Random discrete surfaces and graph exploration processes

Gilles Schaeffer

CNRS / Ecole Polytechnique, Palaiseau, France

Combinatorics

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Combinatorial objects

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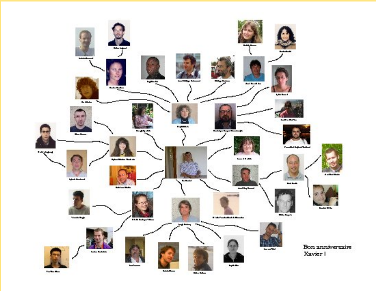
Combinatorial objects

tree like structures

Combinatorics

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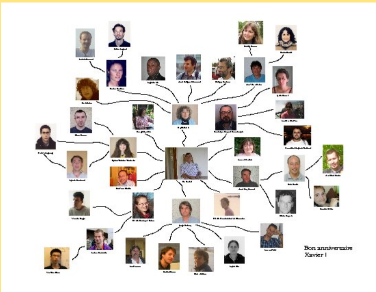
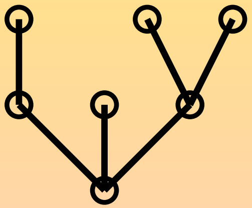



tree like structures

concept of graph



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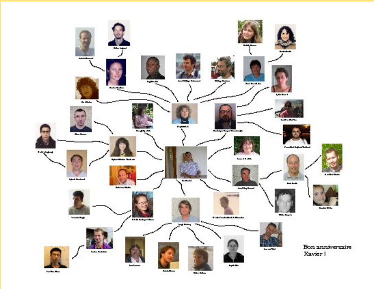

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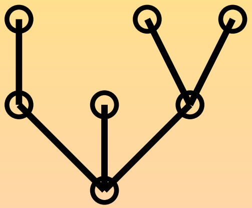

concept of tree

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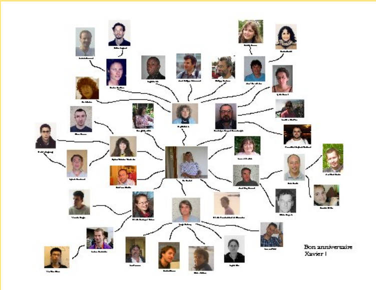


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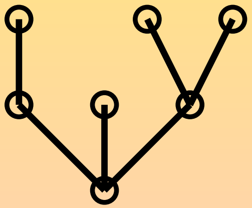
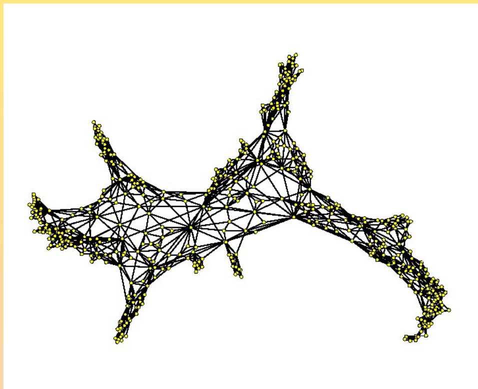
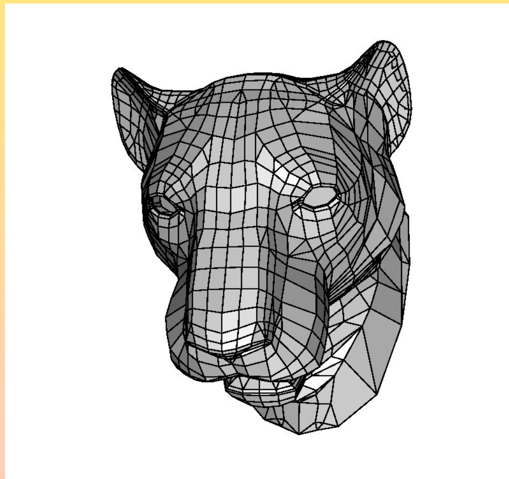


concept of tree

2d discrete structures

(discretized surfaces, meshes,...)



Combinatorics

Combinatorial objects



tree like structures

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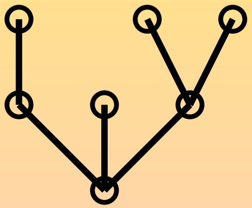

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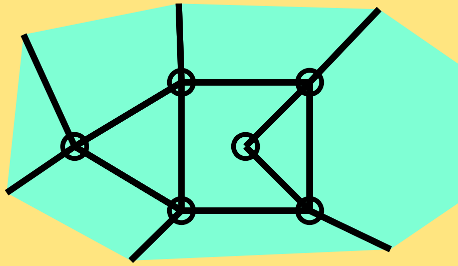

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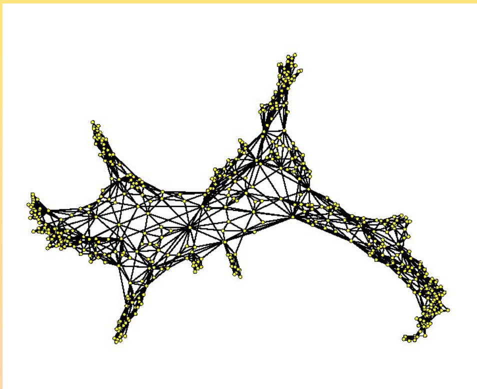
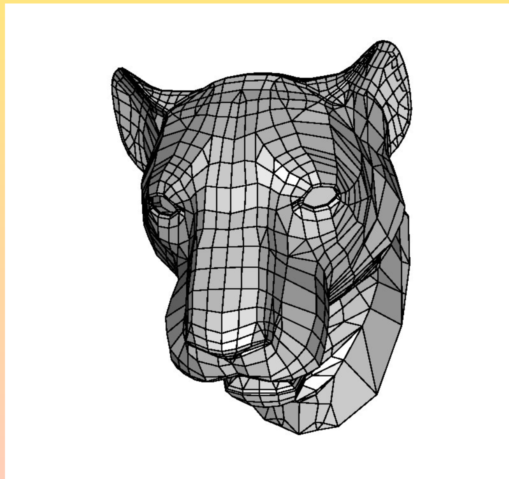
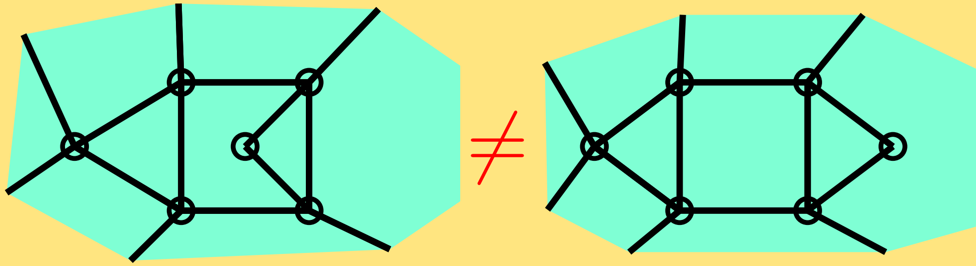


Combinatorics

Combinatorial objects

concept of *map*



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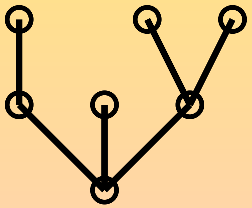

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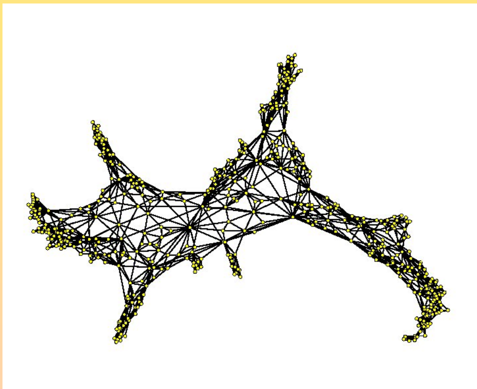

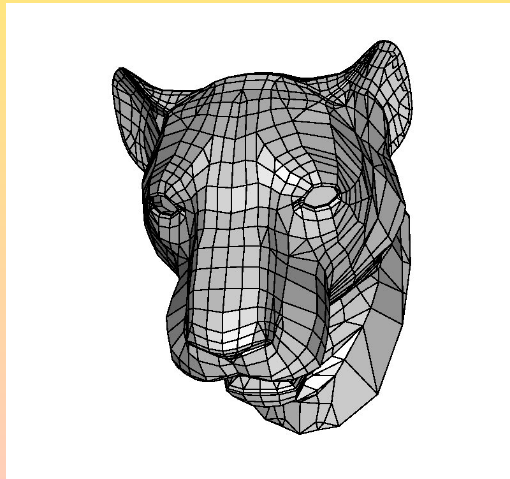

Combinatorics

Combinatorial objects = discrete abstractions of fundamental structures

tree like structures

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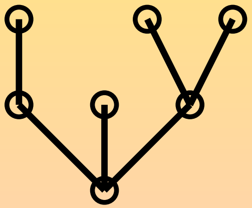

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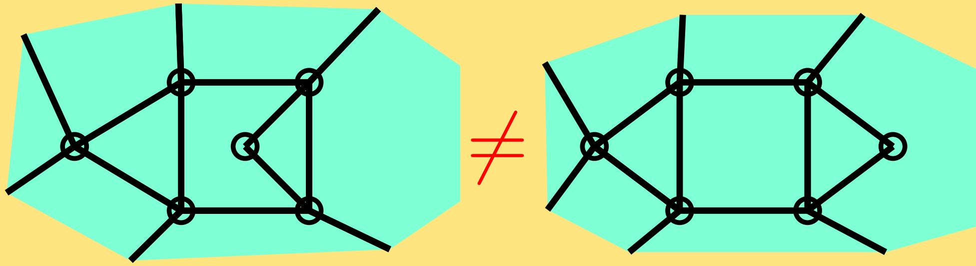

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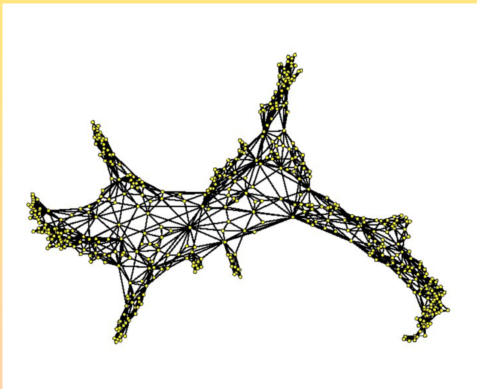

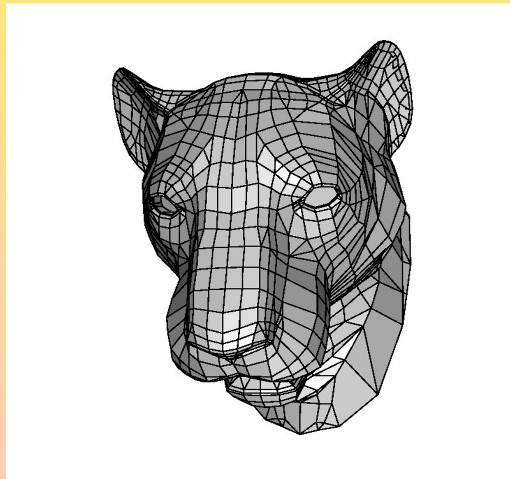

My idea of combinatorics

Elucidate the properties of those fundamental discrete structures that are common to various scientific fields (CS/math/physics/bio).

Algorithmic combinatorics

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and, more specifically of "algorithmic combinatorics"

concentrate on constructive properties and on the algorithmic point of view on structures

Algorithmic combinatorics

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Elucidate the properties of those fundamental discrete structures that are common to various scientific fields (CS/math/physics/bio).

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concentrate on constructive properties and on the algorithmic point of view on structures

The example of trees...

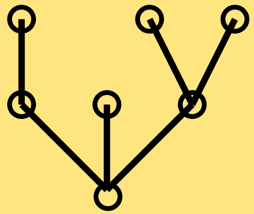
mathematical pt of view: connected graphs without cycle

algorithmic pt of view: recursive description (root; subtrees)
$\Rightarrow$ concept of breadth first or depth first search,
links with context free languages

(... Schützenberger's methodology...)

Exploration algorithms

Tree exploration



breadth first

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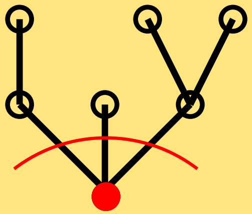

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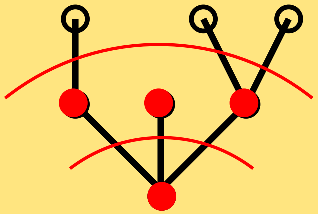

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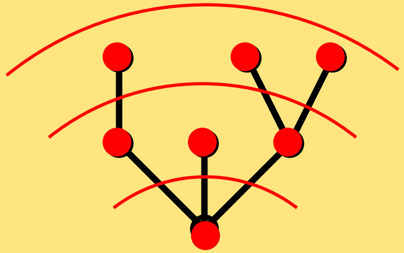

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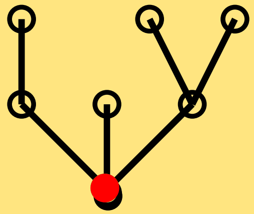

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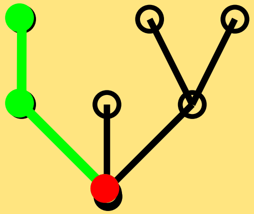

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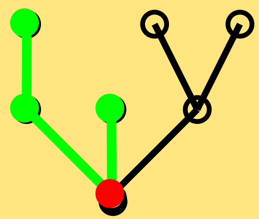

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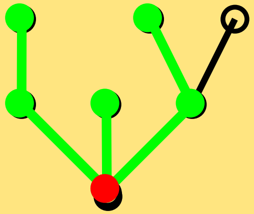

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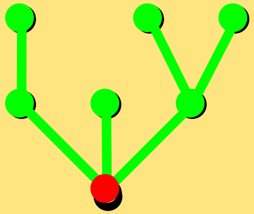

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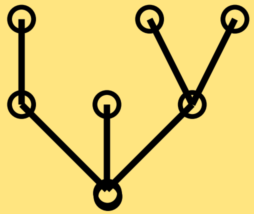

breadth first

depth first

fundamental tools

for instance to encode trees

Exploration algorithms

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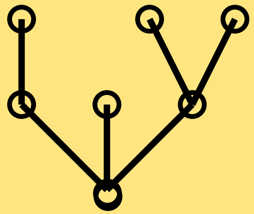

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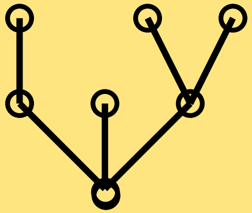

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Exploration algorithms

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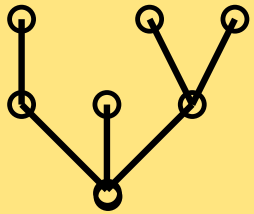

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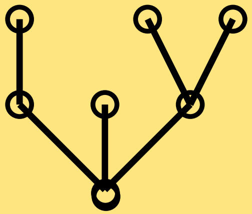
Statement. The set of code words is easy to describe.

More precisely: the language of prefix codes of ordered trees is *context-free*.

Exploration algorithms

Tree exploration



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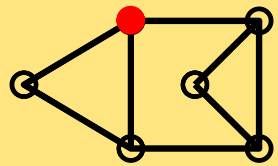

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Exploration algorithms

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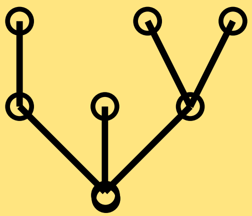

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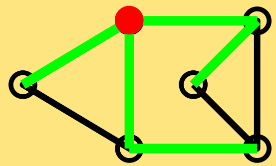

breadth first

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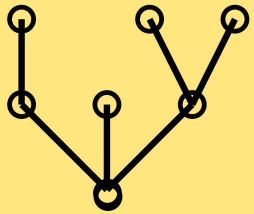
construct a tree along the exploration

Exploration algorithms

Tree exploration



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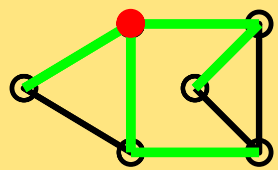

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+ extra info for external edges

⇒ encode graphs by tree-like structures

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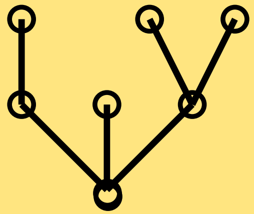

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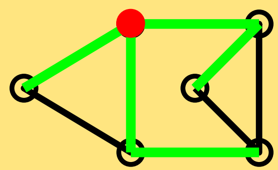

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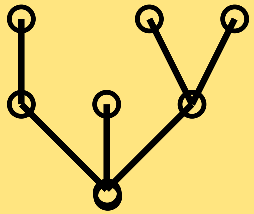
Exploration algorithms

Tree exploration



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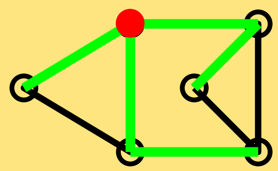

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No good analog of the previous "statement".

Exploration of a map and surface surgery

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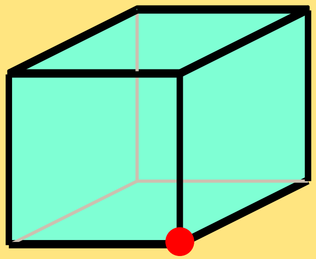

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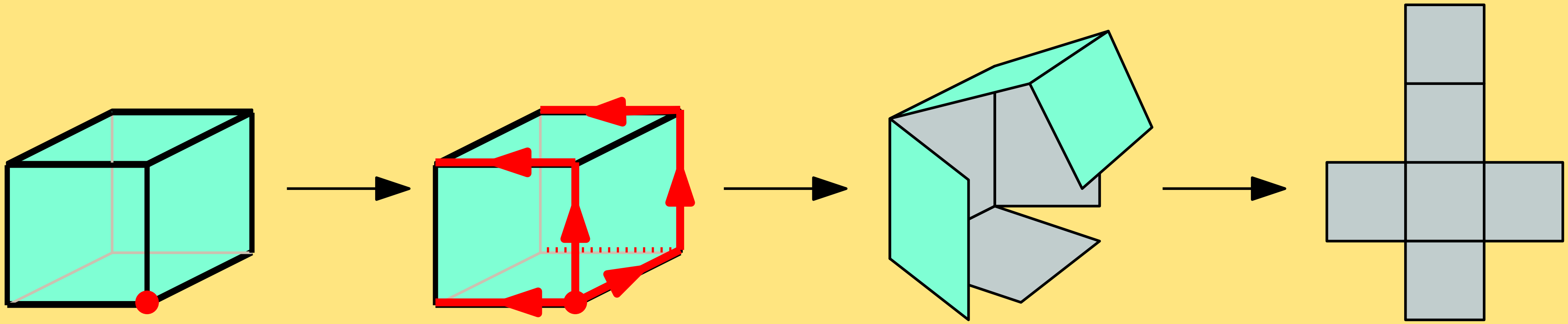

Exploration + cut ⇒ a "net" of the map

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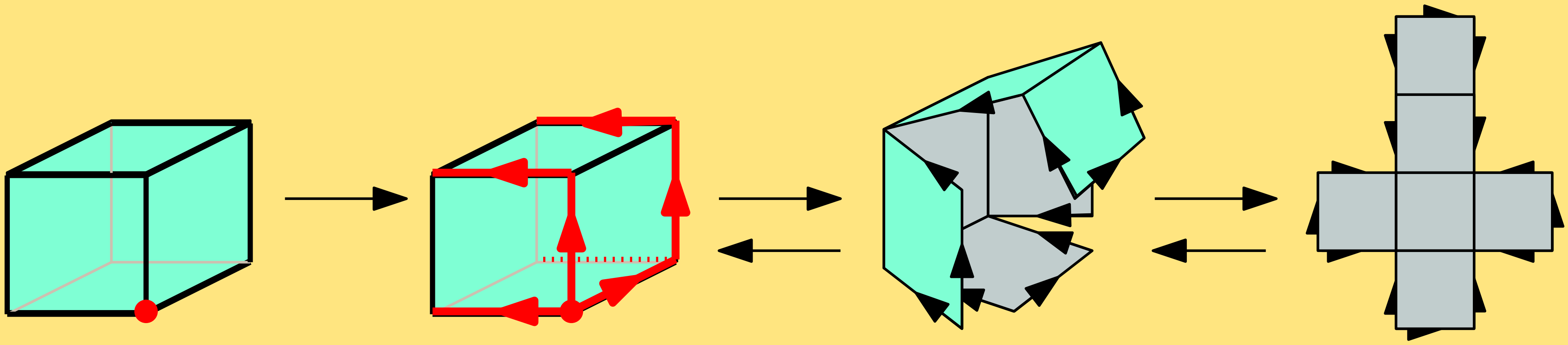

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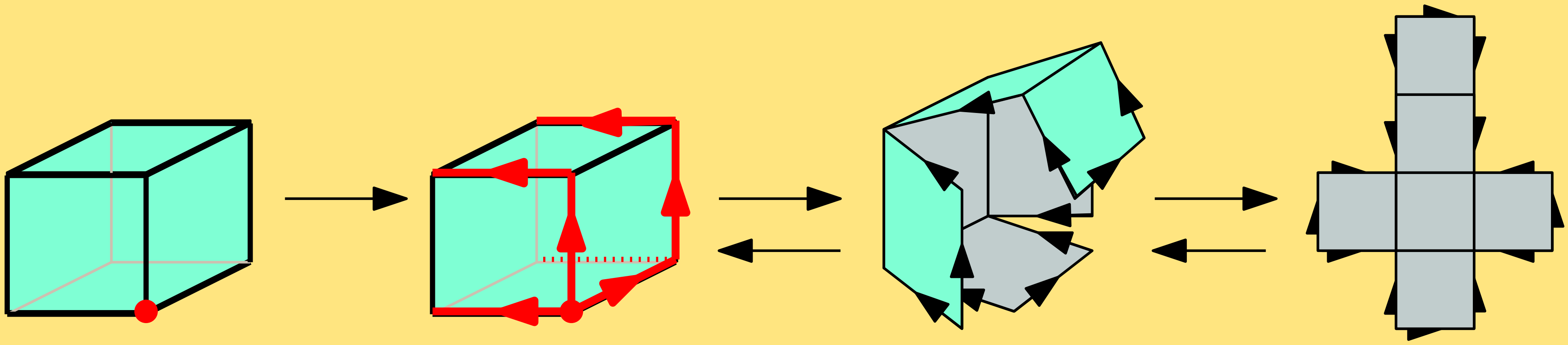

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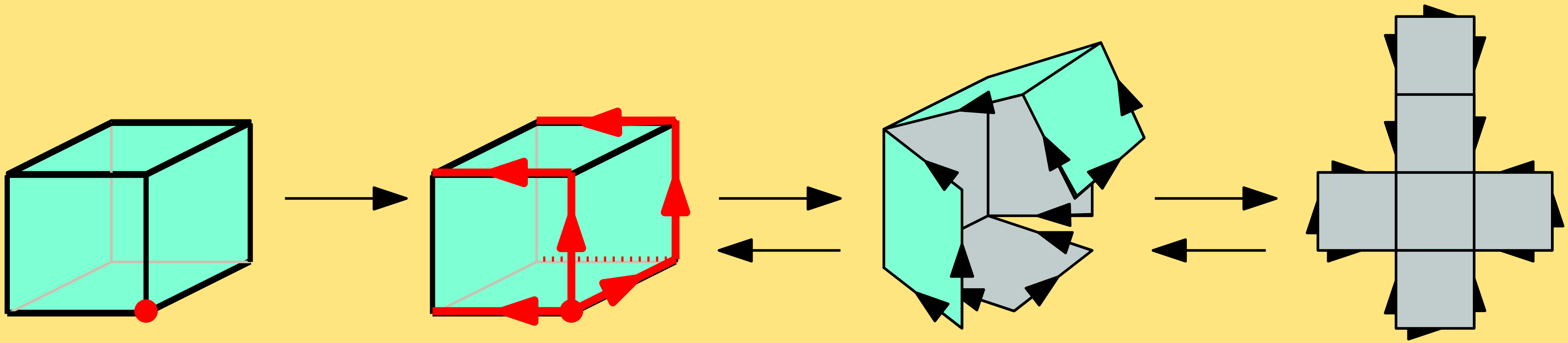

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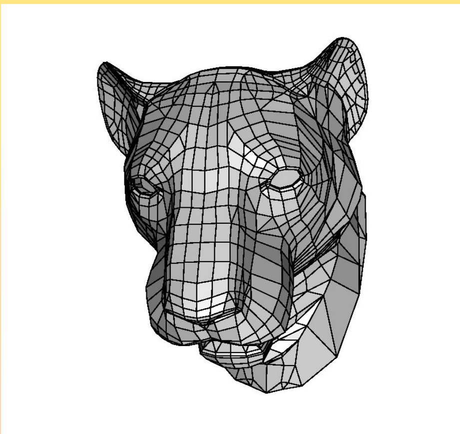
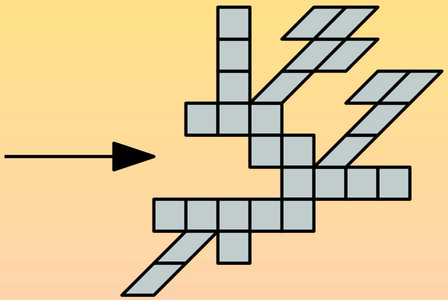


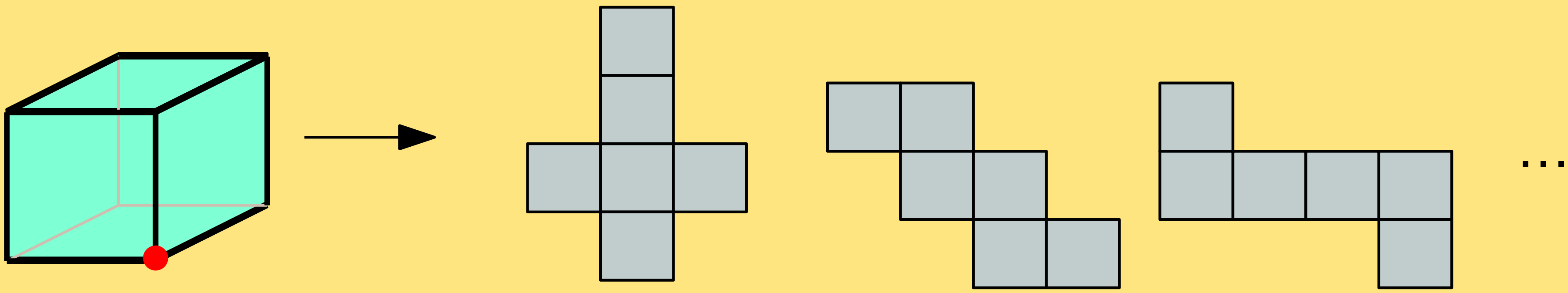


Nets are always trees of polygons

(as long as the surface has no handle)

Exploration algorithms

To a map are associated many different nets



Exploration algorithms

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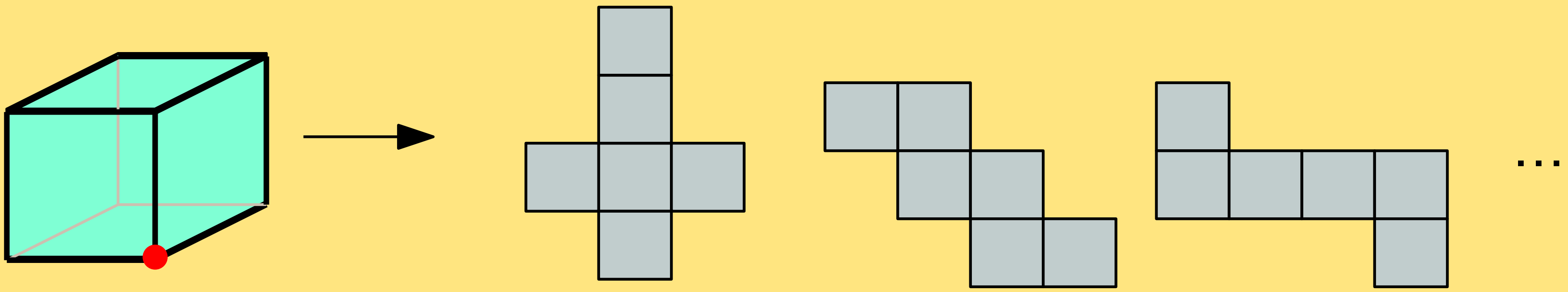

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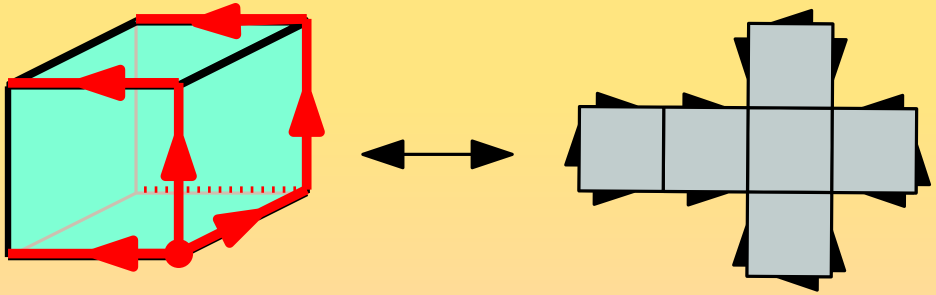

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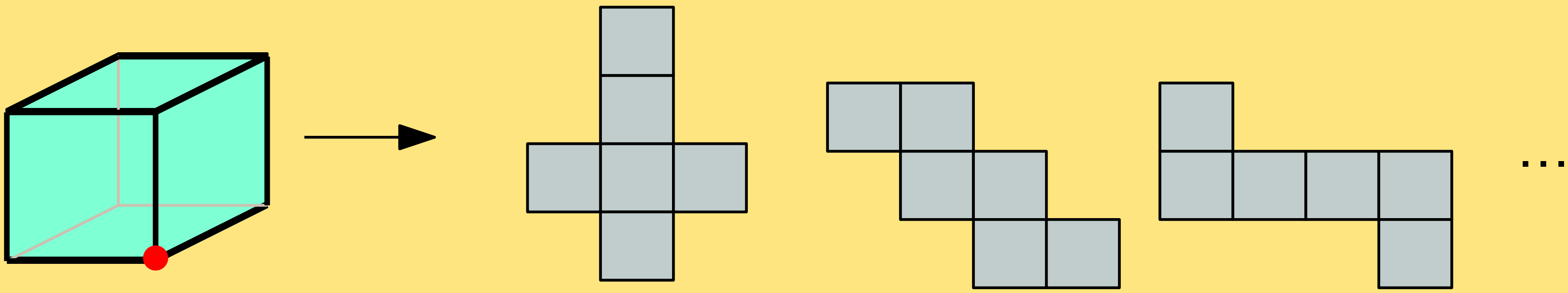

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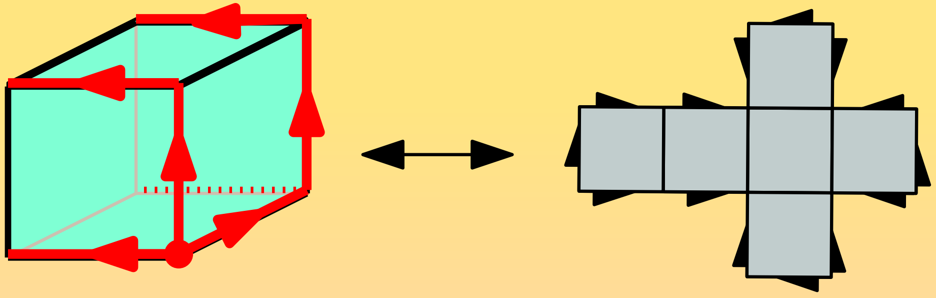

Represent again a map by a tree like structure!

Exploration algorithms

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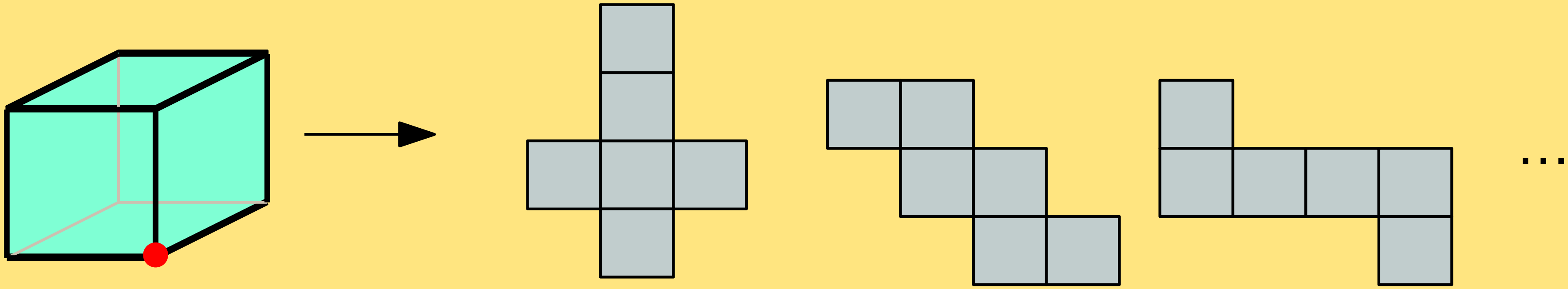

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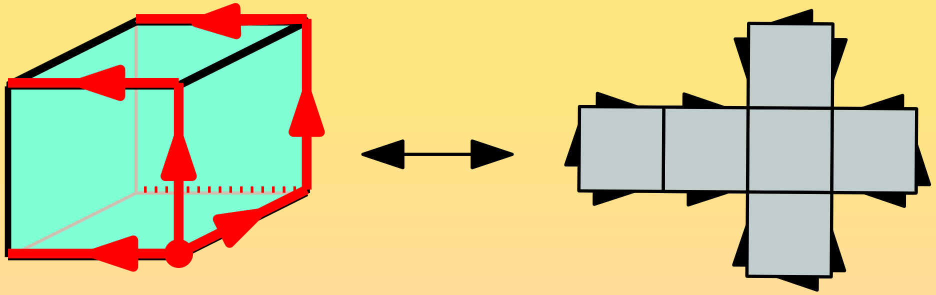
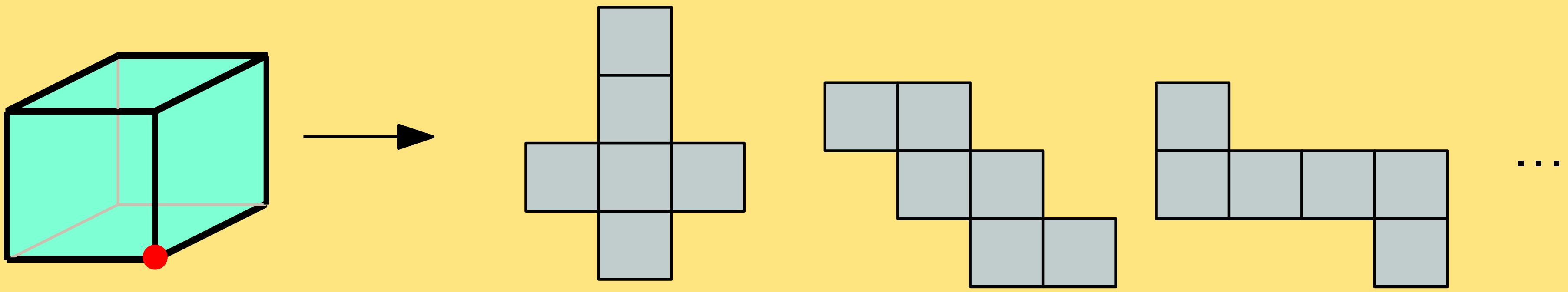


Represent again a map by a tree like structure!

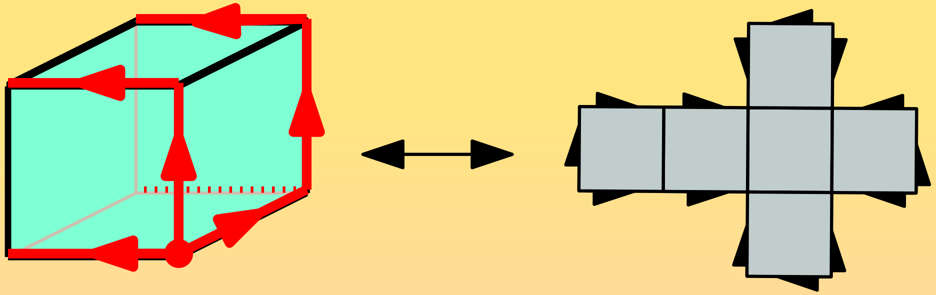
Each exploration algo $\Rightarrow$ a bijection, but what is the set of valid nets?

Exploration algorithms

To a map are associated
many different nets



but a given exploration algorithm associates a canonical net to each map



Represent again a map by a tree like structure!

Each exploration algo $\Rightarrow$ a bijection, but what is the set of valid nets?

Valid nets are easier to describe than exploration trees!

Exploration algorithms

Statement

To many natural families of maps is associated a standard exploration algorithms (breadth first, depth first, Schnyder,...) such that the cut yields *context-free* nets.

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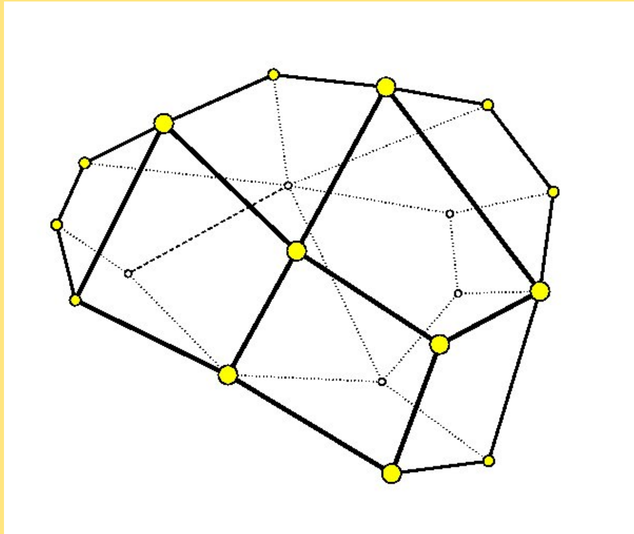
with various types of applications

- optimal encodings and compact data structures for meshes
- random sampling and automatic drawing of graph and map
- enumeration: maps, ramified coverings, alternating knots...
- random discrete surfaces

Application to discrete random surfaces

Planar quadrangulations (quads) as a model of discretized spheres



Let $|Q_n|$ be the set of quads with n faces and X_n be a uniform random quad of Q_n:

$$\Pr(X_n = q) = \frac{1}{|Q_n|}, \quad \forall q \in Q_n$$

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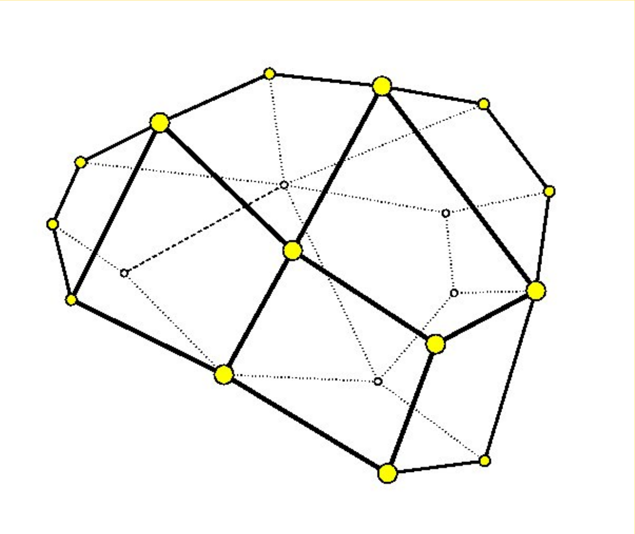

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Lots of results via the celebrated method of topological expansion of matrix integrals (Brezin, Itzykson, Parisi, Zuber, 72).

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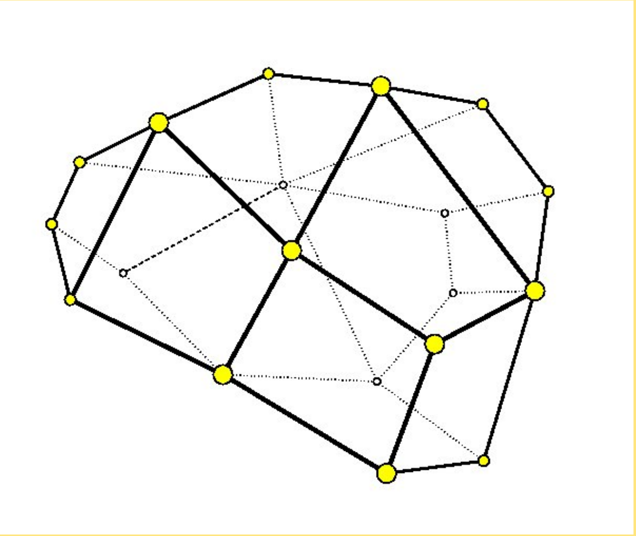

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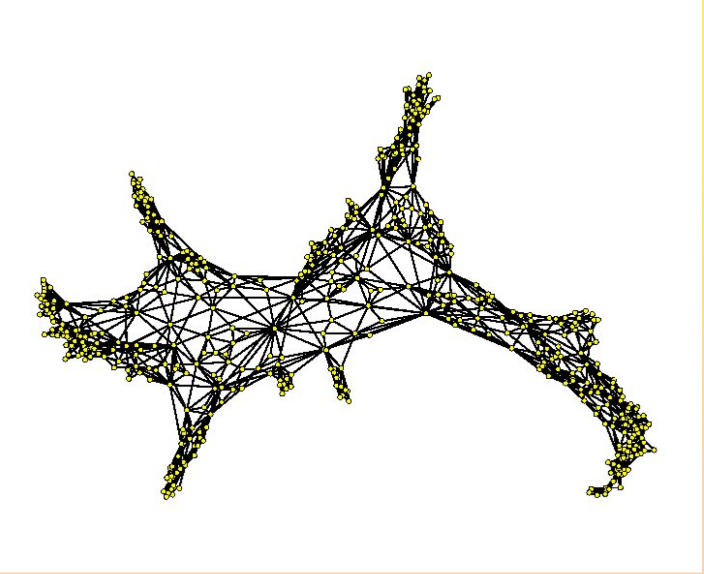

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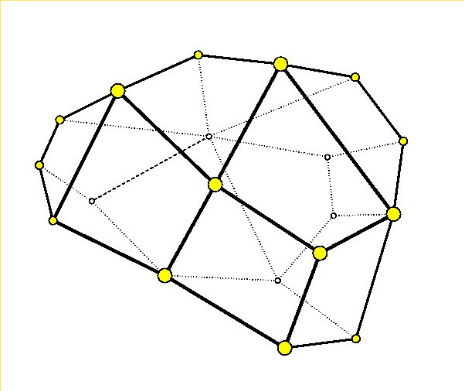
But this approach does not allow to study the intrinsec geometry of these surface!

Quadrangulations via breadth first search

Consider a planar quadrangulation



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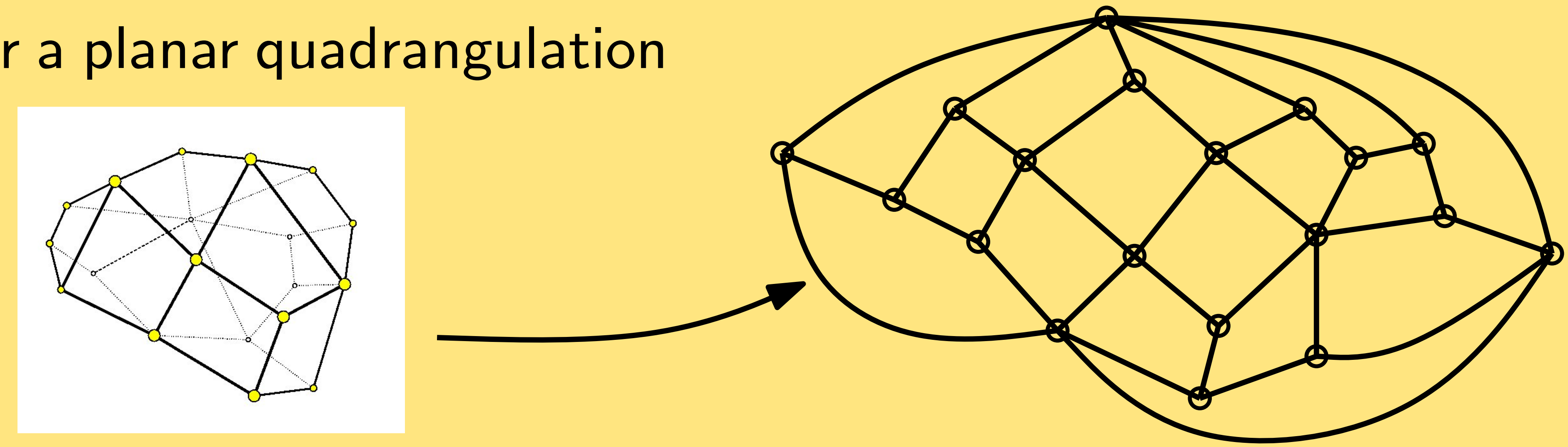

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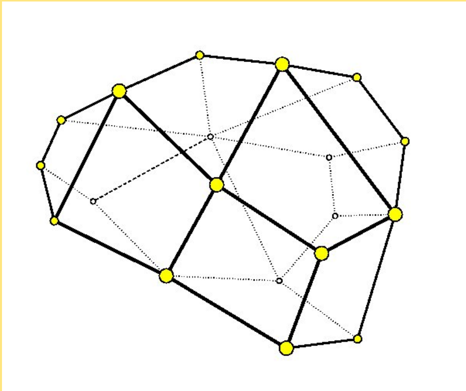
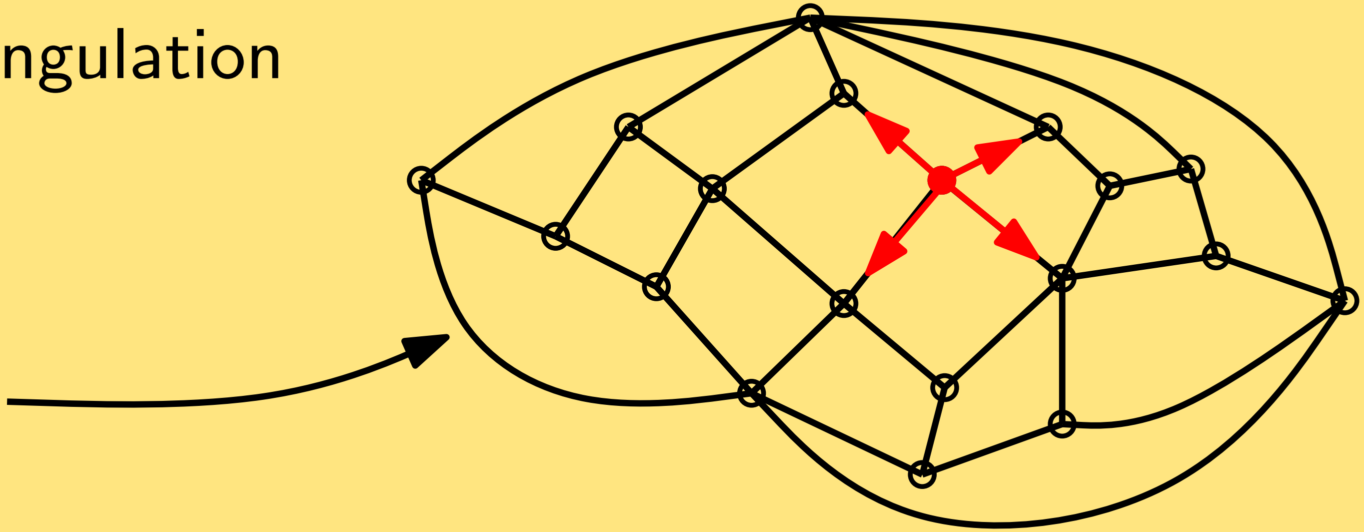
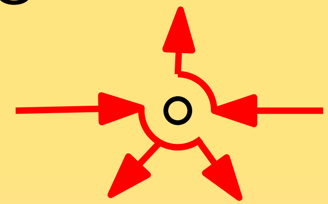




Apply bfs with the rotatoria rule and cut along the flow



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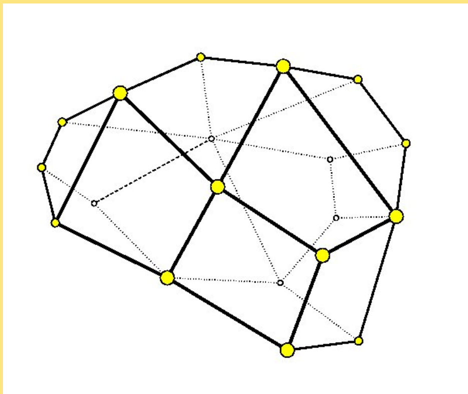

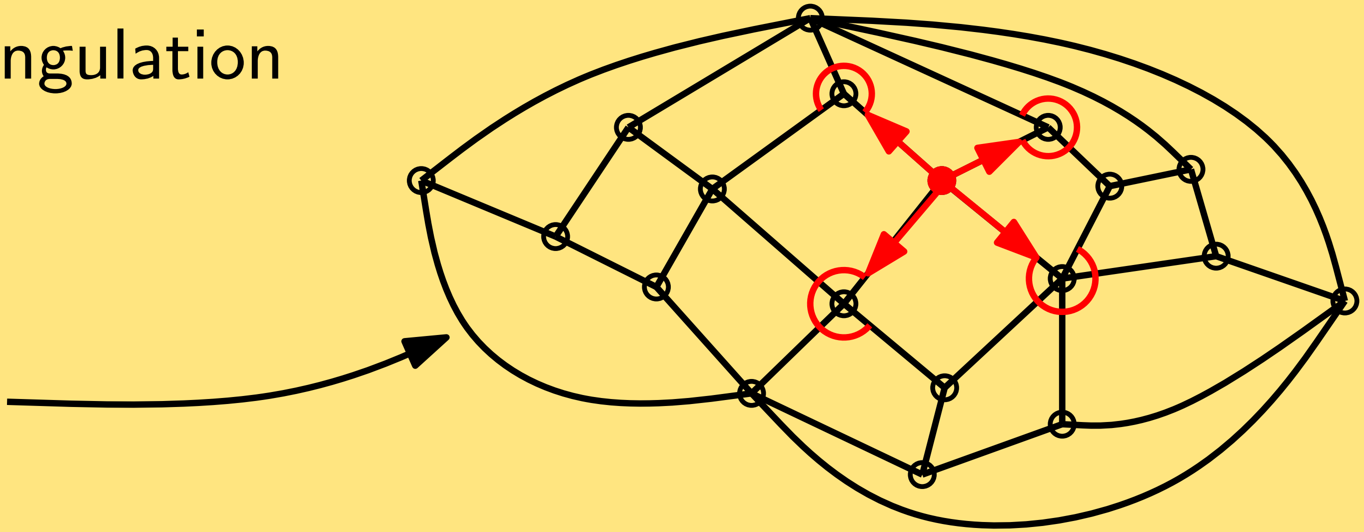

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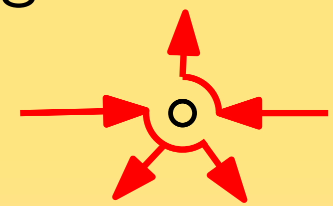

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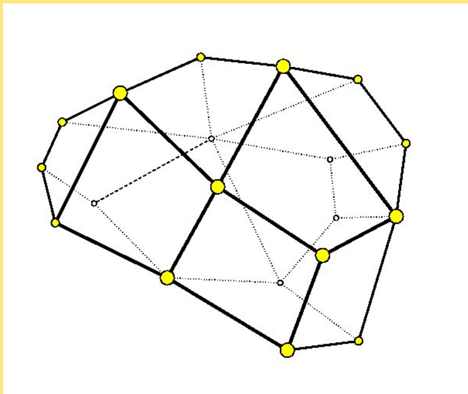

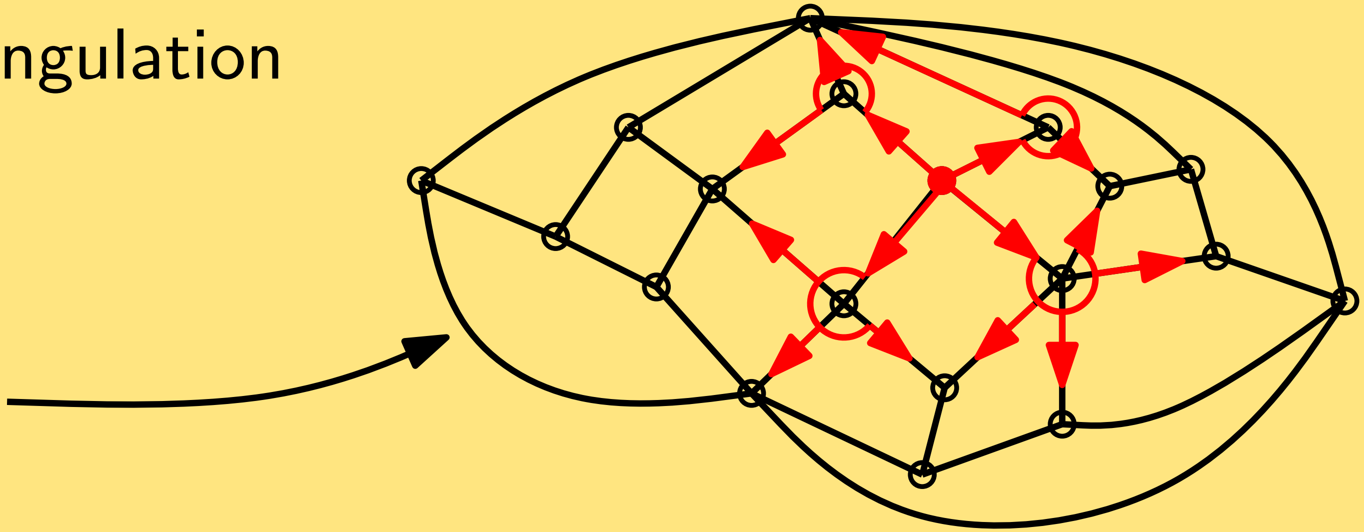

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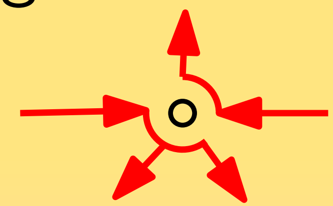

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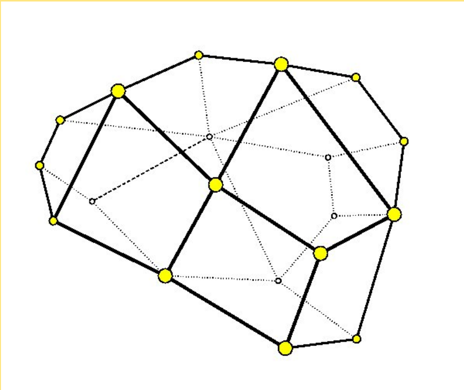

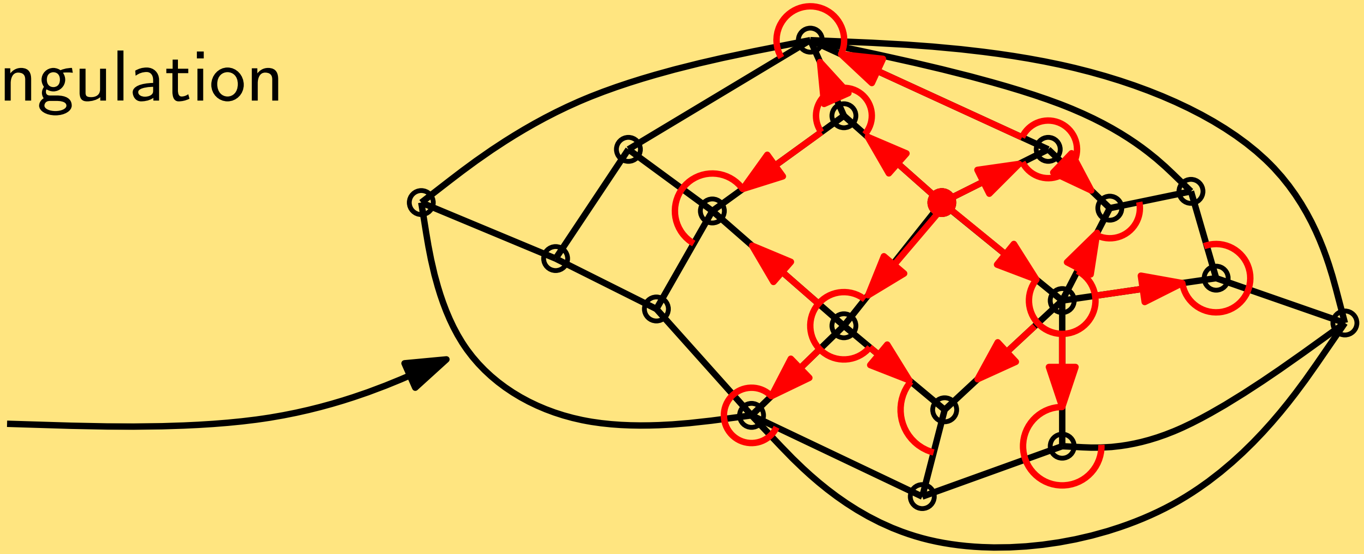

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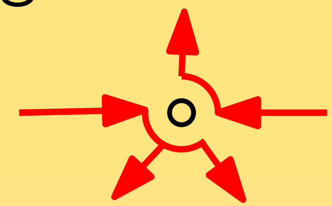

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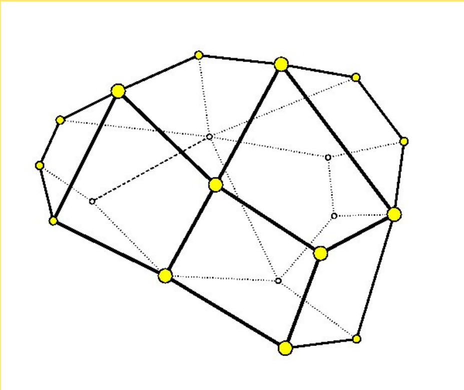

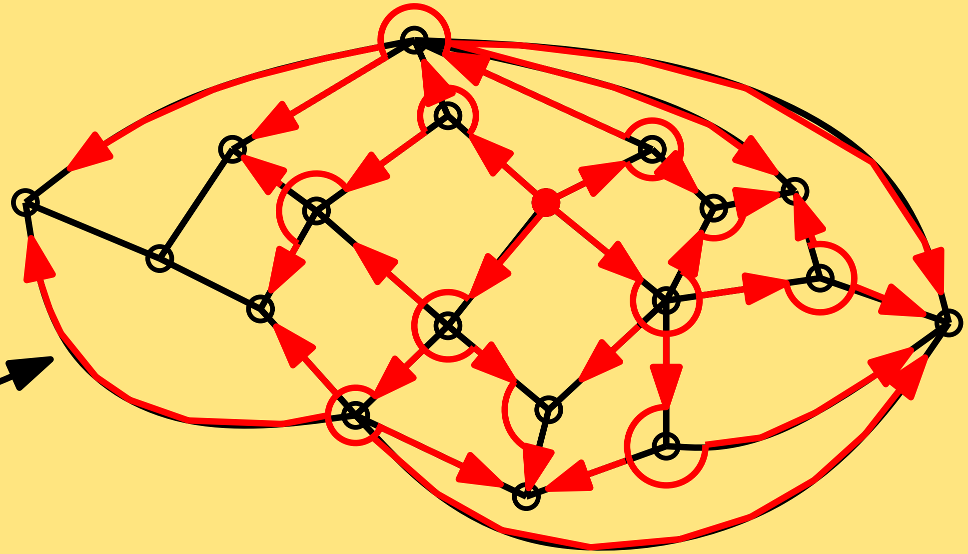

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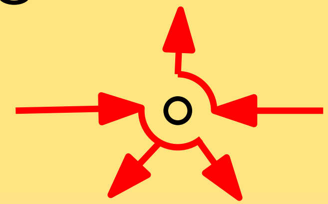

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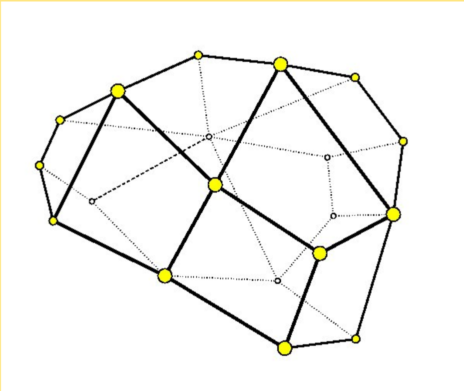

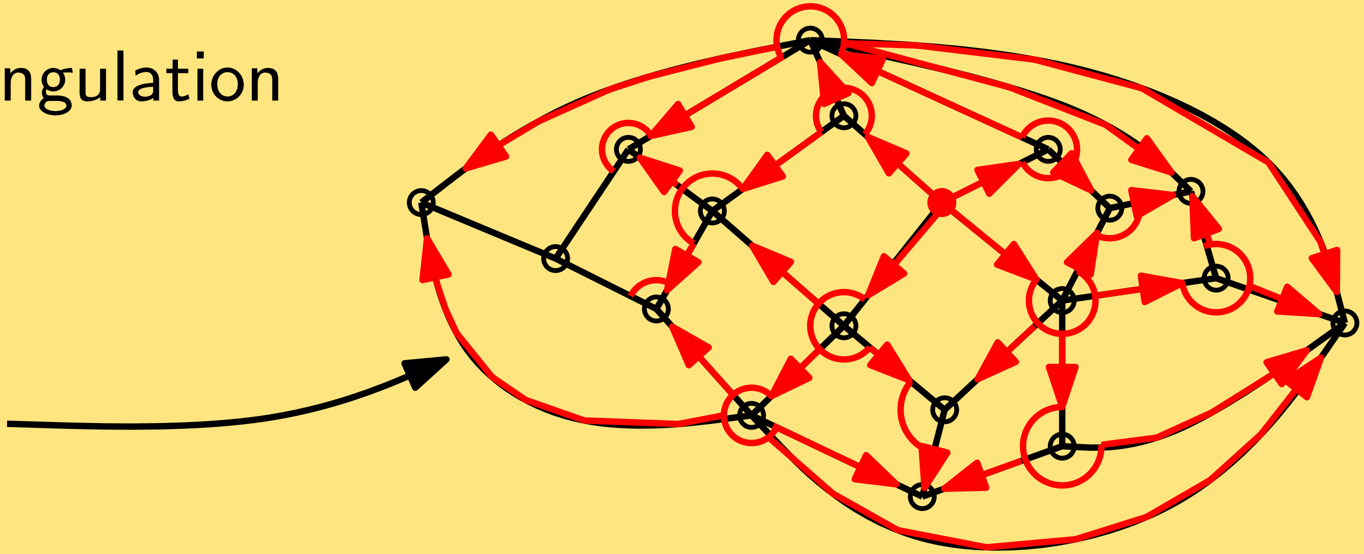

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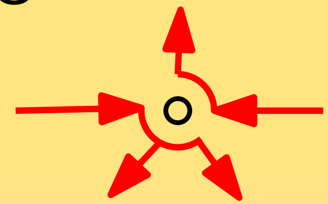

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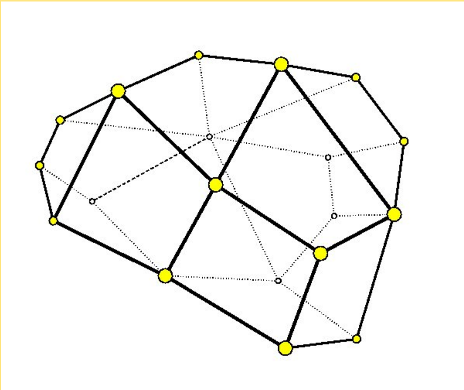

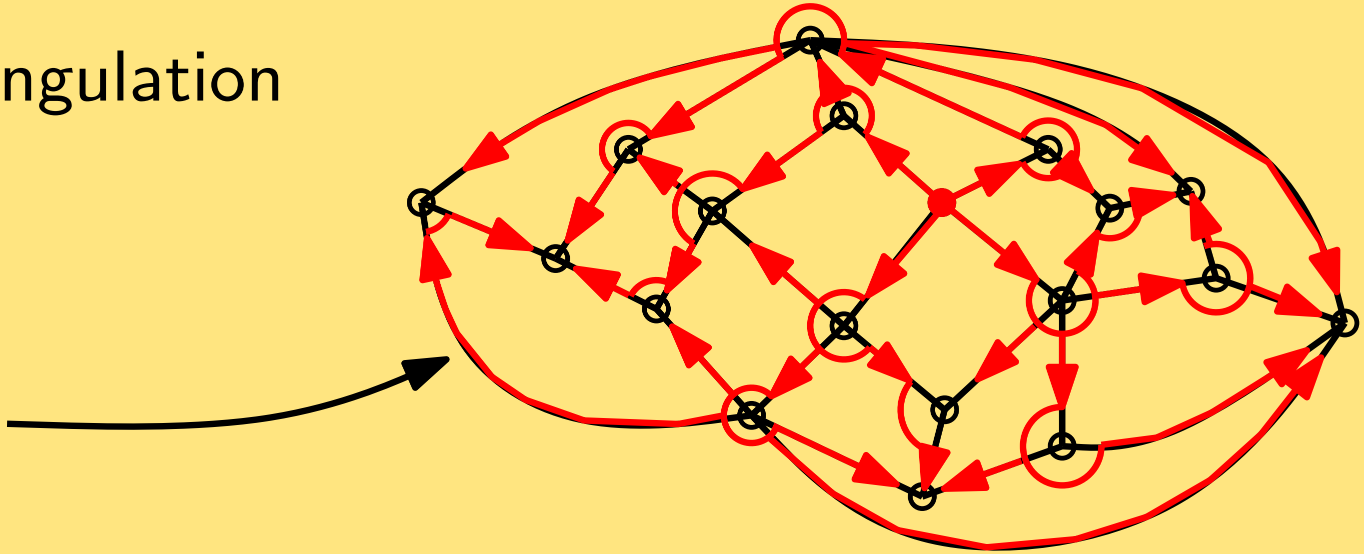

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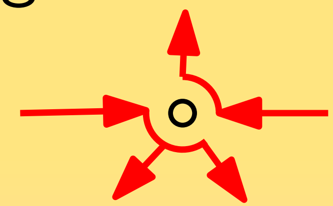

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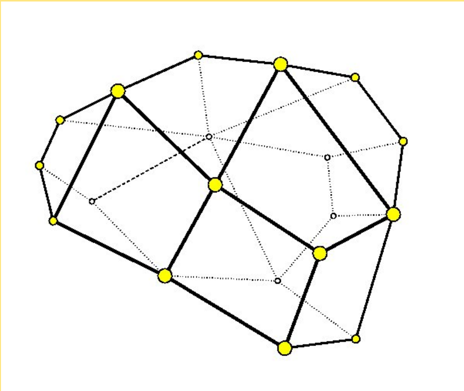

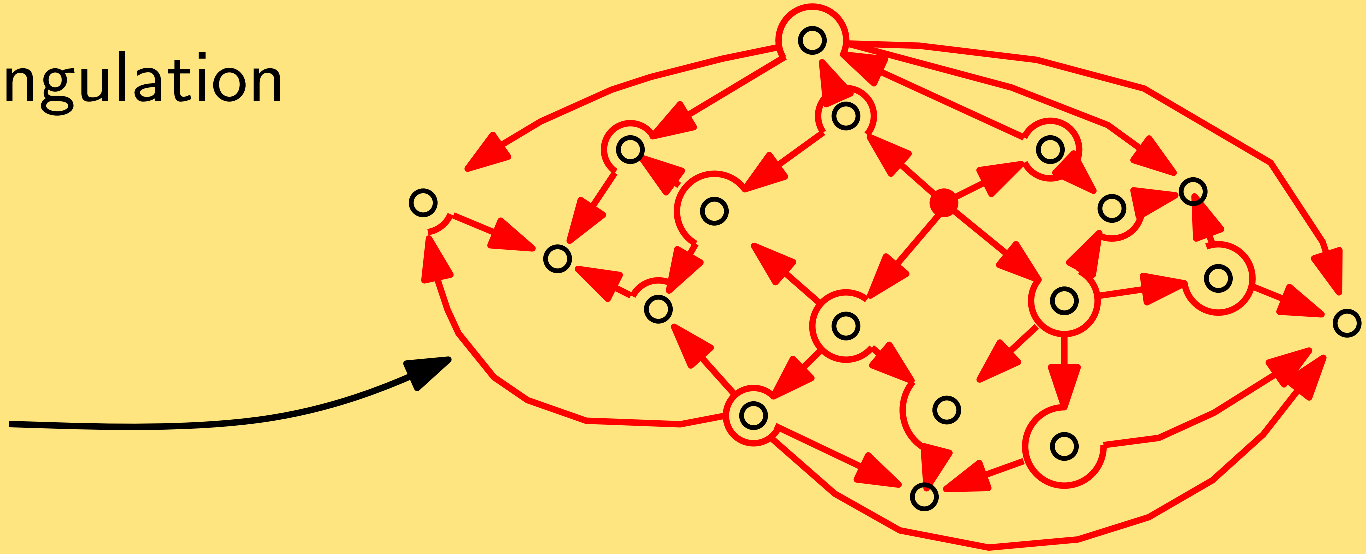

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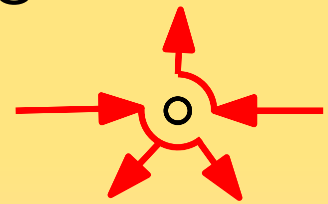

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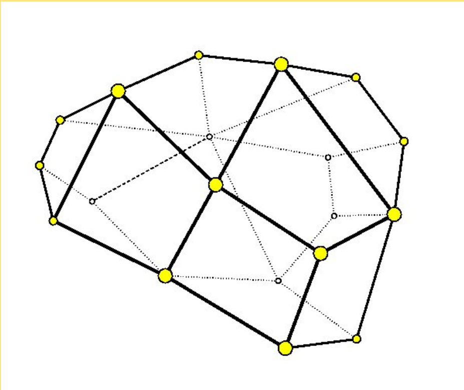

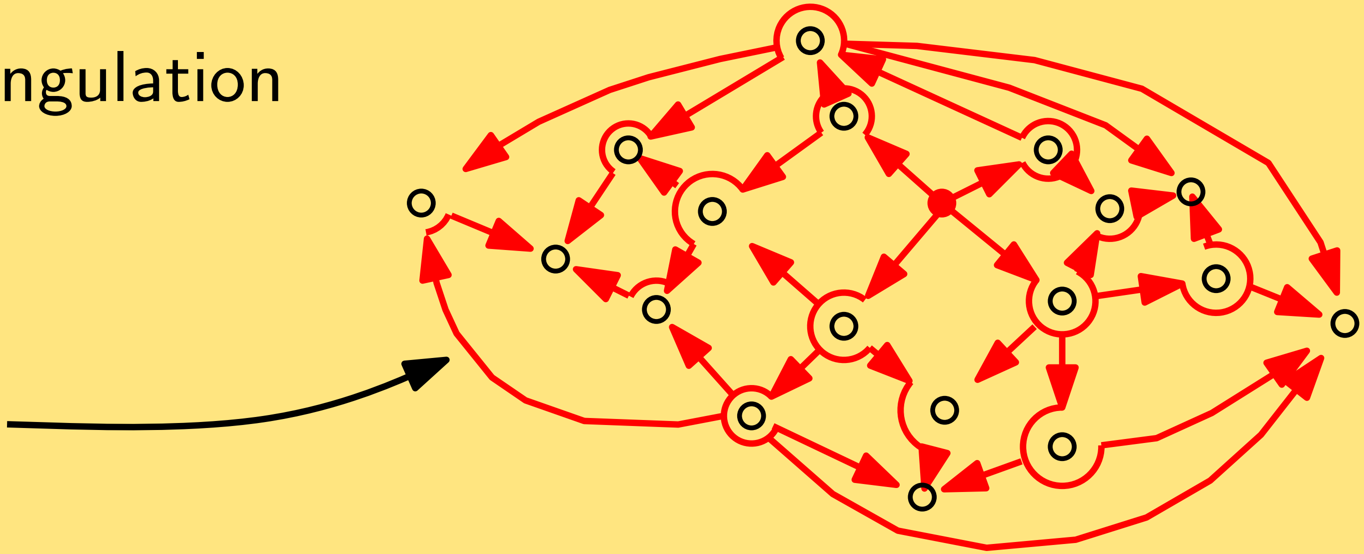

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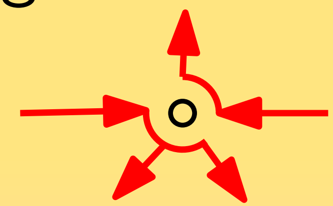
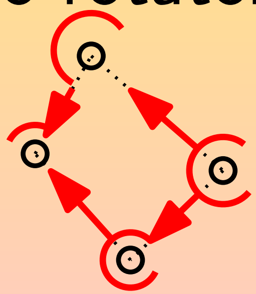


Each face sees exactly two rotatoria



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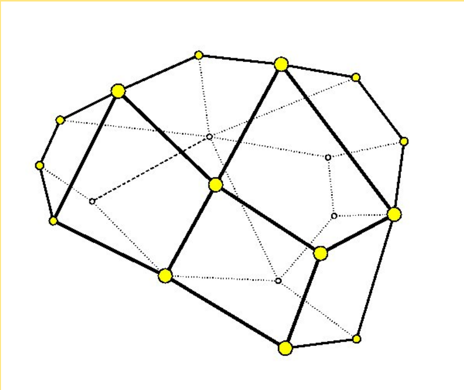

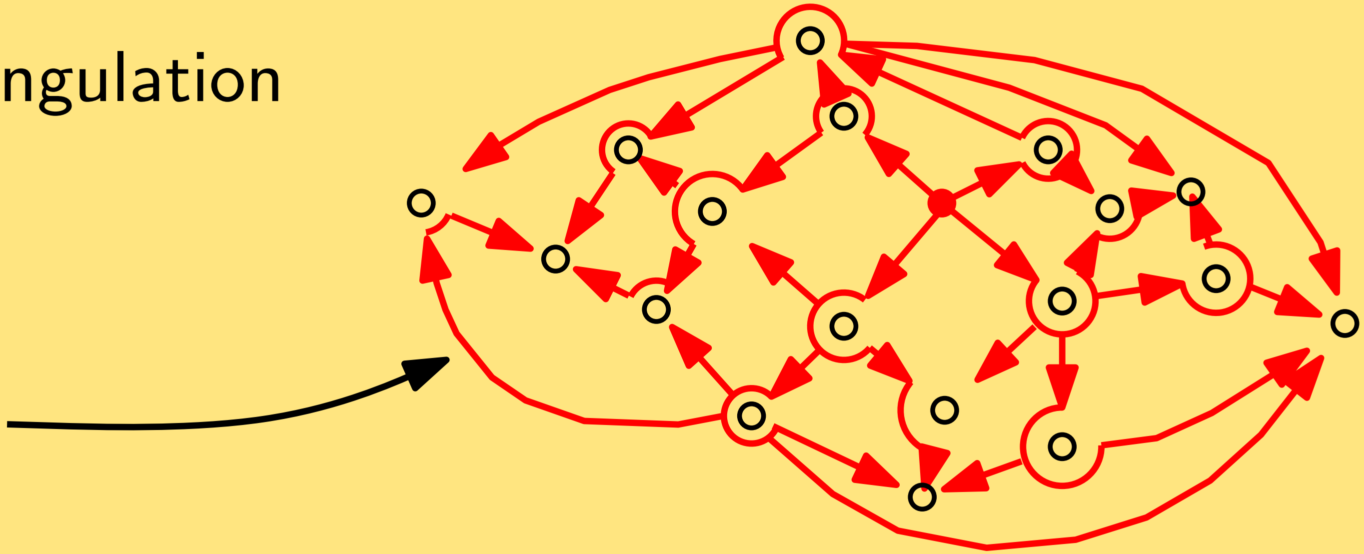

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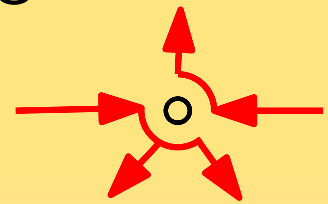

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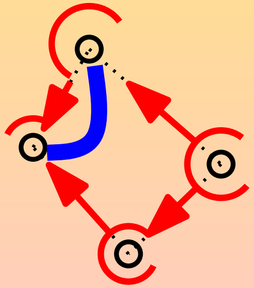

Quadrangulations via breadth first search

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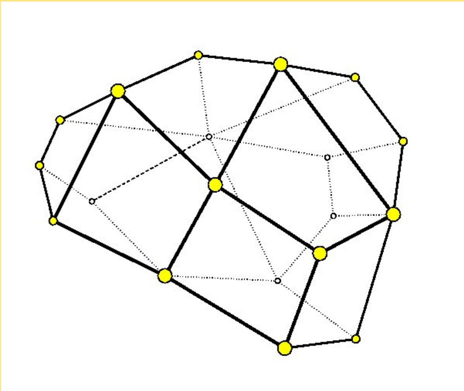

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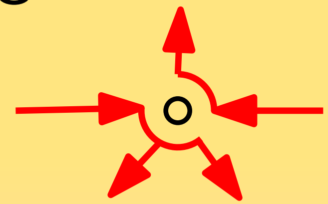

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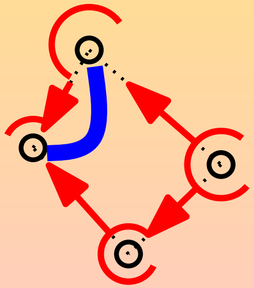

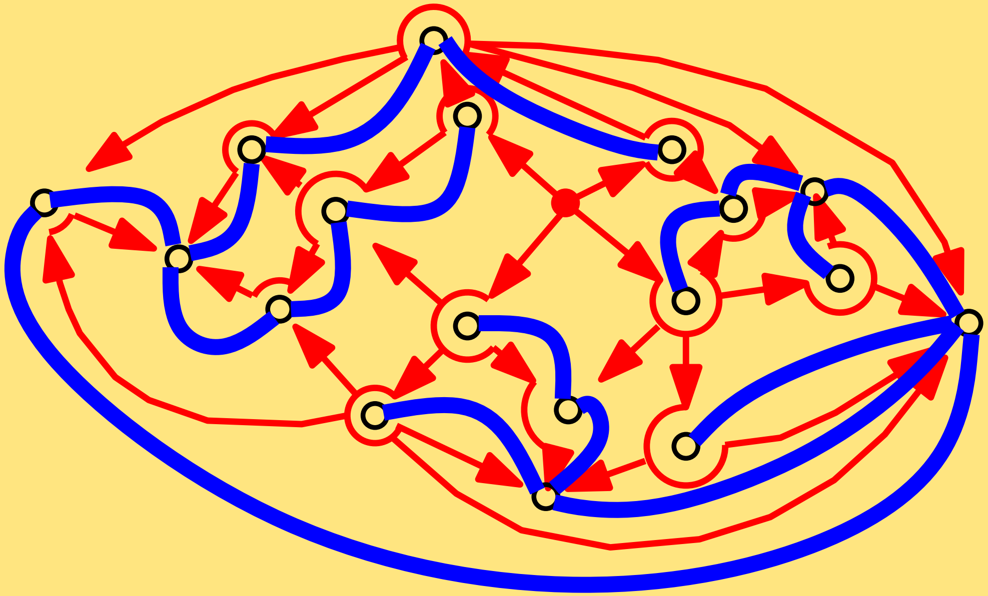

Quadrangulations via breadth first search

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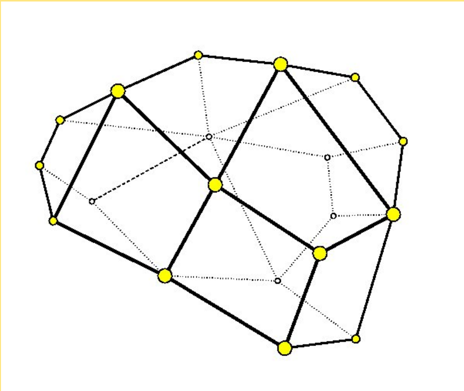

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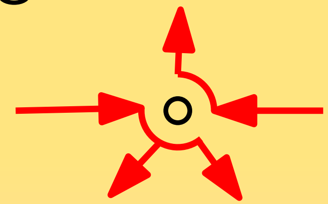

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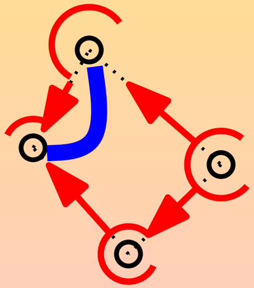

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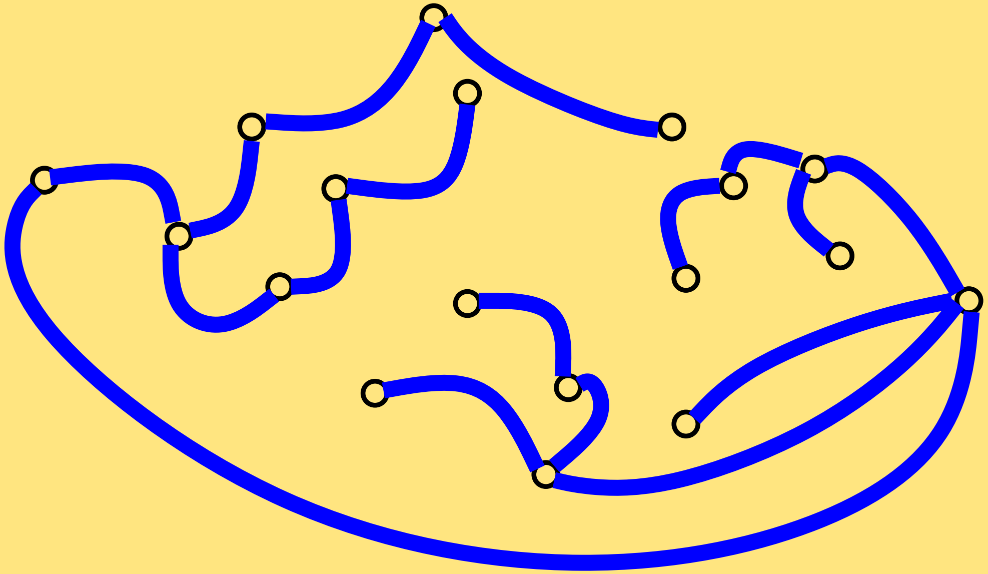
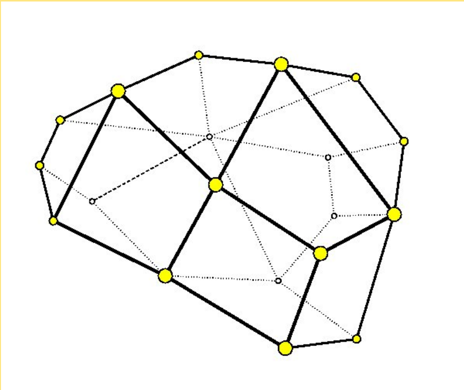


The result is **tree**.

Quadrangulations via breadth first search

Consider a planar quadrangulation



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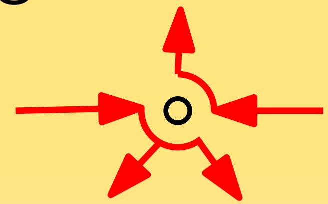

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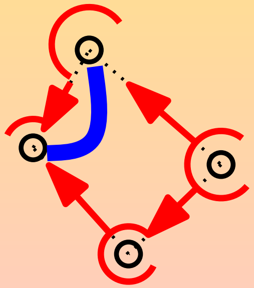

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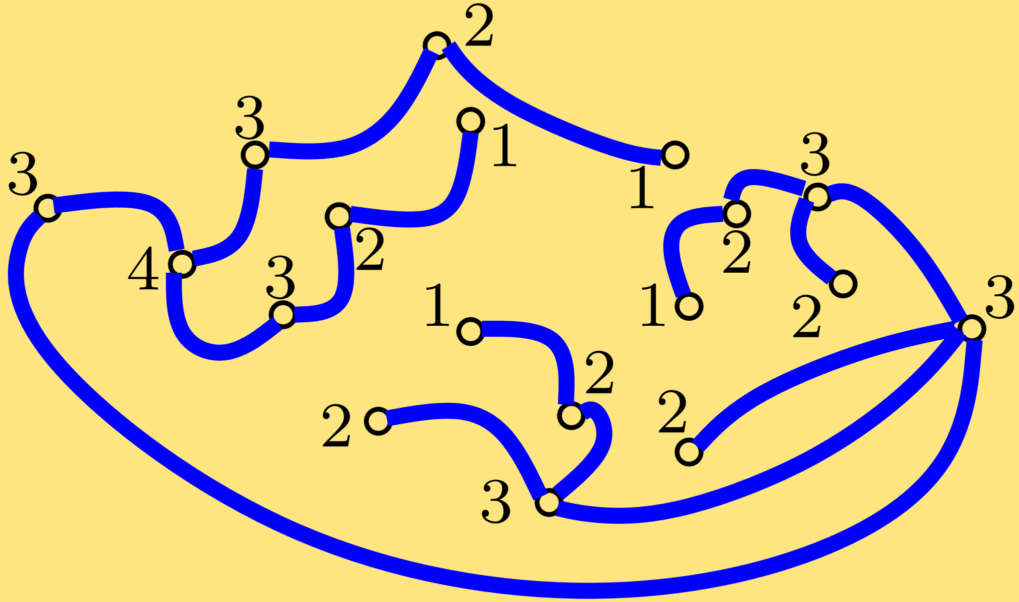

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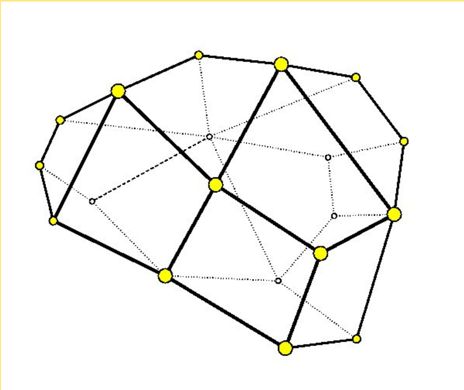
Label vertices by the round at which they were visited by bfs.

Quadrangulations via breadth first search

Consider a planar quadrangulation



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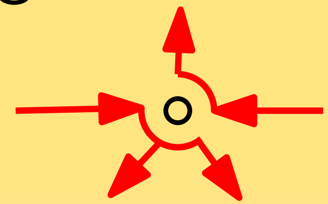

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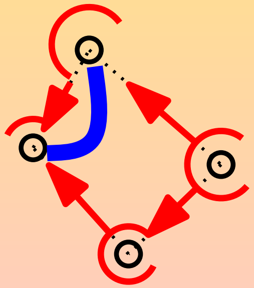
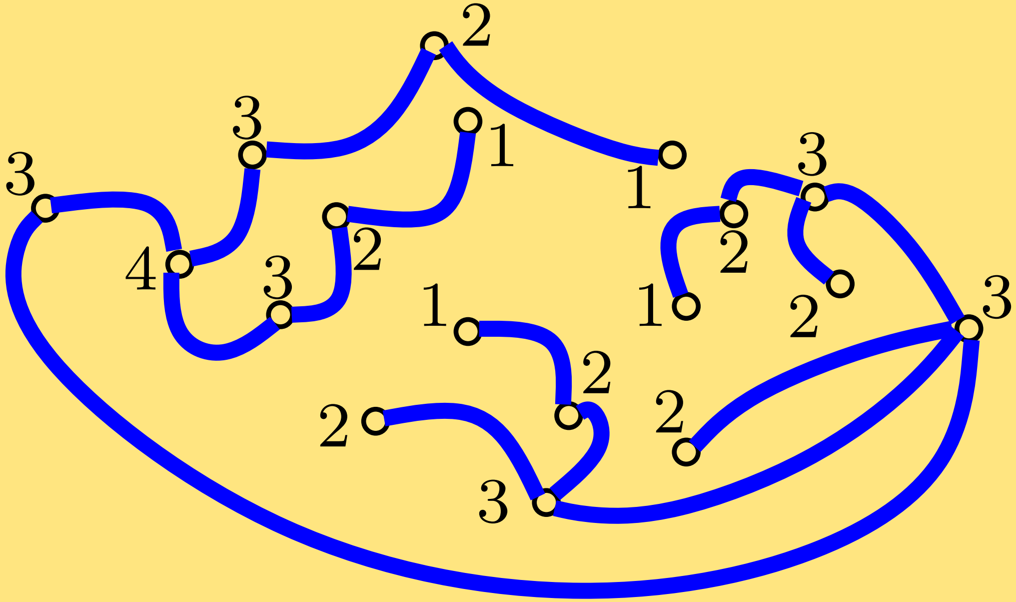


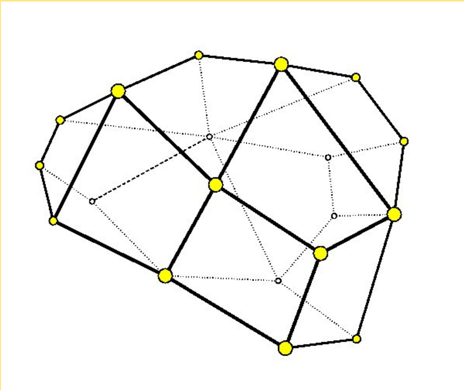


The result is a well labeled **tree**.

Label vertices by the round at which they were visited by bfs.

Quadrangulations via breadth first search

Consider a planar quadrangulation



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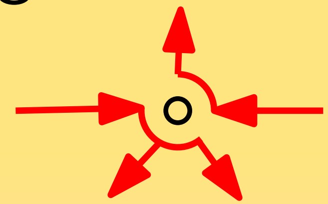

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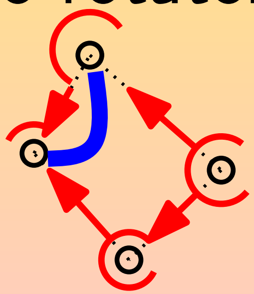
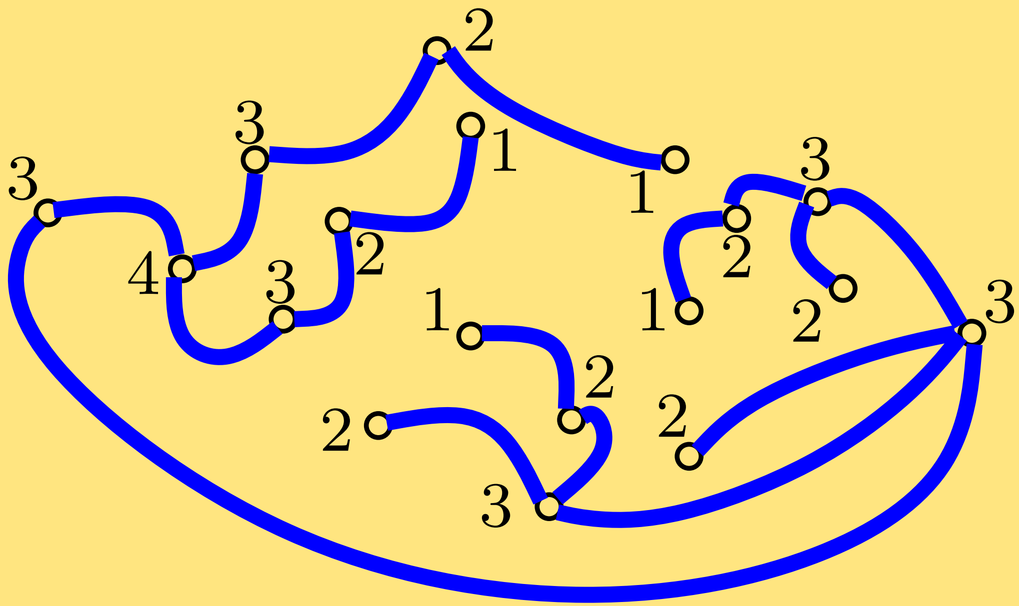




The result is a well labeled **tree**.

Label vertices by the round at which they were visited by bfs.

Theorem. This is a bijection.

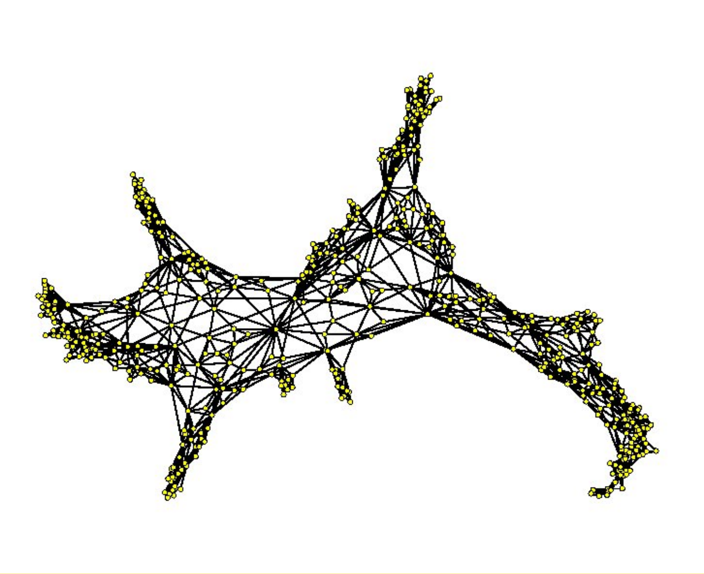
X_n: pointed quads, n faces

$\wr$

T_n: well labeled trees, n vtx

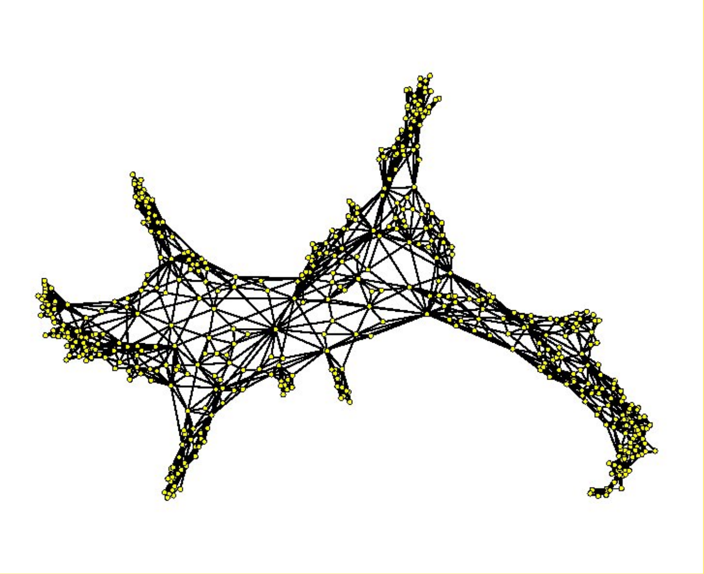
Quadrangulations via breadth first search

use breadth first search to study the geometry



distance between 2 pts = nb of edges on a path

Quadrangulations via breadth first search



use breadth first search to study the geometry

distance between 2 pts = nb of edges on a path

distance from basepoint

= round of exploration by bfs

⇒ breadth first search computes distances:

Quadrangulations via breadth first search

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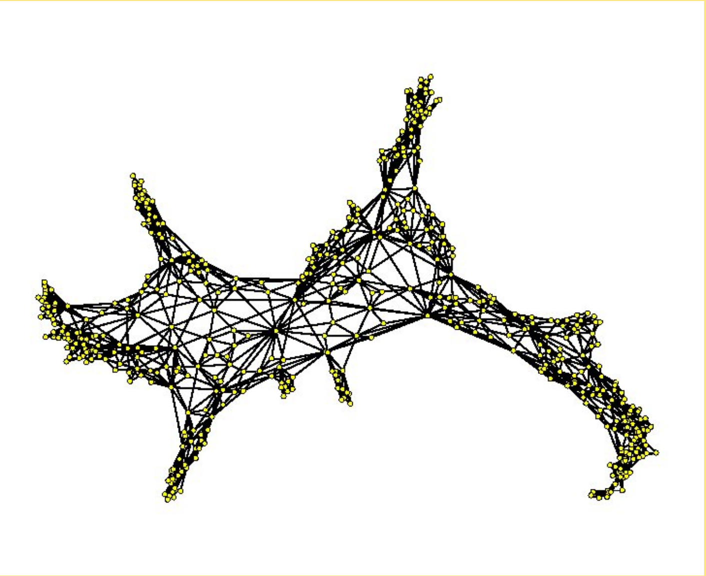

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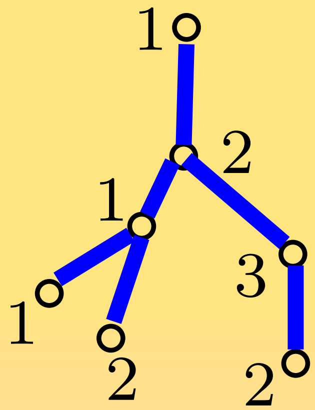

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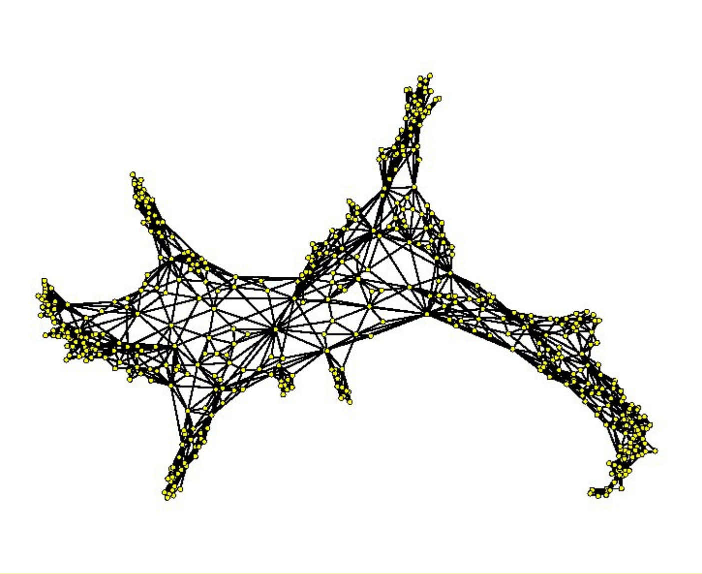

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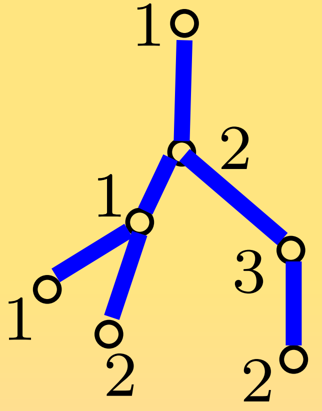
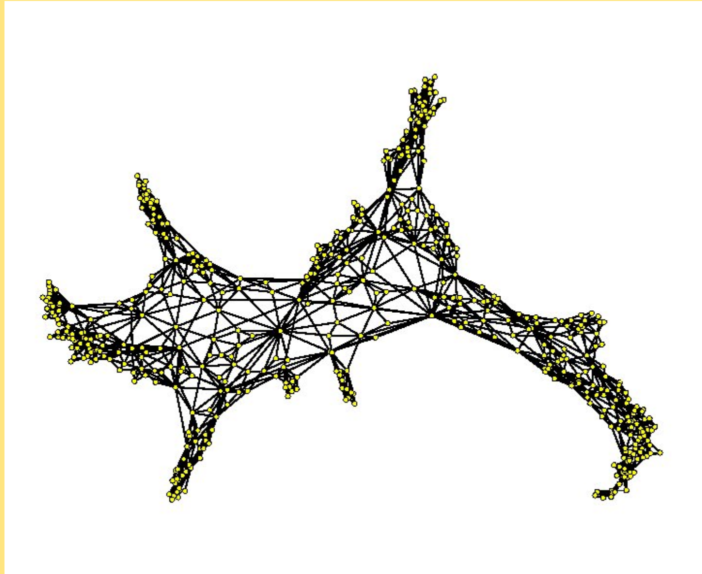


- labels of the tree record distances from the basepoint
- the height of a random tree of size n is $n^{1/2}$
- the random walk of labels on a branch of length ℓ has max about $\ell^{1/2}$

$\Rightarrow$ typical labels are of order $n^{1/4}$.

Quadrangulations via breadth first search

use breadth first search to study the geometry



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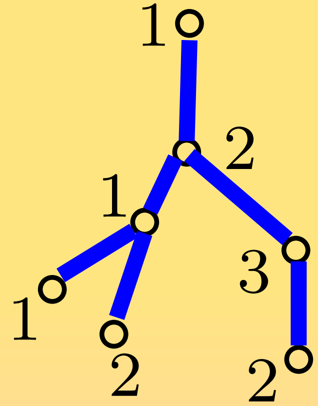

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Theorem (Chassaing-S, 2004).
The distance between 2 random vertices of X_n is of order $n^{1/4}$.

This approach was pursued by Chassaing-Durhuus (2005), Marckert-Mokkadem (2004), Miermond (2005), Weill (2006)... culminating with

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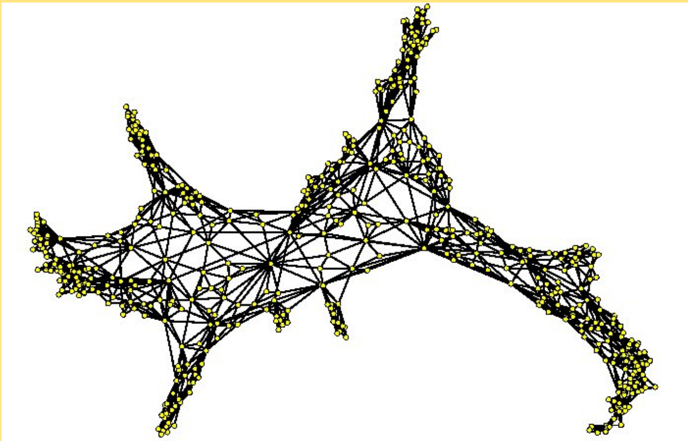

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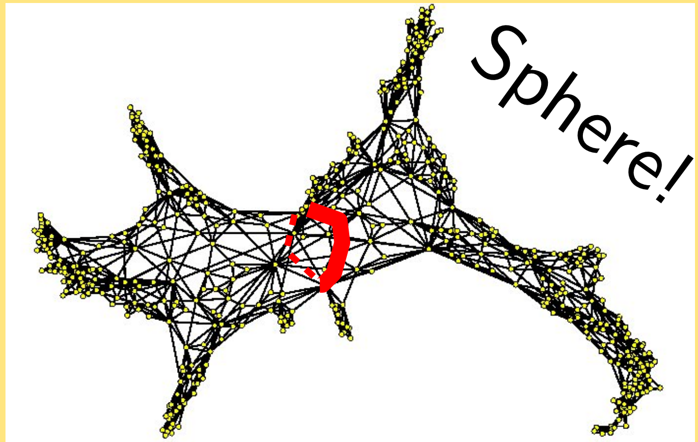

In particular there exists no separating cycle of size $\ll n^{1/4}$.

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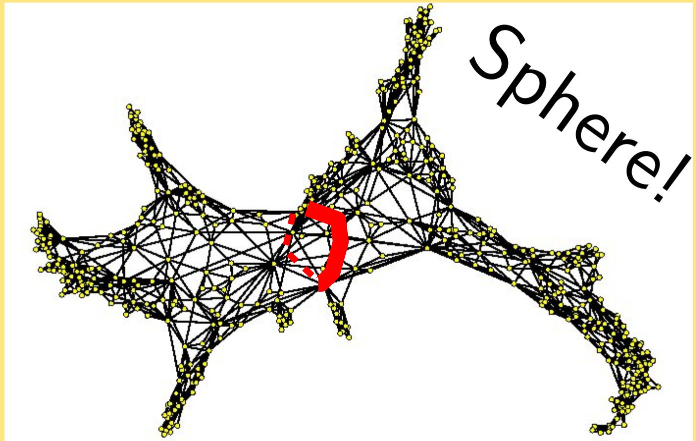

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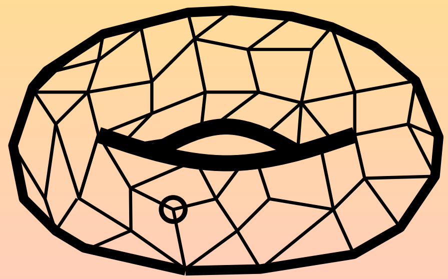
The bfs exploration works also for higer genus surfaces:
Theorem (Chapuy-Marcus-S. 2006) The distance between 2 random vertices of a random quad X_n^g of genus g is of order $n^{1/4}$.



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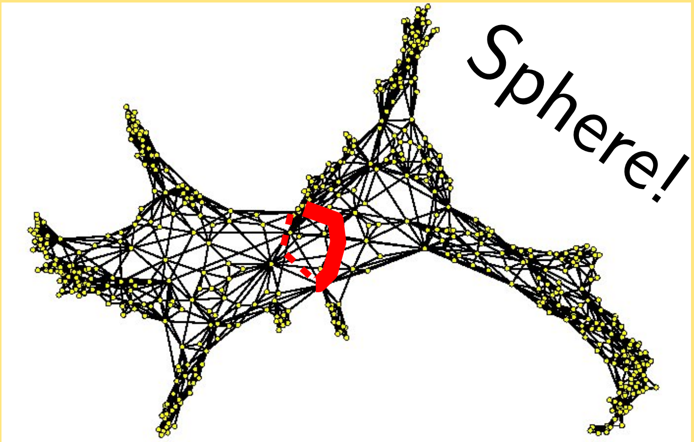

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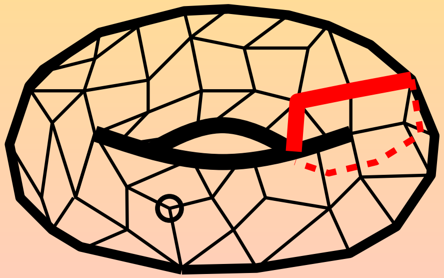
Conjectures.
There is no non-contractible cycles with size $\ll n^{1/4}$.
The rescaled continuum limit exists and has genus g.



A conjecture on random graphs with low genus

Let Y_n^g be a uniform random connected labelled graphs with n vertices that can be embedded on a surface of genus g.

For instance Y_n^0 is a random connected planar graph with n vertices.

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Conjecture. The graph Y_n^g is a.s. composed of a 3-connected graph $Core(Y)$ of size $\Theta(n)$ with edges replaced by small planar networks and with small pending planar components.

Moreover $Core(Y)$ a.s. has minimal genus g and has a unique minimal embedding. The small parts have size $O(n^{2/3})$.

In the rescaled limit, Y_n^g converge to the same continuum random map of genus g as X_n^g.

Cf. McDiarmid, Noy, Steger's talks for proofs...

Many thanks for your attention !

Many thanks to my collaborators!