Individual Discrete Logarithm in GF(p^k)
(last step of the Number Field Sieve algorithm)

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Link with Logjam attack (K. Bhargavan's talk)

Solving actual practical problem:
Given a **fixed** finite field GF(q),

Huge massive precomputation (weeks, months, years)

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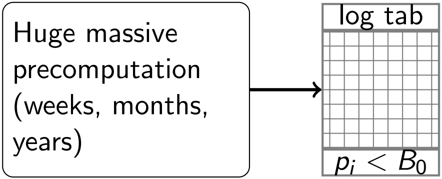

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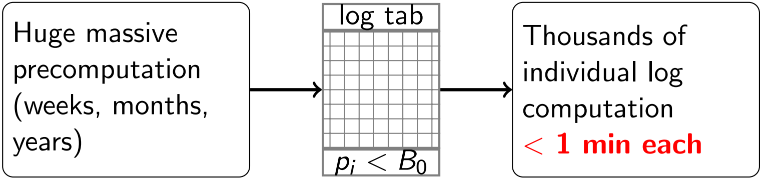

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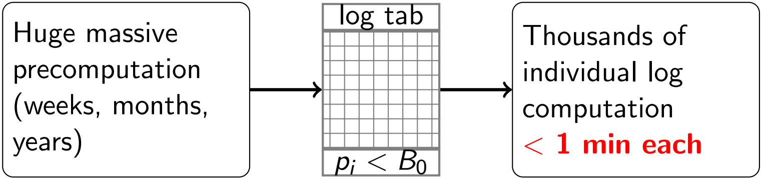

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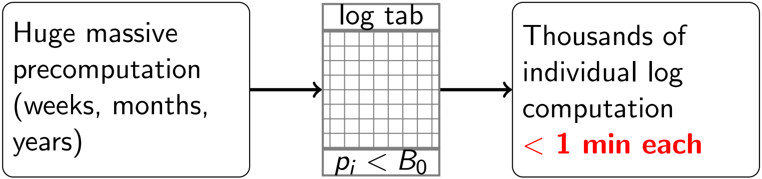

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- Pairing-based cryptosystems: GF(q) = GF(p^2), GF(p^6), GF(p^{12})

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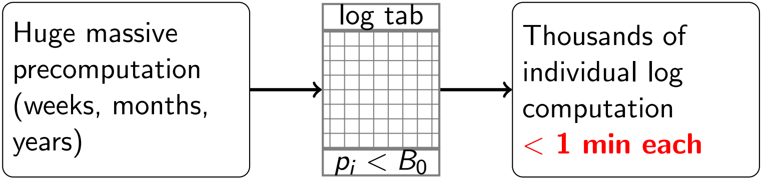

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Could we compute individual discrete logs in GF(p^2), GF(p^6), GF(p^{12}) in **less than 1 min**?

DLP in the target group of pairing-friendly curves

Why DLP in finite fields $\mathbb{F}_{p^2}, \mathbb{F}_{p^3}, \ldots$?

In a subgroup $\mathbb{G} = \langle g \rangle$ of order ℓ,

- $(g, x) \mapsto g^x$ is easy (polynomial time)
- $(g, g^x) \mapsto x$ is (in well-chosen subgroup) hard: DLP.

$$\textbf{pairing:} \quad \begin{array}{ccccc} \mathbb{G}_1 & \times & \mathbb{G}_2 & \to & \mathbb{G}_T \\ \cap & & \cap & & \cap \\ E(\mathbb{F}_p) & & E(\mathbb{F}_{p^k}) & & \mathbb{F}_{p^k}^* \end{array}$$

- where $E/\mathbb{F}_p$ is a *pairing-friendly* curve
- $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$ of large prime order ℓ (generic attacks in $O(\sqrt{\ell})$: take e.g. 256-bit ℓ)
- $1 \leq k \leq 12$ embedding degree: very specific property (specific attacks (NFS): take 3072-bit p^k)

DL records in small characteristic

✗ Small characteristic:

- supersingular curves $E/\mathbb{F}_{2^n}$: $\mathbb{G}_T \subset \mathbb{F}_{2^{4n}}$, $E/\mathbb{F}_{3^m}$: $\mathbb{G}_T \subset \mathbb{F}_{3^{6m}}$

Practical attacks (first one and most recent):

- Hayashi, Shimoyama, Shinohara, Takagi: GF($3^{6\cdot 97}$) (923 bit field) (2012)
- Granger, Kleinjung, Zumbragel: GF(2^{9234}), GF(2^{4404}) (2014)
- Adj, Menezes, Oliveira, Rodríguez-Henríquez: GF(3^{822}), GF(3^{978}) (2014)
- Joux: GF(3^{2395}) (with Pierrot, 2014), GF(2^{6168}) (2013)

Theoretical attacks: Quasi-Polynomial-time Algorithm (QPA)

- [Barbulescu Gaudry Joux Thomé 14]
- [Granger Kleinjung Zumbragel 14]

Common used pairing-friendly curves

✓ Curves over prime fields $E/\mathbb{F}_p$ where QPA does NOT apply (with $\log p \geq \log \ell \approx 256$ bits, s.t. $k \log p \geq 3072$)

- supersingular: $\mathbb{G}_T \subset \mathbb{F}_{p^2}$ ($\log p = 1536$)
- [Miyaji Nakabayashi Takano 01] (MNT): $\mathbb{G}_T \subset \mathbb{F}_{p^3}$ ($\log p = 1024$), $\mathbb{F}_{p^4}$ ($\log p = 768$), $\mathbb{F}_{p^6}$ ($\log p = 512$)
- [Barreto Naehrig 05] (BN): $\mathbb{G}_T \subset \mathbb{F}_{p^{12}}$ ($\log p = 256$, optimal)
- [Kachisa Schaefer Scott 08] (KSS): $\mathbb{G}_T \subset \mathbb{F}_{p^{18}}$ (used for 192-bit security level: 384-bit ℓ, $\log p = 512$, $k \log p = 9216$)

Theoretical attacks in non-small characteristic fields

Variants of NFS, generic fields

- MNFS [Coppersmith 89]: $\mathbb{F}_p$, [Barbulescu Pierrot 14], [Pierrot 15]: $\mathbb{F}_{p^k}$

Specific to pairing target groups, when $p = P(x_0)$, with $\deg P \geq 2$

- [Joux Pierrot 13]
- [Barbulescu Gaudry Kleinjung 15] Tower NFS

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These attacks were not taken into account in the 3072-bit target field recommendation.

Last DL records, with the NFS-DL algorithm

GF(p)	GF(p'^2), $p'^2 = q$ [BGGM15]	
Massive precomputation (d=core-day, y=core-year)		
[Logjam] 512-bit p: 10y	530-bit q: 0.2y + 1.25 GPU d	
[BGIJT14] 596-bit p: 131y	598-bit q: 0.75y + 18 GPU-d	**175× faster**
Individual Discrete Log		
512-bit p: 70s median ✓	530-bit q : few d	**slow**
596-bit p: 2d	600-bit q : few d	**slow**

[Logjam]: see `weakdh.org`
[BGGM15]: Barbulescu, Gaudry, G., Morain
[BGIJT14]: Bouvier, Gaudry, Imbert, Jeljeli, Thomé

This talk:

- Faster **individual** discrete logarithm in $\mathbb{F}_{p^k}$, especially $k = 2, 3, 4, 6$
- Apply to pairing target group $\mathbb{G}_T$
- source code: part of `http://cado-nfs.gforge.inria.fr/`

NFS – Number Field Sieve algorithm

Number Field Sieve algorithm for DL in $\mathbb{F}_{p^k}$

1. *Polynomial selection:*
   compute $f(x)$, $g(x)$ with
   $\varphi = \gcd(f, g) \pmod p$ and
   $\mathbb{F}_{p^k} = \mathbb{F}_p[x]/(\varphi(x))$

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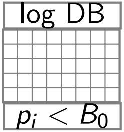
2. *Relation collection*
3. *Linear algebra modulo $\ell \mid p^k - 1$.*

➙ here we know the discrete log of a subset of elements.



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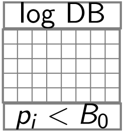

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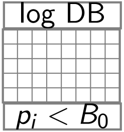

1. *Individual target discrete logarithm*

Number Field Sieve algorithm for DL in $\mathbb{F}_{p^k}$

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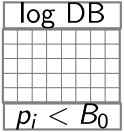


1. **Individual target discrete logarithm for each given DLP instance**

- not so trivial
- this talk: practical improvements very efficient for small k

Individual Discrete Logarithm

Preimage in $\mathbb{Z}[x]/(f(x))$ and ρ map

$$\begin{tikzcd}[column sep=small] & \mathbb{Z}[x] \arrow[dl] \arrow[dr] & \\ \mathbb{Z}[x]/(f(x)) \arrow[dr, "\rho_f\,:\ x \mapsto z"'] & & \mathbb{Z}[y]/(g(y)) \arrow[dl, "\rho_g\,:\ y \mapsto z"] \\ & \mathbb{F}_{p^k} = \mathbb{F}_p[z]/(\varphi(z)) & \end{tikzcd}$$

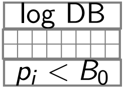
Randomized target $T = t_0 + t_1 X + t_2 X^2 \in \mathbb{F}_{p^3}^* = \mathbb{F}_p[X]/(\varphi(X))$
Simplest choice of preimage $\mathbf{T}$:
$\mathbf{T} = \mathbf{t_0} + \mathbf{t_1} x + \mathbf{t_2} x^2 \in \mathbb{Z}[x]/(f(x))$, with $\mathbf{t_i} \equiv t_i \pmod p$.
We can always choose $\mathbf{T}$ s.t.

- $|\mathbf{t_i}| < p$
- $\deg \mathbf{T} < \deg f$

$$\text{We need } \rho(\mathbf{T}) = T$$

(where ρ is simply a reduction modulo (φ, p) when f (resp. g) is monic)

Individual DL of random target $T_0 \in \mathbb{F}_{p^k}^*$

Given G and a log database s.t. for all $p_i < B_0$, $\log p_i \in$ 

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Given G and a log database s.t. for all $p_i < B_0$, $\log p_i \in$ log DB $p_i < B_0$

1. booting step (a.k.a. smoothing step):
   DO
   1.1 take t at random in $\{1, \ldots, \ell - 1\}$ and set $T = G^t T_0$
   (hence $\log_G(T_0) = \log_G(T) - t$)
   1.2 factorize $\mathrm{Norm}(\mathbf{T}) = \underbrace{q_1 \cdots q_i}_{\text{too large: } B_0 < q_i \leq B_1} \times$(elements in DL database),

   UNTIL $q_i \leq B_1$

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2. Descent strategy: set $\mathcal{S} = \{q_i : \ B_0 < q_i \leq B_1\}$
   while $\mathcal{S} \neq \emptyset$ **do**
   - set $B_j < B_i$
   - find a relation $q_i = \prod_{B_0 < q_j < B_j} q_j \times$ (elements in log DB)
   - $\mathcal{S} \leftarrow \mathcal{S} \setminus \{q_i\} \cup \{q_j\}_{j \in J}$

   end while

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3. log combination to find the individual target DL

Individual DL of random target $T_0 \in \mathbb{F}_{p^k}^*$

log DB

$p_i < B_0$

Given G and a log database s.t. for all $p_i < B_0$, $\log p_i \in$ log DB

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   1.2 factorize $\underbrace{\mathrm{Norm}(\mathbf{T})}_{\text{reduce this}} = \underbrace{q_1 \cdots q_i}_{\text{too large: } B_0 < q_i \leq B_1} \times$(elements in DL database),

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Finding Boots (booting step)

Norm computation

f monic,
$\mathbf{T} = t_0 + t_1 x + \ldots + t_d x^d \in \mathbb{Z}[x]/(f(x))$, $d < \deg f$:

$$\mathrm{Norm}_f(\mathbf{T}) = Res(f, \mathbf{T}) \leq A||\mathbf{T}||_\infty^{\deg f} ||f||_\infty^d$$

with $||f||_\infty = \max_{1 \leq i \leq \deg f} |f_i|$

Example: [MNT01], $k = 3$, $\deg g = 3$, $||g||_\infty = O(p^{1/2})$

$$\begin{aligned}
p &= 908761003790427908077548955758380356675829026531247 \\
\mathbf{T} &= 314159265358979323846264338327950288419716939937510 \\
&\quad +582097494459230781640628620899862803482534211706798x \\
&\quad +214808651328230664709384460955058223172535940812829x^2 \\
f &= 108x^6 + 1116x^5 + 3347x^4 + 2194x^3 - 613x^2 - 468x + 108 \\
g &= 6x^3 + 34809213412360199593267639x^2 + 34809213412360199593267621x - 6
\end{aligned}$$

$\mathrm{Norm}_f(\mathbf{T})(\approx ||\mathbf{T}||_\infty^6 ||f||_\infty^2) =$ **1017**bits $\sim p^6$
$\mathrm{Norm}_g(\mathbf{T})(\approx ||\mathbf{T}||_\infty^3 ||g||_\infty^2) =$ **665**bits $\sim p^4$

Booting step complexity

Given random target $T_0 \in \mathbb{F}_{p^k}^*$, and G a generator of $\mathbb{F}_{p^k}^*$
repeat

1. take t at random in $\{1, \ldots, \ell - 1\}$ and set $T = g^t T_0$
2. factorize $\mathrm{Norm}(\mathbf{T})$

until it is B_1-smooth: $\mathrm{Norm}(\mathbf{T}) = \prod_{q_i \leq B_1} q_i \prod_{p_i \leq B_0} p_i$

L-notation: $Q = p^k$, $L_Q[1/3, \mathbf{c}] = \mathrm{e}^{(\mathbf{c}+o(1))(\log Q)^{1/3} \ (\log \log Q)^{2/3}}$ for $\mathbf{c} > 0$.
Norm factorization done with ECM method, in time $L_{B_1}[1/2, \sqrt{2}]$

Lemma (Booting step running-time)

If $\mathrm{Norm}(\mathbf{T}) \leq Q^e$, *take* $B_1 = L_Q[2/3, (e^2/3)^{1/3}]$, *then the running-time is* $L_Q[1/3, (3e)^{1/3}]$ *(and this is optimal).*

Booting step complexity

- $\mathbb{F}_p$: Norm(preimage) $\leq p = Q$, running-time: $L_Q[1/3, \mathbf{1.44}]$ with $B_1 = L_Q[\mathbf{2/3}, 0.69]$ [Commeine Semaev 06, Barbulescu 13]
- med. char. $\mathbb{F}_{p^k}$, JLSV1 poly. select.: $\deg f = \deg g = k$, $||f||_\infty = ||g||_\infty = O(p^{1/2})$, Norm(preimage) $\leq Q^{3/2}$, running time: $L_Q[1/3, \mathbf{1.65}]$, with $B_1 = L_Q[\mathbf{2/3}, 0.91]$ [Joux Lercier Naccache Thomé 09, Barbulescu Pierrot 14]

field	$\mathbb{F}_p$	$\mathbb{F}_{p^k}$		
polynomial selection		gJL	JLSV_1	Conj
NFS dominating part, c	1.92	1.92	2.42	2.20
$L_Q[\frac{1}{3}, c]$, 512-bit Q	2^{64}	2^{64}	2^{81}	2^{73}
Norm($\mathbf{T}$) $< Q^e$	Q	Q	$Q^{3/2}$	Q
time $L_Q[1/3, c]$, c	1.44	1.44	1.65	1.44
nb of operations, 512-bit Q	2^{48}	2^{48}	2^{55}	2^{48}
q_i bound B_1	2^{90}	2^{90}	2^{118}	2^{90}

Optimizing the Preimage Computation

Preimage optimization

f, $\deg f$, $||f||_\infty$, g, $\deg g$, $||g||_\infty$ are given by the polynomial selection step (NFS-DL step 1)

$$\mathrm{Norm}_f(\mathbf{T}) = Res(f, \mathbf{T}) \leq A||\mathbf{T}||_\infty^{\deg f}||f||_\infty^d$$

To reduce the norm,

- reduce $||\mathbf{T}||_\infty$
- and/or reduce $d = \deg \mathbf{T}$

Previous work

- $\mathbb{F}_p$: Rational Reconstruction. $T \in \mathbb{Z}/p\mathbb{Z}$, $\mathbf{T}$ is an integer $< p$. Rational Reconstruction gives $\mathbf{T} = u/v \pmod{p}$ with $u, v < \sqrt{p}$
  - booting step: we want u,v to be B_1-smooth at the same time, instead of $\mathbf{T}$ to be B_1-smooth. $\mathbf{T}$ is already split into two integers of half size each.
- [Blake Mullin Vanstone 84] Waterloo algorithm in $\mathbb{F}_2[x]$: $\mathbf{T} = U/V = \frac{u_0+\ldots+u_{\lfloor d/2 \rfloor}x^{\lfloor d/2 \rfloor}}{v_0+\ldots+v_{\lfloor d/2 \rfloor}x^{\lfloor d/2 \rfloor}}$ **reduce degree**
- [Joux Lercier Smart Vercauteren 06] in $\mathbb{F}_{p^k}$: $\mathbf{T} = U/V = \frac{u_0+\ldots+u_d x^d}{v_0+\ldots+v_d x^d}$, where $|u_i|, |v_i| \sim p^{1/2}$ **reduce coefficient size**

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How much is the booting step improved?

Booting step: First experiments

Commonly assumed: launched at morning coffee ... finished for afternoon tea.

- $\mathbb{F}_{p^2}$ 600 bits (BGGM15 record) was easy, as fast as for $\mathbb{F}_{p'}$ ($<$ one day)
- $\mathbb{F}_{p^3}$ 400 bits and MNT 508 bits were much slower (days, week)
- $\mathbb{F}_{p^4}$ 400 bits was even worse ($>$ one week)

What happened?

- $\mathbb{F}_{p^3}$: $||\mathbf{T}||_\infty = p$, $\deg f = 6$, [JLSV06] method: $\mathrm{Norm}(\mathbf{T}) \leq Q \to c = 1.44$ (but still much slower)
- $\mathbb{F}_{p^4}$: $||f||_\infty = O(p^{1/2})$, $\mathrm{Norm}(\mathbf{T}) \leq Q^{3/2} \to c = 1.65$

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Because of the constant hidden in the $O()$?

Our solution

Lemma

Let $T \in \mathbb{F}_{p^k}$.
Then $\log(T) = \log(u \cdot T) \pmod{\ell}$ *for any* u *in a proper subfield of* $\mathbb{F}_{p^k}$.

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Then $\log(T) = \log(u \cdot T) \pmod{\ell}$ *for any* u *in a proper subfield of* $\mathbb{F}_{p^k}$.

- $\mathbb{F}_p$ is a proper subfield of $\mathbb{F}_{p^k}$
- target $T = t_0 + t_1 x + \ldots + t_d x^d$
- we divide the target by its leading term:

$$\log(T) = \log(T/t_d) \pmod{\ell}$$

From now on we assume that the target is monic.

Our solution

Lemma

Let $T \in \mathbb{F}_{p^k}$.
Then $\log(T) = \log(u \cdot T) \pmod{\ell}$ for any u in a proper subfield of $\mathbb{F}_{p^k}$.

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Similar technique in pairing computation:
Miller loop denominator elimination [Boneh Kim Lynn Scott 02]

Subfield Cofactor Simplification + LLL

We want to reduce $||\mathbf{T}||_\infty$. Example with $\mathbb{F}_{p^3}$:

- $f = 108x^6 + 1116x^5 + 3347x^4 + 2194x^3 - 613x^2 - 468x + 108$
- $\varphi = x^3 - yx^2 - (y+3)x - 1 \; y \in \mathbb{Z}$
- $\mathbf{T} = t_0 + t_1 x + x^2$
- define $L = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ t_0 & t_1 & 1 & 0 & 0 & 0 \\ \varphi_0 & \varphi_1 & \varphi_2 & 1 & 0 & 0 \\ 0 & \varphi_0 & \varphi_1 & \varphi_2 & 1 & 0 \\ 0 & 0 & \varphi_0 & \varphi_1 & \varphi_2 & 1 \end{bmatrix}$
- LLL(L) outputs a short vector r, linear combination of L's rows.
  $r = \lambda_0 p + \lambda_1 px + \lambda_2 T + \lambda_3 \varphi + \lambda_4 x\varphi + \lambda_5 x^2 \varphi$.
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Subfield Cofactor Simplification + LLL

$$\mathrm{Norm}_f(\mathbf{T}) = Res(f, \mathbf{T}) \le A||\mathbf{T}||_\infty^{\deg f}||f||_\infty^d$$

- $\mathrm{Norm}_f(r) \le ||r||_\infty^6||f||_\infty^5 = O(p^2) = O(Q^{2/3}) < O(Q)$

MNT example: $\log Q = 508$ bits

	$\mathrm{Norm}_f(\mathbf{T})$		$\mathrm{Norm}_g(\mathbf{T})$		$L_Q[1/3, c]$		$q_i \le B_1 =$
	Q^e	bits	Q^e	bits	c	time	$L_Q[\frac{2}{3}, c]$
Nothing	Q^2	1010	$Q^{4/3}$	667	1.58	2^{53}	2^{109}
[JLSV06]	Q	508	$Q^{5/3}$	847	1.44	2^{48}	2^{90}
This work	$\mathbf{Q^{2/3}}$	**340**	$\mathbf{Q}$	**508**	**1.26**	$\mathbf{2^{42}}$	$\mathbf{2^{69}}$

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Faster descent

$\mathbb{F}_{p^4}$: JLSV$_1$ polynomial selection and booting step improvement

$\mathbb{F}_{p^4}$ of 400 bits

[JLSV06] first method: choose f of degree 4 and very small coefficients, and set $g = f + p$. Booting step on f side, with the $\mathbf{T} = U/V$ method.

Relation collection and Linear algebra do not scale well for large p

We use JLSV06 other method: $\deg f = \deg g = k$, $||f||_\infty = ||g||_\infty = p^{1/2}$

$$
\begin{aligned}
p &= 314159265358979323846270891033 \text{ of 98 bits (30 dd)} \\
\ell &= 986960440108935861883490271847705742814406423277877598 0709 \text{ of 192 bits} \\
f &= x^4 - 560499121640472x^3 - 6x^2 + 560499121640472x + 1 \\
g &= 560499121639105x^4 + 4898685125033473x^3 - 3362994729834630x^2 \\
&\quad -4898685125033473x + 560499121639105 \\
\varphi &= g
\end{aligned}
$$

Terribly slow booting step (more than one week)

Terribly slow booting step

- $T = t_0 + t_1 x + t_2 x^2 + x^3$
- define
$$L = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ t_0 & t_1 & t_2 & 1 \end{bmatrix}$$
- dim 4 because $\max(\deg f, \deg g) = 4$
- compute LLL(L), get r, $||r||_\infty \approx p^{3/4}$,
$\mathrm{Norm}_f(r) \approx ||r||_\infty^4 ||f||_\infty^3 \approx p^{9/2} = Q^{9/8}$ of 450 bits!
- Booting step, number of operations: 2^{44}
- Large prime bound B_1 of 82 bits

Terribly slow booting step

- $T = t_0 + t_1 x + t_2 x^2 + x^3$
- define
$$L = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ t_0 & t_1 & t_2 & 1 \end{bmatrix} \quad \leftarrow \text{ could we find something else, } \textit{monic}?$$
- dim 4 because $\max(\deg f, \deg g) = 4$
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Our solution: quadratic subfield cofactor simplification

Lemma

Let $T \in \mathbb{F}_{p^k}$, k even. We can always find $u \in \mathbb{F}_{p^2}$ and $T' \in \mathbb{F}_{p^k}$ such that $T' = u \cdot T$ and T' is of degree $k-2$ instead of $k-1$.

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- define $L = \begin{array}{cl} \left[\begin{array}{cccc} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ t'_0 & t'_1 & 1 & 0 \\ t_0 & t_1 & t_2 & 1 \end{array}\right] & \begin{array}{l} \rho(p) = 0 \in \mathbb{F}_{p^k} \\ \\ T' \\ T \end{array} \end{array}$
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$\mathrm{Norm}_f(r) = ||r||_\infty^4 ||f||_\infty^3 = p^{7/2} = Q^{7/8} < Q$

CATREL Workshop breaking news

Degree-d subfield cofactor simplification

Lemma

Let $T \in \mathbb{F}_{p^k}$, and $d \mid k$, $1 < d < k$. We can always find $u \in \mathbb{F}_{p^d}$ and $T' \in \mathbb{F}_{p^k}$ such that $T' = u \cdot T$ and T' is (monic) of degree $k - d$ instead of $k - 1$.

We use linear algebra to do this in practice:

- find a basis $\{1, U, \ldots, U^{d-1}\}$ of $\mathbb{F}_{p^d} \subset \mathbb{F}_{p^k}$
- solve $(u_0 + u_1 U + \ldots + u_{d-1} U^{d-1}) T = T'$ with $t'_{k-d} = 1, t'_{k-d+1} = \ldots = t'_{k-1} = 0$ i.e. solve modulo p the system

$$\begin{bmatrix} T_{k-d} & (UT)_{k-d} \ldots & (U^d T)_{k-d} \\ T_{k-d+1} & (UT)_{k-d+1} \ldots & (U^d T)_{k-d+1} \\ \vdots & & \vdots \\ T_{k-1} & (UT)_{k-1} \ldots & (U^d T)_{k-1} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- $T' = uT$ is of degree $k - d$

Example: $\mathbb{F}_{p^6}$, subfield $\mathbb{F}_{p^3}$

- $p = 10^{12} + 39 = 1000000000039$
- $f = x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 7$, nothing special
- $\mathbb{F}_{p^6} = \mathbb{F}_p[z]/(f(z))$, and let U be a root of $x^3 + 2$ in $\mathbb{F}_{p^6}$
- random target
  $T = 175247343375X^5 + 606947457111X^4 + 821185152528X^3 + 233479934136X^2 + 286091685405X + 30741878977$
- $u = 123906765420X^5 + 210820130399X^4 + 609700725797X^3 + 529508639774X^2 + 719945573899X + 519020404562$
- $u^{1+p+p^2} = 252075155349 \in \mathbb{F}_p$, meaning $u \in \mathbb{F}_{p^3}$
- $u \cdot T = T' = X^3 + 336056353764X^2 + 69072317659X + 636713253760$
- also find $v \in \mathbb{F}_{p^3}$ s.t. $v \cdot T = T''$ of degree 4

Example: $\mathbb{F}_{p^6}$, subfield $\mathbb{F}_{p^3}$

- Define

$$L = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ t_0'' & t_1'' & t_2'' & 1 & 0 & 0 \\ t_0' & t_1' & t_2' & t_3' & 1 & 0 \\ t_0 & t_1 & t_2 & t_3 & t_4 & 1 \end{bmatrix}$$

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512-bit Q:

- $\mathrm{Norm}(r)$ of 470 bits
- special-q bound of 85 bits
- expected number of operations 2^{47}

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- expected number of operations 2^{47} will still take a week

Finding boots in $\mathbb{F}_{p^k}$, k even

Lemma

Let $T \in \mathbb{F}_{p^k}$, T is not in a proper subfield. We can always find $u \in \mathbb{F}_{p^{k/2}}$ such that $u \cdot T = T'$ with T' of any degree i, where $k/2 \leq i \leq k-1$.

Size of coefficients of r output by LLL:

- Conjugation method: $||r||_\infty = O(p^{1/4})$, $\mathrm{Norm}_f(r) = O(Q^{1/2})$
- JLSV1 method: $||r||_\infty = O(p^{1/2})$, $\mathrm{Norm}_f(r) = O(Q^{1-\frac{1}{2k}}) < O(Q)$

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Could we use this lemma to simplify the descent in small characteristic ?

Summary of results

$\mathbb{G}_T \subset$	$\mathbb{F}_{p^2}$	$\mathbb{F}_{p^3}$	$\mathbb{F}_{p^4}$	$\mathbb{F}_{p^6}$
Norm bound				
prev.	Q [JLSV06]		$Q^{3/2}$ (nothing)	
this work	$Q^{1/2}$	$Q^{2/3}$	$Q^{7/8}$	$Q^{11/12}$
Booting step running time in $L_Q[1/3, c]$				
prev. c (*)	1.44		1.65	
new c	**1.14**	**1.26**	**1.38**	**1.40****
numerical values for a 512-bit Q				
prev. no. of operations	2^{48}		2^{55}	
new no. of operations	$\mathbf{2^{38}}$	$\mathbf{2^{42}}$	$\mathbf{2^{46}}$	$\mathbf{2^{47}}$
q_i bound $B_1 = L_Q[2/3, c']$				
previous B_1	2^{90}		2^{118}	
new $\mathbf{B_1}$	$\mathbf{2^{57}}$	$\mathbf{2^{69}}$	$\mathbf{2^{83}}$	$\mathbf{2^{85}}$

* [CommeineSemaev06, JouxLercierNaccacheThomé09, Barbulescu13, Bar.Pierrot14]

** with cubic subfield simplification

Summary of results

- Asiacrypt 2015, Auckland, New Zealand
- online version HAL 01157378
- `guillevic@lix.polytechnique.fr`

Future work

- optimize the descent: “will it be an algorithm one day?”
- add Barbulescu’s early abort strategy (not me)
- find a pairing-target-group version of JLSV1
- $\mathbb{F}_{p^6}$, $\mathbb{F}_{p^{12}}$

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Be careful with the hidden constant in the $O(\cdot)$

DL record computation in $\mathbb{F}_{p^4}$ of 392 bits (120dd)

Joint work with R. Barbulescu, P. Gaudry, F. Morain

$$
\begin{aligned}
p &= 314159265358979323846270891033 \text{ of 98 bits (30 dd)} \\
\ell &= 986960440108935861883490271847705742814406423277878775980709 \text{ of 192 bits} \\
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&\quad -4898685125033473x + 560499121639105 \\
\varphi &= g \\
G &= x + 3 \in \mathbb{F}_{p^4} \\
T_0 &= 31415926535897x^3 + 93238462643383x^2 + 27950288419716x + 93993751058209
\end{aligned}
$$

$$\mathbf{\log_G(T_0) =}$$

$$\mathbf{1364394725868398385294409072195832018219505919841942570022} \pmod{\ell}$$