ and $P = [\langle P_1; P_2 \rangle, P_3, P_4]$ and Π is

$$
\Pi' \Big\| \mathsf{NELm} \\
\mathsf{q}{\downarrow} \dfrac{[\langle [(R, T), P_1, P_3]; P_2 \rangle, P_4]}{[(R, T), \langle P_1; P_2 \rangle, P_3, P_4]} \quad .
$$

By applying the induction hypothesis to Π', we get

$$
\begin{array}{ccc}
\langle Q_1; Q_2 \rangle & & \\
\Delta_1 \Big\| \mathsf{NELm} \qquad \text{and} & \Pi_1 \Big\| \mathsf{NELm} \qquad \text{and} & \Pi_2 \Big\| \mathsf{NELm} \qquad . \\
P_4 & [(R, T), P_1, P_3, Q_1] & [P_2, Q_2]
\end{array}
$$

As before, we can apply the induction hypothesis again to Π_1 (because the instance of $\mathsf{q}{\downarrow}$ is nontrivial, which means that size$(P_2) > 0$). We get

$$
\begin{array}{ccc}
[P_R, P_T] & & \\
\Delta_2 \Big\| \mathsf{NELm} \qquad \text{and} & \Pi_R \Big\| \mathsf{NELm} \qquad \text{and} & \Pi_T \Big\| \mathsf{NELm} \qquad . \\
[P_1, P_3, Q_1] & [R, P_R] & [T, P_T]
\end{array}
$$

We can now build

$$
\begin{array}{c}
[P_R, P_T] \\
\Delta_2 \Big\| \mathsf{NELm} \\
[P_1, P_3, Q_1] \\
\Pi_2 \Big\| \mathsf{NELm} \\
\mathsf{q}{\downarrow} \dfrac{[\langle [P_1, Q_1]; [P_2, Q_2] \rangle, P_3]}{[\langle P_1; P_2 \rangle, P_3, \langle Q_1; Q_2 \rangle]} \\
\Delta_1 \Big\| \mathsf{NELm} \\
[\langle P_1; P_2 \rangle, P_3, P_4]
\end{array} \quad .
$$

(iii) $\rho = \mathsf{q}{\downarrow}$ and $P = [\langle P_1; P_2 \rangle, P_3, P_4]$ and Π is

$$
\Pi' \Big\| \mathsf{NELm} \\
\mathsf{q}{\downarrow} \dfrac{[\langle P_1; [(R, T), P_2, P_3] \rangle, P_4]}{[(R, T), \langle P_1; P_2 \rangle, P_3, P_4]} \quad .
$$

Analogous to the previous case.

(3) The substructure (R, T) is inside an active structure of the redex of ρ. In this case we have only one possibility:

   (i) $\rho = \mathsf{s}$, $R = (R_1, R_2)$, $T = (T_1, T_2)$, $P = [P_1, P_2]$ and Π is

$$\mathsf{s} \; \frac{\Pi' \| \mathsf{NELm} \atop [([(R_1, T_1), P_1], R_2, T_2), P_2]}{[(R_1, R_2, T_1, T_2), P_1, P_2]} \quad .$$

By applying the induction hypothesis to Π', we get

$$\begin{array}{c} [Q_1, Q_2] \\ \Delta_1 \| \mathsf{NELm} \\ P_2 \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_1 \| \mathsf{NELm} \\ [(R_1, T_1), P_1, Q_1] \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_2 \| \mathsf{NELm} \\ [(R_2, T_2), Q_2] \end{array} \quad .$$

Since the instance of s is nontrivial, we have that at least one of R_2 and T_2 is not $\circ$. Hence, $\mathrm{size}\big([(R_1, T_1), P_1, Q_1]\big) < \mathrm{size}\big([(R_1, R_2, T_1, T_2), P_1, P_2]\big)$. We also have $P_1 \neq \circ$ and $\mathrm{size}\big([(R_2, T_2), Q_2]\big) < \mathrm{size}\big([(R_1, R_2, T_1, T_2), P_1, P_2]\big)$. This means, we can apply the induction hypothesis to Π_1 and Π_2 and get

$$\begin{array}{c} [P_{R_1}, P_{T_1}] \\ \Delta_3 \| \mathsf{NELm} \\ [P_1, Q_1] \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_{R_1} \| \mathsf{NELm} \\ [R_1, P_{R_1}] \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_{T_1} \| \mathsf{NELm} \\ [T_1, P_{T_1}] \end{array} \quad \text{and}$$

$$\begin{array}{c} [P_{R_2}, P_{T_2}] \\ \Delta_4 \| \mathsf{NELm} \\ Q_2 \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_{R_2} \| \mathsf{NELm} \\ [R_2, P_{R_2}] \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_{T_2} \| \mathsf{NELm} \\ [T_2, P_{T_2}] \end{array} \quad .$$

Now let $P_R = [P_{R_1}, P_{R_2}]$ and $P_T = [P_{T_1}, P_{T_2}]$. We can build

$$\begin{array}{c} [P_{R_1}, P_{R_2}, P_{T_1}, P_{T_2}] \\ \Delta_4 \| \mathsf{NELm} \\ [P_{R_1}, P_{T_1}, Q_2] \\ \Delta_3 \| \mathsf{NELm} \\ [P_1, Q_1, Q_2] \\ \Delta_1 \| \mathsf{NELm} \\ [P_1, P_2] \end{array} \qquad \text{and}$$

$$\begin{array}{c} \Pi_{R_1} \| \mathsf{NELm} \\ [R_1, P_{R_1}] \\ \Pi_{R_2} \| \mathsf{NELm} \\ \mathsf{s} \; \dfrac{[(R_1, [R_2, P_{R_2}]), P_{R_1}]}{[(R_1, R_2), P_{R_1}, P_{R_2}]} \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_{T_1} \| \mathsf{NELm} \\ [T_1, P_{T_1}] \\ \Pi_{T_2} \| \mathsf{NELm} \\ \mathsf{s} \; \dfrac{[(T_1, [T_2, P_{T_2}]), P_{T_1}]}{[(T_1, T_2), P_{T_1}, P_{T_2}]} \end{array} \quad .$$

(b) This is very similar to (a). Consider the bottommost rule instance ρ in the proof $\begin{array}{c}\Pi \| \mathsf{NELm} \\ [\langle R; T \rangle, P]\end{array}$. We can assume that the application of ρ is nontrivial. Again, we distinguish between three cases:

(1) The redex of ρ is inside R, T or P. Analogous to (a.1).

(2) The substructure $\langle R;T \rangle$ is inside a passive structure of the redex of ρ. This case is the same as (a.2), with the only difference that the derivation

$$
\begin{array}{c}
[P_R, P_T] \\
\Delta_2 \| \mathsf{NELm} \\
[P_1, P_3, Q_1]
\end{array}
\quad \text{is replaced by} \quad
\begin{array}{c}
\langle P_R; P_T \rangle \\
\Delta_2 \| \mathsf{NELm} \\
[P_1, P_3, Q_1]
\end{array}
\quad .
$$

(3) The substructure $\langle R;T \rangle$ is inside an active structure of the redex of ρ. This case is also similar to (a.3). But here we have that $\rho = \mathsf{q}{\downarrow}$ and there are two possibilities:

(i) $R = \langle R_1; R_2 \rangle$, $P = [\langle P_1; P_2 \rangle, P_3]$ and Π is

$$
\mathsf{q}{\downarrow} \frac{
\begin{array}{c}
\Pi' \| \mathsf{NELm} \\
[\langle [R_1, P_1]; [\langle R_2; T \rangle, P_2] \rangle, P_3]
\end{array}
}{
[\langle R_1; R_2; T \rangle, \langle P_1; P_2 \rangle, P_3]
} \quad .
$$

By applying the induction hypothesis to Π', we get

$$
\begin{array}{c}
\langle Q_1; Q_2 \rangle \\
\Delta_1 \| \mathsf{NELm} \\
P_3
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\Pi_1 \| \mathsf{NELm} \\
[R_1, P_1, Q_1]
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\Pi_2 \| \mathsf{NELm} \\
[\langle R_2; T \rangle, P_2, Q_2]
\end{array}
\quad .
$$

Since the instance of $\mathsf{q}{\downarrow}$ is nontrivial (and therefore at least one of R_1 and P_1 is not $\circ$), we can apply the induction hypothesis again to Π_2 and get

$$
\begin{array}{c}
\langle P_{R_2}; P_T \rangle \\
\Delta_2 \| \mathsf{NELm} \\
[P_2, Q_2]
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\Pi_{R_2} \| \mathsf{NELm} \\
[R_2, P_{R_2}]
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\Pi_T \| \mathsf{NELm} \\
[T, P_T]
\end{array}
\quad .
$$

Now let $P_R = \langle [P_1, Q_1]; P_{R_2} \rangle$. We can build

$$
\mathsf{q}{\downarrow} \frac{
\begin{array}{c}
\langle [P_1, Q_1]; P_{R_2}; P_T \rangle \\
\Delta_2 \| \mathsf{NELm} \\
\langle [P_1, Q_1]; [P_2, Q_2] \rangle
\end{array}
}{
\begin{array}{c}
[\langle P_1; P_2 \rangle, \langle Q_1; Q_2 \rangle] \\
\Delta_1 \| \mathsf{NELm} \\
[\langle P_1; P_2 \rangle, P_3]
\end{array}
}
\quad \text{and} \quad
\mathsf{q}{\downarrow} \frac{
\begin{array}{c}
\Pi_1 \| \mathsf{NELm} \\
[R_1, P_1, Q_1] \\
\Pi_{R_2} \| \mathsf{NELm} \\
\langle [R_1, P_1, Q_1]; [R_2, P_{R_2}] \rangle
\end{array}
}{
[\langle R_1; R_2 \rangle, \langle [P_1, Q_1]; P_{R_2} \rangle]
}
\quad .
$$

(ii) $T = \langle T_1; T_2 \rangle$, $P = [\langle P_1; P_2 \rangle, P_3]$ and Π is

$$
\mathsf{q}{\downarrow} \frac{
\begin{array}{c}
\Pi' \| \mathsf{NELm} \\
[\langle [\langle R; T_1 \rangle, P_1]; [T_2, P_2] \rangle, P_3]
\end{array}
}{
[\langle R; T_1; T_2 \rangle, \langle P_1; P_2 \rangle, P_3]
} \quad .
$$

Similar to (i). $\qquad \qquad \square$

7.3.6   **Lemma (Splitting for Exponentials)**   *Let R and P be any* NEL *structures.*

(a) *If $[!R, P]$ is provable in* NELm*, then there are structures $P_{R1}, \ldots, P_{Rh}$ for some $h \geqslant 0$, such that*

$$\begin{array}{ccc} [?P_{R1}, \ldots, ?P_{Rh}] \\ \parallel \text{NELm} \\ P \end{array} \qquad and \qquad \begin{array}{c} \top \text{NELm} \\ [R, P_{R1}, \ldots, P_{Rh}] \end{array} \quad .$$

(b) *If $[?R, P]$ is provable in* NELm*, then there is a structure P_R, such that*

$$\begin{array}{ccc} !P_R \\ \parallel \text{NELm} \\ P \end{array} \qquad and \qquad \begin{array}{c} \top \text{NELm} \\ [R, P_R] \end{array} \quad .$$

Proof: In the case that $R = \circ$, both statements are trivially true. Let us now assume that $R \neq \circ$. I will proceed by induction on the same measure as in the previous proof.

(a) Consider the bottommost rule instance ρ in the proof $\begin{array}{c} \Pi \top \text{NELm} \\ [!R, P] \end{array}$. We can assume that the application of ρ is nontrivial. There are again the same three cases:

   (1) The redex of ρ is inside R or P. Analogous to (a.1) in the previous proof.

   (2) The substructure $!R$ is inside a passive structure of the redex of ρ. This case is the same as (a.2) in the previous proof, but this time we have

$$\begin{array}{ccc} [?P_{R1}, \ldots, ?P_{Rh}] & & [P_R, P_T] \\ \Delta_2 \parallel \text{NELm} & \text{instead of} & \Delta_2 \parallel \text{NELm} \\ [P_1, P_3, Q_1] & & [P_1, P_3, Q_1] \end{array} \quad .$$

   (3) The substructure $!R$ is inside an active structure of the redex of ρ. There is only one possibility

     (i) $\rho = \mathsf{p}{\downarrow}$, $P = [?P_1, P_2]$ and Π is

$$\mathsf{p}{\downarrow} \frac{\begin{array}{c} \Pi' \top \text{NELm} \\ [![R, P_1], P_2] \end{array}}{[!R, ?P_1, P_2]} \quad .$$

     By applying the induction hypothesis to Π', we get structures $?P_{R2}, \ldots, ?P_{Rh}$ such that

$$\begin{array}{ccc} [?P_{R2}, \ldots, ?P_{Rh}] \\ \parallel \text{NELm} \\ P_2 \end{array} \qquad and \qquad \begin{array}{c} \top \text{NELm} \\ [R, P_1, P_{R2}, \ldots, P_{Rh}] \end{array} \quad .$$

     Now let $P_{R1} = P_1$. We immediately get

$$\begin{array}{c} [?P_1, ?P_{R2}, \ldots, ?P_{Rh}] \\ \parallel \text{NELm} \\ [?P_1, P_2] \end{array} \quad .$$

(b) Consider the bottommost rule instance ρ in the proof $\Pi \| \mathsf{NELm}$ over $[?R, P]$. We can assume that the application of ρ is nontrivial. There are again the same three cases:

(1) The redex of ρ is inside R or P. Analogous to (a.1) in the previous proof.

(2) The substructure $?R$ is inside a passive structure of the redex of ρ. This case is the same as (a.2) in the previous proof, but this time we have

$$
\begin{array}{c} !P_R \\ \Delta_2 \| \mathsf{NELm} \\ [P_1, P_3, Q_1] \end{array}
\qquad \text{instead of} \qquad
\begin{array}{c} [P_R, P_T] \\ \Delta_2 \| \mathsf{NELm} \\ [P_1, P_3, Q_1] \end{array} \quad .
$$

(3) The substructure $?R$ is inside an active structure of the redex of ρ. There is only one possibility

   (i) $\rho = \mathsf{p}{\downarrow}$, $P = [!P_1, P_2]$ and Π is

$$
\mathsf{p}{\downarrow} \frac{\Pi' \| \mathsf{NELm}}{[![R, P_1], P_2]}{[?R, !P_1, P_2]} \quad .
$$

By applying part (a) to Π', we get

$$
\begin{array}{c} [?Q_1, \dots, ?Q_h] \\ \| \mathsf{NELm} \\ P_2 \end{array}
\qquad \text{and} \qquad
\begin{array}{c} \| \mathsf{NELm} \\ [R, P_1, Q_1, \dots, Q_h] \end{array} \quad .
$$

Now let $P_R = [P_1, Q_1, \dots, Q_h]$. We can build:

$$
\begin{array}{c} ![P_1, Q_1, \dots, Q_h] \\ \| \{\mathsf{p}{\downarrow}\} \\ [!P_1, ?Q_1, \dots, ?Q_h] \\ \| \mathsf{NELm} \\ [!P_1, P_2] \end{array} \quad . \qquad \Box
$$

7.3.7 Lemma (Splitting for Atoms) *Let a be any atom and P be any* NEL *structure.*

If there is a proof $\begin{array}{c} \| \mathsf{NELm} \\ [a, P] \end{array}$ *then there is a derivation* $\begin{array}{c} \bar{a} \\ \| \mathsf{NELm} \\ P \end{array}$.

Proof: This is very similar to the previous two proofs. Consider the bottommost rule instance ρ in the proof $\begin{array}{c} \Pi \| \mathsf{NELm} \\ [a, P] \end{array}$, where the application of ρ is nontrivial. There are again the same three cases:

(1) The redex of ρ is inside P. As before.

(2) The atom a is inside a passive structure of the redex of ρ. As before, but this time we have

$$\begin{array}{c} \bar{a} \\ \Delta_2 \| \mathsf{NELm} \\ [P_1, P_3, Q_1] \end{array} \qquad \text{instead of} \qquad \begin{array}{c} [P_R, P_T] \\ \Delta_2 \| \mathsf{NELm} \\ [P_1, P_3, Q_1] \end{array} \quad .$$

(3) The atom a is inside an active structure of the redex of ρ. There is only one possibility: $\rho = \mathsf{ai}{\downarrow}$, $P = [\bar{a}, P_1]$ and Π is

$$\mathsf{ai}{\downarrow} \, \dfrac{\begin{array}{c} \Pi' \| \mathsf{NELm} \\ P_1 \end{array}}{[a, \bar{a}, P_1]} \quad .$$

Then, we immediately get $\begin{array}{c} \bar{a} \\ \| \mathsf{NELm} \\ [\bar{a}, P_1] \end{array}$. $\square$

7.3.2 Context Reduction

The idea of context reduction is to reduce a problem that concerns an arbitrary (deep) context $S\{\ \}$ to a problem that concerns only a shallow context $[\{\ \}, P]$. In the case of cut elimination, for example, we will then be able to apply splitting.

7.3.8 Lemma (Context Reduction) *Let R be a* NEL *structure and $S\{\ \}$ be a context. If $S\{R\}$ is provable in* NELm*, then there is a structure P_R, such that $[R, P_R]$ is provable in* NELm *and such that for every structure X, we have*

$$\begin{array}{c} [X, P_R] \\ \| \mathsf{NELm} \\ S\{X\} \end{array} \qquad or \qquad \begin{array}{c} ![X, P_R] \\ \| \mathsf{NELm} \\ S\{X\} \end{array} \quad .$$

Proof: This proof will be carried out by induction on the context $S\{\ \}$.

(1) $S\{\ \} = \{\ \}$. Then the lemma is trivially true for $P_R = \circ$.

(2) $S\{\ \} = [S'\{\ \}, P]$ for some P, such that $S\{R\}$ is not a proper par.

   (i) $S'\{\ \} = \{\ \}$. Then the lemma is trivially true for $P_R = P$.

   (ii) $S'\{\ \} = (S''\{\ \}, T)$ for some context $S''\{\ \}$ and structure $T \neq \circ$. Then we can apply splitting (Lemma 7.3.5) to the proof of $[(S''\{R\}, T), P]$ and get:

$$\begin{array}{c} [P_S, P_T] \\ \Delta_P \| \mathsf{NELm} \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_S \| \mathsf{NELm} \\ [S''\{R\}, P_S] \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_T \| \mathsf{NELm} \\ [T, P_T] \end{array} \quad .$$

   By applying the induction hypothesis we get P_R such that $\begin{array}{c} \| \mathsf{NELm} \\ [R, P_R] \end{array}$ and for every X

$$\begin{array}{c} [X, P_R] \\ \| \mathsf{NELm} \\ [S''\{X\}, P_S] \end{array} \qquad or \qquad \begin{array}{c} ![X, P_R] \\ \| \mathsf{NELm} \\ [S''\{X\}, P_S] \end{array} \quad .$$

From this we can build:

$$
\mathsf{s}\frac{
\begin{array}{c}
[X, P_R] \\
\Big\|\mathsf{NELm} \\
[S''\{X\}, P_S] \\
\Pi_T \Big\|\mathsf{NELm} \\
[(S''\{X\}, [T, P_T]), P_S]
\end{array}
}{
\begin{array}{c}
[(S''\{X\}, T), P_S, P_T] \\
\Delta_P \Big\|\mathsf{NELm} \\
[(S''\{X\}, T), P]
\end{array}
}
\quad \text{or} \quad
\mathsf{s}\frac{
\begin{array}{c}
![X, P_R] \\
\Big\|\mathsf{NELm} \\
[S''\{X\}, P_S] \\
\Pi_T \Big\|\mathsf{NELm} \\
[(S''\{X\}, [T, P_T]), P_S]
\end{array}
}{
\begin{array}{c}
[(S''\{X\}, T), P_S, P_T] \\
\Delta_P \Big\|\mathsf{NELm} \\
[(S''\{X\}, T), P]
\end{array}
} \quad .
$$

(iii) $S'\{\ \} = \langle S''\{\ \}; T \rangle$ for some context $S''\{\ \}$ and structure $T \neq \circ$. Then we can apply splitting (Lemma 7.3.5) to the proof of $[\langle S''\{R\}; T \rangle, P]$ and get:

$$
\begin{array}{c}
\langle P_S; P_T \rangle \\
\Delta_P \Big\|\mathsf{NELm} \\
P
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\Pi_S \Big\|\mathsf{NELm} \\
[S''\{R\}, P_S]
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\Pi_T \Big\|\mathsf{NELm} \\
[T, P_T]
\end{array}
\quad .
$$

By applying the induction hypothesis we get P_R such that $\begin{array}{c}\Big\|\mathsf{NELm}\\ [R, P_R]\end{array}$ and for every X

$$
\begin{array}{c}
[X, P_R] \\
\Big\|\mathsf{NELm} \\
[S''\{X\}, P_S]
\end{array}
\quad \text{or} \quad
\begin{array}{c}
![X, P_R] \\
\Big\|\mathsf{NELm} \\
[S''\{X\}, P_S]
\end{array}
\quad .
$$

From this we can build:

$$
\mathsf{q}\!\downarrow\frac{
\begin{array}{c}
[X, P_R] \\
\Big\|\mathsf{NELm} \\
[S''\{X\}, P_S] \\
\Pi_T \Big\|\mathsf{NELm} \\
\langle [S''\{X\}, P_S]; [T, P_T] \rangle
\end{array}
}{
\begin{array}{c}
[\langle S''\{X\}; T \rangle, \langle P_S; P_T \rangle] \\
\Delta_P \Big\|\mathsf{NELm} \\
[\langle S''\{X\}; T \rangle, P]
\end{array}
}
\quad \text{or} \quad
\mathsf{q}\!\downarrow\frac{
\begin{array}{c}
![X, P_R] \\
\Big\|\mathsf{NELm} \\
[S''\{X\}, P_S] \\
\Pi_T \Big\|\mathsf{NELm} \\
\langle [S''\{X\}, P_S]; [T, P_T] \rangle
\end{array}
}{
\begin{array}{c}
[\langle S''\{X\}; T \rangle, \langle P_S; P_T \rangle] \\
\Delta_P \Big\|\mathsf{NELm} \\
[\langle S''\{X\}; T \rangle, P]
\end{array}
} \quad .
$$

(iv) $S'\{\ \} = !S''\{\ \}$ for some context $S''\{\ \}$. Then we can apply splitting for the exponentials (Lemma 7.3.6) to the proof of $[!S''\{R\}, P]$ and get:

$$
\begin{array}{c}
[?P_1, \ldots, ?P_h] \\
\Delta_P \Big\|\mathsf{NELm} \\
P
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\Pi_S \Big\|\mathsf{NELm} \\
[S''\{R\}, P_1, \ldots, P_h]
\end{array}
\quad .
$$

By applying the induction hypothesis to Π_S we get P_R such that $\begin{array}{c}\Big\|\mathsf{NELm}\\ [R, P_R]\end{array}$ and for every X

$$
\begin{array}{c}
[X, P_R] \\
\Big\|\mathsf{NELm} \\
[S''\{X\}, P_1, \ldots, P_h]
\end{array}
\quad \text{or} \quad
\begin{array}{c}
![X, P_R] \\
\Big\|\mathsf{NELm} \\
[S''\{X\}, P_1, \ldots, P_h]
\end{array}
\quad .
$$

From this we can build:

$$
\begin{array}{c}
![X, P_R] \\
\|\text{NELm} \\
![S''\{X\}, P_1, \ldots, P_h] \\
\|\{\mathsf{p}\downarrow\} \\
[!S''\{X\}, ?P_1, \ldots, ?P_h] \\
\Delta_P\|\text{NELm} \\
[!S''\{X\}, P]
\end{array} \quad .
$$

(v) $S'\{\ \} = ?S''\{\ \}$ for some context $S''\{\ \}$. Then we can apply splitting for the exponentials (Lemma 7.3.6) to the proof of $[?S''\{R\}, P]$ and get:

$$
\begin{array}{c}
!P_S \\
\Delta_P\|\text{NELm} \\
P
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Pi_S\|\text{NELm} \\
[S''\{R\}, P_S]
\end{array} \quad .
$$

By applying the induction hypothesis to Π_S we get P_R such that $\begin{array}{c}\|\text{NELm}\\ [R, P_R]\end{array}$ and

for every X

$$
\begin{array}{c}
[X, P_R] \\
\|\text{NELm} \\
[S''\{X\}, P_S]
\end{array}
\quad \text{or} \quad
\begin{array}{c}
![X, P_R] \\
\|\text{NELm} \\
[S''\{X\}, P_S]
\end{array} \quad .
$$

From this we can build:

$$
\begin{array}{c}
![X, P_R] \\
\|\text{NELm} \\
\mathsf{p}\downarrow \dfrac{[?S''\{X\}, !P_S]}{![S''\{X\}, P_S]} \\
\Delta_P\|\text{NELm} \\
[?S''\{X\}, P]
\end{array} \quad .
$$

(3) $S\{\ \} = (S'\{\ \}, T)$ for some context $S'\{\ \}$ and $T \neq \circ$. This is a special case of (2.ii) for $P = \circ$.

(4) $S\{\ \} = \langle S'\{\ \}; T \rangle$ for some context $S'\{\ \}$ and $T \neq \circ$. This is a special case of (2.iii) for $P = \circ$.

(5) $S\{\ \} = !S'\{\ \}$ for some context $S'\{\ \}$. This is a special case of (2.iv) for $P = \circ$.

(6) $S\{\ \} = ?S'\{\ \}$ for some context $S'\{\ \}$. This is a special case of (2.v) for $P = \circ$. $\qquad \square$

7.3.3 Elimination of the Up Fragment

In this section, I will first show three Lemmata, which are all easy consequences of splitting and which say that the core up rules of system SNEL are admissible if they are applied in a shallow context $[\{\ \}, P]$. Then I will show how context reduction is used to extend these Lemmata to any context. As a result we get a modular proof of cut elimination because the three core up rules $\mathsf{ai}\uparrow$, $\mathsf{q}\uparrow$, and $\mathsf{p}\uparrow$ are shown to be admissible independently from each other.

7.3.9 Lemma *Let P be a structure and let a be an atom. If $[(a, \bar{a}), P]$ is provable in* NELm, *then P is also provable in* NELm.

Proof: Apply splitting to the proof of $[(a, \bar{a}), P]$. This yields:

$$
\begin{array}{ccccc}
[P_a, P_{\bar{a}}] & & & & \\
\ \|\text{NELm} & \text{and} & \ \|\!\|\text{NELm} & \text{and} & \ \|\!\|\text{NELm} \quad . \\
P & & [a, P_a] & & [\bar{a}, P_{\bar{a}}]
\end{array}
$$

By applying Lemma 7.3.7, we get

$$
\begin{array}{ccc}
\bar{a} & & a \\
\ \|\text{NELm} & \text{and} & \ \|\text{NELm} \quad . \\
P_a & & P_{\bar{a}}
\end{array}
$$

From this we can build:

$$
\begin{array}{c}
\circ\!\downarrow \dfrac{\ \overline{\quad}\ }{\circ} \\[4pt]
\text{ai}\!\downarrow \dfrac{}{[\bar{a}, a]} \\[4pt]
\ \|\text{NELm} \\
[P_a, P_{\bar{a}}] \\
\ \|\text{NELm} \\
P
\end{array} \quad .
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7.3.10 Lemma *Let R, T, U, V and P be any* NEL *structures. If $[(\langle R; U \rangle, \langle T; V \rangle), P]$ is provable in* NELm, *then $[\langle (R, T); (U, V) \rangle, P]$ is also provable in* NELm.

Proof: By applying splitting to the proof of $[(\langle R; U \rangle, \langle T; V \rangle), P]$, we get two structures P_1 and P_2 such that:

$$
\begin{array}{ccccc}
[P_1, P_2] & & & & \\
\ \|\text{NELm} & \text{and} & \ \|\!\|\text{NELm} & \text{and} & \ \|\!\|\text{NELm} \quad . \\
P & & [\langle R; U \rangle, P_1] & & [\langle T; V \rangle, P_2]
\end{array}
$$

By applying splitting again, we get P_R, P_T, P_U and P_V such that

$$
\begin{array}{ccccccc}
\langle P_R; P_U \rangle & & & & & & \\
\ \|\text{NELm} & \text{and} & \ \|\!\|\text{NELm} & \text{and} & \ \|\!\|\text{NELm} & \text{and} & \\
P_1 & & [R, P_R] & & [U, P_U] & &
\end{array}
$$

$$
\begin{array}{ccccc}
\langle P_T; P_V \rangle & & & & \\
\ \|\text{NELm} & \text{and} & \ \|\!\|\text{NELm} & \text{and} & \ \|\!\|\text{NELm} \quad . \\
P_2 & & [T, P_T] & & [V, P_V]
\end{array}
$$

From this we can build:

$$
\begin{array}{c}
\displaystyle\parallel\!\parallel\text{NELm} \\
\langle([R,P_R],[T,P_T]);([U,P_U],[V,P_V])\rangle \\
\parallel\!\parallel\{\text{s}\} \\
\langle[(R,T),P_R,P_T];[(U,V),P_U,P_V]\rangle \\
\parallel\!\parallel\{\text{q}\downarrow\} \\
[\langle(R,T);(U,V)\rangle,\langle P_R,P_U\rangle,\langle P_T,P_V\rangle] \\
\parallel\!\parallel\text{NELm} \\
[\langle(R,T);(U,V)\rangle,P_1,P_2]\quad. \\
\parallel\!\parallel\text{NELm} \\
[\langle(R,T);(U,V)\rangle,P]
\end{array}
\qquad\qquad\square
$$

7.3.11   Lemma   *Let R,T and P be any* NEL *structures. If $[(?R,!T),P]$ is provable in* NELm*, then $[?(R,T),P]$ is also provable in* NELm*.*

Proof: By applying splitting to the proof of $[(?R,!T),P]$, we get two structures P_1 and P_2 such that:

$$
\begin{array}{ccccc}
[P_1,P_2] & & & & \\
\parallel\text{NELm} & \text{and} & \parallel\!\parallel\text{NELm} & \text{and} & \parallel\!\parallel\text{NELm}\quad. \\
P & & [?R,P_1] & & [!T,P_2]
\end{array}
$$

By applying splitting again, we get P_R, $P_{T1},\dots,P_{Tn}$, such that

$$
\begin{array}{ccccc}
!P_R & & & & \\
\parallel\text{NELm} & \text{and} & \parallel\!\parallel\text{NELm} & \text{and} & \\
P_1 & & [R,P_R] & &
\end{array}
$$

$$
\begin{array}{ccccc}
[?P_{T1},\dots,?P_{Tn}] & & & & \\
\parallel\text{NELm} & & \text{and} & & \parallel\!\parallel\text{NELm}\quad. \\
P_2 & & & & [T,P_{T1},\dots,P_{Tn}]
\end{array}
$$

From this we can build:

$$
\begin{array}{c}
\parallel\!\parallel\text{NELm} \\
!([R,P_R],[T,P_{T1},\dots,P_{Tn}]) \\
\parallel\!\parallel\{\text{s}\} \\
![(R,T),P_R,P_{T1},\dots,P_{Tn}] \\
\parallel\!\parallel\{\text{p}\downarrow\} \\
[?(R,T),!P_R,?P_{T1},\dots,?P_{Tn}] \\
\parallel\!\parallel\text{NELm} \\
[?(R,T),P_1,P_2]\quad. \\
\parallel\!\parallel\text{NELm} \\
[?(R,T),P]
\end{array}
\qquad\qquad\square
$$

By the use of context reduction, we can extent the statements of Lemmata 7.3.9–7.3.11 from shallow contexts $[\{\ \},P]$ to arbitrary contexts $S\{\ \}$.

7.3.12   Lemma   *Let R, T, U and V be any structures, let a be an atom and let $S\{\ \}$ be any context. Then we have the following*

- *If* $\dfrac{\|\text{NELm}}{S(a,\bar a)}$, *then* $\dfrac{\|\text{NELm}}{S\{\circ\}}$.

- *If* $\dfrac{\|\text{NELm}}{S(\langle R;U\rangle,\langle T;V\rangle)}$, *then* $\dfrac{\|\text{NELm}}{S\langle (R,T);(U,V)\rangle}$.

- *If* $\dfrac{\|\text{NELm}}{S(?R,!T)}$, *then* $\dfrac{\|\text{NELm}}{S\{?(R,T)\}}$.

Proof: All three statements are proved similarly. I will here show only the third. To the given proof of $S(?R,!T)$ apply context reduction, to get a structure P, such that

$$\frac{\|\text{NELm}}{[(?R,!T),P]}$$

and for all structures X, either

$$\frac{[X,P]}{\|\text{NELm}}\qquad\text{or}\qquad\frac{![X,P]}{\|\text{NELm}}$$
$$S\{X\}\qquad\qquad\qquad S\{X\}$$

In particular, for $X = ?(R,T)$, we have

$$\frac{[?(R,T),P]}{\|\text{NELm}}\qquad\text{or}\qquad\frac{![?(R,T),P]}{\|\text{NELm}}$$
$$S\{?(R,T)\}\qquad\qquad\qquad S\{?(R,T)\}$$

By Lemma 7.3.11 there is a proof $\dfrac{\|\text{NELm}}{[?(R,T),P]}$, from which we get $\dfrac{\|\text{NELm}}{S\{?(R,T)\}}$. $\square$

Now we can very easily give a proof for the cut elimination theorem for system NEL.

Proof of Theorem 7.1.6:   First, we apply to a given proof $\dfrac{\|\text{SNEL}\cup\{\circ\}}{R}$ the second decomposition theorem, which yields

$$\circ\!\downarrow\frac{\quad\quad}{\circ}$$
$$\|\{\text{ai}\downarrow\}$$
$$R_4$$
$$\|\text{SNELc}$$
$$R_3$$
$$\|\{\text{ai}\uparrow\}$$
$$R_2$$
$$\|\{\text{w}\downarrow\}$$
$$R_1$$
$$\|\{\text{b}\downarrow\}$$
$$R$$

for some structures R_1, R_2, R_3 and R_4. The instances of $\mathsf{b}{\uparrow}$ and $\mathsf{w}{\uparrow}$ disappear because their premise is the unit $\circ$ (compare Corollary 4.4.1). In other words, we have

$$
\begin{array}{l}
\Pi \| \mathsf{NELm} \cup \{\mathsf{ai}{\uparrow},\mathsf{q}{\uparrow},\mathsf{p}{\uparrow}\} \\
R_2 \\
\quad \| \{\mathsf{w}{\downarrow}\} \\
R_1 \\
\quad \| \{\mathsf{b}{\downarrow}\} \\
R
\end{array}
\qquad .
$$

All instances of $\mathsf{ai}{\uparrow}$, $\mathsf{q}{\uparrow}$ and $\mathsf{p}{\uparrow}$ are now eliminated from Π, starting with the topmost instance: Let $\rho \in \{\mathsf{ai}{\uparrow}, \mathsf{q}{\downarrow}, \mathsf{p}{\uparrow}\}$. Then we can replace

$$
\begin{array}{l}
\| \mathsf{NELm} \\
\rho \dfrac{S\{W\}}{S\{Z\}} \\
\quad \| \mathsf{NELm}\cup\{\mathsf{ai}{\uparrow},\mathsf{q}{\uparrow},\mathsf{p}{\uparrow}\} \\
R_2 \\
\quad \| \{\mathsf{w}{\downarrow}\} \\
R_1 \\
\quad \| \{\mathsf{b}{\downarrow}\} \\
R
\end{array}
\qquad \text{by} \qquad
\begin{array}{l}
\| \mathsf{NELm} \\
S\{Z\} \\
\quad \| \mathsf{NELm}\cup\{\mathsf{ai}{\uparrow},\mathsf{q}{\uparrow},\mathsf{p}{\uparrow}\} \\
R_2 \\
\quad \| \{\mathsf{w}{\downarrow}\} \\
R_1 \\
\quad \| \{\mathsf{b}{\downarrow}\} \\
R
\end{array}
\qquad ,
$$

by applying Lemma 7.3.12. $\square$

Observe that we are not able to obtain a proof of the shape

$$
\begin{array}{l}
\circ{\downarrow} \dfrac{}{\circ} \\
\quad \| \{\mathsf{ai}{\downarrow}\} \\
R_3 \\
\quad \| \{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{p}{\downarrow}\} \\
R_2 \\
\quad \| \{\mathsf{w}{\downarrow}\} \\
R_1 \\
\quad \| \{\mathsf{b}{\downarrow}\} \\
R
\end{array}
\qquad ,
$$

(compare Proposition 4.4.14), because the rule $\mathsf{ai}{\downarrow}$ cannot be permuted over $\mathsf{q}{\downarrow}$ without introducing instances of $\mathsf{q}{\uparrow}$ (see case (x.c) in the proof of Lemma 7.2.4). This is also the reason, why all attempts to prove cut elimination for BV and NEL via a permutation argument (as done in Section 4.4 for ELS) failed.

However, we have the following:

7.3.13 Proposition *For every proof* $\left\|\begin{array}{c}\text{\scriptsize NEL}\end{array}\right._{R}$ *, there is a proof*

$$
\begin{array}{c}
\Pi\big\|\text{\scriptsize NELm}\\
R_2\\
\big\|\{\text{w}\downarrow\}\\
R_1\\
\big\|\{\text{b}\downarrow\}\\
R
\end{array}\quad,
$$

for some structures R_1 and R_2.

7.4   The Undecidability of System NEL

Since the introduction of linear logic [Gir87a], the complexity of the provability problem of its fragments has been studied. The multiplicative fragment (MLL) is NP-complete [Kan94], the multiplicative additive fragment (MALL) is PSPACE-complete and full propositional linear logic is undecidable [LMSS92]. The decidability of the multiplicative exponential fragment (MELL) is still an open problem. But in a purely noncommutative setting, i.e. in the presence of two mutually dual noncommutative connectives, the multiplicatives and the exponentials are sufficient to get undecidability [LMSS92].

In this section I will show that the provability in system NEL is undecidable. This means that in a mixed commutative and noncommutative system in which there is only one single self-dual noncommutative connective, the multiplicatives and the exponentials alone are sufficient to get undecidability. For showing this, A. Guglielmi proposes in [Gug02a] an encoding of Post's correspondence problem, which makes the noncommutativity correspond to sequential composition of words. Since I was not able to find a complete proof along these lines, I will here use an encoding of two counter machines (two tape nonwriting Turing machines [Min61]), which also has the advantage of being simpler.

The undecidability of NEL shows that it is possible to have an undecidable propositional multiplicative exponential system without relying on the additives of linear logic, two mutually dual noncommutative connectives [LMSS92], or second order quantifiers [LS96].

This shows that the ability of copying information (i.e. reusing the program instructions) was the only missing ingredient for making BV (which is obviously decidable) undecidable. If it turns out that MELL is decidable (as many believe), then the border to undecidability is crossed by the self-dual seq, because it allows to model the tapes of a nonwriting Turing machine. In particular, the seq allows to test whether the head of the Turing machine is at the end of a (semi-infinite) tape, i.e. the zero-test.

In the following, I will first (in Section 7.4.1) introduce two counter machines [Min61, Lam61] and show in Section 7.4.2 how they are encoded in system NEL. The encoding is pretty much inspired by [Kan00], and the proof of its completeness is an easy exercise (done in Section 7.4.3).

However, the proof of the soundness of the encoding is quite different from what has been done so far. There are two reasons for this: First, the simple way of extracting the computation sequence of the machine from the proof of the encoding, as done in [LMSS92, Kan95] for full linear logic, is not possible because the calculus of structures allows more

freedom in applying and permuting rules than the sequent calculus does. And second, the use of phase spaces [Gir87a], as it has been done in [Laf96, LS96, Kan00], is not possible because (so far) there is no phase semantics available for NEL.

The method I will use instead (in Section 7.4.5) is the following. The given proof in system NEL of an encoding of a two-counter machine is first transformed into a certain normal form, which allows to remove the exponentials. The resulting proof in the multiplicative fragment has as conclusion a structure which has the shape of what I call a *weak encoding.* From this proof, I will extract the first computation step of the machine and another proof (in the multiplicative fragment) which has as conclusion again a weak encoding. By an inductive argument it is then possible to obtain the whole computation sequence.

In order to carry out the technical details, I need to prove some facts about system BV, which will be done in Section 7.4.4.

7.4.1   Two Counter Machines

Two counter machines have been introduced by M. Minsky in [Min61] as two tape nonwriting Turing machines. He showed that any (usual) Turing machine can be simulated on a two counter machine. In [Lam61], J. Lambek showed that any recursive function can be computed by an n counter machine, for some number $n \in \mathbf{N}$.

7.4.1   Definition   A *two counter machine* $\mathcal{M}$ is a tuple $(\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$, where $\mathcal{Q}$ is a finite set of *states*, $q_0 \in \mathcal{Q}$ is called the *initial state*, $q_f \in \mathcal{Q}$ is called the *final state*, $n_0, m_0 \in \mathbf{N}$ represent the initial positions of the heads on the two tapes, and $\mathcal{T} \subseteq \mathcal{Q} \times \mathcal{I} \times \mathcal{Q}$ is a finite set of *transitions*, where

$$\mathcal{I} = \{\mathsf{inc1}, \mathsf{dec1}, \mathsf{zero1}, \mathsf{inc2}, \mathsf{dec2}, \mathsf{zero2}\}$$

is the set of possible *instructions*.

7.4.2   Example   The running example in this section will be the following

$$\begin{aligned}\mathcal{M} &= (\{q_0, q_1, q_2\}, q_0, 1, 0, q_1, \mathcal{T}) \quad, \quad \text{where}\\ \mathcal{T} &= \{(q_0, \mathsf{dec2}, q_2), (q_1, \mathsf{dec1}, q_1), (q_0, \mathsf{zero2}, q_1)\} \quad.\end{aligned}$$

7.4.3   Definition   Let $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$ be a two counter machine. A *configuration* of $\mathcal{M}$ is given by a tuple (q, n, m), where $q \in \mathcal{Q}$ is a state and n and m are natural numbers. The configuration (q_0, n_0, m_0) is called *initial configuration*. A configuration (q', n', m') is *reachable in one step* from a configuration (q, n, m), written as

$$(q, n, m) \to (q', n', m') \quad,$$

if one of the following six cases holds:

$$\begin{array}{lllll}(q, \mathsf{inc1}, q') \in \mathcal{T} & \text{and} & n' = n + 1 & \text{and} & m' = m \quad,\\ (q, \mathsf{dec1}, q') \in \mathcal{T} & \text{and} & n > 0, \quad n' = n - 1 & \text{and} & m' = m \quad,\\ (q, \mathsf{zero1}, q') \in \mathcal{T} & \text{and} & n' = n = 0 & \text{and} & m' = m \quad,\\ (q, \mathsf{inc2}, q') \in \mathcal{T} & \text{and} & n' = n & \text{and} & m' = m + 1 \quad,\\ (q, \mathsf{dec2}, q') \in \mathcal{T} & \text{and} & n' = n & \text{and} & m > 0, \quad m' = m - 1 \quad,\\ (q, \mathsf{zero2}, q') \in \mathcal{T} & \text{and} & n' = n & \text{and} & m' = m = 0 \quad.\end{array}$$

Further, a configuration (q', n', m') is *reachable in r steps* from a configuration (q, n, m), written as

$$(q, n, m) \to^r (q', n', m') \quad ,$$

if

- $r = 0$ and $(q', n', m') = (q, n, m)$ or

- $r \geqslant 1$ and there is a configuration (q'', n'', m'') such that $(q, n, m) \to (q'', n'', m'')$ and $(q'', n'', m'') \to^{r-1} (q', n', m')$.

A configuration (q', n', m') is *reachable* from a configuration (q, n, m), written as

$$(q, n, m) \to^* (q', n', m') \quad ,$$

if there is an $r \in \mathbf{N}$ such that $(q, n, m) \to^r (q', n', m')$.

In other words, the relation $\to^*$ is the transitive closure of $\to$.

7.4.4   Example   For the machine in Example 7.4.2, we have for example

$$(q_0, 5, 2) \to (q_2, 5, 1) \quad \text{and} \quad (q_1, 5, 2) \to^4 (q_1, 1, 2) \quad .$$

7.4.5   Definition   A two counter machine $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$ *accepts* a configuration (q, n, m), if

$$(q, n, m) \to^* (q_f, 0, 0) \quad .$$

7.4.6   Example   The machine in Example 7.4.2 accepts for example the configuration $(q_0, 8, 0)$, because $(q_0, 8, 0) \to (q_1, 8, 0) \to^8 (q_1, 0, 0)$. More precisely, it accepts any configuration $(q_0, n, 0)$ for $n \geqslant 0$. In particular it also accepts its initial configuration $(q_0, 1, 0)$.

7.4.7   Theorem   *In general, it is undecidable whether a two counter machine accepts its initial configuration.* [Min61, Lam61]

7.4.8   Remark   In Definition 7.4.1, I defined two counter machines to have only one final state, whereas in the standard textbook definition [Min61, HU79, Koz97], they might have many final states. But this is not a problem, since any two counter machine $\mathcal{M}$ with many final states $q_{f1}, \ldots, q_{fn}$ can be transformed into a two counter machine $\mathcal{M}'$ that has only one final state and that accepts the same configurations, by adding a new state q_f (which will be the new final state) and a transition $(q_{fi}, \texttt{zero2}, q_f)$ for each $i = 1, \ldots, n$.

7.4.2   Encoding Two Counter Machines in NEL Structures

Let a be an atom and $n \in \mathbf{N}$. Then a^n denotes the structure $\langle a; a; \ldots; a \rangle$ with n copies of a. More precisely,

$$\begin{aligned} a^0 &= \circ \\ a^n &= \langle a^{n-1}; a \rangle \quad , \quad \text{for } n \geqslant 1 \quad . \end{aligned}$$

7.4.9   Encoding   Let now a two counter machine $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathscr{T})$ be given. For each state $q \in \mathcal{Q}$, I will introduce a fresh atom, also denoted by q. Further, I will need four atoms a, b, c and d. Without loss of generality, let $\mathcal{Q} = \{q_0, q_1, \ldots, q_z\}$ for some $z \geqslant 0$. Then $q_f = q_i$ for some $i \in \{0, \ldots, z\}$. A configuration (q, n, m) will be encoded by the following structure

$$\langle b; a^n; q; c^m; d \rangle \quad .$$

Since $\mathscr{T}$ is finite, we have $\mathscr{T} = \{t_1, t_2, \ldots, t_h\}$ for some $h \in \mathbf{N}$ (if $\mathscr{T} = \varnothing$, then $h = 0$). For each $k \in \{1, \ldots, h\}$, I will define the structure T_k, that encodes the transition t_k, as follows. For all $i, j \in \{0, \ldots, z\}$,

$$
\begin{array}{llll}
\text{if} & t_k = (q_i, \mathsf{inc1}, q_j) \ , & \text{then} & T_k = (\bar{q}_i, \langle a; q_j \rangle) & , \\
\text{if} & t_k = (q_i, \mathsf{dec1}, q_j) \ , & \text{then} & T_k = (\langle \bar{a}; \bar{q}_i \rangle, q_j) & , \\
\text{if} & t_k = (q_i, \mathsf{zero1}, q_j), & \text{then} & T_k = (\langle \bar{b}; \bar{q}_i \rangle, \langle b; q_j \rangle) & , \\
\text{if} & t_k = (q_i, \mathsf{inc2}, q_j) \ , & \text{then} & T_k = (\bar{q}_i, \langle q_j; c \rangle) & , \\
\text{if} & t_k = (q_i, \mathsf{dec2}, q_j) \ , & \text{then} & T_k = (\langle \bar{q}_i; \bar{c} \rangle, q_j) & , \\
\text{if} & t_k = (q_i, \mathsf{zero2}, q_j), & \text{then} & T_k = (\langle \bar{q}_i; \bar{d} \rangle, \langle q_j; d \rangle) & .
\end{array}
$$

I will say that a structure T *encodes* a transition of $\mathcal{M}$, if $T = T_k$ for some $k \in \{1, \ldots, h\}$. The machine $\mathcal{M}$ is encoded by the structure

$$\mathcal{M}_{\mathrm{enc}} = [?T_1, \ldots, ?T_h, \langle b; a^{n_0}; q_0; c^{m_0}; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle] \quad .$$

The structure $\mathcal{M}_{\mathrm{enc}}$ is called the *encoding* of $\mathcal{M}$.

7.4.10   Example   The machine in Example 7.4.2 is encoded by the structure

$$\mathcal{M}_{\mathrm{enc}} = [?(\langle \bar{q}_0; \bar{c} \rangle, q_2), ?(\langle \bar{a}; \bar{q}_1 \rangle, q_1), ?(\langle \bar{q}_0; \bar{d} \rangle, \langle q_1; d \rangle), \langle b; a; q_0; d \rangle, \langle \bar{b}; \bar{q}_1; \bar{d} \rangle] \quad .$$

7.4.11   Theorem   *A two counter machine $\mathcal{M}$ accepts its initial configuration if and only if its encoding $\mathcal{M}_{\mathrm{enc}}$ is provable in* NEL.

From this we can obtain immediately the following:

7.4.12   Theorem   *Provability in system* NEL *is undecidable.*

Proof: By way of contradiction, assume the existence of a decision procedure for probability in NEL. Then we could decide whether for a given two counter machine $\mathcal{M}$, its encoding $\mathcal{M}_{\mathrm{enc}}$ is provable in NEL. By Theorem 7.4.11, we could therefore decide whether $\mathcal{M}$ accepts its initial configuration. This is a contradiction to Theorem 7.4.7. $\square$

The remaining sections are devoted to the proof of Theorem 7.4.11. As expected, one direction is easy and the other is difficult. Let me start with the easy one.

7.4.3 Completeness of the Encoding

7.4.13 Lemma *Given a two counter machine* $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathscr{T})$.

$$If \quad (q_i, n, m) \to (q_j, n', m') \quad then \quad \begin{array}{c} [?T_1, \ldots, ?T_h, \langle b; a^{n'}; q_j; c^{m'}; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle] \\ \Big\| \mathsf{NEL} \\ [?T_1, \ldots, ?T_h, \langle b; a^n; q_i; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle] \end{array} \quad .$$

Proof: There are six possible cases how the machine $\mathcal{M}$ can go from (q_i, n, m) to (q_j, n', m'):

- The first counter has been incremented: $(q_i, \mathsf{inc1}, q_j) \in \mathscr{T}$ and $n' = n + 1$ and $m' = m$. Then we have $T_k = (\bar{q}_i, \langle a; q_j \rangle)$ for some $k \in \{1, \ldots, h\}$. Now use

$$\mathsf{b}{\downarrow}\cfrac{\mathsf{q}{\downarrow}\cfrac{\mathsf{q}{\downarrow}\cfrac{\mathsf{s}\cfrac{\mathsf{ai}{\downarrow}\cfrac{[?T_1, \ldots, ?T_h, \langle b; a^{n+1}; q_j; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}{[?T_1, \ldots, ?T_h, \langle b; a^n; ([\bar{q}_i, q_i], \langle a; q_j \rangle); c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle b; a^n; [(\bar{q}_i, \langle a; q_j \rangle), q_i]; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle [(\bar{q}_i, \langle a; q_j \rangle), \langle b; a^n; q_i \rangle]; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, (\bar{q}_i, \langle a; q_j \rangle), \langle b; a^n; q_i; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle b; a^n; q_i; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]} \quad .$$

- The first counter has been decremented: $(q_i, \mathsf{dec1}, q_j) \in \mathscr{T}$ and $n > 0$ and $n' = n - 1$ and $m' = m$. Then we have $T_k = (\langle \bar{a}; \bar{q}_i \rangle, q_j)$ for some $k \in \{1, \ldots, h\}$. Now use

$$\mathsf{b}{\downarrow}\cfrac{\mathsf{q}{\downarrow}\cfrac{\mathsf{q}{\downarrow}\cfrac{\mathsf{s}\cfrac{\mathsf{q}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{[?T_1, \ldots, ?T_h, \langle b; a^{n-1}; q_j; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}{[?T_1, \ldots, ?T_h, \langle b; a^{n-1}; ([\bar{q}_i, q_i], q_j); c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle b; a^{n-1}; (\langle [\bar{a}, a]; [\bar{q}_i, q_i] \rangle, q_j); c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle b; a^{n-1}; ([\langle \bar{a}; \bar{q}_i \rangle, \langle a; q_i \rangle], q_j); c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle b; a^{n-1}; [(\langle \bar{a}; \bar{q}_i \rangle, q_j), \langle a; q_i \rangle]; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle [(\langle \bar{a}; \bar{q}_i \rangle, q_j), \langle b; a^n; q_i \rangle]; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, (\langle \bar{a}; \bar{q}_i \rangle, q_j), \langle b; a^n; q_i; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle b; a^n; q_i; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]} \quad .$$

- The first counter has been tested for zero: $(q_i, \mathsf{zero1}, q_j) \in \mathscr{T}$ and $n = n' = 0$ and $m' = m$. Then we have $T_k = (\langle \bar{b}; \bar{q}_i \rangle, \langle b; q_j \rangle)$ for some $k \in \{1, \ldots, h\}$. Now use

$$\mathsf{b}{\downarrow}\cfrac{\mathsf{q}{\downarrow}\cfrac{\mathsf{s}\cfrac{\mathsf{q}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{[?T_1, \ldots, ?T_h, \langle b; q_j; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}{[?T_1, \ldots, ?T_h, \langle ([\bar{q}_i, q_i], \langle b; q_j \rangle); c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle (\langle [\bar{b}, b]; [\bar{q}_i, q_i] \rangle, \langle b; q_j \rangle); c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle ([\langle \bar{b}; \bar{q}_i \rangle, \langle b; q_i \rangle], \langle b; q_j \rangle); c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle [(\langle \bar{b}; \bar{q}_i \rangle, \langle b; q_j \rangle), \langle b; q_i \rangle]; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, (\langle \bar{b}; \bar{q}_i \rangle, \langle b; q_j \rangle), \langle b; q_i; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]}}{[?T_1, \ldots, ?T_h, \langle b; q_i; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]} \quad .$$

The other three cases (where the second counter is concerned) are similar. □

7.4.14   Lemma   *Given a two counter machine $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$.*

$$If \quad (q, n, m) \to^* (q', n', m') \quad then \qquad \begin{array}{c} [?T_1, \ldots, ?T_h, \langle b; a^{n'}; q'; c^{m'}; d\rangle, \langle \bar{b}; \bar{q}_f; \bar{d}\rangle] \\ \Big\| \mathsf{NEL} \\ [?T_1, \ldots, ?T_h, \langle b; a^n; q; c^m; d\rangle, \langle \bar{b}; \bar{q}_f; \bar{d}\rangle] \end{array} \quad .$$

Proof: By definition there is a number $r \in \mathbf{N}$ such that $(q, n, m) \to^r (q', n', m')$. Apply induction on r and Lemma 7.4.13 to get the result. $\qquad\square$

7.4.15   Lemma   *Given a two counter machine $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$. Then there is a proof*

$$\begin{array}{c} \Big\| \mathsf{NEL} \\ [?T_1, \ldots, ?T_h, \langle b; q_f; d\rangle, \langle \bar{b}; \bar{q}_f; \bar{d}\rangle] \end{array} \quad .$$

Proof: Use $\mathsf{w}{\downarrow}$ to eliminate the $?T_i$, and $\mathsf{q}{\downarrow}$ and $\mathsf{ai}{\downarrow}$ for the remaining structure. $\qquad\square$

Now we can prove the first direction of Theorem 7.4.11.

7.4.16   Proposition   *Given a two counter machine $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$.*

$$If \quad (q_0, n_0, m_0) \to^* (q_f, 0, 0) \quad then \qquad \begin{array}{c} \Big\| \mathsf{NEL} \\ \mathcal{M}_{\mathrm{enc}} \end{array} \quad .$$

Proof: Use

$$\begin{array}{c} \Pi \Big\| \mathsf{NEL} \\ [?T_1, \ldots, ?T_h, \langle b; q_f; d\rangle, \langle \bar{b}; \bar{q}_f; \bar{d}\rangle] \\ \Delta \Big\| \mathsf{NEL} \\ [?T_1, \ldots, ?T_h, \langle b; a^{n_0}; q_0; c^{m_0}; d\rangle, \langle \bar{b}; \bar{q}_f; \bar{d}\rangle] \end{array} \quad ,$$

where Π exists by Lemma 7.4.15 and Δ by Lemma 7.4.14. $\qquad\square$

7.4.17   Example   The proof of the encoding in Example 7.4.10 has the following shape:

$$\begin{array}{c} \Big\| \mathsf{NEL} \\ [?(\langle \bar{q}_0; \bar{c}\rangle, q_2), ?(\langle \bar{a}; \bar{q}_1\rangle, q_1), ?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle b; q_1; d\rangle, \langle \bar{b}; \bar{q}_1; \bar{d}\rangle] \\ \Big\| \mathsf{NEL} \\ [?(\langle \bar{q}_0; \bar{c}\rangle, q_2), ?(\langle \bar{a}; \bar{q}_1\rangle, q_1), ?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle b; a; q_1; d\rangle, \langle \bar{b}; \bar{q}_1; \bar{d}\rangle] \\ \Big\| \mathsf{NEL} \\ [?(\langle \bar{q}_0; \bar{c}\rangle, q_2), ?(\langle \bar{a}; \bar{q}_1\rangle, q_1), ?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle b; a; q_0; d\rangle, \langle \bar{b}; \bar{q}_1; \bar{d}\rangle] \end{array}$$

7.4.4   Some Facts about System BV

Before I can show the soundness of the encoding, I need to establish some facts about system BV. The first observation is that if in a provable structure R, some atoms are renamed in such a way that dual atoms are mapped to dual atoms, then the result is still provable. This is made precise as follows.

7.4.18 Definition A set $\mathcal{P}$ of atoms is called *clean* if for all atoms $a \in \mathcal{P}$, we have $\bar{a} \notin \mathcal{P}$. Let $e : \mathcal{P} \to \mathcal{Q}$ be a mapping, where $\mathcal{P}$ and $\mathcal{Q}$ are two clean sets of atoms. If R is a BV structure, then R^e is the structure obtained from R by replacing every atom a by $e(a)$. More precisely, the mapping $(\cdot)^e$ is defined inductively on BV structures as follows:

$$\circ^e \;=\; \circ$$
$$a^e \;=\; \begin{cases} e(a) & \text{if } a \in \mathcal{P} \\ \overline{e(a)} & \text{if } \bar{a} \in \mathcal{P} \\ a & \text{otherwise} \end{cases}$$
$$[R,T]^e \;=\; [R^e, T^e]$$
$$(R,T)^e \;=\; (R^e, T^e)$$
$$\langle R;T \rangle^e \;=\; \langle R^e;T^e \rangle$$
$$\bar{R}^e \;=\; \overline{R^e}$$

7.4.19 Example Let $\mathcal{P} = \{\bar{a}, b\}$ and $\mathcal{Q} = \{c\}$, and let $e(\bar{a}) = e(b) = c$. Then

$$[\langle a;b;c;d \rangle, \langle \bar{a}; \bar{d}; \bar{b}; \bar{a} \rangle]^e = [\langle \bar{c};c;c;d \rangle, \langle c; \bar{d}; \bar{c}; c \rangle] \quad .$$

7.4.20 Lemma *Let $e : \mathcal{P} \to \mathcal{Q}$ be a mapping, where $\mathcal{P}$ and $\mathcal{Q}$ are two clean sets of atoms, and let R be a structure. If R is provable in* BV*, then R^e is also provable in* BV*.*

Proof: Let $\Pi \| \mathsf{BV}$ over R be given. Now let Π^e be the proof obtained from Π by replacing each structure S occurring inside Π by S^e. Each rule application remains valid. Therefore, $\Pi^e \| \mathsf{BV}$ over R^e is a valid proof. $\qquad\square$

7.4.21 Example Let $\mathcal{P} = \{\bar{a}, b\}$ and $\mathcal{Q} = \{c\}$, and let $e(\bar{a}) = e(b) = c$ as above. Let $R = [\langle \bar{a};b;d \rangle, \langle a; \bar{b}; \bar{d} \rangle]$, which is provable in BV. By Lemma 7.4.20, we have

$$
\mathsf{q}{\downarrow}\cfrac{\mathsf{q}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\circ{\downarrow}\cfrac{-}{\circ}}{[d,\bar{d}]}}{\langle [b,\bar{b}];[d,\bar{d}]\rangle}}{\langle [\bar{a},a];[b,\bar{b}];[d,\bar{d}]\rangle}}{\langle [\bar{a},a];[\langle b;d\rangle,\langle\bar{b};\bar{d}\rangle]\rangle}}{[\langle\bar{a};b;d\rangle,\langle a;\bar{b};\bar{d}\rangle]}
\quad\rightsquigarrow\quad
\mathsf{q}{\downarrow}\cfrac{\mathsf{q}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\mathsf{ai}{\downarrow}\cfrac{\circ{\downarrow}\cfrac{-}{\circ}}{[d,\bar{d}]}}{\langle [c,\bar{c}];[d,\bar{d}]\rangle}}{\langle [c,\bar{c}];[c,\bar{c}];[d,\bar{d}]\rangle}}{\langle [c,\bar{c}];[\langle c;d\rangle,\langle\bar{c};\bar{d}\rangle]\rangle}}{[\langle c;c;d\rangle,\langle\bar{c};\bar{c};\bar{d}\rangle]}
\quad .
$$

The converse of Lemma 7.4.20 does in general not hold. For example $R = [\bar{a}, \bar{b}]$ is not provable, but $R^e = [c, \bar{c}]$ is.

7.4.22 Observation If a structure R is provable in BV, then every atom a occurs as often in R as $\bar{a}$. This is easy to see: The only possibility, where an atom a can disappear is an instance of $\mathsf{ai}{\downarrow}$. But then at the same time an atom $\bar{a}$ disappears.

For a given proof Π of a structure R, I will call the *killer* (in Π) of a given occurrence atom a, that occurrence of $\bar{a}$, that vanishes together with it in an instance of $\mathsf{ai}{\downarrow}$. The

situation is trivial, if in R every atom occurs exactly once. For example in the left-hand side proof in Example 7.4.21, the killer of $\bar{b}$ is b. In the right-hand side proof, more care is necessary: The killer of the first occurrence of $\bar{c}$ is the first c. The killer of the second c is the second $\bar{c}$.

7.4.23   Definition   A BV structure R is called a *nonpar structure* if it it does not contain a par structure as substructure, i.e. it is generated by the grammar

$$R ::= \circ \mid a \mid (R, R) \mid \langle R; R \rangle \mid \bar{R} \quad .$$

7.4.24   Lemma   *Let V and P be BV structures, such that $\bar{V}$ is a nonpar structure.*

$$\text{If} \quad \begin{array}{c} \\ \|_{\mathsf{BV}} \\ [\bar{V}, P] \end{array} \quad , \quad \text{then} \quad \begin{array}{c} V \\ \|_{\mathsf{BV}} \\ P \end{array} \quad .$$

Proof: By structural induction on $\bar{V}$. The base case is either trivial (for $V = \circ$) or a case of Lemma 7.3.7 (if V is an atom). For the inductive cases apply splitting (Lemma 7.3.5).   □

7.4.25   Definition   Let R be a BV structure and let a be an atom occurring in R. The atom a is *unique* in R if it occurs exactly once.

7.4.26   Example   In $[\langle \bar{c}; c; d \rangle, \langle c; \bar{c}; \bar{d} \rangle]$, the atoms d and $\bar{d}$ are unique, but c and $\bar{c}$ are not.

7.4.27   Lemma   *Let $V \neq \circ$ be a BV structure and $S\{\ \}$ and $S'\{\ \}$ be two contexts, such that all atoms in V are unique in $S\{V\}$.*

$$\text{If} \quad \begin{array}{c} S'\{V\} \\ \|_{\mathsf{BV}} \\ S\{V\} \end{array} \quad , \quad \text{then} \quad \begin{array}{c} S'\{X\} \\ \|_{\mathsf{BV}} \\ S\{X\} \end{array} \quad \text{for every structure } X \quad .$$

Proof: Pick any atom a inside V, and replace every other atom occurring in V everywhere inside

$$\begin{array}{c} S'\{V\} \\ \Delta \| {\mathsf{BV}} \\ S\{V\} \end{array}$$

by $\circ$. This yields a derivation

$$\begin{array}{c} S'\{a\} \\ \Delta' \| {\mathsf{BV}} \\ S\{a\} \end{array} \quad ,$$

in which the atom a can everywhere be replaced by the structure X.   □

The following lemma will play a crucial role in the proof of the soundness of the encoding of two counter machines.

7.4.28   Lemma   *Let $R = [Z, (\bar{V}, T), \langle U; V; W \rangle]$ be a BV structure, such that $\bar{V}$ is a nonpar structure and all atoms occurring in V are unique in R. If R is provable in BV, then $R' = [Z, \langle U; T; W \rangle]$ is also provable in BV.*

Proof: Let

$$\Pi \| \mathsf{BV}$$
$$[Z, (\bar{V}, T), \langle U; V; W \rangle]$$

be given. By splitting (Lemma 7.3.5), there are structures P and Q such that

$$
\begin{array}{ccccc}
[P, Q] & & & & \\
\Delta \| \mathsf{BV} & \text{and} & \Pi_1 \| \mathsf{BV} & \text{and} & \Pi_2 \| \mathsf{BV} \\
[Z, \langle U; V; W \rangle] & & [\bar{V}, P] & & [T, Q]
\end{array}
$$

By applying Lemma 7.4.24 to Π_1, we get

$$
\begin{array}{c}
V \\
\Delta_1 \| \mathsf{BV} \\
P
\end{array} \qquad ,
$$

from which we can get the following derivation:

$$
\begin{array}{c}
[V, Q] \\
\Delta_1 \| \mathsf{BV} \\
[P, Q] \\
\Delta \| \mathsf{BV} \\
[Z, \langle U; V; W \rangle]
\end{array} \qquad .
$$

Now, we can apply Lemma 7.4.27, which gives us

$$
\begin{array}{c}
[T, Q] \\
\Delta_3 \| \mathsf{BV} \\
[Z, \langle U; T; W \rangle]
\end{array} \qquad ,
$$

Composing Π_2 and Δ_3 yields the desired proof of R'. □

In the following, I will generalise Proposition 6.2.13 (page 187). The notion of negation circuit has already been generalised in Definition 7.2.8. In Section 6.2, I was mainly concerned with balanced structures, i.e. structures in which each atom occurs at most once. But an inspection of the proof of Proposition 6.2.13 shows that this property is only needed for the atoms that participate in the negation circuit. This motivates the following definition.

7.4.29   Definition   A BV structure R *contains a (*balanced*) negation circuit* if R has a quasi-substructure R' which is a negation circuit and every atom that occurs in R' is unique in R.

7.4.30   Proposition   *Let R be a BV structure. If R contains a balanced negation circuit, then R is not provable in BV.*

Proof: Since R contains a balanced negation circuit, we have atoms a_1, $\bar{a}_1$, a_2, $\bar{a}_2$, ..., a_n, $\bar{a}_n$ which are unique in R such that R contains a $\langle a_1, \bar{a}_1, a_2, \bar{a}_2, \ldots, a_n, \bar{a}_n \rangle$-circuit. By

way of contradiction, assume that there is a proof of R. This proof remains valid, if all atoms other that $a_1, \bar{a}_1, a_2, \bar{a}_2, \ldots, a_n, \bar{a}_n$ are replaced by $\circ$ everywhere (some rule instances become trivial and can be removed). This yields a proof

$$\Pi \| \mathsf{BV} \atop [Z_1, Z_2, \ldots, Z_n] \qquad ,$$

for some $n \geqslant 1$, where

- $Z_j = (\bar{a}_j, a_{j+1})$ or $Z_j = \langle \bar{a}_j ; a_{j+1} \rangle$ for every $j = 1, \ldots, n-1$, and

- $Z_n = (\bar{a}_n, a_1)$ or $Z_n = \langle \bar{a}_n ; a_1 \rangle$.

Now, I will proceed by induction on n to show a contradiction. This will be very similar to the proof of Lemma 7.2.6. The induction measure is simpler because the rule $\mathsf{q}{\uparrow}$ cannot occur. (I will again use the convention that $j + 1 = 1$ in the case of $j = n$.)

Base Case: If $n = 1$, then obviously, there is no proof

$$\Pi \| \mathsf{BV} \atop (\bar{a}_1, a_1) \qquad \text{or} \qquad \Pi \| \mathsf{BV} \atop \langle \bar{a}_1 ; a_1 \rangle \qquad .$$

Inductive Case: Suppose there is no proof Π for all $n' < n$. Now consider the bottommost rule instance ρ in

$$\Pi \| \mathsf{BV} \atop [Z_1, Z_2, \ldots, Z_n] \qquad ,$$

where $Z_j = (\bar{a}_j, a_{j+1})$ or $Z_j = \langle \bar{a}_j ; a_{j+1} \rangle$ for every $j = 1, \ldots, n$. Without loss of generality, we can assume that ρ is nontrivial.

(1) $\rho = \circ{\downarrow}$ or $\rho = \mathsf{ai}{\downarrow}$. This is impossible.

(2) $\rho = \mathsf{q}{\downarrow}$ or $\rho = \mathsf{s}$. This case is exactly the same as in the proof of Lemma 7.2.6. The only difference is that we are now talking about proofs instead of derivations with premise $(W_1, \ldots, W_n)$. $\qquad \square$

This proposition is related to provability in BV, as Lemma 7.2.6 is related to derivability. More precisely, I believe, that the converse of Proposition 7.4.30, i.e. a generalisation of Proposition 6.2.16 does also hold.

7.4.31 Conjecture *Let R be a balanced BV structure (i.e. every atom in R is unique) such that for every atom a occurring in R, its dual $\bar{a}$ does also occur. Then R is provable in BV if and only if R does not contain a negation circuit.*

Obviously, there is a close relationship between Conjecture 7.2.12 and Conjecture 7.4.31. In fact, both statements are equivalent, which means that it suffices to prove one of them. I will not go into further details here. I only want to mention that from Conjecture 7.4.31 the equivalence to Ch. Retoré's pomset logic follows more obviously because the definition of a negation circuit corresponds exactly to Ch. Retoré's cordless Æ-circuits [Ret99b].

7.4.5 Soundness of the Encoding

Before I now prove the second (more difficult) direction of Theorem 7.4.11, let me explain where the difficulties arise. The main problem is the freedom of applying rules in the calculus of structures. If a structure is provable in NEL then there is usually not only one way of doing so. In our situation this has the following consequence. The encoding $\mathcal{M}_{\mathrm{enc}}$ of a given machine $\mathcal{M}$ has a certain shape (namely a big par structure containing several why-not structures and two seq structures), which has been preserved while simulating the computation of the machine in the proof of Proposition 7.4.16 (see Example 7.4.17). But this shape does not at all have to be preserved in an arbitrary proof of $\mathcal{M}_{\mathrm{enc}}$.

7.4.32 Example The encoding in Example 7.4.10 could also be proved as follows:

$$
\begin{array}{c}
\Big\| \mathsf{NEL} \\[2pt]
[\langle a; q_0; d\rangle, \langle \bar{a}; \bar{q}_0; \bar{d}\rangle] \\[2pt]
\Big\| \mathsf{NEL} \\[2pt]
[\langle [a; q_0; d], \langle \bar{a}; [?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle \bar{q}_0; \bar{d}\rangle]\rangle\rangle] \\[2pt]
\mathsf{w}{\downarrow}\,\overline{\phantom{[\langle [a, !?(\langle \bar{q}_0; \bar{c}\rangle, q_2)]; q_0; d\rangle, \langle \bar{a}; [?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle \bar{q}_0; \bar{d}\rangle]\rangle]}} \\[-2pt]
\mathsf{q}{\downarrow}\,\overline{[\langle [a, !?(\langle \bar{q}_0; \bar{c}\rangle, q_2)]; q_0; d\rangle, \langle \bar{a}; [?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle \bar{q}_0; \bar{d}\rangle]\rangle]}} \\[-2pt]
[?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle [a, !?(\langle \bar{q}_0; \bar{c}\rangle, q_2)]; q_0; d\rangle, \langle \bar{a}; \bar{q}_0; \bar{d}\rangle] \\[2pt]
\Big\| \mathsf{NEL} \\[2pt]
\mathsf{q}{\downarrow}\,\overline{[?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle [a, !?(\langle \bar{q}_0; \bar{c}\rangle, q_2)]; q_0; d\rangle, \langle [\bar{q}_1, ?(\langle \bar{a}; \bar{q}_1\rangle, q_1)]; \bar{d}\rangle]} \\[-2pt]
\mathsf{p}{\downarrow}\,\overline{[?(\langle \bar{a}; \bar{q}_1\rangle, q_1), ?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle [a, !?(\langle \bar{q}_0; \bar{c}\rangle, q_2)]; q_0; d\rangle, \langle \bar{q}_1; \bar{d}\rangle]} \\[-2pt]
\mathsf{q}{\downarrow}\,\overline{[?(\langle \bar{a}; \bar{q}_1\rangle, q_1), ?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle [a, ?(\langle \bar{q}_0; \bar{c}\rangle, q_2)]; q_0; d\rangle, \langle \bar{q}_1; \bar{d}\rangle]} \\[-2pt]
\mathsf{ai}{\downarrow}\,\overline{[?(\langle \bar{q}_0; \bar{c}\rangle, q_2), ?(\langle \bar{a}; \bar{q}_1\rangle, q_1), ?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle a; q_0; d\rangle, \langle \bar{q}_1; \bar{d}\rangle]} \\[-2pt]
\mathsf{q}{\downarrow}\,\overline{[?(\langle \bar{q}_0; \bar{c}\rangle, q_2), ?(\langle \bar{a}; \bar{q}_1\rangle, q_1), ?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle [b, \bar{b}]; [\langle a; q_0; d\rangle, \langle \bar{q}_1; \bar{d}\rangle]\rangle]} \\[-2pt]
[?(\langle \bar{q}_0; \bar{c}\rangle, q_2), ?(\langle \bar{a}; \bar{q}_1\rangle, q_1), ?(\langle \bar{q}_0; \bar{d}\rangle, \langle q_1; d\rangle), \langle b; a; q_0; d\rangle, \langle \bar{b}; \bar{q}_1; \bar{d}\rangle]
\end{array}
\quad,
$$

in which the computation steps of the corresponding two counter machine are not obvious.

The basic idea of what follows is: if there is a proof

$$
\Pi \Big\| \mathsf{NEL} \atop \mathcal{M}_{\mathrm{enc}} \quad,
$$

then there is also a proof

$$
\Pi' \Big\| \mathsf{NEL} \atop \mathcal{M}_{\mathrm{enc}} \quad,
$$

that preserves (to some extend) the shape of the encoding and from which we can extract the computation of the machine. For this, the notion of weak encoding is introduced.

7.4.33 Definition Let $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$ a two counter machine. Then a BV structure W is called a *weak encoding* of $\mathcal{M}$, if

$$
W = [U_1, \dots, U_r, \langle b; a^n; q; c^m; d\rangle, \langle \bar{b}; \bar{q}_f; \bar{d}\rangle] \quad,
$$

for some $r, n, m \geqslant 0$ and $q \in \mathcal{Q}$, where the structures $U_1, \ldots, U_r$ encode transitions of $\mathcal{M}$, i.e. for every $l \in \{1, \ldots, r\}$, we have that $U_l = T_k$ for some $k \in \{1, \ldots, h\}$.

Observe that in a weak encoding W of a machine $\mathcal{M}$, some transitions T_k might occur many times and some might not occur at all.

7.4.34 Lemma *Given a two counter machine $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$.*

$$
\text{If} \quad \overset{\overline{\parallel}\mathsf{NEL}}{\mathcal{M}_{\text{enc}}} \quad , \text{ then there is a weak encoding } W \text{ of } \mathcal{M}, \text{ such that} \quad \overset{\overline{\parallel}\mathsf{BV}}{\underset{\mathcal{M}_{\text{enc}}}{\overset{W}{\parallel\{\mathsf{w}\downarrow,\mathsf{b}\downarrow\}}}} \quad .
$$

Proof: By Proposition 7.3.13, we have a proof

$$
\overset{\Pi\overline{\parallel}\mathsf{NELm}}{\underset{\underset{\mathcal{M}_{\text{enc}}}{\Delta\parallel\{\mathsf{w}\downarrow,\mathsf{b}\downarrow\}}}{\mathcal{M}'_{\text{enc}}}} \quad .
$$

Let W be the structure, which is obtained from $\mathcal{M}'_{\text{enc}}$ by removing all exponentials, and let Π' be the proof obtained from Π by removing the exponentials from each structure occurring inside Π. By this manipulation, all rule instances remain valid, except for the promotion rule, which becomes trivial:

$$
\mathsf{p}\downarrow \frac{S\{![R, T]\}}{S[!R, ?T]} \quad \leadsto \quad \mathsf{p}\downarrow' \frac{S[R, T]}{S[R, T]} \quad ,
$$

and can therefore be omitted. This means that Π' is valid proof of W in system BV. Further, $\mathcal{M}'_{\text{enc}}$ does not contain any $!$ because $\mathcal{M}_{\text{enc}}$ is $!$-free. Therefore, there is a derivation

$$
\overset{W}{\underset{\mathcal{M}'_{\text{enc}}}{\parallel\{\mathsf{w}\downarrow,\mathsf{b}\downarrow\}}} \quad \text{because of} \quad \mathsf{b}\downarrow \frac{\mathsf{w}\downarrow \dfrac{S\{R\}}{[?R, R]}}{S\{?R\}} \quad .
$$

Hence, we have $\overset{W}{\underset{\mathcal{M}_{\text{enc}}}{\parallel\{\mathsf{w}\downarrow,\mathsf{b}\downarrow\}}}$. $\qquad \square$

7.4.35 Example In our example, we get

$$
W = [(\langle \bar{a}; \bar{q}_1 \rangle, q_1), (\langle \bar{q}_0; \bar{d} \rangle, \langle q_1; d \rangle), \langle b; a; q_0; d \rangle, \langle \bar{b}; \bar{q}_1; \bar{d} \rangle] \quad .
$$

The following lemma is nothing but an act of bureaucracy. The idea is to rename the atoms $q_0, \ldots, q_z$ that encode the states of the machine in such a way that each new atom occurs only once. This will then simplify the extraction of the computation sequence from the proof.

7.4.36 Lemma *Let* $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathscr{T})$ *be a two counter machine, let* $W = [U_1, \ldots, U_r, \langle b; a^n; q; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]$ *be a weak encoding of* $\mathcal{M}$*, and let* $\mathcal{P} = \{p_0, \ldots, p_r\}$ *be a clean set of* $r+1$ *fresh atoms. If* W *is provable in* BV*, then there is a mapping* $e : \mathcal{P} \to \mathcal{Q}$ *and a structure*

$$\tilde{W} = [\tilde{U}_1, \ldots, \tilde{U}_r, \langle b; a^n; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle] \quad ,$$

such that

(1) $\tilde{W}$ *is provable in* BV*,*

(2) *all atoms* $p_0, \bar{p}_0, \ldots, p_r, \bar{p}_r$ *occur exactly once in* $\tilde{W}$*,*

(3) *for every* $l \in \{1, \ldots, r\}$*, the atoms* $\bar{p}_{l-1}$ *and* p_l *occur inside* $\tilde{U}_l$*,*

(4) $\tilde{W}^e = W$*, and*

(5) *for every* $l \in \{1, \ldots, r\}$*, we have* $\tilde{U}_l^e = U_{l'}$ *for some* $l' \in \{1, \ldots, r\}$*,*

Proof: Let $\mathcal{O} = \{o_0, \ldots, o_r\}$ be another clean set of $r+1$ fresh atoms. The structure W contains $r+1$ occurrences of atoms $q \in \mathcal{Q}$ and $r+1$ occurrences of atoms q' with $\bar{q}' \in \mathcal{Q}$ (because each U_l for $l = 1, \ldots, r$ contains exactly one $q \in \mathcal{Q}$ and one q' with $\bar{q}' \in \mathcal{Q}$). Since W is provable, each such q and q' must have its killer inside W. Now let W' be obtained from W by replacing each such q and its killer by o_l and $\bar{o}_l$, respectively, for some $l = 0, \ldots, r$, such that each $o \in \mathcal{O}$ is used exactly once. Then W' is also provable because the replacement can be continued to the proof Π of W. This also yields a mapping $f : \mathcal{O} \to \mathcal{Q}$ with $f(o) = q$ if an occurrence of q has been replaced by o. We now have

$$W' = [U_1', \ldots, U_r', \langle b; a^n; o_l; c^m; d \rangle, \langle \bar{b}; \bar{o}_{l'}; \bar{d} \rangle]$$

for some $l, l' \in \{0, \ldots, r\}$. Further, all atoms $o_0, \bar{o}_0, \ldots, o_r, \bar{o}_r$ occur exactly once in W'. The atom $\bar{o}_l$ must occur inside a U_{s_1}' for some $s_1 \in \{1, \ldots, r\}$ (i.e. $l \neq l'$). Otherwise the atoms $o_0, \bar{o}_0, \ldots, o_{l-1}, \bar{o}_{l-1}, o_{l+1}, \bar{o}_{l+1}, \ldots, o_r, \bar{o}_r$ would form a negation circuit inside $[U_1', \ldots, U_r']$, which is by Proposition 7.4.30 a contradiction to the provability of W'. Now let W_0' be obtained from W' by replacing o_l and $\bar{o}_l$ by p_0 and $\bar{p}_0$, respectively. Let o_{l_1} be the atom from $\mathcal{O}$ that occurs inside U_{s_1}'. Again, we have that $\bar{o}_{l_1}$ must occur inside U_{s_2}' for some $s_2 \in \{1, \ldots, r\}$ (i.e. $l_1 \neq l'$), because otherwise there would be a negation circuit inside $[U_1', \ldots, U_r']$. Let W_1' be obtained from W_0' by replacing o_{l_1} and $\bar{o}_{l_1}$ by p_1 and $\bar{p}_1$, respectively. Repeat this to get the structures $W_2', \ldots, W_r'$. This also defines a bijective mapping $g : \mathcal{O} \to \mathcal{P}$ with $g(o) = p$ if o has been replaced by p. Now let $\tilde{W} = W_r'$ and $e(p) = f(g^{-1}(p))$. Further let $\tilde{U}_1 = U_{s_1}'^g$, $\tilde{U}_2 = U_{s_2}'^g$, and so on. Then $\tilde{W}$ is provable in BV, because W' is provable in BV. Further, all atoms $p_0, \bar{p}_0, \ldots, p_r, \bar{p}_r$ occur exactly once in $\tilde{W}$ because all atoms $o_0, \bar{o}_0, \ldots, o_r, \bar{o}_r$ occur exactly once in W'. The replacement of atoms is done in such a way that for every $l \in \{1, \ldots, r\}$, the atoms $\bar{p}_{l-1}$ and p_l occur inside $\tilde{U}_l$ and $\tilde{W}^e = W$. $\square$

7.4.37 Example For the weak encoding in Example 7.4.35, we get

$$\tilde{W} = [\,\underbrace{(\langle \bar{p}_0; \bar{d} \rangle, \langle p_1; d \rangle)}_{\tilde{U}_1}, \underbrace{(\langle \bar{a}; \bar{p}_1 \rangle, p_2)}_{\tilde{U}_2}, \langle b; a; p_0; d \rangle, \langle \bar{b}; \bar{p}_2; \bar{d} \rangle\,] \quad ,$$

with $e(p_0) = q_0$ and $e(p_1) = e(p_2) = q_1$.

The following lemma is the core of the proof of the soundness of the encoding.

7.4.38   Lemma   *Let $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$ a two counter machine and let $W = [U_1, \ldots, U_r, \langle b; a^n; q; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]$ be a weak encoding of $\mathcal{M}$.*

$$\text{If} \qquad \mathop{\|}_W \mathsf{BV} \qquad \text{then} \quad (q, n, m) \to^r (q_f, 0, 0) \quad .$$

Proof: By induction on r:

Base case: If $r = 0$ then $W = [\langle b; a^n; q; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]$. This is only provable if $n = m = 0$ and $q = q_f$, i.e. if $W = [\langle b; q_f; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle]$. We certainly have that $(q_f, 0, 0) \to^0 (q_f, 0, 0)$.

Inductive case: By Lemma 7.4.36, there is a set $\mathcal{P} = \{p_0, \ldots, p_r\}$ of $r + 1$ fresh atoms, a mapping $e : \mathcal{P} \to \mathcal{Q}$ and a provable structure

$$\tilde{W} = [\tilde{U}_1, \ldots, \tilde{U}_r, \langle b; a^n; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle] \quad ,$$

with $\tilde{W}^e = W$, and such that the killer $\bar{p}_0$ of p_0 is inside $\tilde{U}_1$. Now we have six cases:

(1) $\tilde{U}_1 = (\bar{p}_0, \langle a; p_1 \rangle)$. Then

$$\tilde{W} \quad = \quad [\, \tilde{U}_2, \ldots, \tilde{U}_r, \underbrace{(\bar{p}_0, \langle a; p_1 \rangle)}_{\tilde{U}_1}, \langle b; a^n; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle \,] \quad .$$

Since $\tilde{W}$ is provable in BV, we have (by Lemma 7.4.28) that

$$\tilde{W}' \quad = \quad [\, \tilde{U}_2, \ldots, \tilde{U}_r, \langle b; a^n; a; p_1; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle \,]$$

is also provable. Let now $W' = \tilde{W}'^e$ and assume $e(p_1) = q'$. Then

$$W' \quad = \quad [\, U_1, \ldots, U_{l-1}, U_{l+1}, \ldots, U_r, \langle b; a^{n+1}; q'; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle \,] \quad ,$$

for some $l \in \{1, \ldots, r\}$. We have that W' is a weak encoding of $\mathcal{M}$ because W is a weak encoding of $\mathcal{M}$. By Lemma 7.4.20, W' is provable in BV. Hence, we can apply the induction hypothesis and get

$$(q', n + 1, m) \to^{r-1} (q_f, 0, 0) \quad .$$

Further, we have that

$$U_l = \tilde{U}_1^e = (\bar{q}, \langle a; q' \rangle) \quad .$$

Therefore $(q, \mathsf{inc1}, q') \in \mathcal{T}$. Hence

$$(q, n, m) \to (q', n + 1, m) \quad ,$$

which gives us

$$(q, n, m) \to^r (q_f, 0, 0) \quad .$$

(2) $\tilde{U}_1 = (\langle \bar{a}; \bar{p}_0 \rangle, p_1)$. Then

$$\tilde{W} \quad = \quad [\ \underbrace{(\langle \bar{a}; \bar{p}_0 \rangle, p_1)}_{\tilde{U}_1}, \tilde{U}_2, \dots, \tilde{U}_r, \langle b; a^n; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle\ ] \quad .$$

Mark inside $\tilde{W}$ the atom $\bar{a}$ inside $\tilde{U}_1$ by $\bar{a}^{\bullet}$ and its killer by $a^{\bullet}$. By way of contradiction, assume now that $a^{\bullet}$ occurs inside $\tilde{U}_l = (\bar{p}_{l-1}, \langle a^{\bullet}; p_l \rangle)$ for some $l \in \{2, \dots, r\}$. This means that

$$\tilde{W} \quad = \quad [\ \underbrace{(\langle \bar{a}^{\bullet}; \bar{p}_0 \rangle, p_1)}_{\tilde{U}_1}, \tilde{U}_2, \dots, \tilde{U}_{l-1}, \underbrace{(\bar{p}_{l-1}, \langle a^{\bullet}; p_l \rangle)}_{\tilde{U}_l}, \tilde{U}_{l+1}, \dots, \tilde{U}_r,$$
$$\langle b; a^n; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle\ ] \quad .$$

But then $\tilde{W}$ contains a negation circuit:

$$[(\bar{a}^{\bullet}, p_1), (\bar{p}_1, p_2), \dots, (\bar{p}_{l-1}, a^{\bullet})] \quad ,$$

which is (by Proposition 7.4.30) a contradiction to the provability of $\tilde{W}$. Hence, the atom $a^{\bullet}$ must occur inside the encoding of the configuration, which means that $n > 0$. Further, we have that

$$\tilde{W} \quad = \quad [\ \underbrace{(\langle \bar{a}^{\bullet}; \bar{p}_0 \rangle, p_1)}_{\tilde{U}_1}, \tilde{U}_2, \dots, \tilde{U}_r, \langle b; a^{n'}; a^{\bullet}; a^{n''}; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle\ ] \quad ,$$

for some n', n'' with $n = n' + 1 + n''$. I will now show that $n'' = 0$. For this, assume by way of contradiction, that $n'' > 0$. Mark the first atom a in $a^{n''}$ by a° and its killer by $\bar{a}^{\circ}$. Then $\bar{a}^{\circ}$ must occur inside $\tilde{U}_k = (\langle \bar{a}^{\circ}; \bar{p}_{k-1} \rangle, p_k)$ for some $k \in \{2, \dots, r\}$. Then we have that

$$\tilde{W} \quad = \quad [\ \underbrace{(\langle \bar{a}^{\bullet}; \bar{p}_0 \rangle, p_1)}_{\tilde{U}_1}, \tilde{U}_2, \dots, \tilde{U}_{k-1}, \underbrace{(\langle \bar{a}^{\circ}; \bar{p}_{k-1} \rangle, p_k)}_{\tilde{U}_k}, \tilde{U}_{k+1}, \dots, \tilde{U}_r,$$
$$\langle b; a^{n'}; a^{\bullet}; a^{\circ}; a^{n''-1}; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle\ ] \quad .$$

But then $\tilde{W}$ contains a negation circuit:

$$[\langle a^{\bullet}; a^{\circ} \rangle, \langle \bar{a}^{\circ}; \bar{p}_{k-1} \rangle, (p_{k-1}, \bar{p}_{k-2}), \dots, (p_2, \bar{p}_1), (p_1, \bar{a}^{\bullet})] \quad ,$$

which is (by Proposition 7.4.30) a contradiction to the provability of $\tilde{W}$. Hence, the atom a° cannot exist, which means that $n'' = 0$ and $n' = n - 1$. This means that

$$\tilde{W} \quad = \quad [\ \tilde{U}_2, \dots, \tilde{U}_r, \underbrace{(\langle \bar{a}^{\bullet}; \bar{p}_0 \rangle, p_1)}_{\tilde{U}_1}, \langle b; a^{n-1}; a^{\bullet}; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle\ ] \quad .$$

Since this is provable in BV, we have (by Lemma 7.4.28) that

$$\tilde{W}' \quad = \quad [\ \tilde{U}_2, \dots, \tilde{U}_r, \langle b; a^{n-1}; p_1; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle\ ]$$

is also provable. Let now $W' = \tilde{W}'^e$ and $e(p_1) = q'$. Then

$$W' \quad = \quad [\, U_1, \ldots, U_{l-1}, U_{l+1}, \ldots, U_r, \langle b; a^{n-1}; q'; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle \,] \quad,$$

for some $l \in \{1, \ldots, r\}$. As before, W' is a weak encoding of $\mathcal{M}$ and (by Lemma 7.4.20) provable in BV. Hence, we can apply the induction hypothesis and get

$$(q', n-1, m) \rightarrow^{r-1} (q_f, 0, 0) \quad.$$

Further, we have that

$$U_l = \tilde{U}_1^e = (\langle \bar{a}; \bar{q} \rangle, q') \quad.$$

Therefore $(q, \mathtt{dec1}, q') \in \mathscr{T}$. Since we also have $n > 0$, we have

$$(q, n, m) \rightarrow (q', n-1, m) \quad,$$

which gives us

$$(q, n, m) \rightarrow^r (q_f, 0, 0) \quad.$$

(3) $\tilde{U}_1 = (\langle \bar{b}; \bar{p}_0 \rangle, \langle b; p_1 \rangle)$. Then

$$\tilde{W} \quad = \quad [\, \underbrace{(\langle \bar{b}; \bar{p}_0 \rangle, \langle b; p_1 \rangle)}_{\tilde{U}_1}, \tilde{U}_2, \ldots, \tilde{U}_r, \langle b; a^n; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle \,] \quad.$$

Mark inside $\tilde{W}$ the atom $\bar{b}$ inside $\tilde{U}_1$ by $\bar{b}^\bullet$ and its killer by $b^\bullet$. By way of contradiction, assume now that $b^\bullet$ occurs inside $\tilde{U}_l = (\langle \bar{b}; \bar{p}_{l-1} \rangle, \langle b^\bullet; p_l \rangle)$ for some $l \in \{2, \ldots, r\}$ (it can certainly not be inside $\tilde{U}_1$). Then we have that

$$\tilde{W} \quad = \quad [\, \underbrace{(\langle \bar{b}^\bullet; \bar{p}_0 \rangle, \langle b; p_1 \rangle)}_{\tilde{U}_1}, \tilde{U}_2, \ldots, \tilde{U}_{l-1}, \underbrace{(\langle \bar{b}; \bar{p}_{l-1} \rangle, \langle b^\bullet; p_l \rangle)}_{\tilde{U}_l}, \tilde{U}_{l+1}, \ldots, \tilde{U}_r,$$
$$\langle b; a^n; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle \,] \quad.$$

But then we have a negation circuit inside $\tilde{W}$:

$$[(\bar{b}^\bullet, p_1), (\bar{p}_1, p_2), \ldots, (\bar{p}_{l-1}, b^\bullet)] \quad,$$

which is (by Proposition 7.4.30) a contradiction to the provability of $\tilde{W}$. Hence, the atom $b^\bullet$ must occur inside the encoding of the configuration. This means that

$$\tilde{W} \quad = \quad [\, \underbrace{(\langle \bar{b}^\bullet; \bar{p}_0 \rangle, \langle b; p_1 \rangle)}_{\tilde{U}_1}, \tilde{U}_2, \ldots, \tilde{U}_r, \langle b^\bullet; a^n; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle \,] \quad.$$

I will now show that $n = 0$. For this, assume by way of contradiction, that $n > 0$. Mark the first atom a in a^n by a° and its killer by $\bar{a}^\circ$. Then $\bar{a}^\circ$ must occur inside $\tilde{U}_k = (\langle \bar{a}^\circ; \bar{p}_{k-1} \rangle, p_k)$ for some $k \in \{2, \ldots, r\}$. This means that

$$\tilde{W} \quad = \quad [\, \underbrace{(\langle \bar{b}^\bullet; \bar{p}_0 \rangle, \langle b; p_1 \rangle)}_{\tilde{U}_1}, \tilde{U}_2, \ldots, \tilde{U}_{k-1}, \underbrace{(\langle \bar{a}^\circ; \bar{p}_{k-1} \rangle, p_k)}_{\tilde{U}_k}, \tilde{U}_{k+1}, \ldots, \tilde{U}_r,$$
$$\langle b^\bullet; a^\circ; a^{n-1}; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle \,] \quad.$$

But then $\tilde{W}$ contains a negation circuit:

$$[\langle b^\bullet; a^\circ \rangle, \langle \bar{a}^\circ; \bar{p}_{k-1} \rangle, (p_{k-1}, \bar{p}_{k-2}), \ldots, (p_2, \bar{p}_1), (p_1, \bar{b}^\bullet)] \quad ,$$

which is (by Proposition 7.4.30) a contradiction to the provability of $\tilde{W}$. Hence, the atom a° cannot exist, which means that $n = 0$. This means that

$$\tilde{W} \quad = \quad [\ \tilde{U}_2, \ldots, \tilde{U}_r, \underbrace{(\langle \bar{b}^\bullet; \bar{p}_0 \rangle, \langle b; p_1 \rangle)}_{\tilde{U}_1}, \langle b^\bullet; p_0; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle \ ] \quad .$$

Since this is provable in BV, we have (by Lemma 7.4.28) that

$$\tilde{W}' \quad = \quad [\ \tilde{U}_2, \ldots, \tilde{U}_r, \langle b; p_1; c^m; d \rangle, \langle \bar{b}; \bar{p}_r; \bar{d} \rangle \ ]$$

is also provable. Let now $W' = \tilde{W}'^e$ and $e(p_1) = q'$. Then

$$W' \quad = \quad [\ U_1, \ldots, U_{l-1}, U_{l+1}, \ldots, U_r, \langle b; q'; c^m; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle \ ] \quad ,$$

for some $l \in \{1, \ldots, r\}$. As before, W' is a weak encoding of $\mathcal{M}$ and (by Lemma 7.4.20) provable in BV. Hence, we can apply the induction hypothesis and get

$$(q', 0, m) \rightarrow^{r-1} (q_f, 0, 0) \quad .$$

Further, we have that

$$U_l = \tilde{U}_1^e = (\langle \bar{b}; \bar{q} \rangle, \langle b; q' \rangle) \quad .$$

Therefore $(q, \mathtt{zero1}, q') \in \mathcal{T}$. Since we also have $n = 0$, we have

$$(q, 0, m) \rightarrow (q', 0, m) \quad ,$$

which gives us

$$(q, n, m) \rightarrow^r (q_f, 0, 0) \quad .$$

(4) $\tilde{U}_1 = (\bar{p}_0, \langle p_1; c \rangle)$. Similar to (1).

(5) $\tilde{U}_1 = (\langle \bar{p}_0; \bar{c} \rangle, p_1)$. Similar to (2).

(6) $\tilde{U}_1 = (\langle \bar{p}_0; \bar{d} \rangle, \langle p_1; d \rangle)$. Similar to (3). $\qquad \square$

7.4.39 **Proposition** *Given a two counter machine* $\mathcal{M} = (\mathcal{Q}, q_0, n_0, m_0, q_f, \mathcal{T})$.

$$If \quad \underset{\mathcal{M}_{\mathrm{enc}}}{\overset{\|\mathrm{NEL}}{}} \quad then \quad (q_0, n_0, m_0) \rightarrow^* (q_f, 0, 0) \quad .$$

Proof: First apply Lemma 7.4.34 to get

$$\underset{\mathcal{M}_{\mathrm{enc}}}{\overset{\|\mathrm{BV}}{\underset{\|\{\mathsf{w}\downarrow, \mathsf{b}\downarrow\}}{W}}} \quad ,$$

where W is a weak encoding of $\mathcal{M}$. Since the rules $\mathsf{w}\downarrow$ and $\mathsf{b}\downarrow$ cannot modify the substructure $\langle b; a^{n_0}; q_0; c^{m_0}; d \rangle$ of $\mathcal{M}_{\mathrm{enc}}$, this substructure must still be present in W. Hence, we have that

$$W = [U_1, \ldots, U_r, \langle b; a^{n_0}; q_0; c^{m_0}; d \rangle, \langle \bar{b}; \bar{q}_f; \bar{d} \rangle] \quad ,$$

for some $r \geqslant 0$. By Lemma 7.4.38, we have that $(q_0, n_0, m_0) \rightarrow^* (q_f, 0, 0)$. $\qquad \square$

Proof of Theorem 7.4.11: For the first direction use Proposition 7.4.16 and for the second direction use Proposition 7.4.39. $\qquad \square$

7.5   Discussion

The results presented in this chapter were the original motivation for this work, namely, to extend A. Guglielmi's system BV by the exponentials of linear logic, prove cut elimination for the new system, and show that it is undecidable. Proving cut elimination turned out to be more difficult than expected in the beginning. The techniques of decomposition and splitting had to be developed for that. The impact of decomposition on cut elimination has already been discussed before. The technique of splitting is the most general technique for proving cut elimination in the calculus of structures that has been developed so far. The technique of permuting rules, as it has been used in Chapter 4, seems to fail when more than two binary connectives are present, as it is the case in full linear logic and in NEL. Using semantical arguments, as done in [BT01], cannot be applied if no (truth value) semantics is available, as in the case of BV and NEL. The traditional method of the sequent calculus, as done in the first proof of Theorem 3.4.1 on page 42, cannot be applied if no sequent calculus system is available, as in the case of BV and NEL.

Also the proof of the undecidability of system NEL offered some unexpected difficulties because the known techniques of using phase semantics [Laf96, LS96, Kan00] or extracting the computation sequence from a sequent calculus proof [LMSS92, Kan95] could not be applied. In fact, finding a proof of the undecidability of NEL was the original motivation for inventing the notion of negation circuit. Only afterwards I realised the equivalence to Ch. Retoré's cordless Æ-circuits [Ret99b].

There is another interesting observation to make about the undecidability result. In the case of propositional linear logic, the additives are added to MELL in order to get undecidability. If the additives are added to MLL, which is NP-complete, then the complexity of the provability problem jumps to PSPACE. In the case of NEL, the *seq* is added to MELL to get undecidability. But if we add *seq* to MLL, then the complexity class is not increased. Provability of BV is still in NP. This means that *seq* is in some sense weaker than the additives. This observation is also justified by the fact that the two decomposition theorems for SELS (Section 4.3) could be generalised to the case where *seq* is added, but (so far) not to the case where the additives are added.

8

Open Problems

In this chapter I have collected observations, conjectures, and vague sketched research problems, that are related to the results presented in this thesis and that I consider important. I believe that they deserve further investigation but I did not have the time to go deeper into the matter. (The order in which they are listed here is randomly chosen and does not imply any preference order.)

8.1 Quantifiers

The calculus of structures is also able to deal with quantifiers. The rules for classical first order predicate logic have already been shown in [BT01, Brü03b]. The rules for the first order quantifiers in linear logic are very similar:

$$\mathsf{u_1}{\downarrow}\,\frac{S\{\forall x.[R,T]\}}{S[\forall x.R, \exists x.T]} \qquad \text{and} \qquad \mathsf{n_1}{\downarrow}\,\frac{S\{R\{x \leftarrow t\}\}}{S\{\exists x.R\}} \quad,$$

where the De Morgan laws and the equations

$$\forall x.R = R = \exists x.R \qquad \text{if } x \text{ is not free in } R$$

are added to the equational theory. The two advantages over the sequent calculus rules that already occur in the classical case are also present in linear logic: First, both rules are sound in the sense that the premise implies the conclusion, without any further quantifications, and second, there is no need for a proviso saying that the variable x is not free in the conclusion of the rule (the proviso has moved to the equational theory).

This pattern can also be continued to the second order propositional quantifiers, where the rules are similar. If we add the quantifiers to the language in the obvious way, together with the equations for the De Morgan laws and

$$\forall a.R = R = \exists a.R \quad , \quad \text{if } a \text{ is not free in } R,$$

we can extend system SLS by the following rules:

$$\mathsf{u_2\downarrow}\,\frac{S\{\forall a.[R,T]\}}{S[\forall a.R,\exists a.T]}\qquad,\qquad \mathsf{u_2\uparrow}\,\frac{S(\exists a.R,\forall a.T)}{S\{\exists a.(R,T)\}}\qquad,$$

$$\mathsf{n_2\downarrow}\,\frac{S\{R\{T\leftarrow a\}\}}{S\{\exists a.R\}}\qquad,\qquad \mathsf{n_2\uparrow}\,\frac{S\{\forall a.R\}}{S\{R\{T\leftarrow a\}\}}\qquad.$$

Again, observe that there is no proviso saying that the propositional variable a is not allowed to be free in the context, as it is the case in the sequent calculus.

As already observed in [BT01, Brü03b], it is possible even under the presence of the quantifiers to reduce contraction to its atomic version, by adding the rules

$$\mathsf{l_3\downarrow}\,\frac{S\{\!\!|\forall a.R,\forall a.T|\!\!\}}{S\{\forall a.\{\!\!|R,T|\!\!\}\}}\qquad,\qquad \mathsf{l_3\uparrow}\,\frac{S\{\exists a.(\!\!|R,T|\!\!)\}}{S(\!\!|\exists a.R,\exists a.T|\!\!)}\qquad,$$

$$\mathsf{l_4\downarrow}\,\frac{S\{\!\!|\exists a.R,\exists a.T|\!\!\}}{S\{\exists a.\{\!\!|R,T|\!\!\}\}}\qquad,\qquad \mathsf{l_4\uparrow}\,\frac{S\{\forall a.(\!\!|R,T|\!\!)\}}{S(\!\!|\forall a.R,\forall a.T|\!\!)}\qquad.$$

However, even if contraction is atomic, the new system can no longer be called local because in the rule $\mathsf{n_2\downarrow}$ a structure of arbitrary size is introduced.

The question that arises now is whether the various properties of proofs and derivations that I presented in this thesis are still valid. In particular, what happens with splitting, context reduction and decomposition?

8.2   A General Recipe for Designing Rules

The core part of each system in the calculus of structures can be given by a very simple recipe [Gug02c].

Consider the core rules of the systems shown in this work. Let us start with the rules for the binary connectives.

$$\mathsf{d\downarrow}\,\frac{S(\!\!|[R,U],[T,V]|\!\!)}{S[(\!\!|R,T|\!\!),\{\!\!|U,V|\!\!\}]}\qquad,\qquad \mathsf{d\uparrow}\,\frac{S(\{\!\!|R,U|\!\!\},(\!\!|T,V|\!\!))}{S\{\!\!|(R,T),(U,V)|\!\!\}}\qquad,$$

$$\mathsf{q\downarrow}\,\frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]}\qquad,\qquad \mathsf{q\uparrow}\,\frac{S(\langle R;U\rangle,\langle T;V\rangle)}{S\langle(R,T);(U,V)\rangle}\qquad.$$

They follow a certain scheme, which can (by a little abuse of notation) be written as:

$$\mathsf{\times\downarrow}\,\frac{S\{[R,U]\bullet[T,V]\}}{S[R\bullet T,U\circ V]}\quad\text{and}\quad \mathsf{\times\uparrow}\,\frac{S(R\circ U,T\bullet V)}{S\{(R,T)\circ(U,V)\}}\qquad,$$

where $\bullet$ and $\circ$ are two binary connectives that are (by De Morgan) dual to each other. For example conjunction and disjunction. It has to be postulated which one is $\bullet$ (usually

conjunction) and remains in the premise of the rule. Observe that also the switch follows this scheme because it is a special case of the two rules

$$\mathsf{s}{\downarrow} \frac{S([R,U],[T,V])}{S[(R,T),U,V]} \quad \text{and} \quad \mathsf{s}{\uparrow} \frac{S([R,U],T,V)}{S[(R,T),(U,V)]} \quad,$$

and can mimic both of them.

The rules for the exponentials and the quantifiers (see previous section) follow the same scheme:

$$\mathsf{p}{\downarrow} \frac{S\{![R,T]\}}{S[!R,?T]} \quad , \quad \mathsf{p}{\uparrow} \frac{S(?R,!T)}{S\{?(R,T)\}} \quad,$$

$$\mathsf{u_2}{\downarrow} \frac{S\{\forall a.[R,T]\}}{S[\forall a.R, \exists a.T]} \quad , \quad \mathsf{u_2}{\uparrow} \frac{S(\exists a.R, \forall a.T)}{S\{\exists a.(R,T)\}} \quad.$$

Furthermore, this pattern does also work for modal logics [SS03]. We get the following rules:

$$\mathsf{p}{\downarrow} \frac{S\{\Box[R,T]\}}{S[\Box R, \Diamond T]} \quad , \quad \mathsf{p}{\uparrow} \frac{S(\Diamond R, \Box T)}{S\{\Diamond(R,T)\}} \quad,$$

which are valid in all normal modal logics. This has already been observed in [Dun95].

This scheme is responsible for the possible reduction of the interaction rules to their atomic version. It plays also a crucial role in the cut elimination argument because splitting and context reduction is always possible (and the proof becomes very simple) if only the core fragment is considered.

The rules that are problematic are the noncore rules. Since the interaction fragment is fixed in every system, and the core fragment is determined by the recipe described above, it is only the noncore fragment that determines the behaviour of the connectives and modalities.

A closer inspection of system SLLS shows that also the noncore rules follow a certain pattern. In particular, all the medial rules have the same shape. However, there are also rules which do not have the "medial shape", for example the atomic thinning and contraction rules. The desire to have a single rule scheme which governs all fragments—interaction, core and noncore—led to the discovery of *subatomic logic* in [Gug02d]. (The reader should be warned that the research on subatomic logic is still at a very experimental stage.)

8.3 The Relation between Decomposition and Cut Elimination

There is a close relationship between the second decomposition theorem and cut elimination, in the sense that they imply each other, provided the system is "well-designed". The question is, of course, to find the right definition of "well-designed". In order to state the second decomposition theorem, we must be able to distinguish between the following six fragments:

<div align="center">

interaction down   interaction up

core down   core up

noncore down   noncore up

</div>

Let me now call the noncore of a system "well-designed" if every proof $\left\|\substack{\text{noncore up}}\right.$ is

trivial and every proof $\left\|\substack{\text{down fragment}}\right.$ can be decomposed into

$$
R
$$

$$
\begin{array}{c} R \\ \left\|\substack{\text{noncore up}} \right. \\ R \end{array}
$$

$$
\begin{array}{c} \left\|\substack{\text{interaction down}\,\cup\,\text{core down}}\right. \\ R' \\ \left\|\substack{\text{noncore down}}\right. \\ R \end{array}.
$$

Then we have the following:

8.3.1 Observation A system with a "well-designed" noncore admits a cut elimination result if and only if the second decomposition theorem holds for it.

Proof: I will only sketch the proof here. For the first direction proceed exactly as in the proofs of Theorems 5.3.2 and 5.3.25. The separation of interaction from the core is always possible. (Observe that both fragments of the core are needed. It is in general not possible to separate interaction down from core down.) For the other direction proceed similar as in the case of system SNEL. As mentioned before, splitting and context reduction (and therefore also cut elimination) do always hold if there is no noncore to consider. □

The definition of "well-designed" that I gave is is certainly not a good definition. First, for pragmatic reasons "well-designed" should imply cut elimination and second, the two conditions are too sophisticated for a basic definition. Anyway, what I want to show here is that cut elimination and the second decomposition theorem are based on the same underlying principles. The question is, what exactly those principles are. It might well be that *subatomic logic* [Gug02d] can help to unveil them.

8.4 Controlling the Nondeterminism

The systems presented in this thesis are not very well suited for proof search. The reason is the huge amount of nondeterminism intrinsically connected to the calculus of structures. It should therefore be a topic of future research to find ways of controlling this nondeterminism. This might be possible by means of the permutability of rules, as it already happened in the sequent calculus [And92, Mil96], but it might also be possible by totally different methods. A. Guglielmi suggested to use the technique of splitting.

8.5 The Equivalence between System BV and Pomset Logic

I have already discussed this problem in Chapter 7. Let me summarise here, what has already been achieved and where the difficulty lies.

There is a one-to-one correspondence between A. Guglielmi's BV structures and Ch. Retoré's series-parallel R&B proof structures, which I explained in Chapter 6. For more detailed information consult [Gug02e], where the correspondence between BV structures and relation webs is explained. From a graph theoretical viewpoint, relation webs and series-parallel R&B proof structures are the same objects.

In Proposition 7.4.30, I have shown that a given BV structure is not provable if it contains a (balanced) negation circuit. In other words, it is not provable if the corresponding

series-parallel R&B proof structure is not a proof net according to Ch. Retoré's correctness criterion.

It therefore only remains to show the other direction, namely, for every series-parallel R&B proof structure that is a proof net, the corresponding BV structure is provable. The problem lies in actually constructing the proof. The method that I employed in Section 6.2 cannot be used because in the presence of *seq* it is no longer true that an arbitrary pair of atoms can be picked and "moved closer together". The method used in [BdGR97] cannot be employed because the *times* does not behave the same way as the *seq* with respect to the *par*. More precisely, the rule

$$\mathsf{ws}\ \frac{S([R,U],[T,V])}{S[(R,T),(U,V)]}$$

is unsound (see also [Gug02e] and [Ret99b]).

8.6 The Decidability of MELL

The question whether the provability in MELL is decidable or not is probably the best-known open problem in the area of linear logic. Apart from the fact that I believe that it is decidable, I cannot contribute much to this problem. However, I think it should be added it to this list because it is related to this thesis in two ways. First, I devoted a whole chapter (in fact, the longest chapter of this thesis) to multiplicative exponential linear logic, and second, the undecidability of NEL increases the significance of that problem.

8.7 The Equivalence of Proofs in the Calculus of Structures

In the calculus of structures, rules can be applied with more freedom than in the sequent calculus. They have a smaller granularity and a larger applicability. Consequently, there are many proofs in the calculus of structures that correspond to the same sequent calculus proof, because of the different possible sequencing of rules, and because rules in the calculus of structures have smaller granularity and larger applicability. This means that the general problem of the sequent calculus, where proofs exhibit a lot of redundancy in the sense that inactive formulae are copied many times and that two sequent proofs can represent "essentially" the same proof up to the order of rule applications, is also present in the calculus of structures (even in a more drastic form).

For the sequent calculus of linear logic, proof nets have been developed [Gir87a], which identify proofs that vary only in some irrelevant permutation of rules. This works particularly well for multiplicative linear logic. The questions that arises now is whether the theory of proof nets can also be used to identify proofs in the calculus of structures, or whether we can find other methods for identifying proofs in the calculus of structures that can then be used also for other logics (like classical logic) for which a theory proof nets has not yet been developed.

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Index

absorption, 32
accepts, 233
active, 87
additive, 5, 32
additive group, 19, 32
additives, 18
admissible, 30
Æ-circuit, 185, 187, 240
atom, 17, 25, 180, 198
atomic cocontraction, 147
atomic cointeraction, 33
atomic contraction, 147, 152
atomic cothinning, 149
atomic cut, 33
atomic interaction, 33, 160
atomic thinning, 149, 173
axiom, 18, 28

balanced, 186, 213
balanced negation circuit, 239
basic context, 108, 204
bottom, 17, 19, 26
Brouwer-Heyting-Kolmogorov interpretation, 4

calculus of structures, 9
CCS, 6
chain, 105
   !-chain, 101, 104
   ?-chain, 101, 104
characteristic number, 106
circuit, 213
classical logic, 4, 148, 149, 154, 167
clean, 237
coabsorption, 32
coadditive, 32
cocircuit, 213
cocontraction, 32
cointeraction, 30
commutative case, 22

conclusion, 18, 19, 28, 29
configuration, 232
connected, 102
connectives
   $\invamp$, 17
   $\oplus$, 17
   $\otimes$, 17
   $\&$, 17
conservative extension, 201
constants, 17, 26
   $\bot$, 17, 26
   1, 17, 26
   0, 17, 26
   $\top$, 17, 26
contains
   a balanced negation circuit, 239
   a circuit, 187, 214
   a cocircuit, 187, 214
   a negation circuit, 187, 214
context, 27
   n-ary, 101
   LS, 27
context reduction, 44, 68, 215, 216, 224, 251
   for LLS, 151
   for LS, 69
   for NELm, 224
contraction, 20, 32, 144, 152
contractum, 87
copromotion, 32
core, 34, 199, 250
coseq, 198
cothinning, 32
coweakening, 32
creation, 96
croissant, 5
   infinite supply, 6
Curry-Howard-isomorphism, 1
cut, 19, 30

263