A logical basis for quantum evolution and entanglement

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Dedicated to Jim Lambek on the occasion of his 90th birthday.

Abstract. We reconsider *discrete quantum causal dynamics* where quantum systems are viewed as discrete structures, namely directed acyclic graphs. In such a graph, events are considered as vertices and edges depict propagation between events. Evolution is described as happening between a special family of spacelike slices, which were referred to as *locative slices*. Such slices are not so large as to result in acausal influences, but large enough to capture nonlocal correlations.
In our logical interpretation, edges are assigned logical formulas in a special logical system, called BV, an instance of a *deep inference system*. We demonstrate that BV, with its mix of commutative and noncommutative connectives, is precisely the right logic for such analysis. We show that the commutative tensor encodes (possible) entanglement, and the noncommutative seq encodes causal precedence. With this interpretation, the locative slices are precisely the derivable strings of formulas. Several new technical results about BV are developed as part of this analysis.

1 Introduction

The subject of this paper is the analysis of the evolution of quantum systems. Such systems may be protocols such as *quantum teleportation* [1]. But we have a more general notion of system in mind. Of course the key to the success of the teleportation protocol is the possibility of entanglement of particles. Our analysis will provide a syntactic way of describing and analyzing such entanglements, and their evolution in time.

This subject started with the idea that since the monoidal structure of the category of Hilbert spaces, i.e. the tensor product, provides a basis for understanding entanglement, the more general theory of monoidal categories could provide a more abstract and general setting. The idea of using general monoidal categories in place of the specific category of Hilbert spaces can be found in a number of sources, most notably [2], where it is shown that the notion of a symmetric compact closed dagger monoidal category is the correct level of abstraction to encode and prove the correctness of protocols. Subsequent work in this area can be found in [3], and the references therein.

A natural step in this program is to use the logic underlying monoidal categories as a syntactic framework for analyzing such quantum systems. But more than that is possible. While a logic does come with a syntax, it also has a built-in notion of dynamics, given by the cut-elimination procedure. In intuitionistic logic, the syntax is given by simply-typed λ-calculus, and dynamics is then given by β-reduction [4]. In linear logic, the syntax for specifying proofs is given by *proof nets* [5]. Cut-elimination takes the form of a local graph rewriting system.

In [6], it is shown that causal evolution in a discrete system can be modelled using monoidal categories. The details are given in the next section, but one begins with a directed, acyclic graph, called a *causal graph*. The nodes of the graph represent events, while the edges represent flow of particles between events. The dynamics is represented by assigning to each edge an object in a monoidal category and each vertex a morphism with domain the tensor of the incoming edges and codomain the tensor of the outgoing edges. Evolution is described as happening between a special family of spacelike slices, which were referred to as *locative slices*. Locative slices differ from the *maximal slices* of Markopolou [7]. Locative slices are not so large as to result in acausal influences, but large enough to capture nonlocal correlations.

In a longer unpublished version of [6], see [8], a first logical interpretation of this semantics is given. We assign to each edge a (linear) logical formula, typically an atomic formula. Then a vertex is assigned a sequent, saying that the conjunction (linear tensor) of the incoming edges entails the disjunction (linear par) of the outgoing edges. One uses logical deduction via the cut-rule to model the evolution of the system. There are several advantages to this logical approach. Having two connectives, as opposed to the single tensor, allows for more subtle encoding. We can use the linear par to indicate that two particles are (potentially) entangled, while linear tensor indicates two unentangled particles. Application of the cut-rule is a purely local phenomenon, so this logical approach seems to capture quite nicely the interaction between the local nature of events and the nonlocal nature of entanglement. But the earlier work ran into the problem that it could not handle all possible examples of evolution. Several specific examples were given. The problem was that over the course of a system evolving, two particles which had been unentangled can become entangled due to an event that is nonlocal to either. The simple linear logic calculus had no effective way to encode this situation. A solution was proposed, using something the authors called *entanglement update*, but it was felt at the time that more subtle encoding, using more connectives, should be possible.

Thus enters the new system of logics which go under the general name *deep inference*. Deep inference is a new methodology in proof theory, introduced in [9] for expressing the logic BV, and subsequently developed to the point that all major logics can be expressed with deep-inference proof systems (see [10] for a complete overview). Deep inference is more general than traditional Gentzen proof theory because proofs can be freely composed by the logical operators, instead of having a rigid formula-directed tree structure. This induces a new symmetry, which can be exploited for achieving locality of inference rules, and which is not generally achievable with Gentzen methods. Locality, in turn, makes it possible to use new methods, often with a geometric flavour, in the normalisation theory of proof systems.

Remarkably, the additional expressive power of deep inference turns out to be precisely what is needed to fully encode the sort of discrete quantum evolution that the first paper attempted to describe. The key is the noncommutativity of the added connective seq. This gives a method of encoding causal precedence directly into the syntax in a way that the original encoding of [6] using only linear logic lacked. This is the content of Theorem 4, which asserts that there is a precise correspondence between locative slices and derivable strings of formulas in the BV logic. This technical result is of independent interest beyond its use here.

2 Evolving quantum systems along directed acyclic graphs

In earlier work [6], the basis of the representation of quantum evolution was the graph of events and causal links between them. An event could be one of the following: a unitary evolution of some subsystem, an interaction of a subsystem with a classical device (a measurement) or perhaps just the coming together or splitting apart of several spatially separated subsystems. Events will be depicted as vertices of a directed graph. The edges of the graph will represent a physical flow between the different events. The vertices of the graph are then naturally labelled with operators representing the corresponding events. We assume that there are no causal cycles; the underlying graph has to be a directed acyclic graph (DAG).

A typical dag is shown in Fig 1. The square boxes, the vertices of the dag, are events where interaction occurs. The labelled edges represent fragments of the system under scrutiny moving through spacetime. At vertex 3, for example, the components c and d come together, interact and fly apart as g and h. Each labelled edge has associated with it a Hilbert space and the state of the subsystem is represented by some density matrix. Each edge thus corresponds to a density matrix and each vertex to a physical interaction.

These dags of events could be thought of as *causal graphs* as they are an evident generalization of the causal sets of Sorkin [11]. A causal set is simply a poset, with the partial order representing causal precedence. A causal graph encodes much richer structure. So in a causal graph, we ask: What are the allowed physical effects? On physical grounds, the most general transformation of density matrices is a *completely positive, trace non-increasing map* or *superoperator* for short; see, for example, Chapter 8 of [1].

Density matrices are not just associated with edges, they are associated with larger, more distributed, subsystems as well. We need some basic terminology

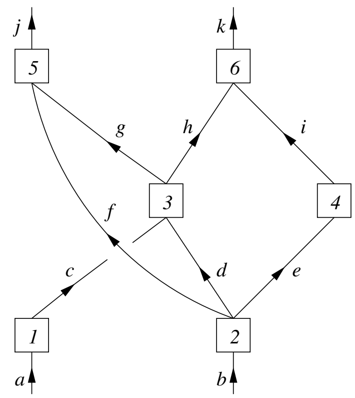

Fig. 1. A dag of events

associated with dags which brings out the causal structure more explicitly. We say that an edge e *immediately precedes* f if the target vertex of e is the source vertex of f. We say that e *precedes* f, written $e \preceq f$ if there is a chain of immediate precedence relations linking e and f, in short, "precedes" is the transitive closure of "immediately precedes". This is not quite a partial order, because we have left out reflexivity, but concepts like chain (a totally ordered subset) and antichain (a completely unordered subset) work as in partial orders.

We use the word "slice" for an antichain in the precedence order. The word is supposed to be evocative of "spacelike slice" as used in relativity, and has exactly the same significance.

A density matrix is a description of a part of a system. Thus it makes sense to ask about the density matrix associated with a part of a system that is not localized at a single event. In our dag of figure 1 we can, for example, ask about the density matrix of the portion of the system associated with the edges d, e and f. Thus density matrices can be associated with arbitrary slices. Note that it makes no sense to ask for the density matrix associated with a subset of edges that is not a slice.

The Hilbert space associated with a slice is the tensor product of the Hilbert spaces associated with the edges. Given a density matrix, say ρ, associated with, for example, the slice d, e, f, we get the density matrix for the subslice d, e by taking the partial trace over the dimensions associated with the Hilbert space f.

One can now consider a framework for evolution. One possibility, considered in [7], is to associate data with *maximal* slices and propagate from one slice to the next. Here, maximal means that to add any other vertex would destroy the antichain property. One then has to prove by examining the details of each dynamical law that the evolution is indeed causal. For example, one would like to show that the event at vertex 4 does not affect the density matrix at edge j. With data being propagated on maximal slices this does not follow automatically. One can instead work with local propagation; one keeps track of the density matrices on the individual edges only. This is indeed guaranteed to be causal, unfortunately it loses some essential nonlocal correlations. For example, the density matrices associated with the edges h and i will not reflect the fact that there might be nonlocal correlation or "entanglement" due to their common origin in the event at vertex 2. One needs to keep track of the density matrix on the slice i, h and earlier on d, e.

The main contribution of [6] was to identify a class of slices, called *locative* slices, that were large enough to keep track of all non-local correlations but "small enough" to guarantee causality.

Definition 1. *A* locative slice *is obtained as the result of taking any subset of the initial edges (all of which are assumed to be independent) and then propagating through edges without ever discarding an edge.*

In our running example, the initial slices are $\{a\}$, $\{b\}$ and $\{a, b\}$,. Just choosing for example the initial edge a as initial slice, and propagating from there gives the locatives slices $\{a\}$, $\{c\}$, $\{g, h\}$, $\{j, h\}$, $\{g, k\}$, and $\{j, h\}$.

In fact, the following is a convenient way of presenting the locative slices and their evolution[6].

$$
\begin{tikzcd}
 & \{j,k\} & \\
\{j,h\} \arrow[ur] & & \{g,k\} \arrow[ul] \\
 & \{g,h\} \arrow[ul] \arrow[ur] & \\
 & \{c\} \arrow[u] & \\
 & \{a\} \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{j,k\} & \\
\{j,h,i\} \arrow[ur] & & \{f,g,k\} \arrow[ul] \\
 & \{f,g,h,i\} \arrow[ul] \arrow[ur] & \\
\{f,g,h,e\} \arrow[ur] & & \{f,d,i\} \arrow[ul] \\
 & \{f,d,e\} \arrow[ul] \arrow[ur] & \\
 & \{b\} \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{j,k\} & \\
\{j,h,i\} \arrow[ur] & & \{f,g,k\} \arrow[ul] \\
 & \{f,g,h,i\} \arrow[ul] \arrow[ur] & \\
\{g,h,f,e\} \arrow[ur] & & \{c,f,d,i\} \arrow[ul] \\
 & \{c,f,d,e\} \arrow[ul] \arrow[ur] \arrow[uu] & \\
 & \{a,b\} \arrow[u] &
\end{tikzcd}
$$

Examples of non-locative slices are c, d, e and g, h, i and g, k. The intuition behind the concept of locativity is that one never discards information (by computing partial traces) when tracking the density matrices on locative slices. This is what allows them to capture all the non-local correlations.

The prescription for computing the density matrix on a given slice, say e, given the density matrices on the incoming slices and the superoperators at the vertices is to evolve from the minimal locative slice in the past of e to the minimal locative slice containing e. Any choice of locative slices in between may be used. The main results that we proved in [6] were that the density matrix so computed is (a) independent of the choice of the slicing (covariance) and (b) only events to the causal past can affect the density matrix at e (causality). Thus the dag and the slices form the geometrical structure and the density matrices and superoperators form the dynamics.

[6] We thank an anonymous referee for this presentation.

3 A First Logical View of Quantum Causal Evolution

3.1 The Logic of Directed Acyclic Graphs

One of the common interpretations of a dag is as generating a simple logic. (For readers not familiar with the approach to logic discussed here, we recommend [12].) The nodes of the dag are interpreted as logical sequents of the form:

$$A_1, A_2, \ldots, A_n \vdash B_1, B_2, \ldots, B_m$$

Here $\vdash$ is the logical entailment relation. Our system will have only one inference rule, called the *Cut rule*, which states:

$$\frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Sequent rules should be interpreted as saying that if one has derived the two sequents above the line, then one can infer the sequent below the line. Proofs in the system always begin with *axioms*. Axioms are of the form $A_1, A_2, \ldots, A_n \vdash B_1, B_2, \ldots, B_m$, where $A_1, A_2, \ldots, A_n$ are the incoming edges of some vertex in our dag, and $B_1, B_2, \ldots, B_m$ will be the outgoing edges. There will be one such axiom for each vertex in our dag. For example, consider Figure 1. Then we will have the following axioms:

$$a \overset{1}{\vdash} c \quad b \overset{2}{\vdash} d, e, f \quad c, d \overset{3}{\vdash} g, h \quad e \overset{4}{\vdash} i \quad f, g \overset{5}{\vdash} j \quad h, i \overset{6}{\vdash} k$$

where we have labelled each entailment symbol with the name of the corresponding vertex. The following is an example of a deduction in this system of the sequent $a, b \vdash f, g, h, i$.

$$\dfrac{\dfrac{b \vdash d, e, f \quad \dfrac{a \vdash c \quad c, d \vdash g, h}{a, d \vdash g, h}}{a, b \vdash e, f, g, h} \qquad e \vdash i}{a, b \vdash f, g, h, i}$$

Categorically, one can show that a dag canonically generates a free *polycategory* [13], which can be used to present an alternative formulation of the structures considered here.

3.2 The Logic of Evolution

We need to make the link between derivability in our logic and locativity. This is not completely trivial. One could, naively, define a set Δ of edges to be *derivable* if there is a deduction in the logic generated by G of $\Gamma \vdash \Delta$ where Γ is a set of initial edges. But this fails to capture some crucial examples. For example, consider the dag underlying the system in Figure 2. Corresponding to this dag, we get the following basic morphisms (axioms):

$$a \vdash b, c \quad b \vdash d \quad c \vdash e \quad d, e \vdash f.$$

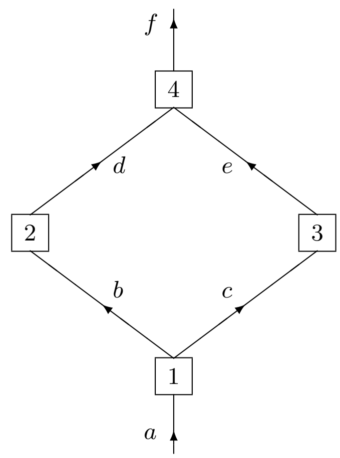

Fig. 2.

Evidently, the set $\{f\}$ is a locative slice, and yet the sequent $a \vdash f$ is not derivable. The sequent $a \vdash d, e$ is derivable, and one would like to cut it against $d, e \vdash f$, but one is only allowed to cut a single formula. Such "multicuts" are expressly forbidden, as they lead to undesirable logical properties [14].

Physically, the reason for this problem is that the sequent $d, e \vdash f$ does not encode the information that the two states at d and e are correlated. It is precisely the fact that they are correlated that implies that one would need to use a multicut. To avoid this problem, one must introduce some notation, specifically a syntax for specifying such correlations. We will use the logical connectives of the multiplicative fragment of linear logic to this end [5]. The multiplicative disjunction of linear logic, denoted $\parr$ and called the *par* connective, will express such nonlocal correlations.

In our example, we will write the sequent corresponding to vertex 4 as $d \parr e \vdash f$ to express the fact that the subsystems associated with these two edges are possibly entangled through interactions in their common past.

Note that whenever two (or more) subsystems emerge from an interaction, they are correlated. In linear logic, this is reflected by the following rule called the (right) *Par rule*:

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \parr B}$$

Thus we can always introduce the symbol for correlation in the right hand side of the sequent.

Notice that we can cut along a compound formula without violating any logical rules. So in the present setting, we would have the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{a \vdash b, c \quad b \vdash d}{a \vdash c, d} \qquad c \vdash e}{a \vdash d, e}}{a \vdash d \parr e} \qquad d \parr e \vdash f}{a \vdash f}$$

All the cuts in this deduction are legitimate; instead of a multicut we are cutting along a compound formula in the last step. So the first step in modifying our general prescription is to extend our dag logic, which originally contained only the cut rule, to include the connective rules of linear logic.

The above logical rule determines how one introduces a par connective on the righthand side of a sequent. For the lefthand side, one introduces pars in the axioms by the following general prescription.

Given a vertex in a multigraph, we suppose that it has incoming edges $a_1, a_2, \ldots, a_n$ and outgoing edges $b_1, b_2, \ldots, b_m$. In the previous formulation, this vertex would have been labelled with the axiom $\Gamma = a_1, a_2, \ldots, a_n \vdash b_1, b_2, \ldots, b_m$. We will now introduce several pars ($\parr$) on the lefthand side to indicate entanglements of the sort described above. Begin by defining a relation $\sim$ by saying $a_i \sim a_j$ if there is an initial edge c and directed paths from c to a_i and from c to a_j. This is not an equivalence relation, but one takes the equivalence relation generated by the relation $\sim$. Call this new relation $\cong$. This relation partitions the set Γ into a set of equivalence classes. One then "pars" together the elements of each equivalence class, and this determines the structure of the lefthand side of our axiom. For example, consider vertices 5 and 6 in Figure 1. Vertex 5 would be labelled by $f \parr g \vdash j$ and vertex 6 would be labelled by $h \parr i \vdash k$. On the other hand, vertex 3 would be labelled by $c, d \vdash g, h$.

Just as the par connective indicates the existence of past correlations, we use the more familiar tensor symbol $\otimes$, which is also a connective of linear logic, to indicate the lack of nonlocal correlation. This connective also has a logical rule:

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \otimes B}$$

But we note that unlike in ordinary logic, this rule can only be applied in situations that are physically meaningful.

Definition 2. *$\pi \colon \Gamma \vdash \Delta$ and $\pi' \colon \Gamma' \vdash \Delta'$ are* spacelike separated *if the following two conditions are satisfied:*

- *Γ and Γ' are disjoint subsets of the set of initial edges.*
- *The edges which make up Δ and Δ' are pairwise spacelike separated.*

In our extended dag logic, we will only allow the tensor rule to be applied when the two deductions are space like separated.

Summarizing, to every dag G we associate its "logic", namely the edges are considered as formulas and vertices are axioms. We have the usual linear

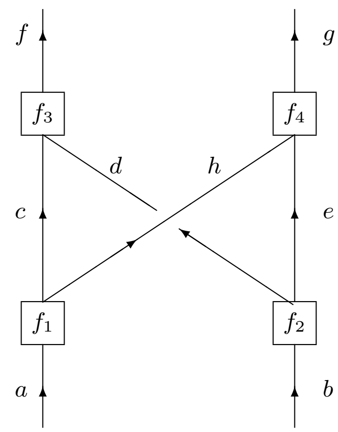

Fig. 3.

logical connective rules, including the cut rule which in our setting is interpreted physically as propagation. The par connective denotes correlation, and the tensor lack of correlation. Note that every deduction in our system will conclude with a sequent of the form $\Gamma \vdash \Delta$, where Γ is a set of initial edges.

Now one would like to modify the definition of derivability to say that a set of edges Δ is *derivable* if in our extended dag logic, one can derive a sequent $\Gamma \vdash \hat{\Delta}$ such that the list of edges appearing in $\hat{\Delta}$ was precisely Δ, and Γ is a set of initial edges. However this is still not sufficient as an axiomatic approach to capturing all locative slices. We note the example in Figure 3.

Evidently the slice $\{f, g\}$ is locative, but we claim that it cannot be derived even in our extended logic. To this directed graph, we would associate the following axioms:

$$a \vdash c, h \quad b \vdash d, e \quad c, d \vdash f \quad h, e \vdash g$$

Note that there are no correlations between c and d or between h and e. Thus no $\parr$-combinations can be introduced. Now if one attempts to derive $a, b \vdash f, g$, we proceed as follows:

$$\dfrac{\dfrac{a \vdash c, h \quad b \vdash d, e}{a, b \vdash c \otimes d, h, e} \quad \dfrac{c, d \vdash f}{c \otimes d \vdash f}}{a, b \vdash h, e, f}$$

At this point, we are unable to proceed. Had we attempted the symmetric approach tensoring h and e together, we would have encountered the same problem.

The problem is that our logical system is still missing one crucial aspect, and that is that correlations develop dynamically as the system evolves, or equivalently as the deduction proceeds. We note that this logical phenomenon is reflected in physically occurring situations. But a consequence is that our axioms must change dynamically as well. This seems to be a genuinely new logical principle.

We give the following definition.

Definition 3. *Suppose we have a deduction π of the sequent $\Gamma \vdash \Delta$ in the logic associated to the dag G, and that T is a vertex in G to the future or acausal to the edges of the set Δ with a and b among the incoming edges of T. Then a and b are* correlated *with respect to π if there exist outgoing edges c and d of the proof π and directed paths from c to a and from d to b.*

So the point here is that when performing a deduction, one does not assign an axiom to a given vertex until it is necessary to use that axiom in the proof. Then one assigns that axiom using this new notion of correlation and the equivalence relation defined above. This prescription reflects the physical reality that entanglement of local quantum subsystems could develop as a result of a distant interaction between some other subsystems of the same quantum system. We are finally able to give the following crucial definition:

Definition 4. *A set Δ of edges in a dag G is said to be* derivable *if there is a deduction in the logic associated to G of $\Gamma \vdash \hat{\Delta}$ where $\hat{\Delta}$ is a sequence of formulas whose underlying set of edges is precisely Δ and where Γ is a set of initial edges, in fact the set of initial edges to the past of Δ.*

Theorem 1. *A set of edges is derivable if and only if it is locative. More specifically, if there is a deduction of $\Gamma \vdash \hat{\Delta}$ as described above, then Δ is necessarily locative. Conversely, given any locative slice, one can find such a deduction.*

Proof. Recall that a locative slice L is obtained from the set of initial edges in its past by an inductive procedure. At each step, we choose arbitrarily a minimal vertex u in the past of L, remove the incoming edges of u and add the outgoing edges. This step corresponds to the application of a cut rule, and the method we have used of assigning the par connective to the lefthand side of an axiom ensures that it is always a legal cut. The tensor rule is necessary in order to combine spacelike separated subsystems in order to prepare for the application of the cut rule.

Thus we have successfully given an axiomatic logic-based approach to describing evolution. In summary, to find the density matrix associated to a locative slice Δ, one finds a set of linear logic formulas whose underlying set of atoms is Δ and a deduction of $\Gamma \vdash \hat{\Delta}$ where Γ is as above.

4 Using Deep Inference to Capture Locativity

In the previous sections we explained the approach of [6], using as key unit of deduction a sequent $a_1, \ldots, a_k \vdash b_1, \ldots, b_l$ meaning that the slice $\{b_1, \ldots, b_l\}$ is *reachable* from $\{a_1, \ldots, a_k\}$ by firing a number of events (vertices). However, this approach is not able to entirely capture the notion of locative slices, because correlations develop dynamically as the system evolves, or equivalently, as the deduction proceeds. Thus, we had to let axioms evolve dynamically.

The deep reason behind this problem is that the underlying logic is multiplicative linear logic (MLL): The sequent above represents the formula $a_1 \otimes \cdots \otimes a_k \multimap b_1 \parr \cdots \parr b_l$ or equivalently $a_1^\perp \parr \cdots \parr a_k^\perp \parr b_1 \parr \cdots \parr b_l$, i.e., the logic is not able see the aspect of *time* in the causality. For this reason we propose to use the logic BV, which is essentially MLL (with mix) enhanced by a third binary connective $\triangleleft$ (called *seq* or *before*) which is associative and non-commutative and self-dual, i.e., the negation of $A \triangleleft B$ is $A^\perp \triangleleft B^\perp$. It is this non-commutative connective, which allows us to properly capture quantum causality.

Of course, we are interested in expressing our logic in a deductive system that admits a complete cut-free presentation. In this case, as we briefly argue in the following, the adoption of deep inference is necessary to deal with a self-dual non-commutative logical operator.

4.1 Review of BV and Deep Inference

The significance of deep inference systems was discussed in the introduction. We note now that within the range of the deep-inference methodology, we can define several formalisms, *i.e.* general prescriptions (like the sequent calculus or natural deduction) on how to design proof systems. The first, and conceptually simplest, formalism that has been defined in deep inference is called the *calculus of structures*, or *CoS*, and this is what we adopt in this paper and call "deep inference". In fact, the fine proof-theoretic points about the various deep inference formalisms are not relevant to this paper.

The proof theory of deep inference is now well developed for classical [15], intuitionistic [16,17], linear [18,19] and modal [20,21] logics. More relevant to us, there is an extensive literature on BV and commutative/non-commutative linear logics containing BV. We cannot here provide a tutorial on BV, so we refer to its literature. In particular, [9] provides the semantic motivation and intuition behind BV, together with examples of its use. In [22], Tiu shows that deep inference is necessary for giving a cut-free deductive system for the logic BV. Kahramanoğulları proves that System BV is NP-complete [23].

We now proceed to define system BV, quickly and informally. The inference rules are:

$$\mathsf{ai}{\downarrow}\,\frac{F\{\circ\}}{F\{a \parr a^\perp\}} \qquad \mathsf{s}\,\frac{F\{A \otimes [B \parr C]\}}{F\{(A \otimes B) \parr C\}} \qquad \mathsf{ai}{\uparrow}\,\frac{F\{a \otimes a^\perp\}}{F\{\circ\}}$$

$$\mathsf{q}{\downarrow}\,\frac{F\{[A \parr C] \triangleleft [B \parr D]\}}{F\{\langle A \triangleleft B\rangle \parr \langle C \triangleleft D\rangle\}} \qquad \mathsf{q}{\uparrow}\,\frac{F\{\langle A \triangleleft B\rangle \otimes \langle C \triangleleft D\rangle\}}{F\{(A \otimes C) \triangleleft (B \otimes D)\}}$$

They have to be read as ordinary rewrite rules acting on the formulas inside arbitrary contexts $F\{\ \}$. Note that we push negation via DeMorgan equalities to the atoms, and thus, all contexts are positive. The letters A, B, C, D stand for arbitrary formulas and a is an arbitrary atom. Formulas are considered equal modulo the associativity of all three connectives $\parr$, $\triangleleft$, and $\otimes$, the commutativity of the two connectives $\parr$ and $\otimes$, and the unit laws for $\circ$, which is unit to all three connectives, i.e., $A = A \parr \circ = A \otimes \circ = A \triangleleft \circ = \circ \triangleleft A$.

Since, in our experience, working modulo equality is a sticky point of deep inference, we invite the reader to meditate on the following examples which are some of the possible instances of the $\mathsf{q}\!\downarrow$ rule:

$$\mathsf{q}\!\downarrow \frac{\langle [a \parr c] \lhd [b \parr d] \rangle \parr e}{\langle a \lhd b \rangle \parr \langle c \lhd d \rangle \parr e}, \quad \mathsf{q}\!\downarrow \frac{[\langle a \lhd b \rangle \parr c \parr e] \lhd d}{\langle a \lhd b \rangle \parr \langle c \lhd d \rangle \parr e}, \quad \mathsf{q}\!\downarrow \frac{\langle c \lhd d \lhd a \lhd b \rangle \parr e}{\langle a \lhd b \rangle \parr \langle c \lhd d \rangle \parr e}.$$

By referring to the previously defined $\mathsf{q}\!\downarrow$ rule scheme, we can see that the second instance above is produced by taking $F\{\ \} = \{\ \}$, $A = \langle a \lhd b \rangle \parr e$, $B = \circ$, $C = c$ and $D = d$, and the third instance is produced by taking $F\{\ \} = \{\ \} \parr e$, $A = c \lhd d$, $B = \circ$, $C = \circ$ and $D = a \lhd b$. The best way to understand the rules of BV is to learn their intuitive meaning, which is explained by an intuitive "space-temporal" metaphor in [9].

The set of rules $\{\mathsf{ai}\!\downarrow, \mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\}$ is called SBV, and the set $\{\mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{q}\!\downarrow\}$ is called BV. We write

$$\begin{array}{c} A \\ \Delta \Big\| \mathsf{SBV} \\ B \end{array}$$

to denote a derivation Δ from premise A to conclusion B using SBV, and we do analogously for BV.

Much like in the sequent calculus, we can consider BV a cut-free system, while SBV is essentially BV plus a cut rule. The two are related by the following theorem.

Theorem 2. *For all formulas A and B, we have*

$$\begin{array}{c} A \\ \Big\| \mathsf{SBV} \\ B \end{array} \quad \textit{if and only if} \quad \begin{array}{c} \circ \\ \Big\| \mathsf{BV} \\ A^\perp \parr B \end{array} .$$

Again, all the details are explained in [9]. Let us here only mention that the usual cut elimination is a special case of Theorem 2, for $A = \circ$. Then it says that a formula B is provable in BV iff it is provable in SBV.

Observation 3 *If a formula A is provable in BV, then every atom a occurs as often in A as $a^\perp$. This is easy to see: the only possibility for an atom a to disappear is in an instance of $\mathsf{ai}\!\downarrow$; but then at the same time an atom $a^\perp$ disappears.*

Definition 5. *A BV formula Q is called a* negation cycle *if there is a nonempty set of atoms $\mathscr{P} = \{a_0, a_2, \dots, a_{n-1}\}$, such that no two atoms in $\mathscr{P}$ are dual, $i \neq j$ implies $a_i \neq a_j$, and such that $Q = Z_0 \parr \cdots \parr Z_{n-1}$, where, for every $j = 0, \dots, n-1$, we have $Z_j = a_j \otimes a^\perp_{j+1 \pmod n}$ or $Z_j = a_j \lhd a^\perp_{j+1 \pmod n}$. We say that a formula P* contains a negation cycle *if there is a negation cycle Q such that*

- *Q can be obtained from P by replacing some atoms in P by $\circ$, and*

– *all the atoms that occur in Q occur only once in P.*

Example 1. The formula $(a \otimes c \otimes [d^\perp \parr b]) \parr c^\perp \parr \langle b^\perp \lhd [a^\perp \parr d] \rangle$ contains a negation cycle $(a \otimes b) \parr \langle b^\perp \lhd a^\perp \rangle = (a \otimes \circ \otimes [\circ \parr b]) \parr \circ \parr \langle b^\perp \lhd [a^\perp \parr \circ] \rangle$.

Proposition 1. *Let A be a* BV *formula. If P contains a negation cycle, then P is not provable in* BV.

A proof of this propostion can be found in [24, Proposition 7.4.30]. A symmetric version of this proposition has been shown for SBV in [25, Lemma 5.20].

4.2 Locativity Via BV

Let us now come back to dags. A vertex $v \in \mathscr{V}$ in such a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is now encoded by the formula

$$V = (a_1^\perp \otimes \cdots \otimes a_k^\perp) \lhd [b_1 \parr \cdots \parr b_l]$$

where $\{a_1, \ldots, a_k\} = \mathsf{target}^{-1}(v)$ is the set of edges having their target in v, and $\{b_1, \ldots, b_l\} = \mathsf{source}^{-1}(v)$ is the set of edges having their source in v. For a slice $\mathscr{S} = \{e_1, \ldots, e_n\} \subseteq \mathscr{E}$ we define its encoding to be the formula $S = e_1 \parr \cdots \parr e_n$.

Lemma 1. *Let $(\mathscr{V}, \mathscr{E})$ be a dag, let $\mathscr{S} \subseteq \mathscr{E}$ be a slice, let $v \in \mathscr{V}$ be such that $\mathsf{target}^{-1}(v) \subseteq \mathscr{S}$, and let $\mathscr{S}'$ be the propagation of $\mathscr{S}$ through v. Then there is a derivation*

$$\begin{array}{c} S \otimes V \\ \Big\| \, \mathsf{SBV} \\ S' \end{array} \tag{1}$$

where V, S, and S' are the encodings of v, $\mathscr{S}$, and $\mathscr{S}'$, respectively.

Proof. Assume $\mathsf{source}^{-1}(v) = \{b_1, \ldots, b_l\}$ and $\mathsf{target}^{-1}(v) = \{a_1, \ldots, a_k\}$ and $\mathscr{S} = \{e_1, \ldots, e_m, a_1, \ldots, a_k\}$. Then $\mathscr{S}' = \{e_1, \ldots, e_m, b_1, \ldots, b_l\}$. Now we can construct

$$\begin{array}{rc}
 & [e_1 \parr \cdots \parr e_m \parr a_1 \parr \cdots \parr a_k] \otimes \langle (a_1^\perp \otimes \cdots \otimes a_k^\perp) \lhd [b_1 \parr \cdots \parr b_l] \rangle \\ \hline
\mathsf{s} & e_1 \parr \cdots \parr e_m \parr ([a_1 \parr \cdots \parr a_k] \otimes \langle (a_1^\perp \otimes \cdots \otimes a_k^\perp) \lhd [b_1 \parr \cdots \parr b_l] \rangle) \\ \hline
\mathsf{q}\!\uparrow & e_1 \parr \cdots \parr e_m \parr \langle ([a_1 \parr \cdots \parr a_k] \otimes a_1^\perp \otimes \cdots \otimes a_k^\perp) \lhd [b_1 \parr \cdots \parr b_l] \rangle \\ \hline
\mathsf{s} & e_1 \parr \cdots \parr e_m \parr \langle ([(a_1 \otimes a_1^\perp) \parr a_2 \parr \cdots \parr a_k] \otimes \cdots \otimes a_k^\perp) \lhd [b_1 \parr \cdots \parr b_l] \rangle \\ \hline
\mathsf{ai}\!\uparrow & e_1 \parr \cdots \parr e_m \parr \langle ([a_2 \parr \cdots \parr a_k] \otimes a_2^\perp \otimes \cdots \otimes a_k^\perp) \lhd [b_1 \parr \cdots \parr b_l] \rangle \\ \hline
\mathsf{s} & \vdots \\ \hline
\mathsf{ai}\!\uparrow & e_1 \parr \cdots \parr e_m \parr \langle ([a_{k-1} \parr a_k] \otimes a_{k-1}^\perp \otimes a_k^\perp) \lhd [b_1 \parr \cdots \parr b_l] \rangle \\ \hline
\mathsf{s} & e_1 \parr \cdots \parr e_m \parr \langle ([(a_{k-1} \otimes a_{k-1}^\perp) \parr a_k] \otimes a_k^\perp) \lhd [b_1 \parr \cdots \parr b_l] \rangle \\ \hline
\mathsf{ai}\!\uparrow & e_1 \parr \cdots \parr e_m \parr \langle (a_k \otimes a_k^\perp) \lhd [b_1 \parr \cdots \parr b_l] \rangle \\ \hline
\mathsf{ai}\!\uparrow & e_1 \parr \cdots \parr e_m \parr \langle \circ \lhd [b_1 \parr \cdots \parr b_l] \rangle \\ \hline
= & e_1 \parr \cdots \parr e_m \parr b_1 \parr \cdots \parr b_l
\end{array} \quad ,$$

as desired.

Lemma 2. *Let $(\mathscr{V}, \mathscr{E})$ be a dag, let $\mathscr{S}, \mathscr{S}' \subseteq \mathscr{E}$ be slices, such that $\mathscr{S}'$ is reachable from $\mathscr{S}$ by firing a number of events (vertices). Then there is a derivation*

$$\begin{array}{c} S \otimes V_1 \otimes \cdots \otimes V_n \\ \Big\| \,\mathsf{SBV} \\ S' \end{array} \qquad (2)$$

where $V_1, \ldots, V_n$ encode $v_1, \ldots, v_n \in \mathscr{V}$ (namely, the vertices through which the slices are propagated), and S, S' encode $\mathscr{S}$, $\mathscr{S}'$.

Proof. If $\mathscr{S}'$ is reachable from $\mathscr{S}$ then there is an $n \geq 0$ and slices $\mathscr{S}_0, \ldots, \mathscr{S}_n \subseteq \mathscr{E}$ and vertices $v_1, \ldots, v_n \in \mathscr{V}$ such that for all $i \in \{1, \ldots, n\}$ we have that $\mathscr{S}_i$ is the propagation of $\mathscr{S}_{i-1}$ through v_i, and $\mathscr{S} = \mathscr{S}_0$ and $\mathscr{S}' = \mathscr{S}_n$. Now we can apply Lemma 1 n times to get the derivation (2).

Lemma 3. *Let $(\mathscr{V}, \mathscr{E})$ be a dag, let S and S' be the encodings of $\mathscr{S}, \mathscr{S}' \subseteq \mathscr{E}$, where $\mathscr{S}$ is a slice. Further, let $V_1, \ldots, V_n$ be the encodings of $v_1, \ldots, v_n \in \mathscr{V}$. If there is a proof*

$$\begin{array}{c} \Pi \,\Big\|\, \mathsf{BV} \\ V_1^\perp \parr \cdots \parr V_n^\perp \parr S^\perp \parr S' \end{array}$$

then $\mathscr{S}'$ is a slice reachable from $\mathscr{S}$ and $v_1, \ldots, v_n$ are the vertices through which it is propagated.

Proof. By induction on n. If $n = 0$, we have a proof of $S^\perp \parr S'$. Since $S^\perp$ contains only negated propositional variables, and S' only non-negated ones, we have that every atom in S' has its killer in $S^\perp$. Therefore $\mathscr{S}' = \mathscr{S}$. Let now $n \geq 1$. We can assume that $S' = e_1 \parr \cdots \parr e_m$, and that for every $i \in \{1, \ldots, n\}$ we have $V_i^\perp = [a_{i1} \parr \cdots \parr a_{ik_i}] \lhd (b_{i1}^\perp \otimes \cdots \otimes b_{il_i}^\perp)$. i.e., $\mathsf{target}^{-1}(v_i) = \{a_{i1}, \ldots, a_{ik_i}\}$ and $\mathsf{source}^{-1}(v_i) = b_{i1}, \ldots, b_{il_i}$. Now we claim that there is an $i \in \{1, \ldots, n\}$ such that $\{b_{i1}, \ldots, b_{il_i}\} \subseteq \{e_1, \ldots, e_m\}$. In other words, there is a vertex among the $v_1, \ldots, v_n$, such that all its outgoing edges are in $\mathscr{S}'$. For showing this claim assume by way of contradiction that every vertex among $v_1, \ldots, v_n$ has an outgoing edge that does not appear in $\mathscr{S}'$, i.e., for all $i \in \{1, \ldots, n\}$, there is an $s_i \in 1, \ldots, l_i$ with $b_{is_i} \notin \{e_1, \ldots, e_m\}$. By Observation 3, we must have that for every $i \in \{1, \ldots, n\}$ there is a $j \in \{1, \ldots, n\}$ with $b_{is_i} \in \{a_{j1}, \ldots, a_{jk_j}\}$, i.e., the killer of $b_{is_i}^\perp$ occurs as incoming edge of some vertex v_j. Let $\mathsf{jump}\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a function that assigns to every i such a j (there might be many of them, but we pick just one). Now let $i_1 = 1$, $i_2 = \mathsf{jump}(i_1)$, $i_3 = \mathsf{jump}(i_2)$, and so on. Since there are only finitely many V_i, we have an p and q with $p \leq q$ and $i_{q+1} = i_p$. Let us take the minimal such q, i.e., $i_p, \ldots, i_q$ are all different. Inside the proof Π above, we now replace everywhere all atoms by $\circ$, except for $b_{i_p}, b_{i_p}^\perp, \ldots, b_{i_q}, b_{i_q}^\perp$. By this, the proof remains valid and has conclusion

$$\langle b_{i_q} \lhd b_{i_p}^\perp \rangle \parr \langle b_{i_p} \lhd b_{i_{p+1}}^\perp \rangle \parr \cdots \parr \langle b_{i_{q-1}} \lhd b_{i_q}^\perp \rangle \quad ,$$

which is a contradiction to Proposition 1. This finishes the proof of the claim.

Now we can, without loss of generality, assume that v_n is the vertex with all its outgoing edges in $\mathscr{S}'$, i.e., $\{b_{n1}, \ldots, b_{nl_n}\} \subseteq \{e_1, \ldots, e_m\}$, and (again without loss of generality) $e_1 = b_{n1}, \ldots, e_{l_n} = b_{nl_n}$. Our proof Π looks therefore as follows:

$$\begin{array}{c} \Pi \Big\| \mathsf{BV} \\ V_1^\perp \parr \cdots \parr V_{n-1}^\perp \parr S^\perp \parr \underbrace{\langle [a_{n1} \parr \cdots \parr a_{nk_n}] \lhd (b_{n1}^\perp \otimes \cdots \otimes b_{nl_n}^\perp) \rangle}_{V_n^\perp} \parr S' \end{array}$$

where $S' = b_{n1} \parr \cdots \parr b_{nl_n} \parr e_{l_{n+1}} \parr \cdots \parr e_m$. In Π we can now replace the atoms $b_{n1}, b_{n1}^\perp, \ldots, b_{nl_n}, b_{nl_n}^\perp$ everywhere by $\circ$. This yields a valid proof

$$\begin{array}{c} \Pi' \Big\| \mathsf{BV} \\ V_1^\perp \parr \cdots \parr V_{n-1}^\perp \parr S^\perp \parr a_{n1} \parr \cdots \parr a_{nk_n} \parr e_{l_{n+1}} \parr \cdots \parr e_m \end{array}$$

to which we can apply the induction hypothesis, from which we can conclude that

$$\mathscr{S}'' = \{a_{n1}, \ldots, a_{nk_n}, e_{l_{n+1}}, \ldots, e_m\}$$

is a slice that is reachable from S. Clearly $\mathscr{S}'$ is the propagation of $\mathscr{S}''$ through v_n, and therefore it is a slice and reachable from $\mathscr{S}$.

Theorem 4. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a dag. A subset $\mathscr{S} \subseteq \mathscr{E}$ is a locative slice if and only if there is a derivation*

$$\begin{array}{c} I \otimes V_1 \otimes \ldots \otimes V_n \\ \Big\| \mathsf{SBV} \\ S \end{array} ,$$

where S is the encoding of $\mathscr{S}$, and I is the encoding of a subset of the initial edges, and $V_1, \ldots, V_n$ encode $v_1, \ldots, v_n \in \mathscr{V}$.

Proof. The "only if" direction follows immediately from Lemma 2. For the "if" direction, we first apply Theorem 2, and then Lemma 3.

5 Conclusion

Having a logical syntax also leads to the possibility of discussing semantics; this would be a mathematical universe in which the logical structure can be interpreted. This has the potential to be of great interest in the physical systems we are considering here, where one would want to calculate such things as expectation values. As in any categorical interpretation of a logic, one needs a category with appropriate structure to support the logical connectives and model the inference rules. The additional logical connectives of BV allows for more subtle encodings than can be expressed in a compact closed category.

The structure of BV leads to interesting category-theoretic considerations [26]. One must find a category with the following structure:

– $*$-autonomous structure, i.e. the category must be symmetric, monoidal closed and self-dual.
– an additional (noncommutative) monoidal structure commuting with the above duality.
– coherence isomorphisms necessary to interpret the logic, describing the interaction of the various tensors.

Such categories are called BV-*categories* in [26]. Of course, trivial examples abound. One can take the category Rel of sets and relations, modelling all three monoidal structures as one. Similarly the category of (finite-dimensional) Hilbert spaces, or any symmetric compact closed category would suffice. But what is wanted is a category in which the third monoidal structure is genuinely noncommutative.

While this already poses a significant challenge, we are here faced with the added difficulty that we would like the category to have some physical significance, to be able to interpret the quantum events described in this paper. Fortunately, work along these lines has already been done. See [26].

That paper considers the category of Girard's *probabilistic coherence spaces* PCS, introduced in [27]. While Girard demonstrates the $*$-autonomous structure, the paper [26] shows that the category properly models the additional noncommutative tensor of BV. We note that the paper [27] also has a notion of *quantum coherence space*, where analogous structure can be found.

Roughly, a probabilistic coherence space is a set X equipped with a set of generalized measures, i.e. functions to the set of nonnegative reals. These are called the *allowable* generalized measures. The set must be closed with respect to the double dual operation, where duality is determined by *polarity*, where we say that two generalized measures on X are polar, written $f \perp g$, if

$$\sum_{x\in X} f(x)g(x) \leq 1$$

The noncommutative connective is then modelled by the formula:

$$\begin{aligned} A \oslash B = \{\ \textstyle\sum_{i=1}^{n} f_i \otimes g_i \mid & f_i \text{ is an allowable measure on } A \text{ and} \\ & \textstyle\sum_{i=1}^{n} g_i \text{ is an allowable measure on } B\ \} \end{aligned}$$

Note the lack of symmetry in the definition. Both the categories of probabilistic and quantum coherence spaces will likely provide physically interesting semantics of the discrete quantum dynamics presented here. We hope to explore this in future work.

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