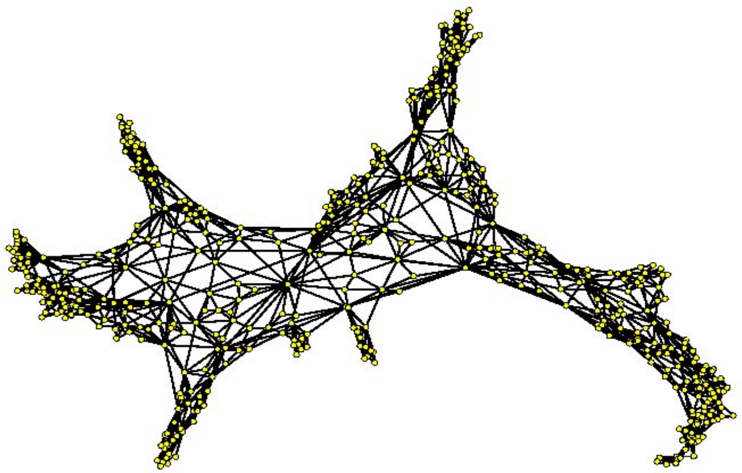
Obvious and hidden tree structures in 2d-triangulations

from algebraic generating functions
to random surfaces

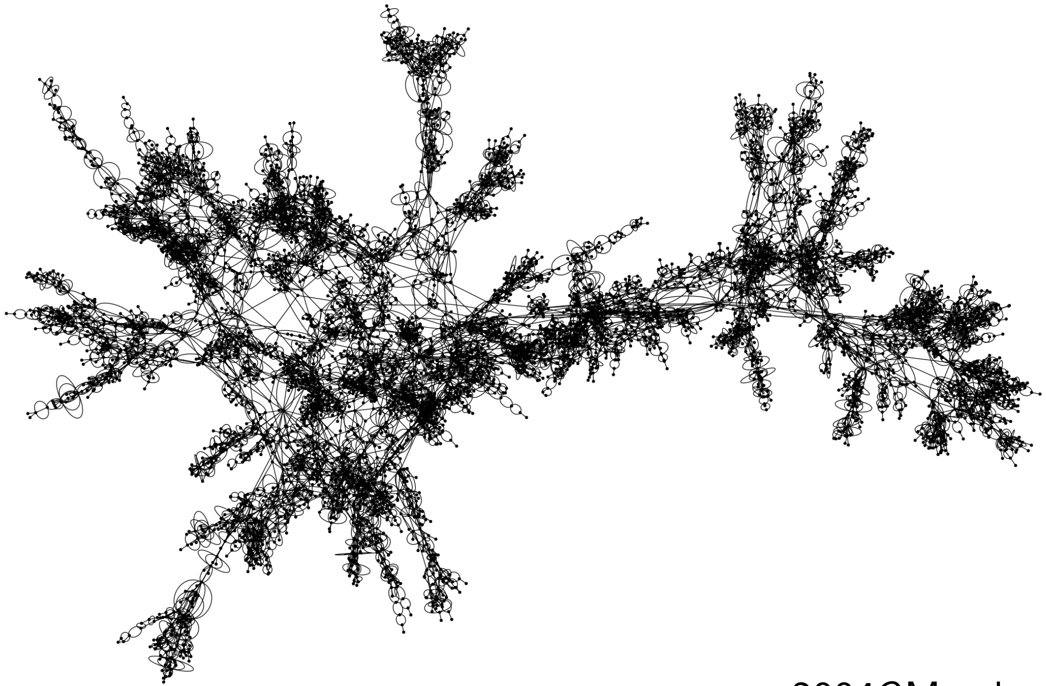
GILLES SCHAEFFER CNRS & École Polytechnique
Supported by ERC RStG 208471 "ExploreMaps"

Quantum gravity in Orsay, march 2013

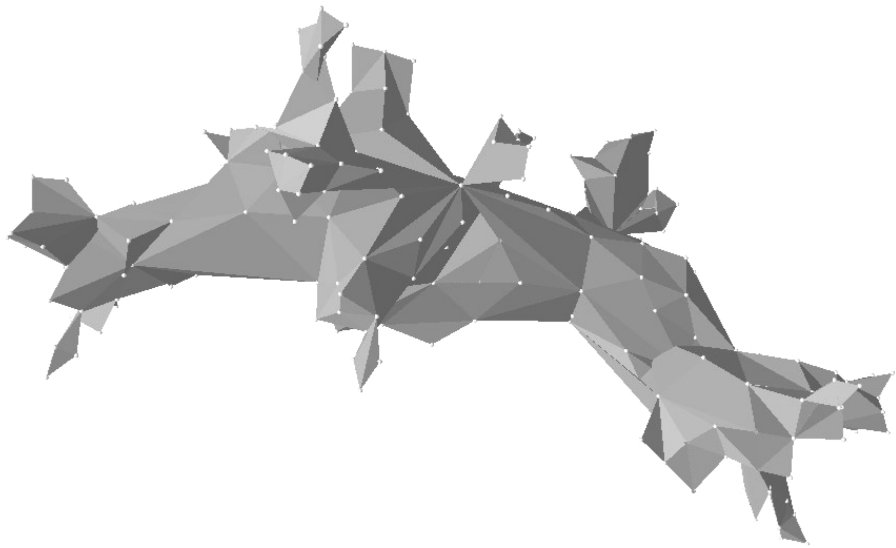
random triangulations
as random surfaces



1998@Schaeffer



2004@Marckert



2009@Chapuy

from the combinatorial literature
(equivalent picts in physics literature)

Counting maps and triangulations

Trees, independence, algebraic generating series

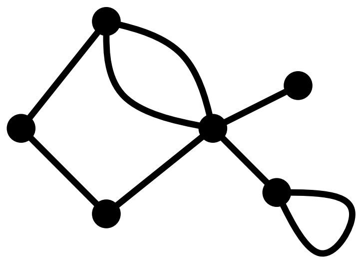
Stack triangulations

General 2d triangulations

Realizers

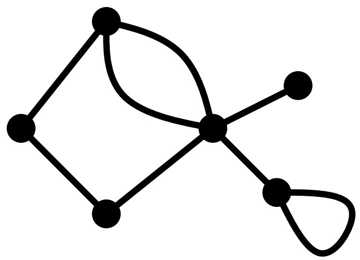
Plane graphs and planar maps

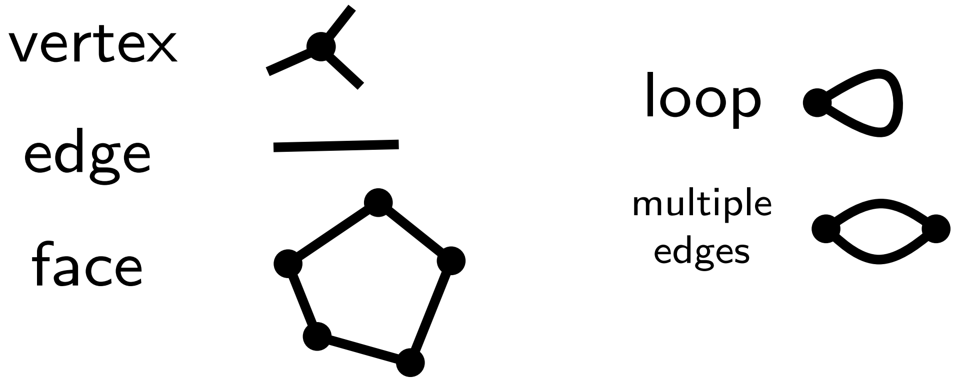
Plane graph = $\left\{ \begin{array}{l} \text{Embedding of a connected graph} \\ \text{in the plane} \end{array} \right.$



Plane graphs and planar maps

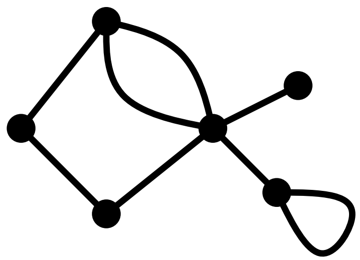
Plane graph $=$ $\left\{ \begin{array}{l} \text{Embedding of a connected graph} \\ \text{in the plane} \end{array} \right.$

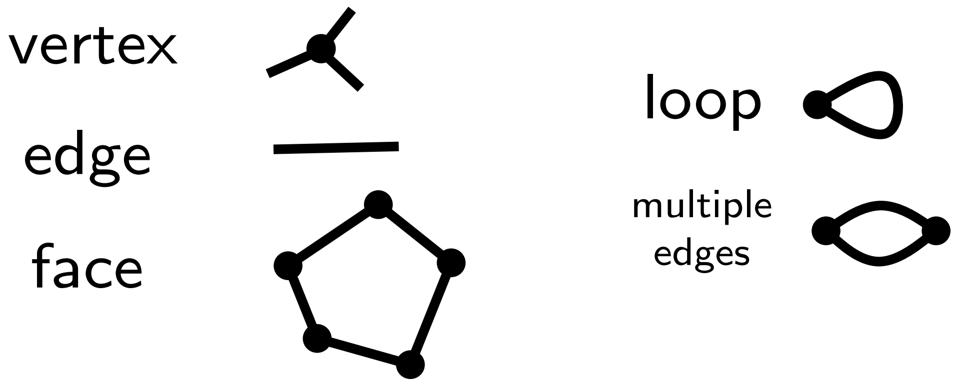


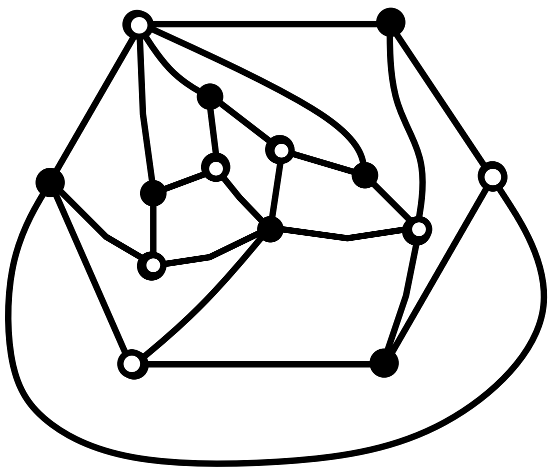


Plane graphs and planar maps

Plane graph $=$ $\left\{ \begin{array}{l} \text{Embedding of a connected graph} \\ \text{in the plane} \end{array} \right.$

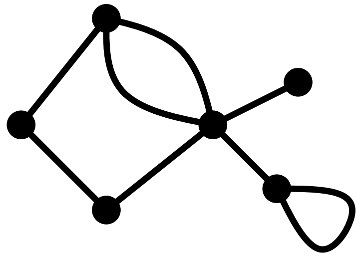


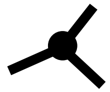




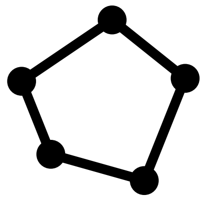
Plane graphs and planar maps

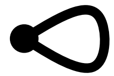
Plane graph $=$ { Embedding of a connected graph in the plane

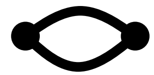


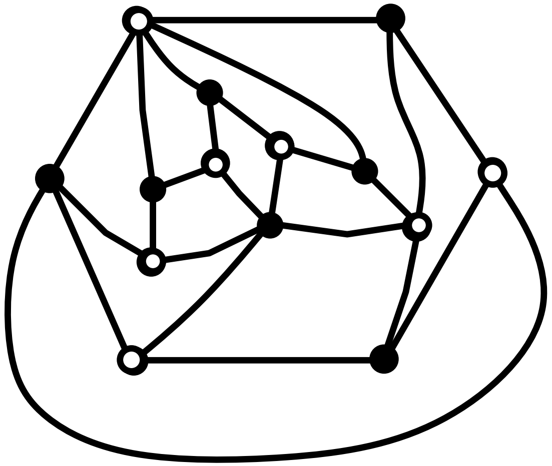
vertex 

edge 

face 

loop 

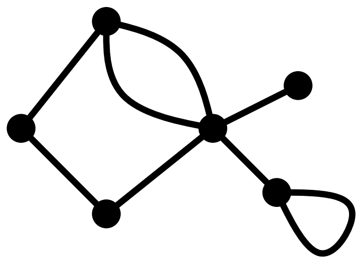
multiple edges 



vertex or face degree = nb of "corners"

Plane graphs and planar maps

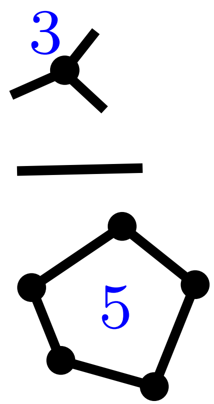
Plane graph = { Embedding of a connected graph in the plane

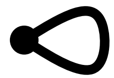


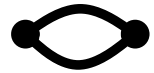
vertex

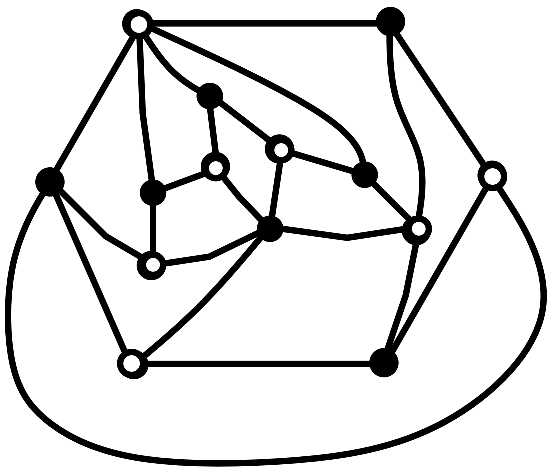
edge

face



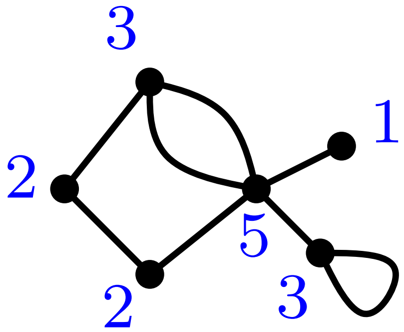
loop 

multiple edges 



vertex or face degree = nb of "corners"

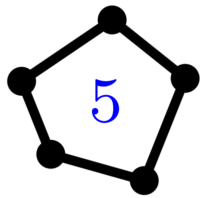
Plane graphs and planar maps

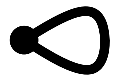


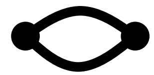
Plane graph = { Embedding of a connected graph in the plane

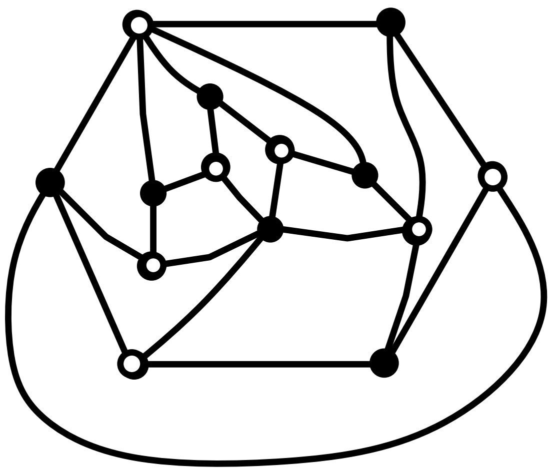
vertex 

edge 

face 

loop 

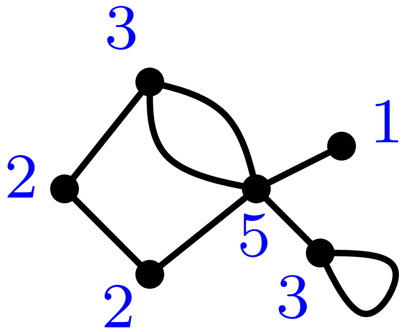
multiple edges 



vertex or face degree = nb of "corners"

Plane graphs and planar maps

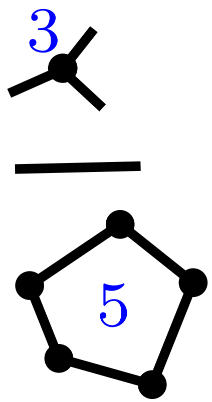
Plane graph $=$ { Embedding of a connected graph in the plane

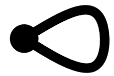


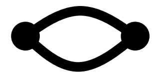
vertex

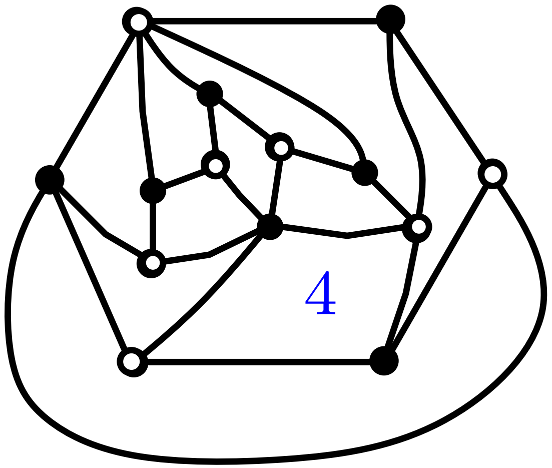
edge

face



loop 

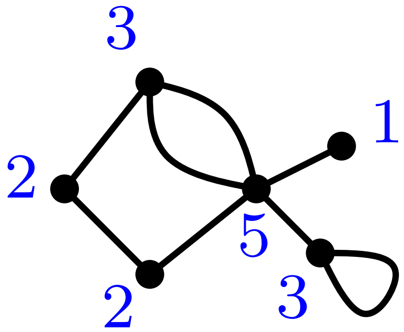
multiple edges 



vertex or face degree $=$ nb of "corners"

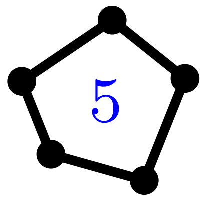
Plane graphs and planar maps

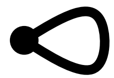
Plane graph = $\left\{ \begin{array}{l} \text{Embedding of a connected graph} \\ \text{in the plane} \end{array} \right.$

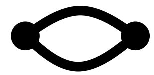


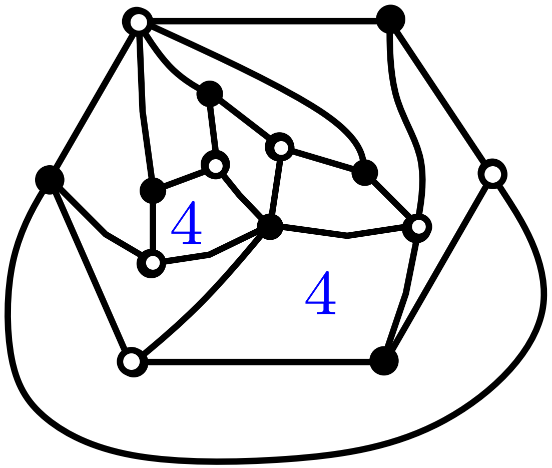
vertex 

edge 

face 

loop 

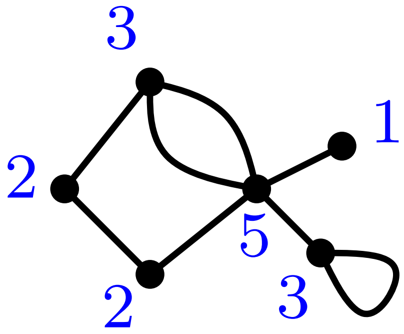
multiple edges 



vertex or face degree = nb of "corners"

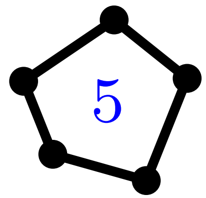
Plane graphs and planar maps

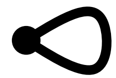
Plane graph $=$ $\left\{ \begin{array}{l} \text{Embedding of a connected graph} \\ \text{in the plane} \end{array} \right.$

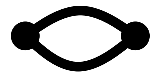


vertex 

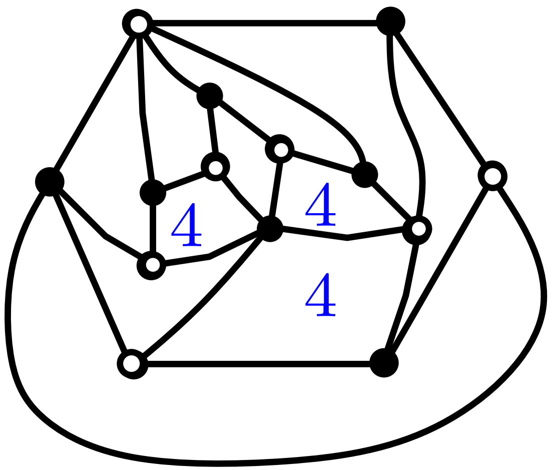
edge 

face 

loop 

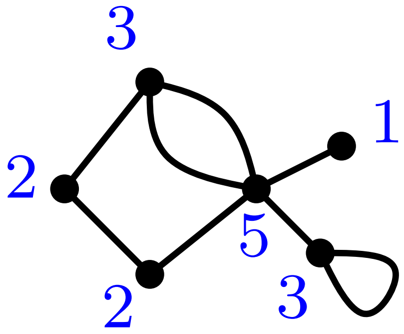
multiple edges 

vertex or face degree $=$ nb of "corners"



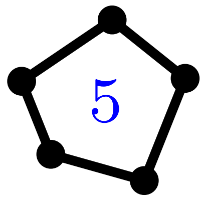
Plane graphs and planar maps

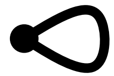
Plane graph $=$ { Embedding of a connected graph in the plane

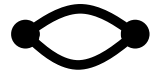


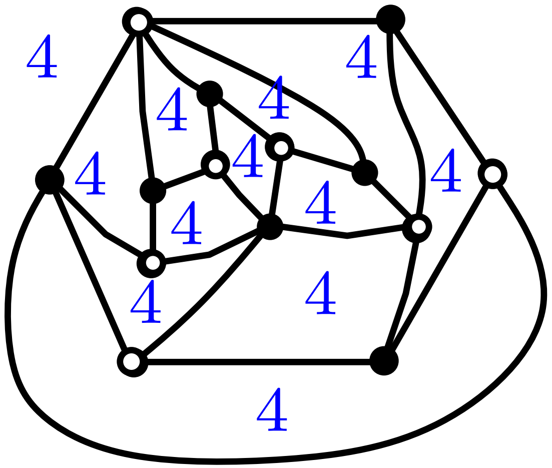
vertex 

edge 

face 

loop 

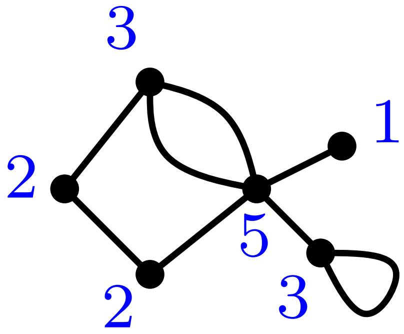
multiple edges 



vertex or face degree $=$ nb of "corners"

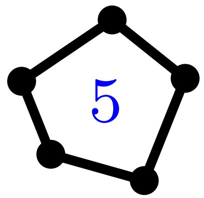
Plane graphs and planar maps

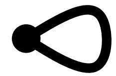
Plane graph = { Embedding of a connected graph in the plane

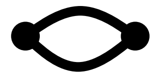


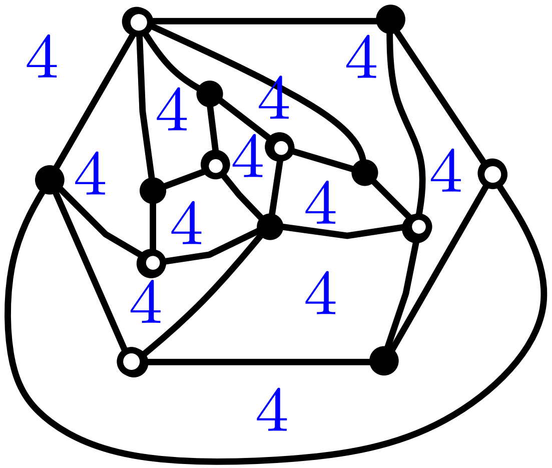
vertex 

edge 

face 

loop 

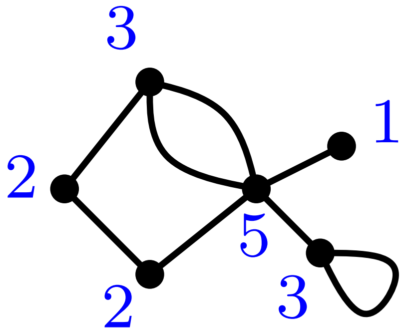
multiple edges 



plane quadrangulation

vertex or face degree = nb of "corners"

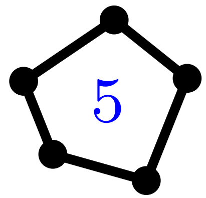
Plane graphs and planar maps

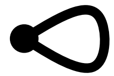


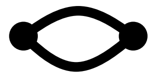
Plane graph = { Embedding of a connected graph in the plane or on the sphere

vertex 

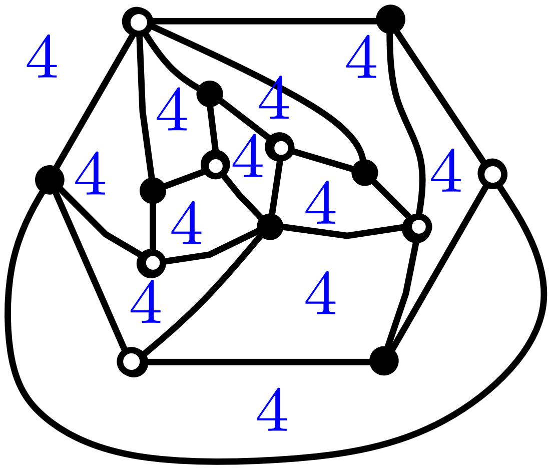
edge 

face 

loop 

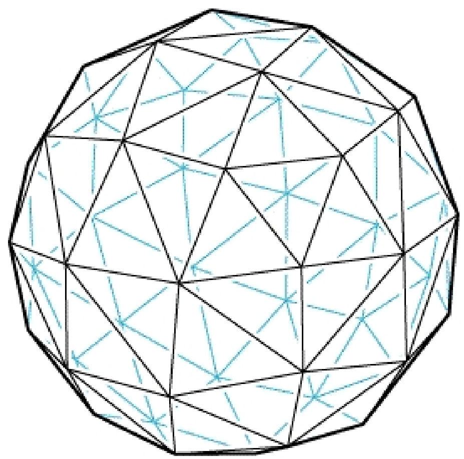
multiple edges 

vertex or face degree = nb of "corners"



plane quadrangulation

sphere triangulation



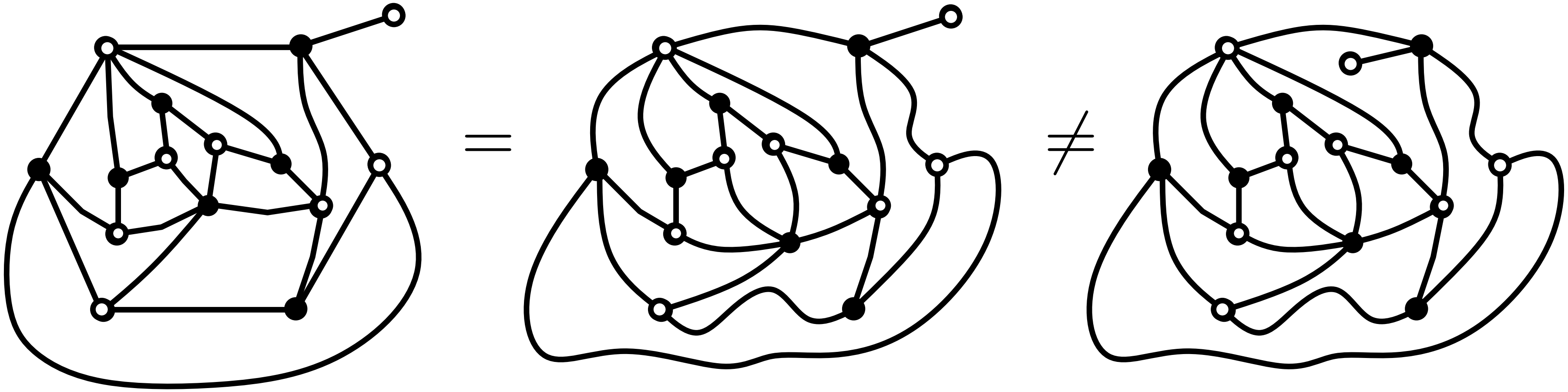
Plane graphs and planar maps

$$\text{Plane graph} = \left\{ \begin{array}{l} \text{Embedding of a connected graph} \\ \text{in the plane} \end{array} \right.$$

Plane graphs and planar maps

planar map = Plane graph = Embedding of a connected graph in the plane

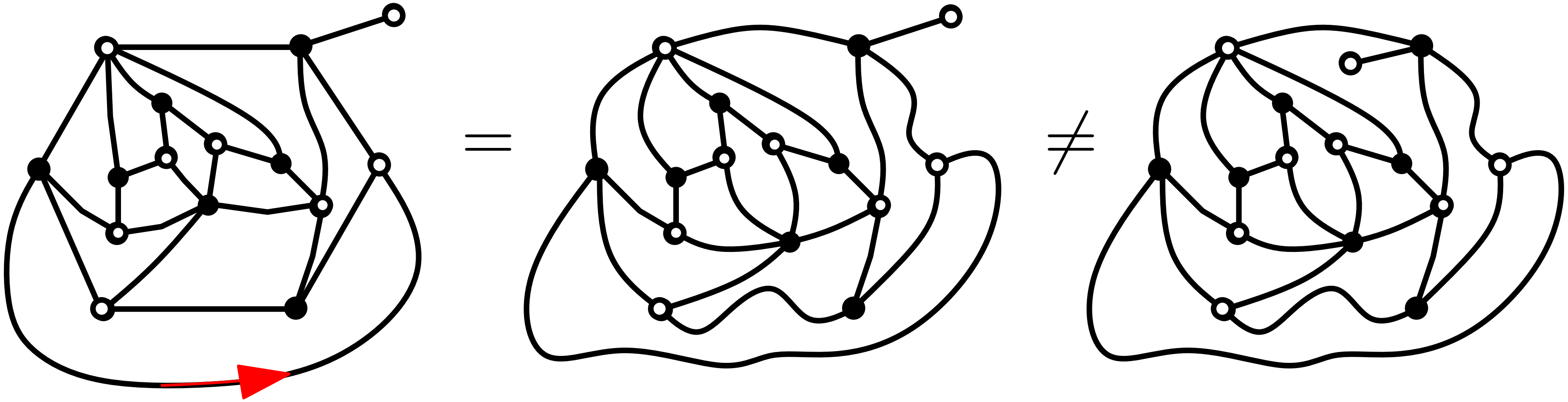
considered up to homeomorphisms of the plane



Plane graphs and planar maps

planar map = { Plane graph = { Embedding of a connected graph in the plane ; considered up to homeomorphisms of the plane }

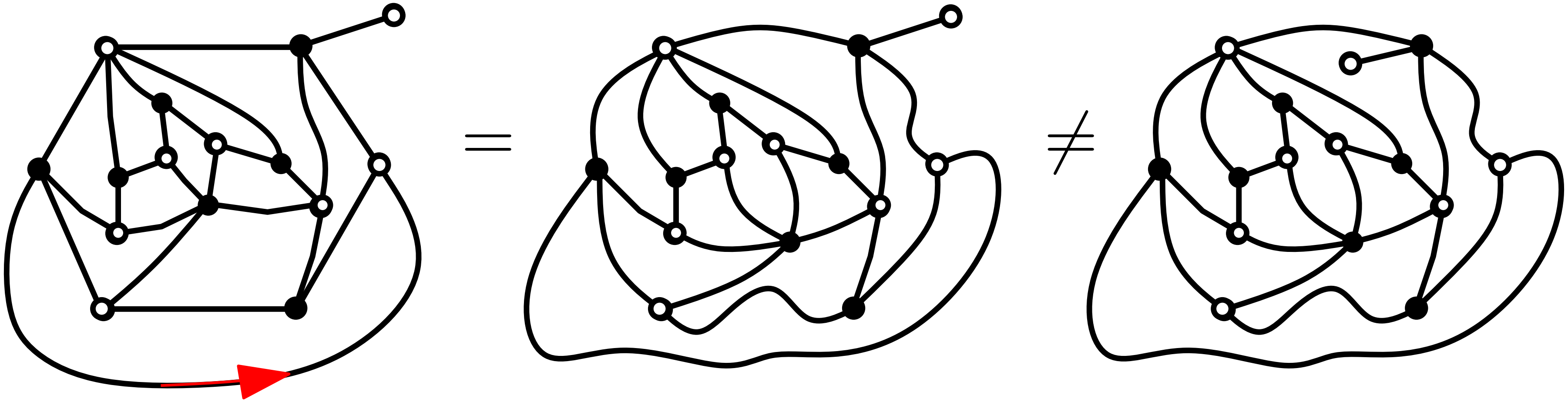
rooted planar map = with one marked edge (to kill symmetries)



Plane graphs and planar maps

planar map = { Plane graph = { Embedding of a connected graph in the plane } considered up to homeomorphisms of the plane }

rooted planar map = with one marked edge (to kill symmetries)

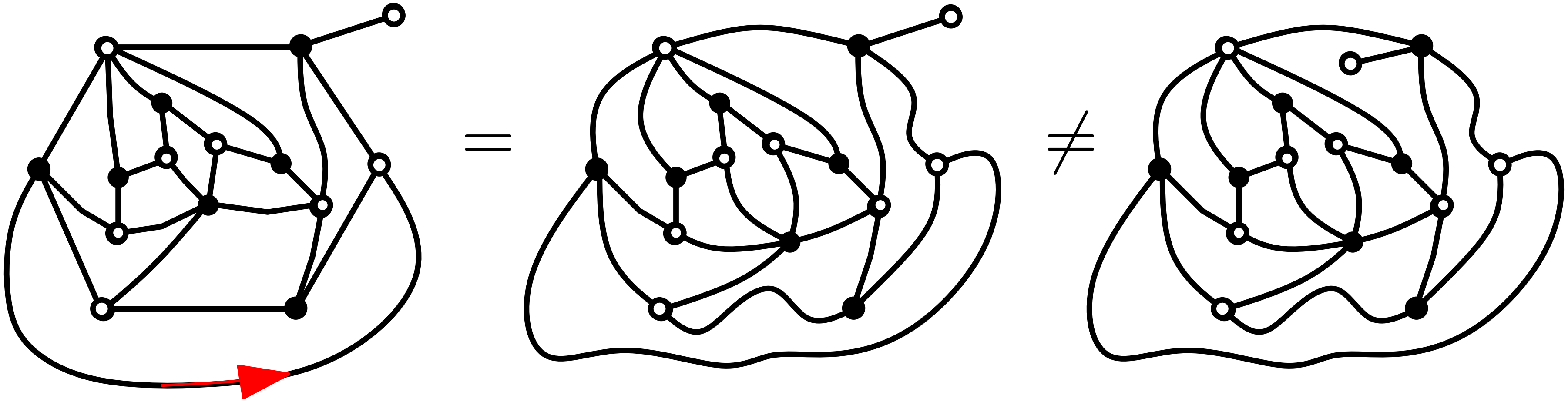


planar maps are discrete (combinatorial) structures

Plane graphs and planar maps

planar map = { Plane graph = { Embedding of a connected graph in the plane ; considered up to homeomorphisms of the plane }

rooted planar map = with one marked edge (to kill symmetries)



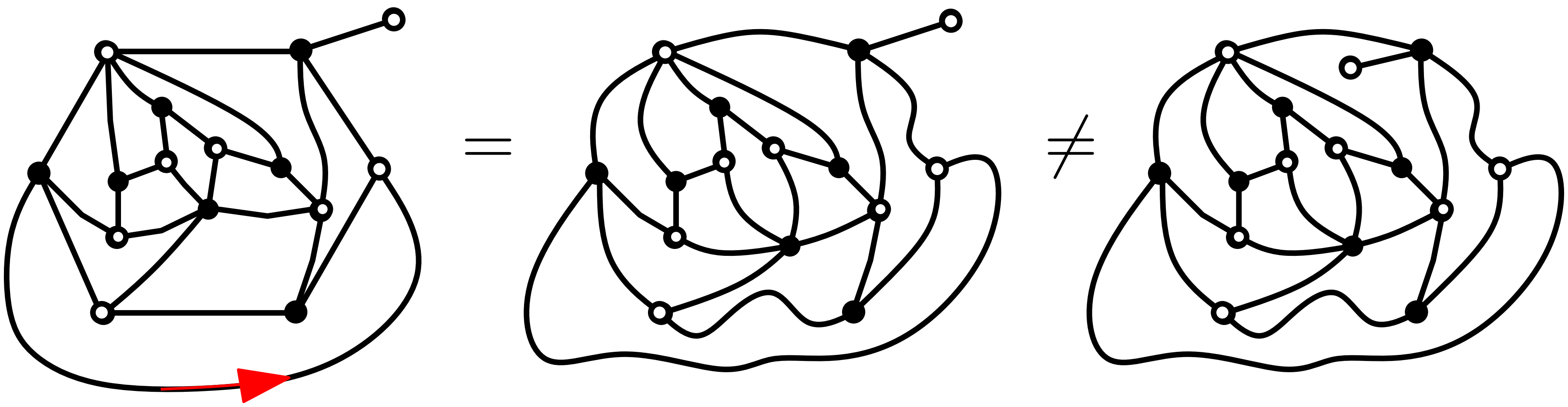
planar maps are discrete (combinatorial) structures

there are finitely many maps with given number of edges/vertices/faces

Plane graphs and planar maps

planar map = { Plane graph = { Embedding of a connected graph in the plane } considered up to homeomorphisms of the plane }

rooted planar map = with one marked edge (to kill symmetries)



planar maps are discrete (combinatorial) structures

there are finitely many maps with given number of edges/vertices/faces

my main activity is to count these things... also on more general surfaces

but not much to say about 3d or higher until now...

Counting / enumerative combinatorics

A set $\mathcal{A}$ of combinatorial structures, endowed with a size: $\mathcal{A} \to \mathbb{N}$, $a \mapsto |a|$.
We assume $\mathcal{A}_n = \{a \in A \mid |a| = n\}$ finite for all n.

The counting problem is to compute $a_n = |\mathcal{A}_n|$, $n \geq 0$

The generating function (gf) of the family $\mathcal{A}$ according to the size is

$$A(t) = \sum_{n \geq 0} a_n t^n = \sum_{a \in \mathcal{A}} t^{|a|}.$$

Counting maps

Why do people care about counting maps ?

- Tutte *et al.* (1962$\longrightarrow$ 2013, decompositions and functionnal equations)

  first with the idea to prove the 4 color theorem

Counting maps

Why do people care about counting maps ?

- Tutte *et al.* (1962$\longrightarrow$ 2013, decompositions and functionnal equations)

  first with the idea to prove the 4 color theorem

Theorem (Tutte, 1963): Let $\mathcal{Q}_n = \{$rooted quadrangulations with n faces$\}$

and let $Q(t) = \sum_{q\in\mathcal{Q}_n} t^{|q|}$ be the gf where $|q| = \#$faces of q.

$$Q(t) = 1 + 2t + 9t^2 + \dots$$

Then $Q(t)$ is solution of the system $\begin{cases} Q(t) = R(t) - tR(t)^3 \\ R(t) = 1 + 3tR(t)^2 \end{cases}$

so that $Q(t) = \frac{(1-12t)^{3/2}-1+18t}{54t^2}$ and $|\mathcal{Q}_n| = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$.

Counting maps

Why do people care about counting maps ?

- Tutte *et al.* (1962→ 2013, decompositions and functionnal equations)
  first with the idea to prove the 4 color theorem

Theorem (Tutte, 1963): Let $\mathcal{Q}_n = \{$rooted quadrangulations with n faces$\}$

and let $Q(t) = \sum_{q \in \mathcal{Q}_n} t^{|q|}$ be the gf where $|q| = \#$faces of q.

$$Q(t) = 1 + 2t + 9t^2 + \ldots$$

Then $Q(t)$ is solution of the system $\begin{cases} Q(t) = R(t) - tR(t)^3 \\ R(t) = 1 + 3tR(t)^2 \end{cases}$

so that $Q(t) = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2}$ and $|\mathcal{Q}_n| = \frac{2}{n+2} \frac{3^n}{n+1} \binom{2n}{n}$.

a posteriori justification: a beautiful formula !

Counting maps

Why do people care about counting maps ?

- Tutte *et al.* (1962→ 2013, decompositions and functionnal equations)
  first with the idea to prove the 4 color theorem

Theorem (Tutte, 1963): Let $\mathcal{Q}_n = \{$rooted quadrangulations with n faces$\}$

and let $Q(t) = \sum_{q\in\mathcal{Q}_n} t^{|q|}$ be the gf where $|q| = \#$faces of q.

$$Q(t) = 1 + 2t + 9t^2 + \dots$$

Then $Q(t)$ is solution of the system $\left\{ \begin{array}{l} Q(t) = R(t) - tR(t)^3 \\ R(t) = 1 + 3tR(t)^2 \end{array} \right.$ algebraic eqs

so that $Q(t) = \frac{(1-12t)^{3/2}-1+18t}{54t^2}$ and $|\mathcal{Q}_n| = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$. explicit formula

a posteriori justification: a beautiful formula !

Counting maps

Why do people care about counting maps ?

- Tutte *et al.* (1962$\rightarrow$ 2013, decompositions and functionnal equations)
  first with the idea to prove the 4 color theorem

A lot of analogous results for other families of maps $\mathcal{F}$:

-a dozain of nice counting formulas for the $|\mathcal{F}_n|$

- many more results of algebraicness of gfs

Counting maps

Why do people care about counting maps ?

- Tutte *et al.* (1962$\rightarrow$ 2013, decompositions and functionnal equations)
  first with the idea to prove the 4 color theorem

A lot of analogous results for other families of maps $\mathcal{F}$:

-a dozain of nice counting formulas for the $|\mathcal{F}_n|$

- many more results of algebraicness of gfs

- Cori, Vauquelin *et al.* (70/80's $\rightarrow$ 2012, bijections with trees)
  to *explain* the nice formulas and algebraicness

Counting maps

Why do people care about counting maps?

- Brezin, Itzykson, Parisi, Zuber, *et al.* (1978$\rightarrow$ 2013, matrix integrals)

  Key remark (t'Hooft): Perturbative expansion of hermician matrix integrals lead to map generating functions...

  $\Rightarrow$ powerful tools and wide extension of Tutte's counting results

  $\longrightarrow$ Eynard's talk

here is how I explain to my collegues that physicists are interested by this: counting maps is a first step in studying the uniform distribution on maps of size n, which happens to be an interesting model of random surface.

Ising model on square lattice = toy model of matter

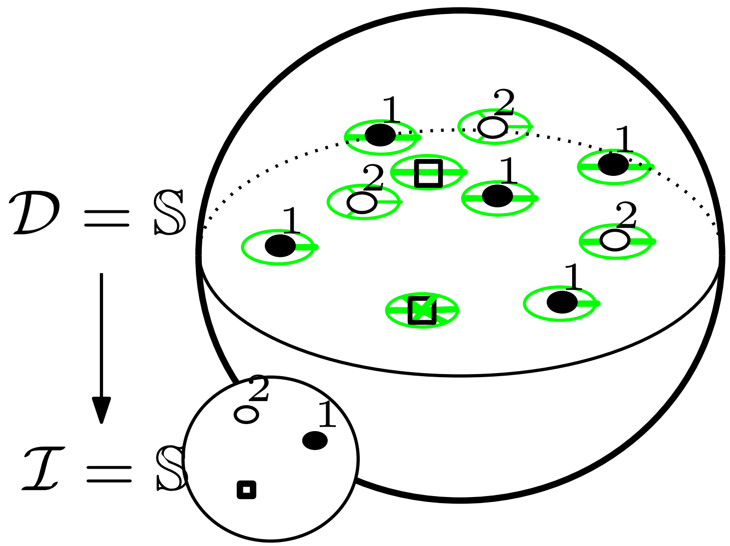
Ising model on random maps "=" 2d quantum geometry coupled with matters

$\longrightarrow$ Loll, Budds, Bouttier's talks

Counting maps

Why do people care about counting maps?

- Goulden, Jackson *et al.* (80's $\longrightarrow$ 2012, characters of the symmetric group) for Hurwitz problem: counting ramified covers of the sphere by itself



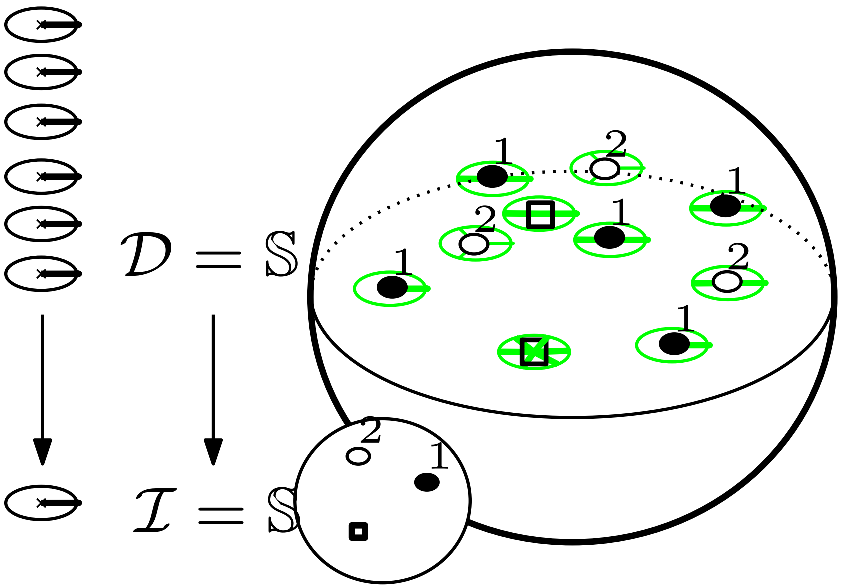
Hurwitz formula (1894): $h_n(\lambda) = n^{\ell-3} \cdot (n+\ell-2)! \cdot \prod_{i\geq 1} \frac{1}{\ell_i!}\left(\frac{i^i}{i!}\right)^{\ell_i}$

$\#\{$covers with $n+\ell-2$ simple ramifications and 1 of type $\lambda = 1^{\ell_1} \dots n^{\ell_n}\}$

Counting maps

Why do people care about counting maps?

- Goulden, Jackson *et al.* (80's $\to$ 2012, characters of the symmetric group) for Hurwitz problem: counting ramified covers of the sphere by itself



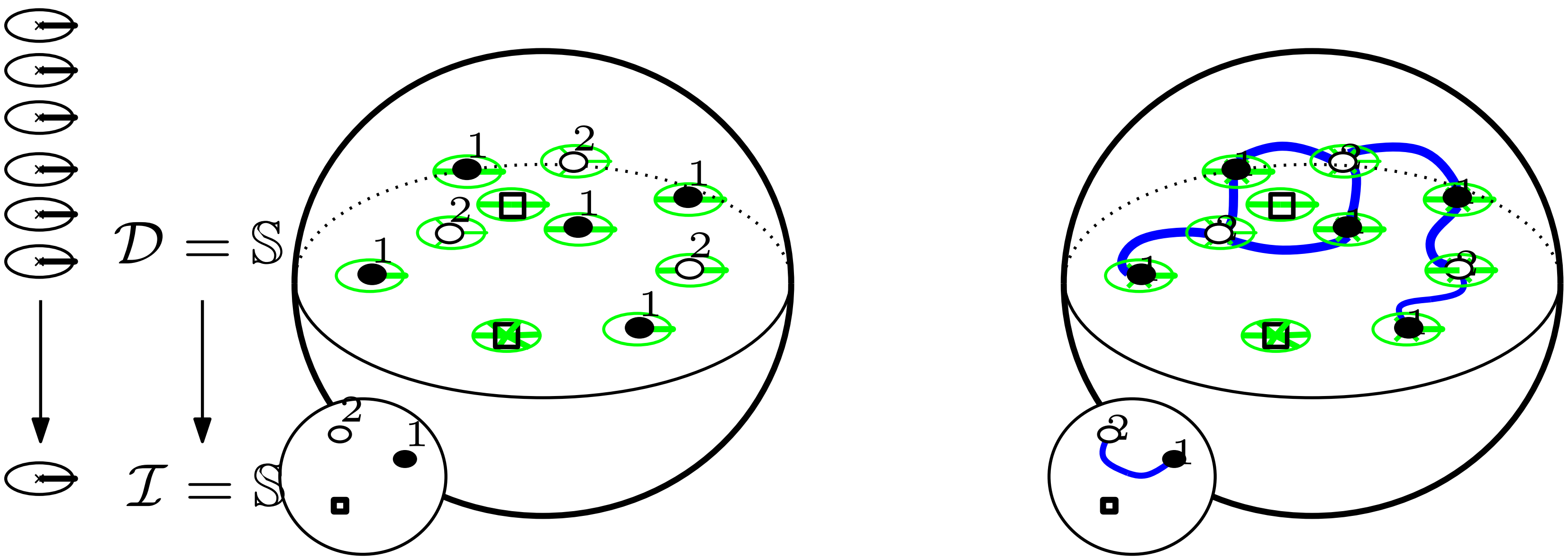
Hurwitz formula (1894): $h_n(\lambda) = n^{\ell-3} \cdot (n+\ell-2)! \cdot \prod_{i\geq 1} \frac{1}{\ell_i!} \left(\frac{i^i}{i!}\right)^{\ell_i}$

$\#\{$covers with $n+\ell-2$ simple ramifications and 1 of type $\lambda = 1^{\ell_1} \dots n^{\ell_n}\}$

Counting maps

Why do people care about counting maps?

- Goulden, Jackson *et al.* (80's $\to$ 2012, characters of the symmetric group) for Hurwitz problem: counting ramified covers of the sphere by itself



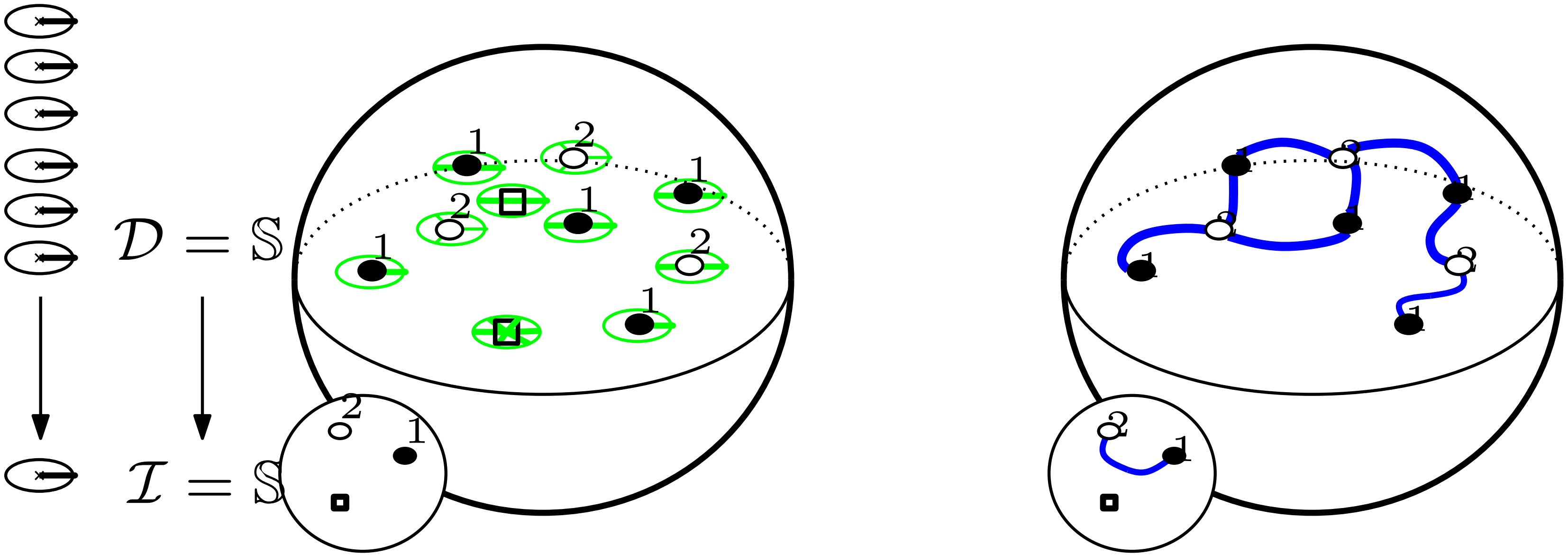
Hurwitz formula (1894): $h_n(\lambda) = n^{\ell-3} \cdot (n+\ell-2)! \cdot \prod_{i\geq 1} \frac{1}{\ell_i!}\left(\frac{i^i}{i!}\right)^{\ell_i}$

$\#\{$covers with $n+\ell-2$ simple ramifications and 1 of type $\lambda = 1^{\ell_1}\dots n^{\ell_n}\}$

Counting maps

Why do people care about counting maps?

- Goulden, Jackson *et al.* (80's $\rightarrow$ 2012, characters of the symmetric group) for Hurwitz problem: counting ramified covers of the sphere by itself



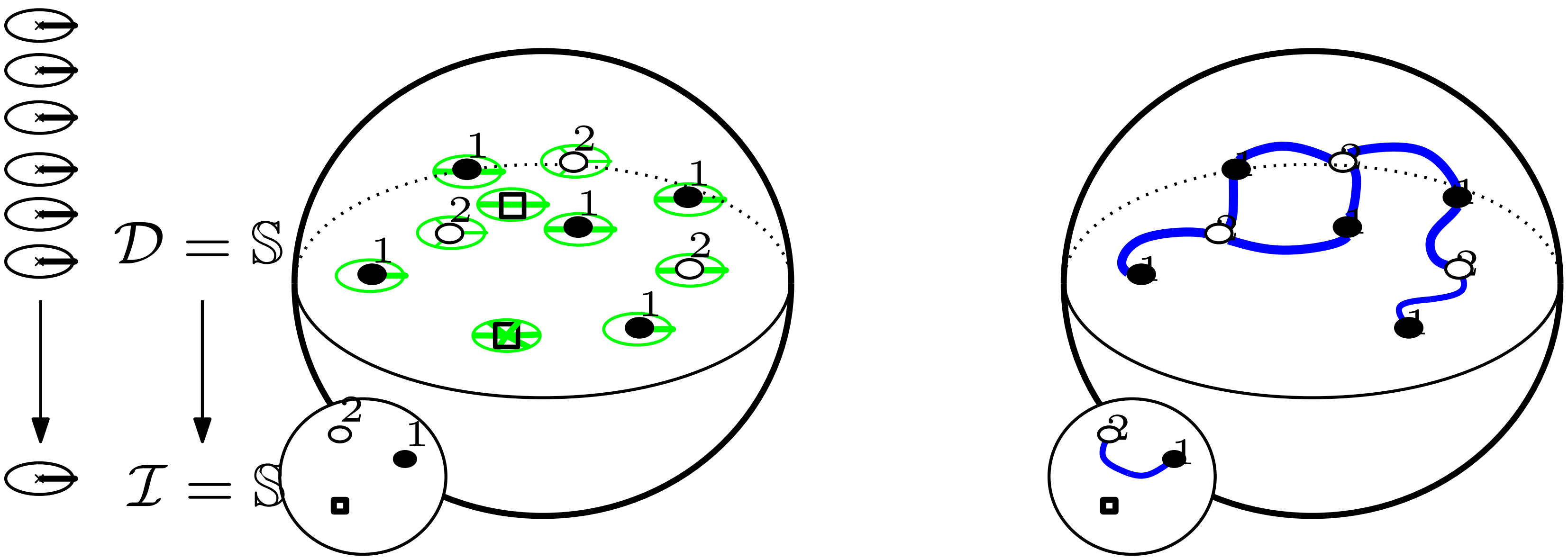
Hurwitz formula (1894): $h_n(\lambda) = n^{\ell-3} \cdot (n+\ell-2)! \cdot \prod_{i\geq 1} \frac{1}{\ell_i!}\left(\frac{i^i}{i!}\right)^{\ell_i}$

$\#\{$covers with $n+\ell-2$ simple ramifications and 1 of type $\lambda = 1^{\ell_1} \ldots n^{\ell_n}\}$

Counting maps

Why do people care about counting maps?

- Goulden, Jackson *et al.* (80's $\rightarrow$ 2012, characters of the symmetric group) for Hurwitz problem: counting ramified covers of the sphere by itself



Hurwitz formula (1894): $h_n(\lambda) = n^{\ell-3} \cdot (n+\ell-2)! \cdot \prod_{i\geq 1} \frac{1}{\ell_i!}\left(\frac{i^i}{i!}\right)^{\ell_i}$

$\#\{$covers with $n+\ell-2$ simple ramifications and 1 of type $\lambda = 1^{\ell_1} \ldots n^{\ell_n}\}$

This is what I have been doing for the last 3 years!

An alternative (but equivalent in the limit) model of 2d quantum geometry.

Counting maps and triangulations

Trees, independence, algebraic generating series

Stack triangulations

General 2d triangulations

Realizers

Rational and algebraic series in combinatorics

A formal power series $A(t)$ is algebraic (on $\mathbb{Q}(t)$) if it satisfies a (nontrivial) polynomial equation: $P(t, A(t)) = 0$.

It is rational if it can be writen as $A(t) = \frac{P(t)}{Q(t)}$ with $P(t)$ and $Q(t)$ polynomials.

A family of combinatorial structures is algebraic or rational if its gf is.

Why do we care about rational and algebraic series

Good closure properties ($+, \times, /$, derivative, composition) and efficient computational tools (partial fraction decomposition, Puisieux expansion, elimination, resultant, Gröbner,...)

Coefficients can be computed in linear time from the equation.

Algebraicness of a series can be guessed from first coefficients of its expansion (for instance using the tools `gfun` and `Maple`)

The asymptotic expansion of coefficients can be determine almost automatically $a_n \sim \frac{\kappa}{\Gamma(d+1)}\rho^{-n}n^d$
with κ and ρ some algebraic constants on $\mathbb{Q}$ and $d \in \mathbb{Q} \setminus \{-1, -2, \ldots\}$

Algebraic series and combinatorics

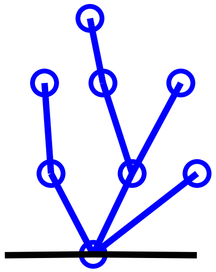
Construction	Numbers	Series
Union: $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$a_n = b_n + c_n$	$A(t) = B(t) + C(t)$
Product: $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\|\alpha\| = \|(\beta, \gamma)\| = \|\beta\| + \|\gamma\|$	$a_n = \sum_{i=0}^{n} b_i c_{n-i}$	$A(t) = B(t) \cdot C(t)$

Algebraic series and combinatorics

Construction	Numbers	Series
Union: $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$a_n = b_n + c_n$	$A(t) = B(t) + C(t)$
Product: $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert\alpha\rvert = \lvert(\beta, \gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a_n = \sum_{i=0}^{n} b_i c_{n-i}$	$A(t) = B(t) \cdot C(t)$

A typical example:

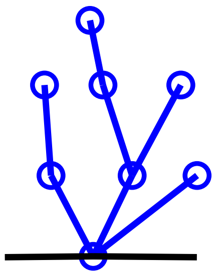
ordered trees



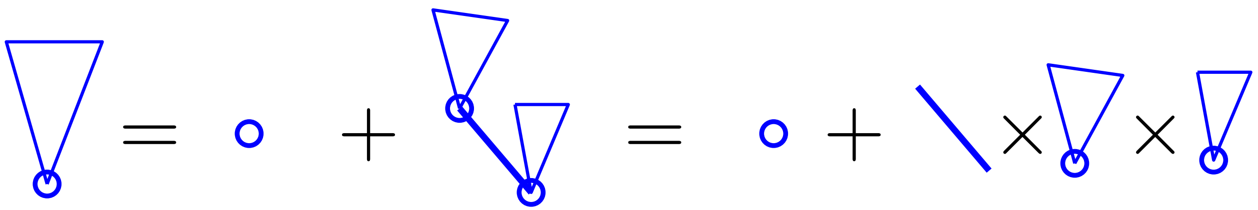
Algebraic series and combinatorics

Construction	Numbers	Series
Union: $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$a_n = b_n + c_n$	$A(t) = B(t) + C(t)$
Product: $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert\alpha\rvert = \lvert(\beta, \gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a_n = \sum_{i=0}^{n} b_i c_{n-i}$	$A(t) = B(t) \cdot C(t)$

A typical example:
ordered trees



Let $A(t)$ be the generating function of ordered trees according to the number of edges

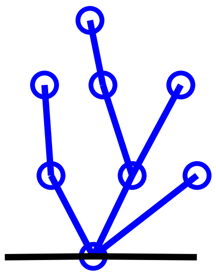


$$\Rightarrow \quad A(t) = 1 + tA(t)^2$$

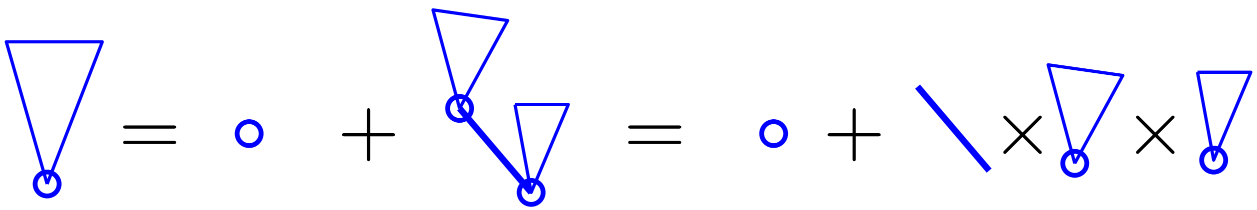
Algebraic series and combinatorics

Construction	Numbers	Series
Union: $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$a_n = b_n + c_n$	$A(t) = B(t) + C(t)$
Product: $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert\alpha\rvert = \lvert(\beta,\gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a_n = \sum_{i=0}^{n} b_i c_{n-i}$	$A(t) = B(t) \cdot C(t)$

A typical example:
ordered trees



Let $A(t)$ be the generating function of ordered trees according to the number of edges



$$\Rightarrow \quad A(t) = 1 + tA(t)^2$$

As a result: $A(t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} t^n$,

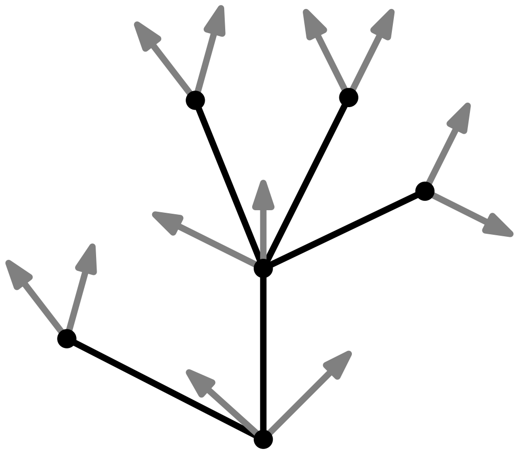
i.e. ordered trees are counted by Catalan numbers

Algebraic series and combinatorics

Construction	Numbers	Series
Union: $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$a_n = b_n + c_n$	$A(t) = B(t) + C(t)$
Product: $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert\alpha\rvert = \lvert(\beta,\gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a_n = \sum_{i=0}^{n} b_i c_{n-i}$	$A(t) = B(t) \cdot C(t)$

another example:

2-leaf trees

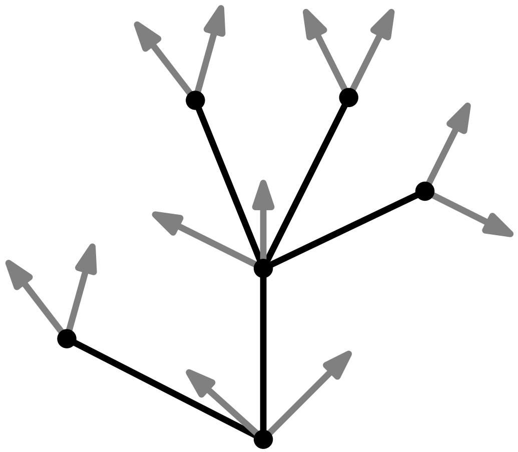


Algebraic series and combinatorics

Construction	Numbers	Series
Union: $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$a_n = b_n + c_n$	$A(t) = B(t) + C(t)$
Product: $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert\alpha\rvert = \lvert(\beta, \gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a_n = \sum_{i=0}^{n} b_i c_{n-i}$	$A(t) = B(t) \cdot C(t)$

another example:
2-leaf trees

Let $C_2(t)$ be the gf of plane trees with 2 leaves per inner vertex, according to the number of inner edges

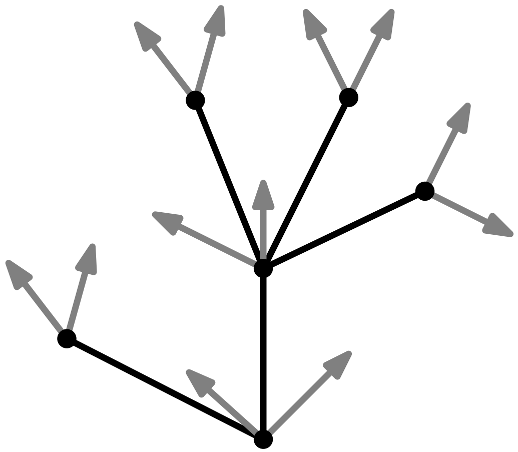


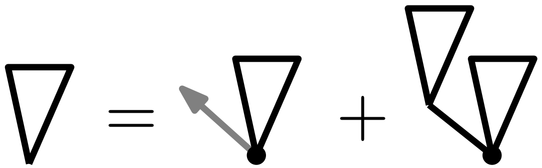
Algebraic series and combinatorics

Construction	Numbers	Series
Union: $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$a_n = b_n + c_n$	$A(t) = B(t) + C(t)$
Product: $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert\alpha\rvert = \lvert(\beta,\gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a_n = \sum_{i=0}^{n} b_i c_{n-i}$	$A(t) = B(t) \cdot C(t)$

another example:
2-leaf trees

Let $C_2(t)$ be the gf of plane trees with 2 leaves per inner vertex, according to the number of inner edges



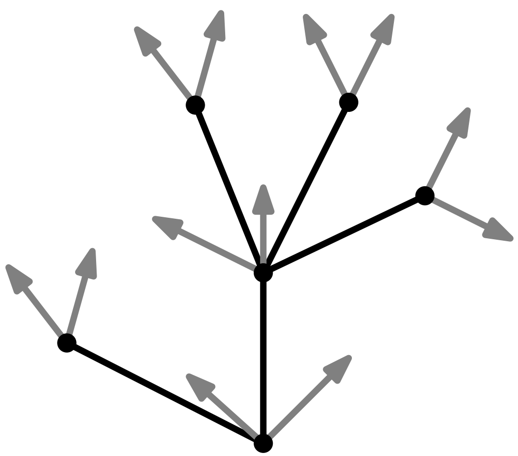


$$
\begin{aligned}
C_2(z) &= C_1(z) + zC_2(z)^2 \\
C_1(z) &= C_0(z) + zC_1(z)C_2(z) \\
C_0(z) &= 1 + zC_0(z)C_2(z)
\end{aligned}
$$

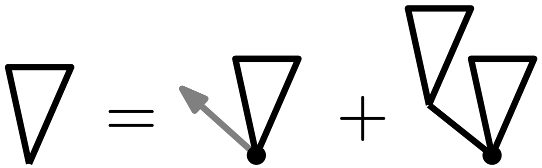
Algebraic series and combinatorics

Construction	Numbers	Series
Union: $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$a_n = b_n + c_n$	$A(t) = B(t) + C(t)$
Product: $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert\alpha\rvert = \lvert(\beta, \gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a_n = \sum_{i=0}^{n} b_i c_{n-i}$	$A(t) = B(t) \cdot C(t)$

another example:
2-leaf trees



Let $C_2(t)$ be the gf of plane trees with 2 leaves per inner vertex, according to the number of inner edges



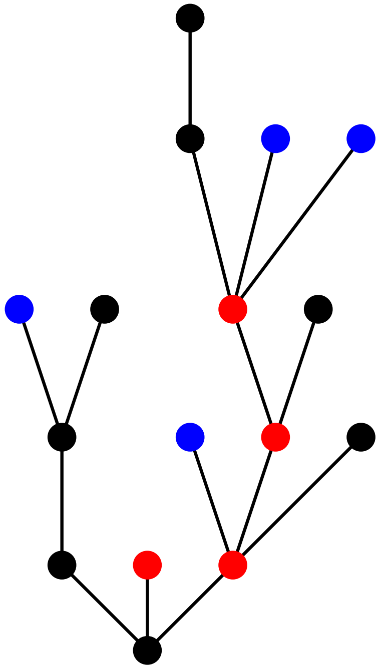
$$\begin{aligned} C_2(z) &= C_1(z) + zC_2(z)^2 \\ C_1(z) &= C_0(z) + zC_1(z)C_2(z) \\ C_0(z) &= 1 + zC_0(z)C_2(z) \end{aligned}$$

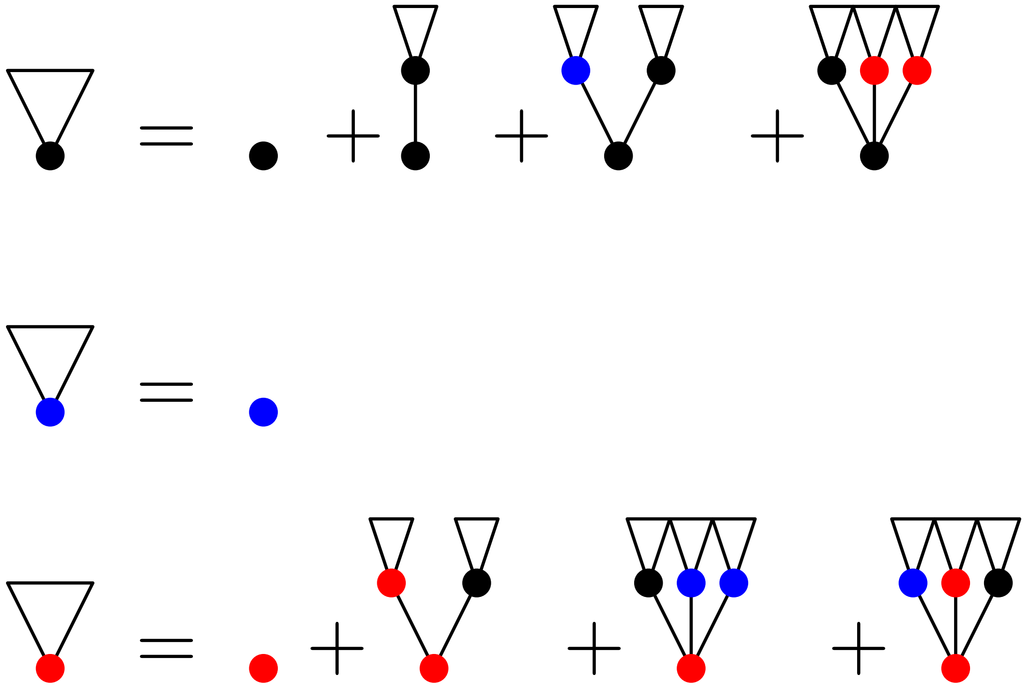
$$C_0 = \frac{1}{1-zC_2},\ C_1 = \frac{1}{(1-zC_2)^2},\ C_2 = \frac{1}{(1-zC_2)^3}$$

As a result: $C_2(z) = \sum_{n\geq 0} \frac{1}{3n+1}\binom{4n}{n} z^n$.

Combinatorial interpretation: $\mathbb{N}$-algebraic structures

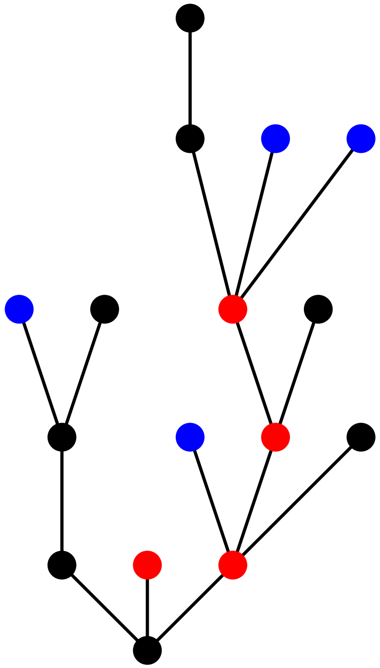
$\mathbb{N}$-algebraic structures = families that can be defined by algebraic specification (aka context-free grammars)

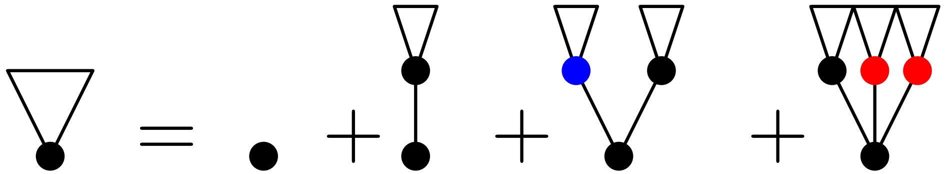




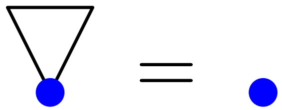
Combinatorial interpretation: $\mathbb{N}$-algebraic structures

$\mathbb{N}$-algebraic structures = families that can be defined by algebraic specification (aka context-free grammars)

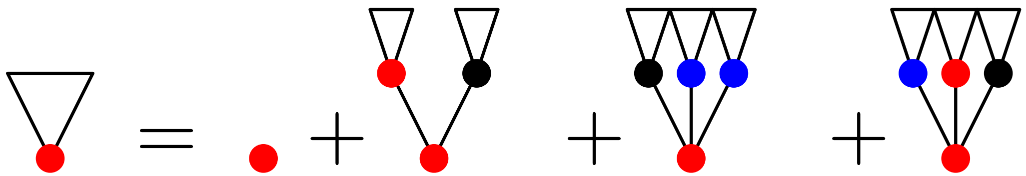




$$N(t) = 1 + tN(t) + t^2 B(t)N(t) + t^3 N(t)R(t)^2$$



$$B(t) = 1$$

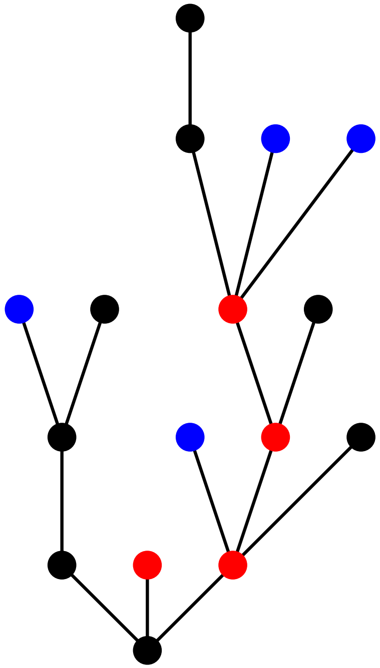


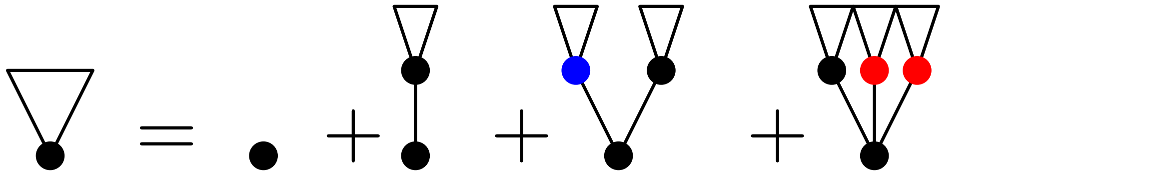
$$R(t) = 1 + t^2 R(t)N(t) + t^3 N(t)B(t)^2 + t^3 B(t)R(t)N(t)$$

$\mathbb{N}$-algebraic structures clearly have algebraic generating functions

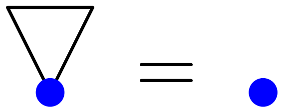
Combinatorial interpretation: $\mathbb{N}$-algebraic structures

$\mathbb{N}$-algebraic structures = families that can be defined by algebraic specification
(aka context-free grammars)

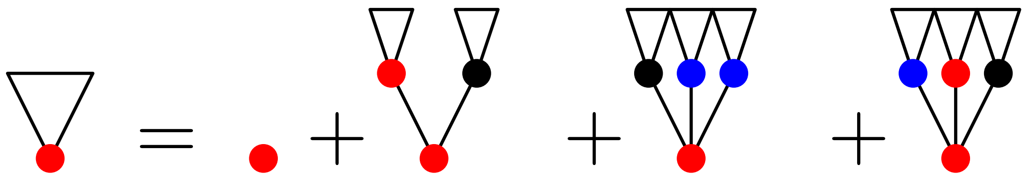




$$N(t) = 1 + tN(t) + t^2 B(t)N(t) + t^3 N(t)R(t)^2$$



$$B(t) = 1$$



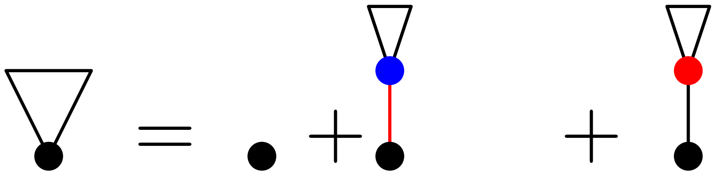
$$R(t) = 1 + t^2 R(t)N(t) + t^3 N(t)B(t)^2 + t^3 B(t)R(t)N(t)$$

$\mathbb{N}$-algebraic structures clearly have algebraic generating functions

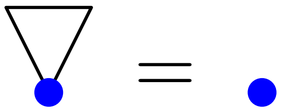
Intuition: $\mathbb{N}$-algebraic structures are tree-like, with independant subtrees (conditionally to the root colors)

Combinatorial interpretation: $\mathbb{N}$-rational structures

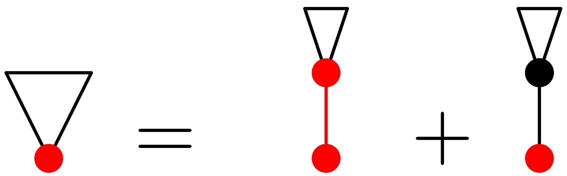
When all equations are linear, the structures are called $\mathbb{N}$-rationnal, and they have rational generating series.



$$N(t) = 1 + tB(t) + tR(t)$$



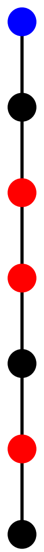
$$B(t) = 1$$

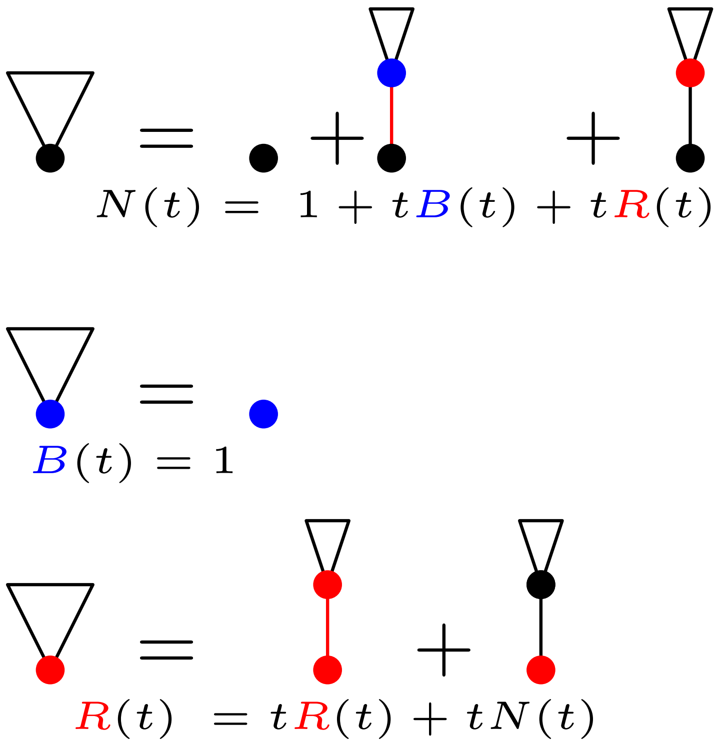


$$R(t) = tR(t) + tN(t)$$

Combinatorial interpretation: $\mathbb{N}$-rational structures

When all equations are linear, the structures are called $\mathbb{N}$-rationnal, and they have rational generating series.

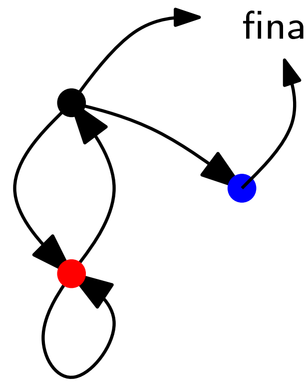




$$N(t) = 1 + tB(t) + tR(t)$$

$$B(t) = 1$$

$$R(t) = tR(t) + tN(t)$$



Intuition: $\mathbb{N}$-rational structures have a linear structure and their growth is controled by a finite number of states (finite state machine)

Validity of the combinatorial intuition

The intuition is "always correct" for rational structures

combinatorial structure + rational gf "$\Rightarrow$" $\mathbb{N}$-rational structure

(empirical implication, one can create ad-hoc conterexamples)

On the contrary there are many examples of combinatorial structures that are algebraic but display no natural tree-like structure

(cf Bousquet-Mélou ICM06)

combinatorial structure + algebraic gf "$\not\Rightarrow$" $\mathbb{N}$-algebraic structure

$\Rightarrow$ the bijective problem

give combinatorial explanations for algebraic gf *i.e.* prove $\mathbb{N}$-algebraicness

to understand better algebraic structures

in other terms, when the gf is algebraic

one would like to make explicit a tree-like structure

Counting maps and triangulations

Trees, independence, algebraic generating series

Stack triangulations

General 2d triangulations

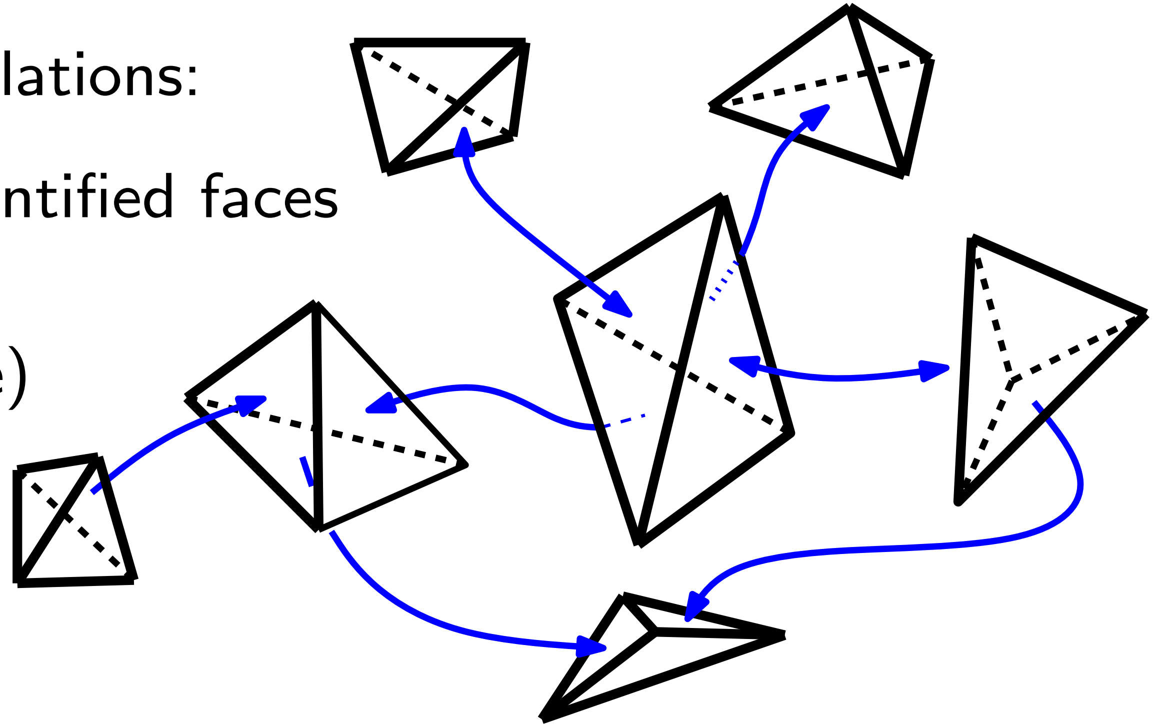
Realizers

Stack-triangulations as 3d triangulations

A "trivial" model of 3d-triangulations:

a set of tetrahedra, pairs of identified faces

each face belong to:
one identification (internal face)
or none (boundary face)



n tetrahedra $\rightarrow 4n$ faces

if connected, at least $n-1$ indentifications, at most $2n+2$ boundary faces

A stack-triangulation is a connected 3d-triangulation with n tetrahedra and $2n+2$ free faces (aka a maximal boundary 3d-triangulation)

Equivalently a 3d-triangulation is stack if it has a tree-like structure.

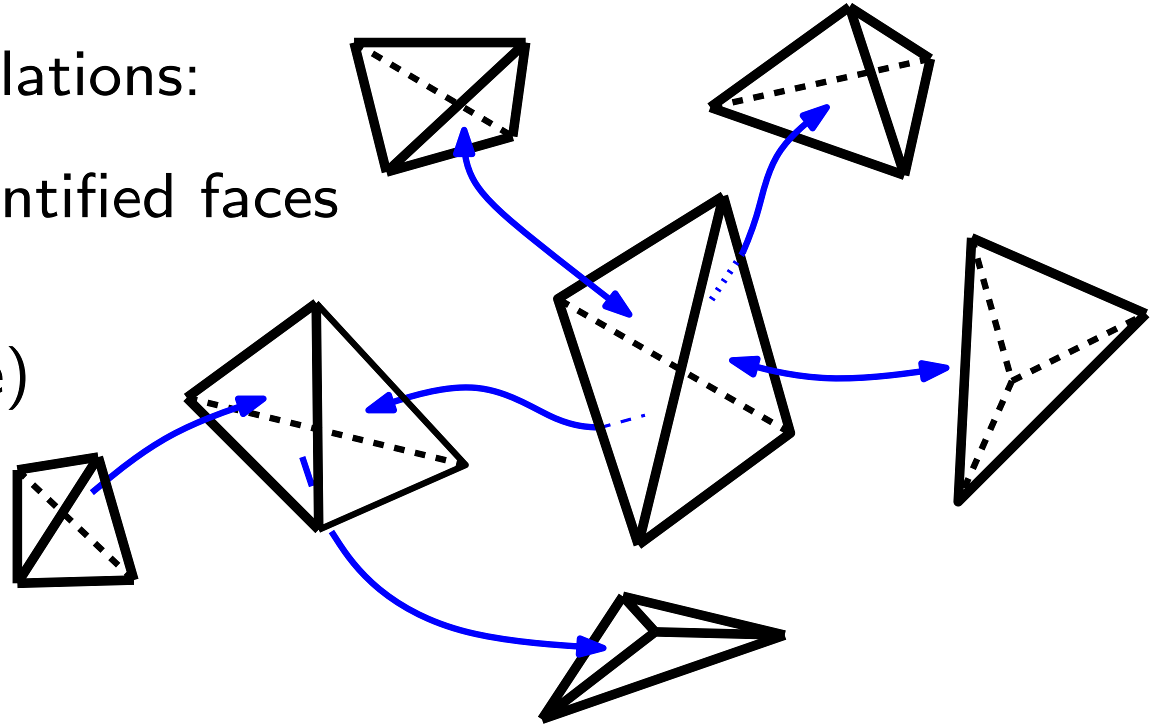
The boundary of a stack-triangulation is topologically a sphere

Stack-triangulations as 3d triangulations

A "trivial" model of 3d-triangulations:

a set of tetrahedra, pairs of identified faces

each face belong to:
one identification (internal face)
or none (boundary face)



n tetrahedra $\longrightarrow 4n$ faces

if connected, at least $n-1$ indentifications, at most $2n+2$ boundary faces

A stack-triangulation is a connected 3d-triangulation with n tetrahedra and $2n+2$ free faces (aka a maximal boundary 3d-triangulation)

Equivalently a 3d-triangulation is stack if it has a tree-like structure.

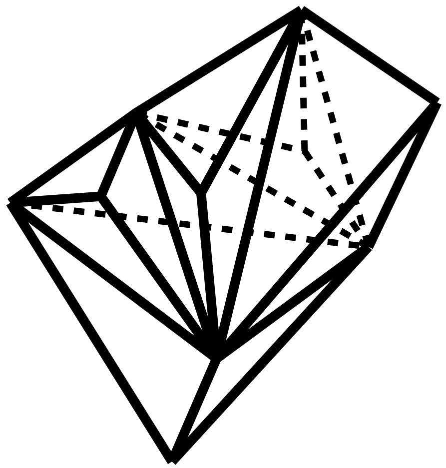
The boundary of a stack-triangulation is topologically a sphere

Stack-triangulations as 3d triangulations

A "trivial" model of 3d-triangulations:

a set of tetrahedra, pairs of identified faces

each face belong to:
one identification (internal face)
or none (boundary face)



n tetrahedra $\rightarrow 4n$ faces

if connected, at least $n-1$ indentifications, at most $2n+2$ boundary faces

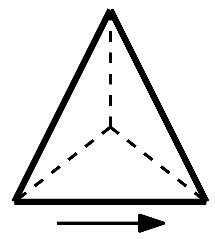
A stack-triangulation is a connected 3d-triangulation with n tetrahedra and $2n+2$ free faces (aka a maximal boundary 3d-triangulation)

Equivalently a 3d-triangulation is stack if it has a tree-like structure.

The boundary of a stack-triangulation is topologically a sphere

The tree structure of stack-triangulations

Take a boundary triangle and root it.

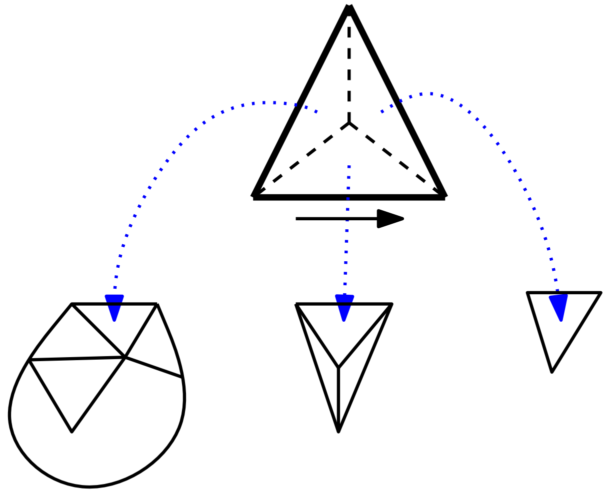


The tree structure of stack-triangulations

Take a boundary triangle and root it.

The other three triangles of the root tetrahedra are either boundary or internal.

Detach the internal ones and take the opposite triangles as root triangles of the corresponding subtriangulations.

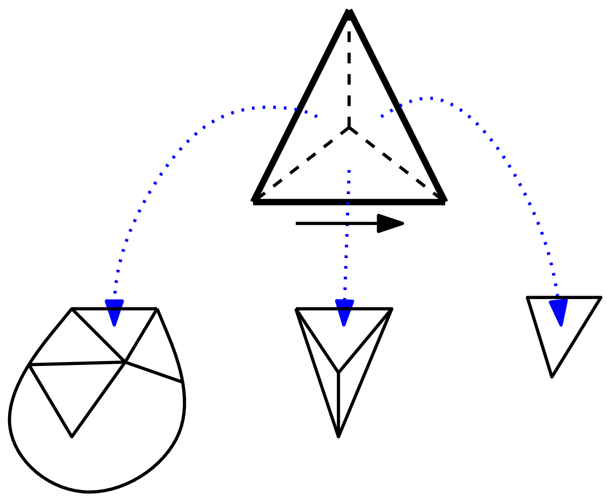


The tree structure of stack-triangulations

Take a boundary triangle and root it.

The other three triangles of the root tetrahedra are either boundary or internal.

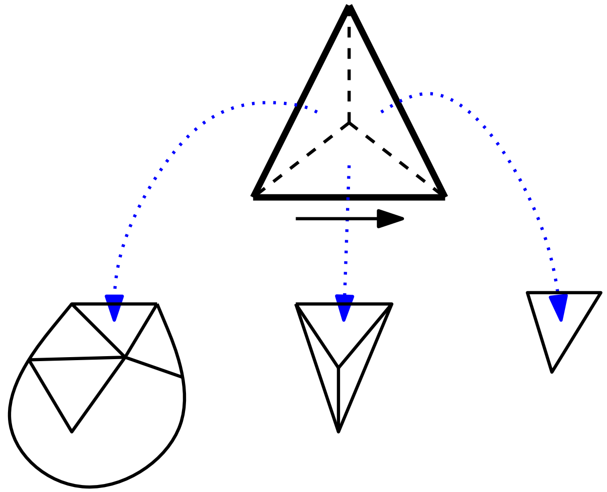
Detach the internal ones and take the opposite triangles as root triangles of the corresponding subtriangulations.



This correspondence is invertible.

The tree structure of stack-triangulations

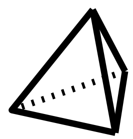
Take a boundary triangle and root it.



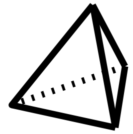
The other three triangles of the root tetrahedra are either boundary or internal.

Detach the internal ones and take the opposite triangles as root triangles of the corresponding subtriangulations.

This correspondence is invertible.

This yields a bijective decomposition: $\mathcal{T} \equiv$  $\cdot \left(\Delta + \mathcal{T} \right)^3$

Counting stack-triangulations

Stack-triangulations are specified by the eq: $\mathcal{T} \equiv$  $\cdot \left(\Delta + \mathcal{T} \right)^3$

which translate into the algebraic equation $T(z) = z(1 + T(z))^3$

Hence the number of rooted stack-triangulations with n tetrahedra is:

$$2i\pi[z^n]T(z) = \int \frac{T(z)}{z^{n+1}} dz = \int \frac{t}{z(t)^{n+1}} z'(t)dt = \int \frac{(1-2t)(1+t)^{3n-1}}{t^n} dt$$

$$[z^n]T(z) = [t^{n-1}](1+t)^{3n-1} - 2[t^{n-2}](1+t)^{3n-1} = \binom{3n-1}{n-1} - 2\binom{3n-1}{n-2} = \frac{1}{2n+1}\binom{3n}{n}$$

These numbers are the number of ternary trees.

In this case there is no real surprise.

Large random stack-triangulations

The uniform distribution on stack-triangulations of size n yields the uniform distribution on ternary trees with n nodes.

Ternary trees (as all "simple" families of trees) converge upon rescaling by a factor $\sqrt{n}$ to the continuum random tree (aka Brownian tree).

The geometry is of glying tetrahedra is trivial however because vertices get multiply identified. Yet Albenque and Marckert (2005) have proved that the convergence hold in the sense of Gromov Hausdorf (cf. Legall's talk).

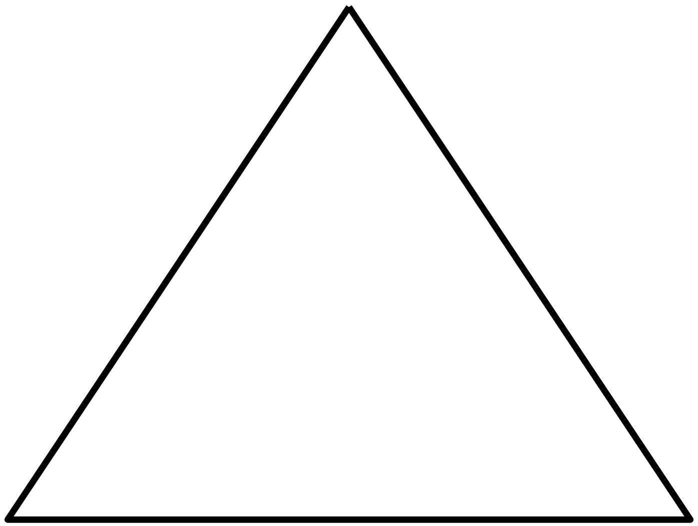
Stack-triangulations as 2d triangulations

Stack-triangulations can be projected on their boundary, and in the plane upon putting a point of a boundary face at infinity.

A 2d-triangulation is (the projection of) a stack triangulation if

- either it is a tetrahedra
- or it contains a vertex of degree 3 and the removal of this vertex and the incident edges is again a stack-triangulation.

Conversely all stack-triangulations can be constructed from an original outer triangle by iteratively subdividing triangles in 3.



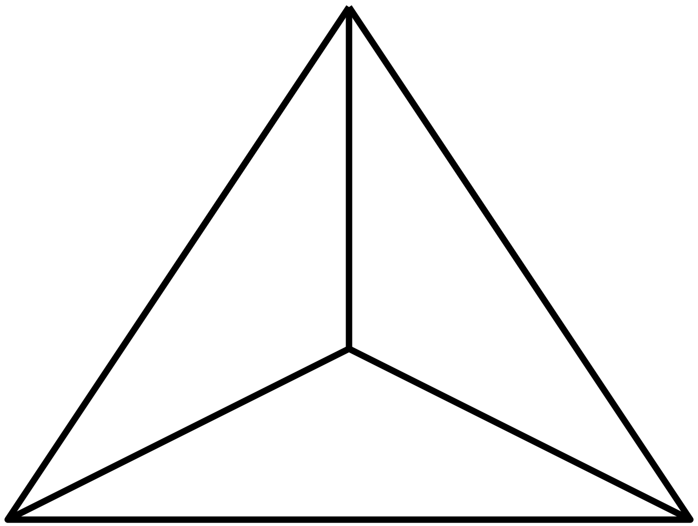
Stack-triangulations as 2d triangulations

Stack-triangulations can be projected on their boundary, and in the plane upon putting a point of a boundary face at infinity.

A 2d-triangulation is (the projection of) a stack triangulation if

- either it is a tetrahedra
- or it contains a vertex of degree 3 and the removal of this vertex and the incident edges is again a stack-triangulation.

Conversely all stack-triangulations can be constructed from an original outer triangle by iteratively subdividing triangles in 3.



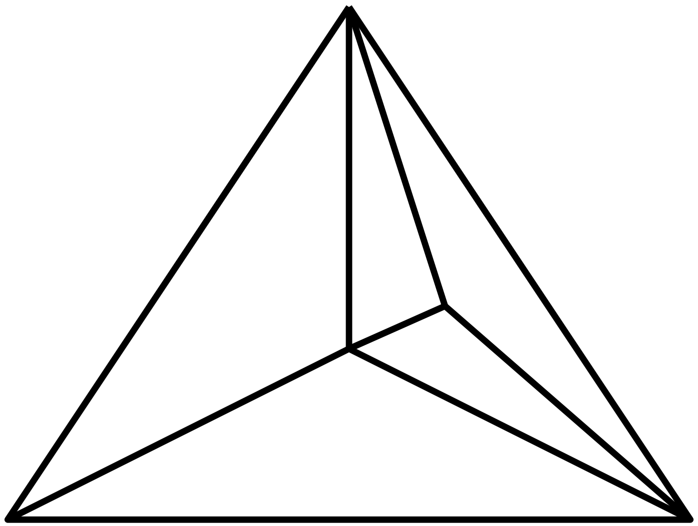
Stack-triangulations as 2d triangulations

Stack-triangulations can be projected on their boundary, and in the plane upon putting a point of a boundary face at infinity.

A 2d-triangulation is (the projection of) a stack triangulation if

- either it is a tetrahedra
- or it contains a vertex of degree 3 and the removal of this vertex and the incident edges is again a stack-triangulation.

Conversely all stack-triangulations can be constructed from an original outer triangle by iteratively subdividing triangles in 3.



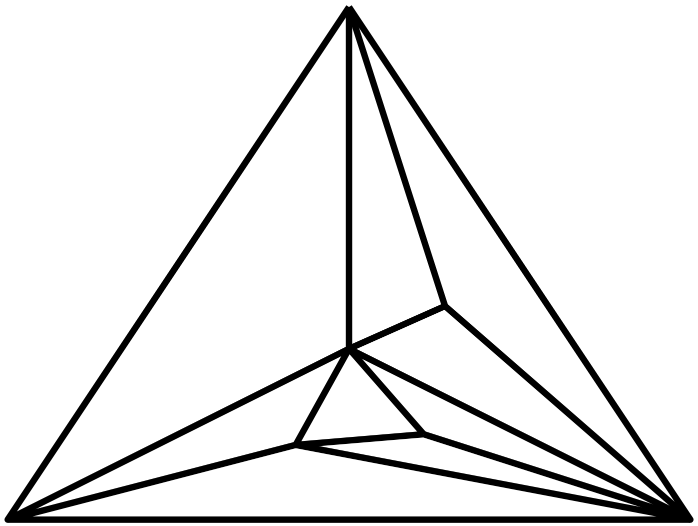
Stack-triangulations as 2d triangulations

Stack-triangulations can be projected on their boundary, and in the plane upon putting a point of a boundary face at infinity.

A 2d-triangulation is (the projection of) a stack triangulation if

- either it is a tetrahedra
- or it contains a vertex of degree 3 and the removal of this vertex and the incident edges is again a stack-triangulation.

Conversely all stack-triangulations can be constructed from an original outer triangle by iteratively subdividing triangles in 3.



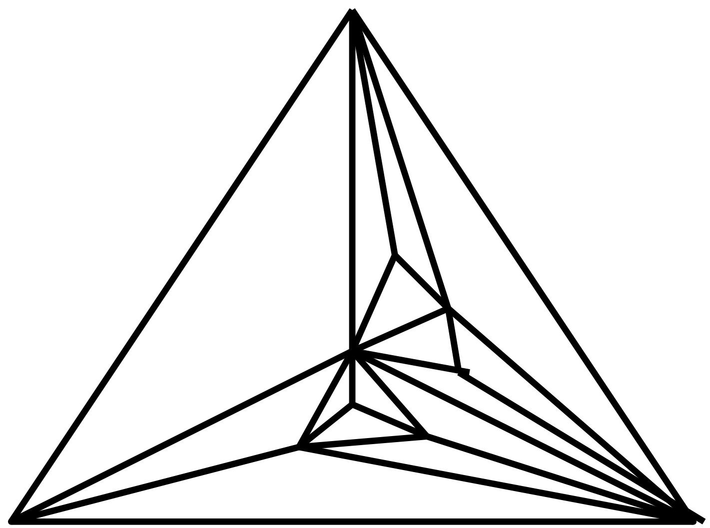
Stack-triangulations as 2d triangulations

Stack-triangulations can be projected on their boundary, and in the plane upon putting a point of a boundary face at infinity.

A 2d-triangulation is (the projection of) a stack triangulation if

- either it is a tetrahedra
- or it contains a vertex of degree 3 and the removal of this vertex and the incident edges is again a stack-triangulation.

Conversely all stack-triangulations can be constructed from an original outer triangle by iteratively subdividing triangles in 3.



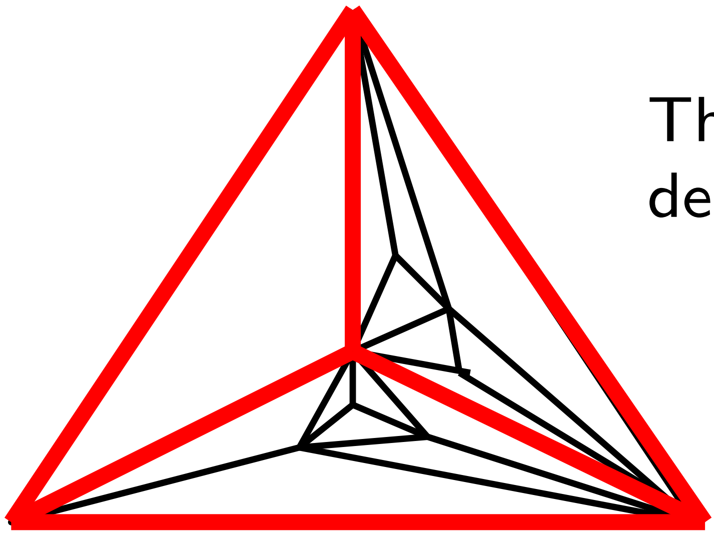
Stack-triangulations as 2d triangulations

Stack-triangulations can be projected on their boundary, and in the plane upon putting a point of a boundary face at infinity.

A 2d-triangulation is (the projection of) a stack triangulation if

- either it is a tetrahedra
- or it contains a vertex of degree 3 and the removal of this vertex and the incident edges is again a stack-triangulation.

Conversely all stack-triangulations can be constructed from an original outer triangle by iteratively subdividing triangles in 3.



The $\mathbb{N}$-algebraic structure arises from a decomposition in 3 independant regions.

Stack-triangulations as 2d triangulations

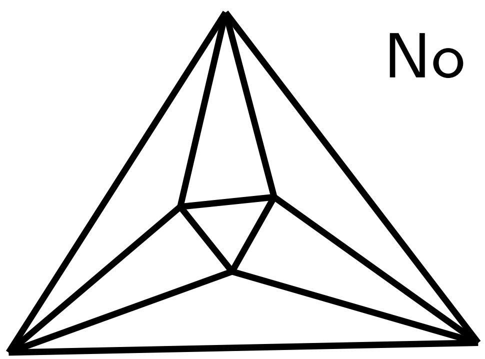
Stack-triangulations can be projected on their boundary, and in the plane upon putting a point of a boundary face at infinity.

A 2d-triangulation is (the projection of) a stack triangulation if

- either it is a tetrahedra
- or it contains a vertex of degree 3 and the removal of this vertex and the incident edges is again a stack-triangulation.

Conversely all stack-triangulations can be constructed from an original outer triangle by iteratively subdividing triangles in 3.

Not all 2d-triangulations are stack-triangulations!



No vertex of degree 3 here...

Are general random triangulations of size n much different from random stack triangulations of size n?

Counting maps and triangulations

Trees, independence, algebraic generating series

Stack triangulations

General 2d triangulations

Realizers

Counting general triangulations

Tutte has found by computations that

$$\mathcal{T}_n = \{ \text{ rooted triangulations with } 2n \text{ faces}\} = \frac{2(4n-3)!}{n!(3n-1)!}$$

Equivalently the generating function is algebraic

$$T(z) = A(z) - 2A(z)^2 \text{ where } A(z) = \frac{z}{(1-A(z))^3} = C_2(z).$$

This formula can also be obtained from BIPZ matrix integral approach
(although not directly: the direct computation yields triangulations with multiple edges;
one needs to perform a simple renormalization to eliminate these multiple edges).

Counting general triangulations

Tutte has found by computations that

$$\mathcal{T}_n = \{ \text{ rooted triangulations with } 2n \text{ faces}\} = \frac{2(4n-3)!}{n!(3n-1)!}$$

Equivalently the generating function is algebraic

$$T(z) = A(z) - 2A(z)^2 \text{ where } A(z) = \frac{z}{(1-A(z))^3} = C_2(z).$$

This formula can also be obtained from BIPZ matrix integral approach
(although not directly: the direct computation yields triangulations with multiple edges;
one needs to perform a simple renormalization to eliminate these multiple edges).

According to our general discussion we expect a tree like structure.
More precisely we should expect 2-leaf trees.

Counting general triangulations

Tutte has found by computations that

$$\mathcal{T}_n = \{ \text{ rooted triangulations with } 2n \text{ faces}\} = \frac{2(4n-3)!}{n!(3n-1)!}$$

Equivalently the generating function is algebraic

$$T(z) = A(z) - 2A(z)^2 \text{ where } A(z) = \frac{z}{(1-A(z))^3} = C_2(z).$$

This formula can also be obtained from BIPZ matrix integral approach
(although not directly: the direct computation yields triangulations with multiple edges;
one needs to perform a simple renormalization to eliminate these multiple edges).

According to our general discussion we expect a tree like structure.

More precisely we should expect 2-leaf trees.

But this tree structure is not obvious on the triangulations...

Counting general triangulations

Tutte has found by computations that

$$\mathcal{T}_n = \{ \text{ rooted triangulations with } 2n \text{ faces}\} = \frac{2(4n-3)!}{n!(3n-1)!}$$

Equivalently the generating function is algebraic

$$T(z) = A(z) - 2A(z)^2 \text{ where } A(z) = \frac{z}{(1-A(z))^3} = C_2(z).$$

This formula can also be obtained from BIPZ matrix integral approach
(although not directly: the direct computation yields triangulations with multiple edges;
one needs to perform a simple renormalization to eliminate these multiple edges).

According to our general discussion we expect a tree like structure.
More precisely we should expect 2-leaf trees.

But this tree structure is not obvious on the triangulations...

2 strategies: • find "$\mathbb{N}$-algebraic" canonical trees inside the structures

• give a direct $\mathbb{N}$-algebraic decomposition (Bouttier-Guitter 2013)

Counting general triangulations

Tutte has found by computations that

$$\mathcal{T}_n = \{ \text{ rooted triangulations with } 2n \text{ faces}\} = \frac{2(4n-3)!}{n!(3n-1)!}$$

Equivalently the generating function is algebraic

$$T(z) = A(z) - 2A(z)^2 \text{ where } A(z) = \frac{z}{(1-A(z))^3} = C_2(z).$$

This formula can also be obtained from BIPZ matrix integral approach
(although not directly: the direct computation yields triangulations with multiple edges;
one needs to perform a simple renormalization to eliminate these multiple edges).

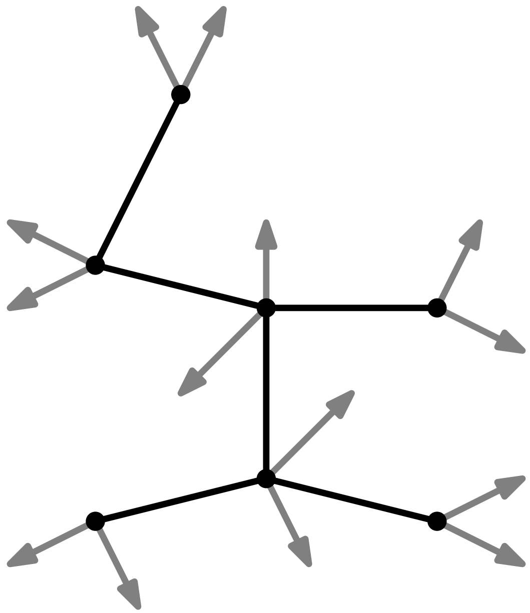
According to our general discussion we expect a tree like structure.
More precisely we should expect 2-leaf trees.

But this tree structure is not obvious on the triangulations...

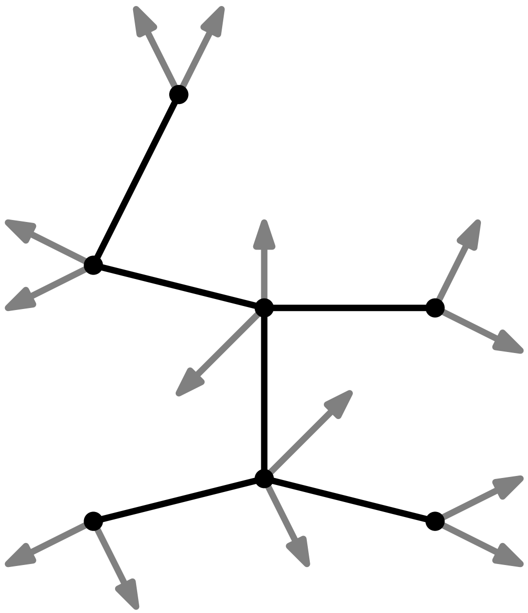
2 strategies:
- find "$\mathbb{N}$-algebraic" canonical trees inside the structures
- give a direct $\mathbb{N}$-algebraic decomposition (Bouttier-Guitter 2013)

From trees to triangulations

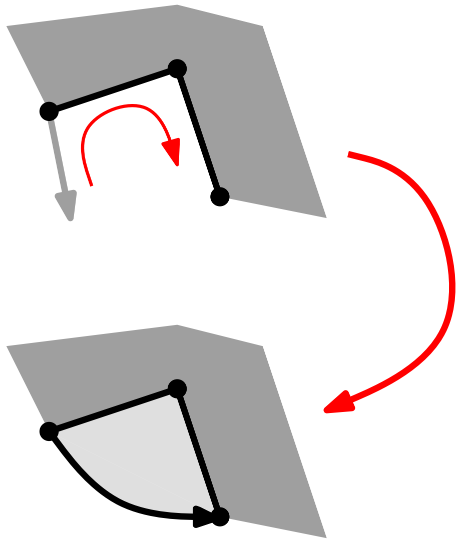
From trees to triangulations



From trees to triangulations

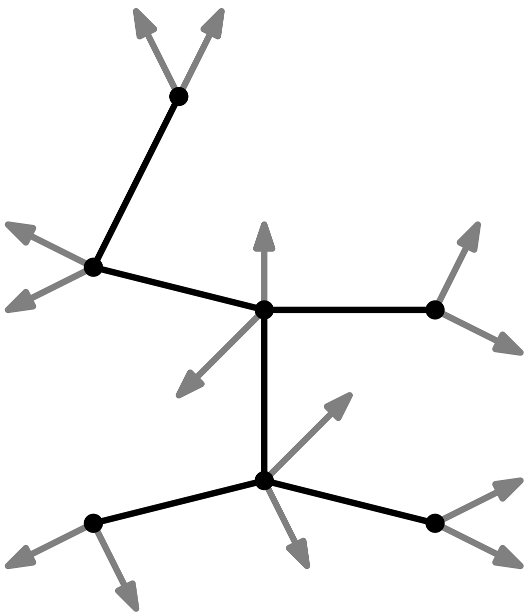
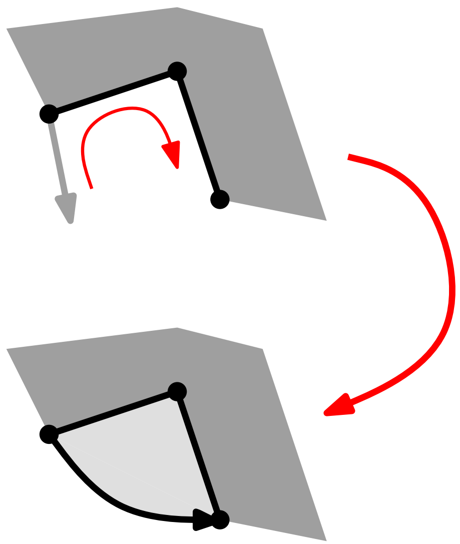
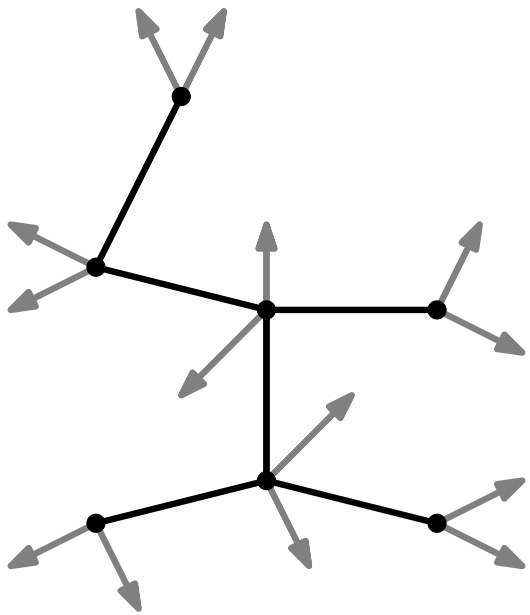


local closure rule



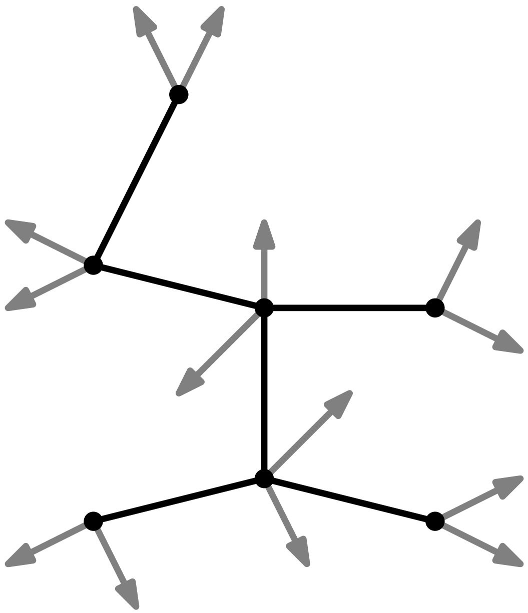
From trees to triangulations

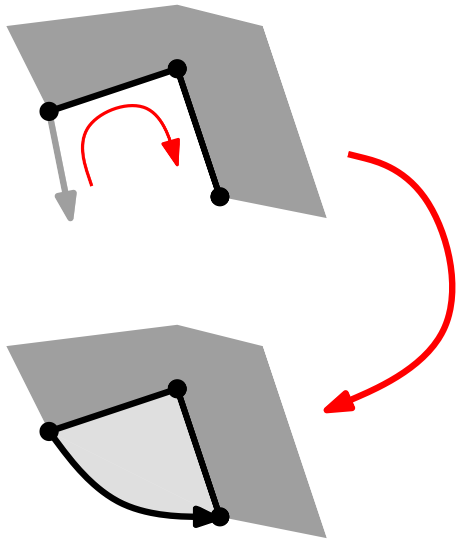
local closure rule

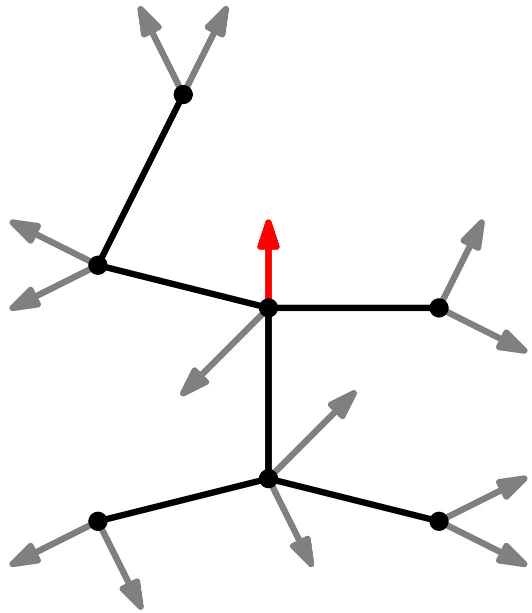




From trees to triangulations

local closure rule

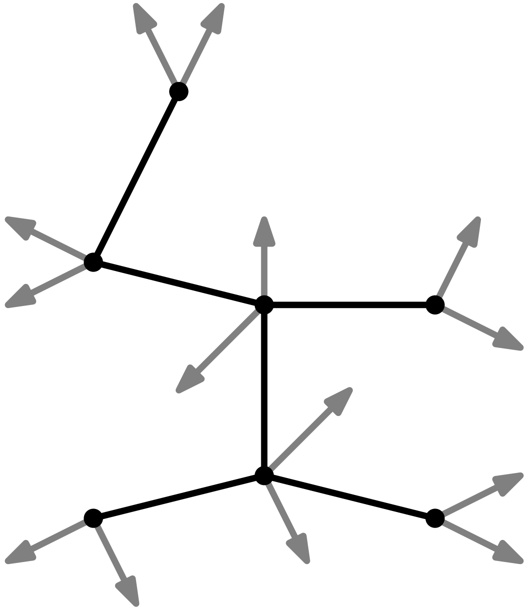


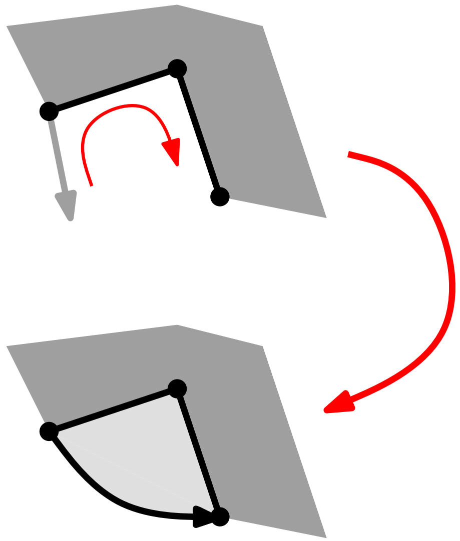


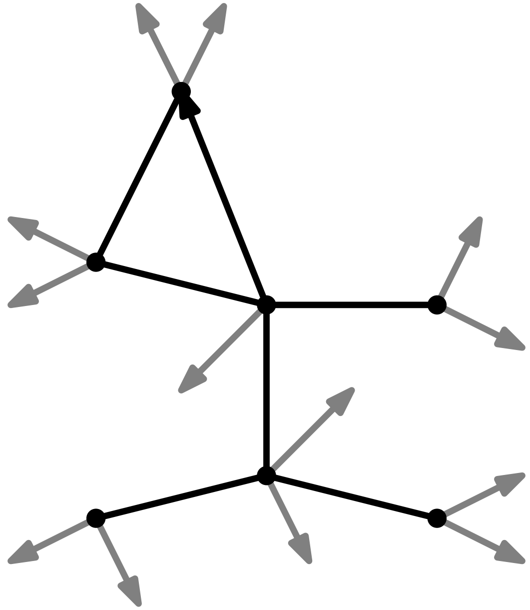


From trees to triangulations

local closure rule

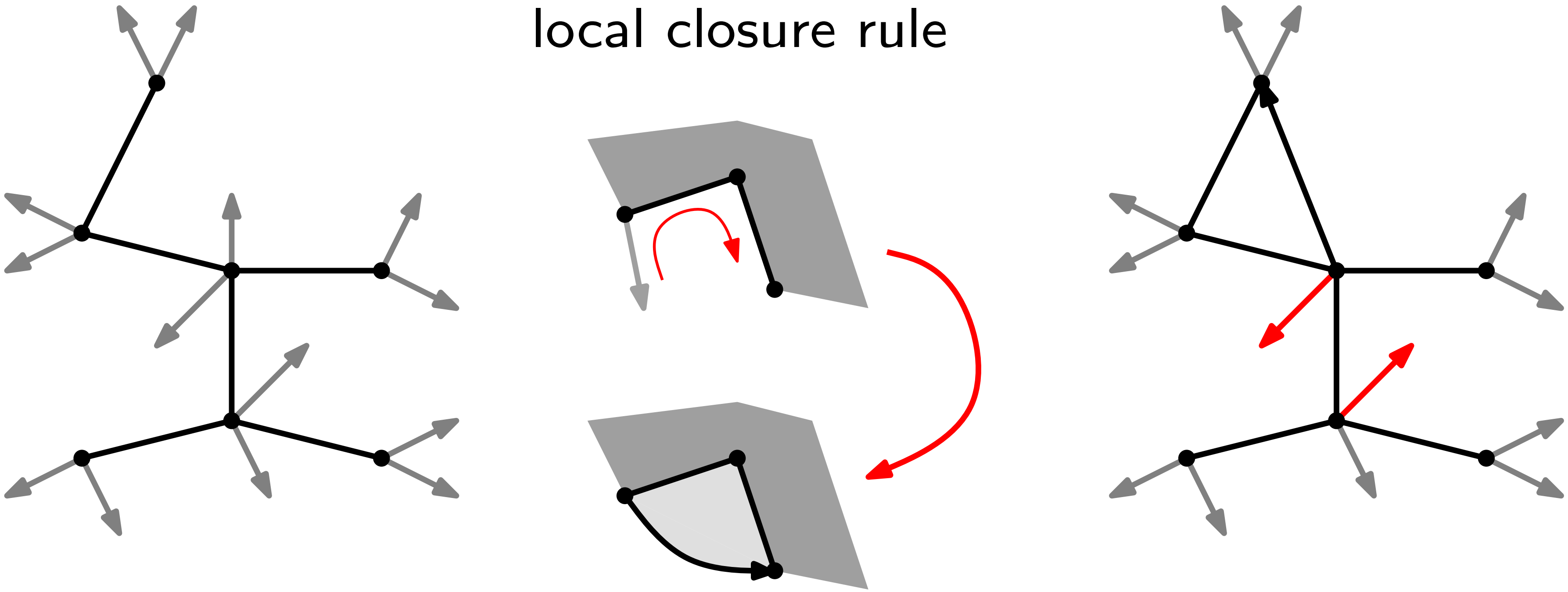






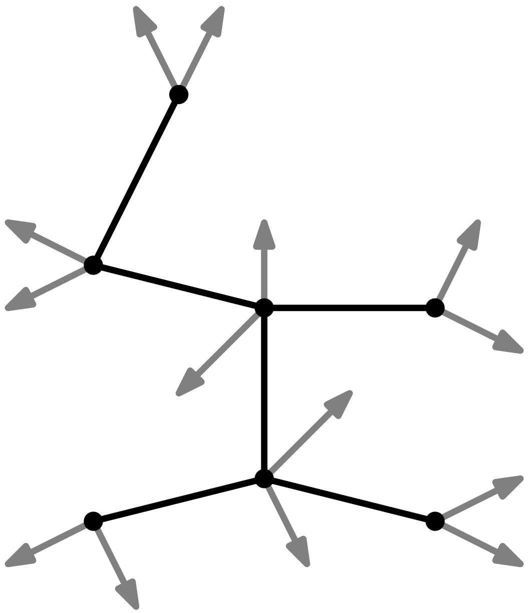
From trees to triangulations

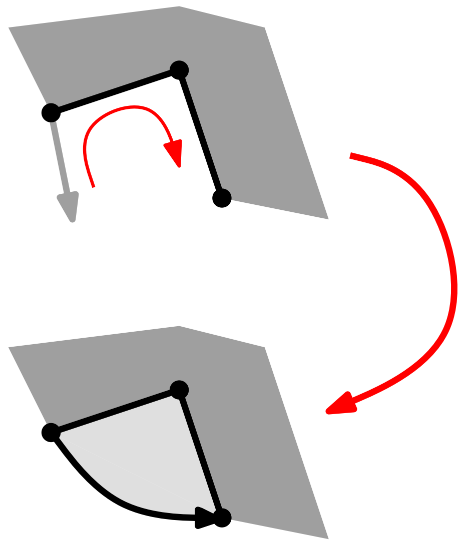
local closure rule

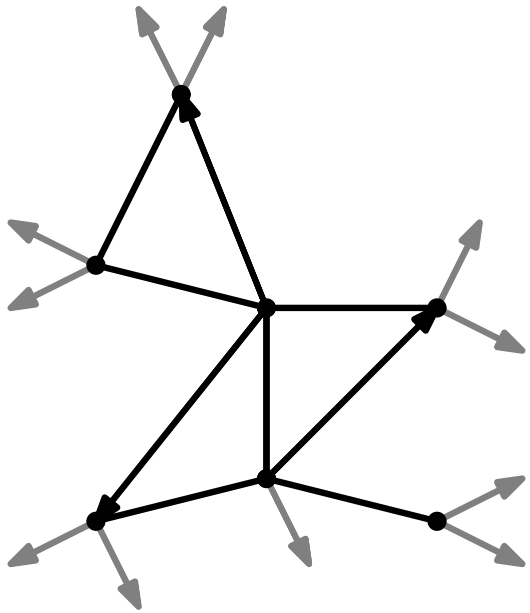


From trees to triangulations

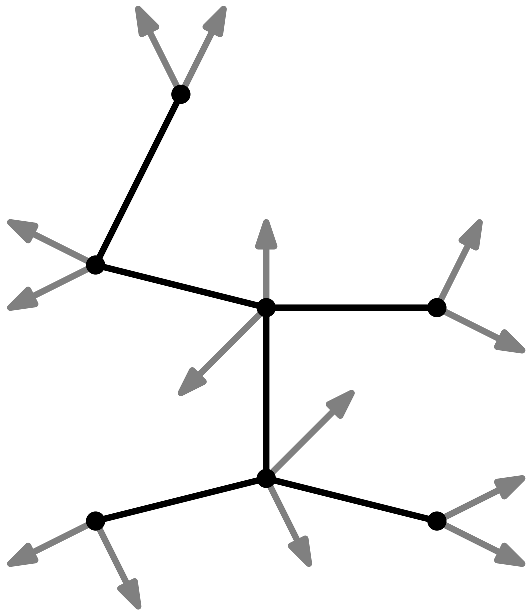
local closure rule



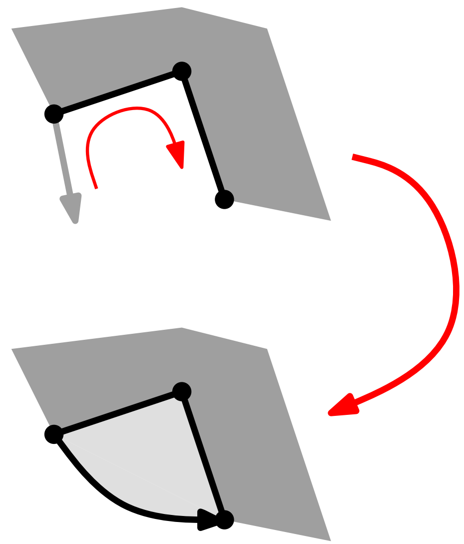


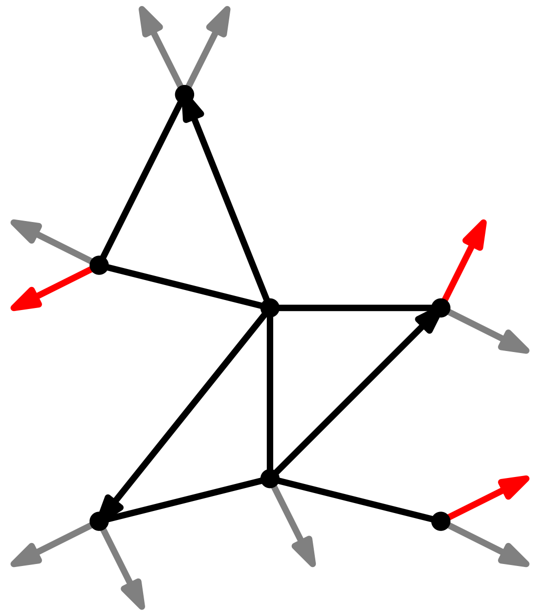


From trees to triangulations



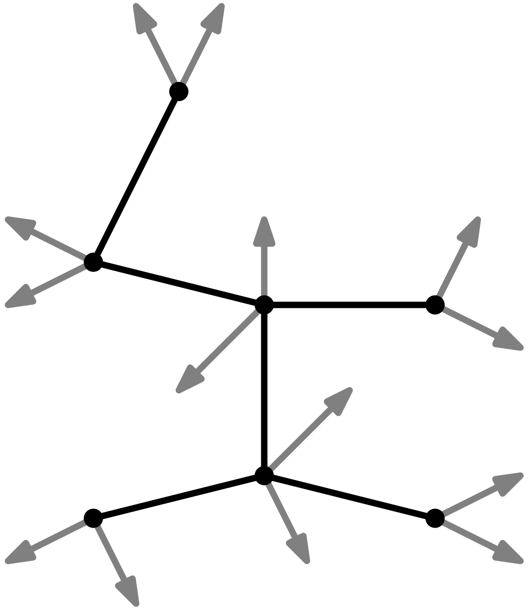
local closure rule

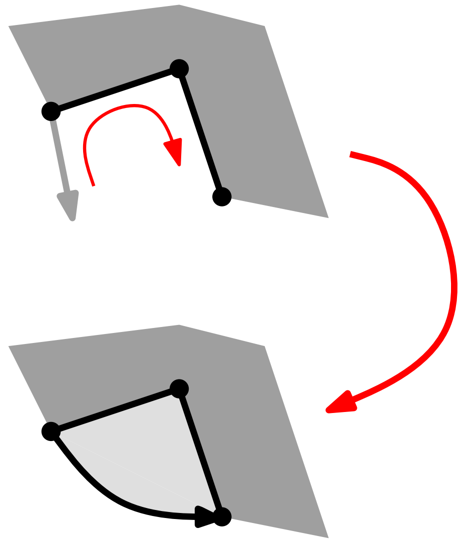


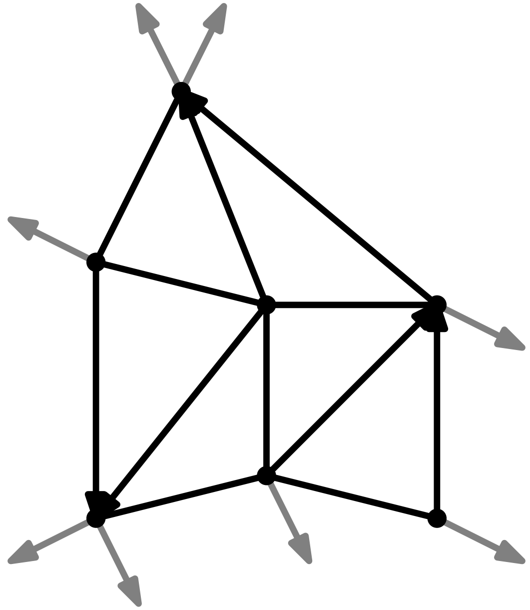


From trees to triangulations

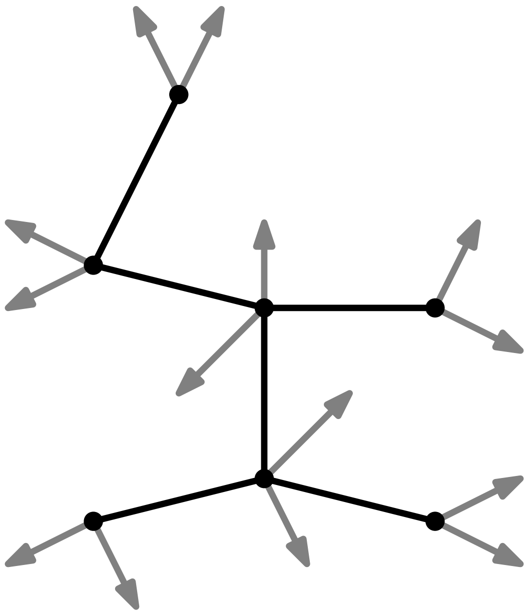
local closure rule



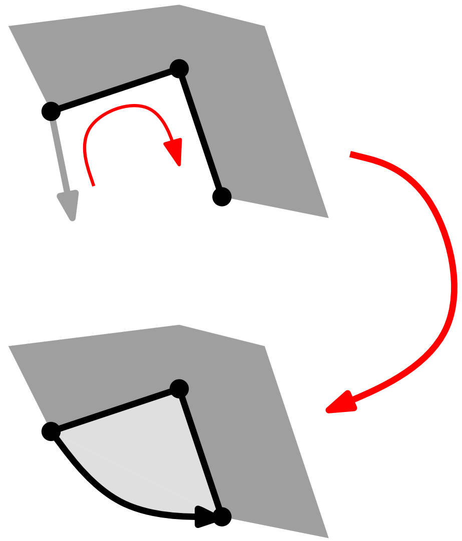


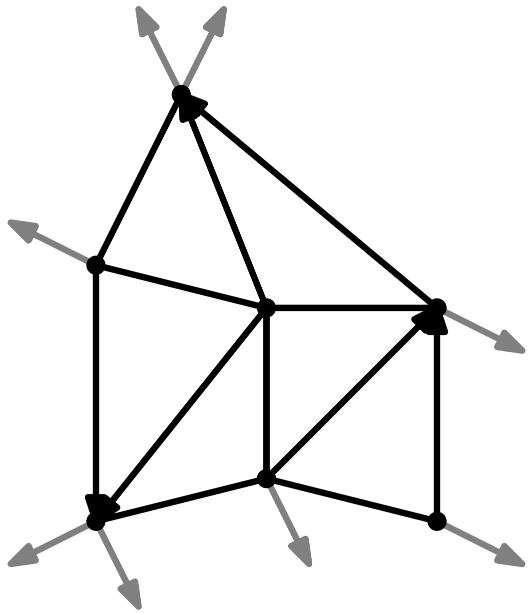


From trees to triangulations

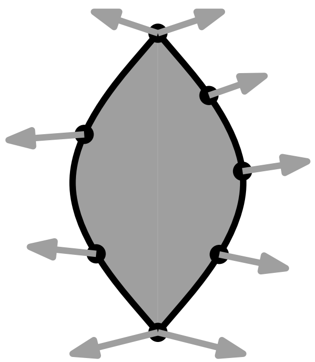


local closure rule

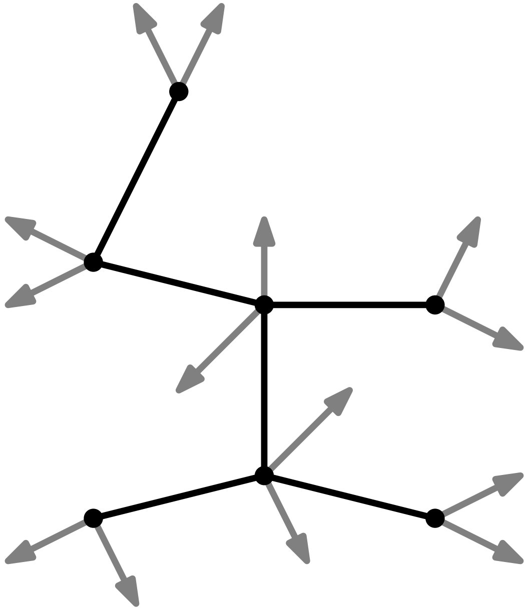




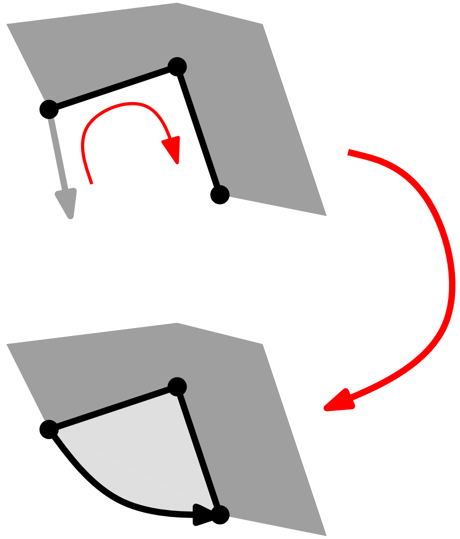
When closure stops,
the result looks like:

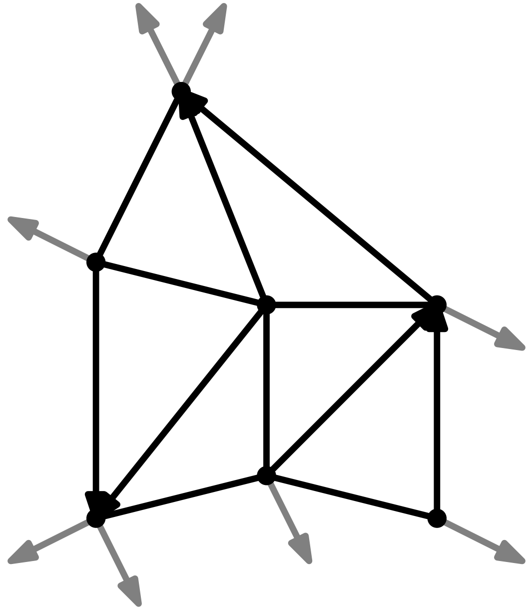


From trees to triangulations



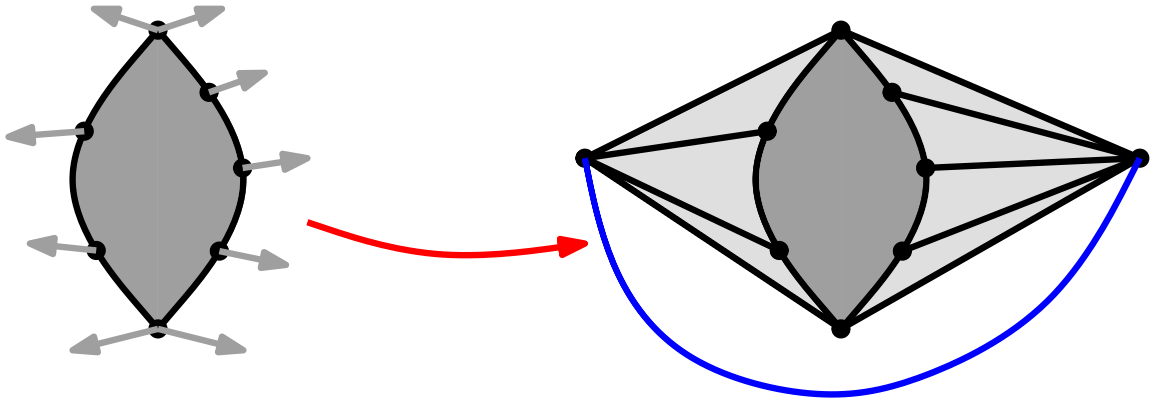
local closure rule



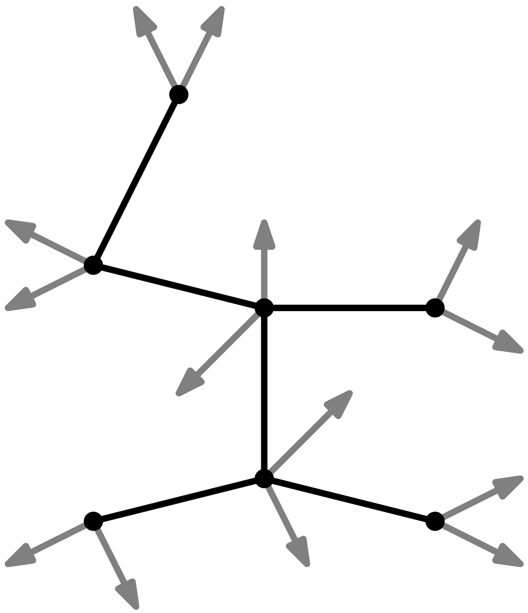


When closure stops,
the result looks like:

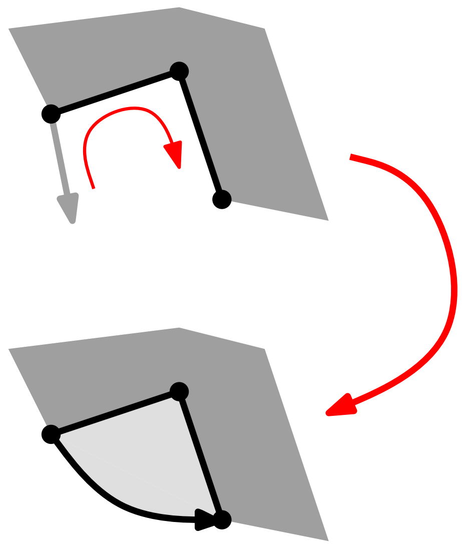
add two fans of triangles
and an edge

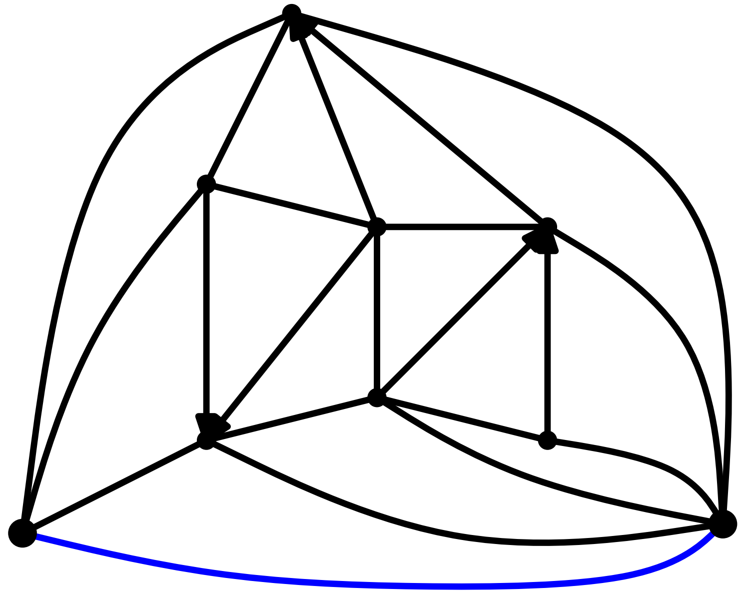


From trees to triangulations

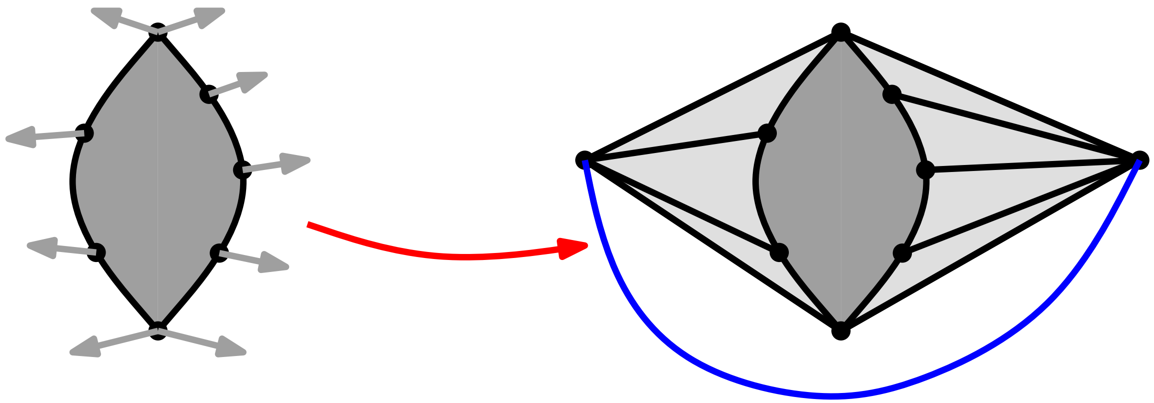


local closure rule



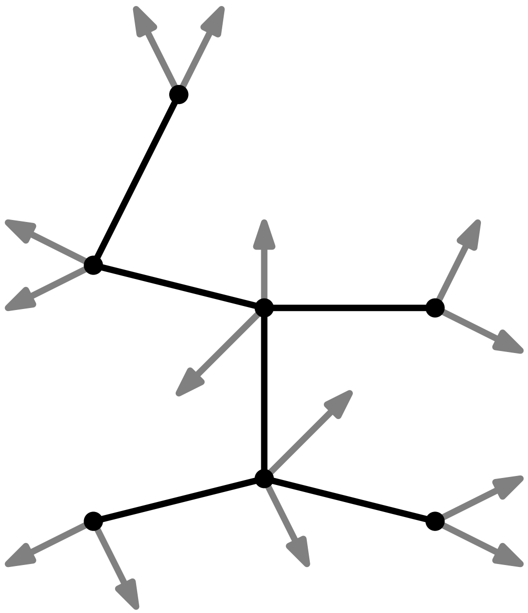


When closure stops,
the result looks like:

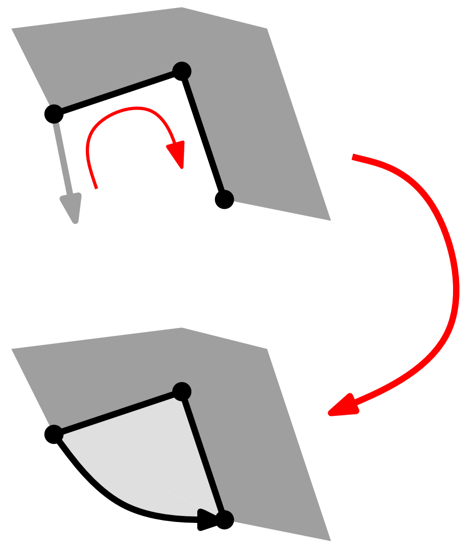


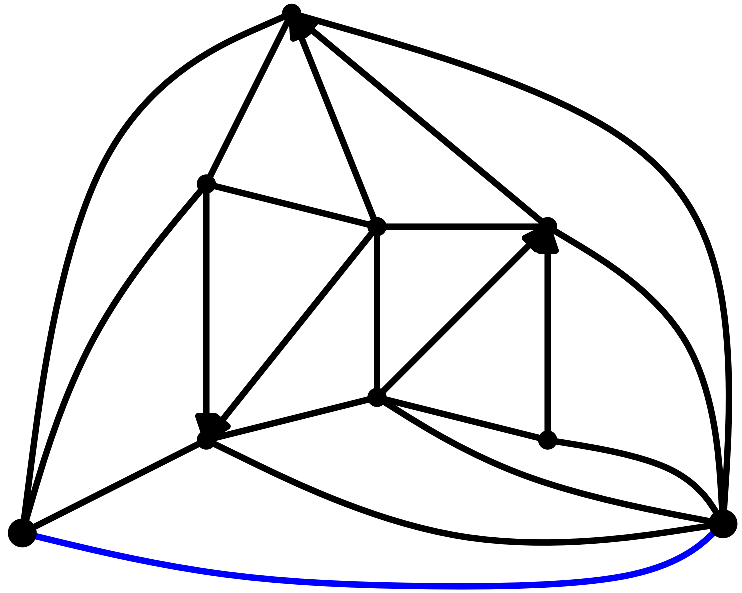
add two fans of triangles
and an edge

From trees to triangulations



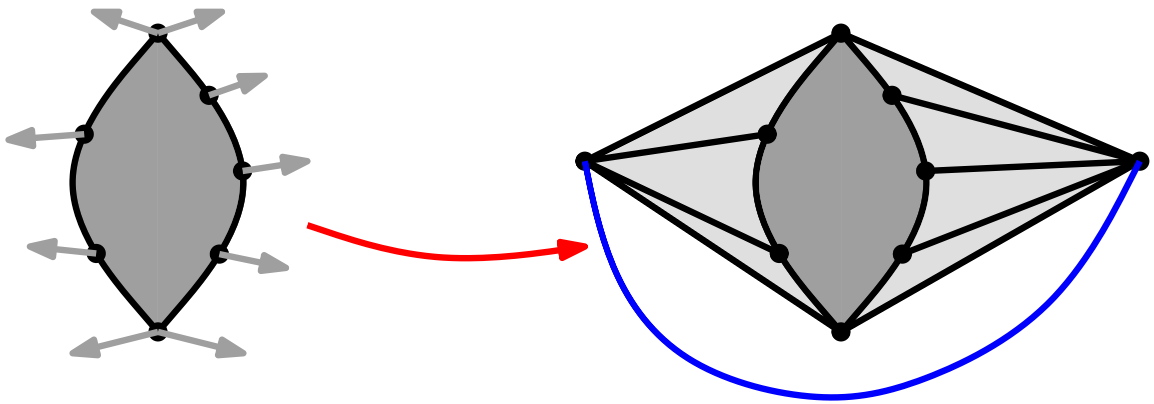
local closure rule





When closure stops,
the result looks like:

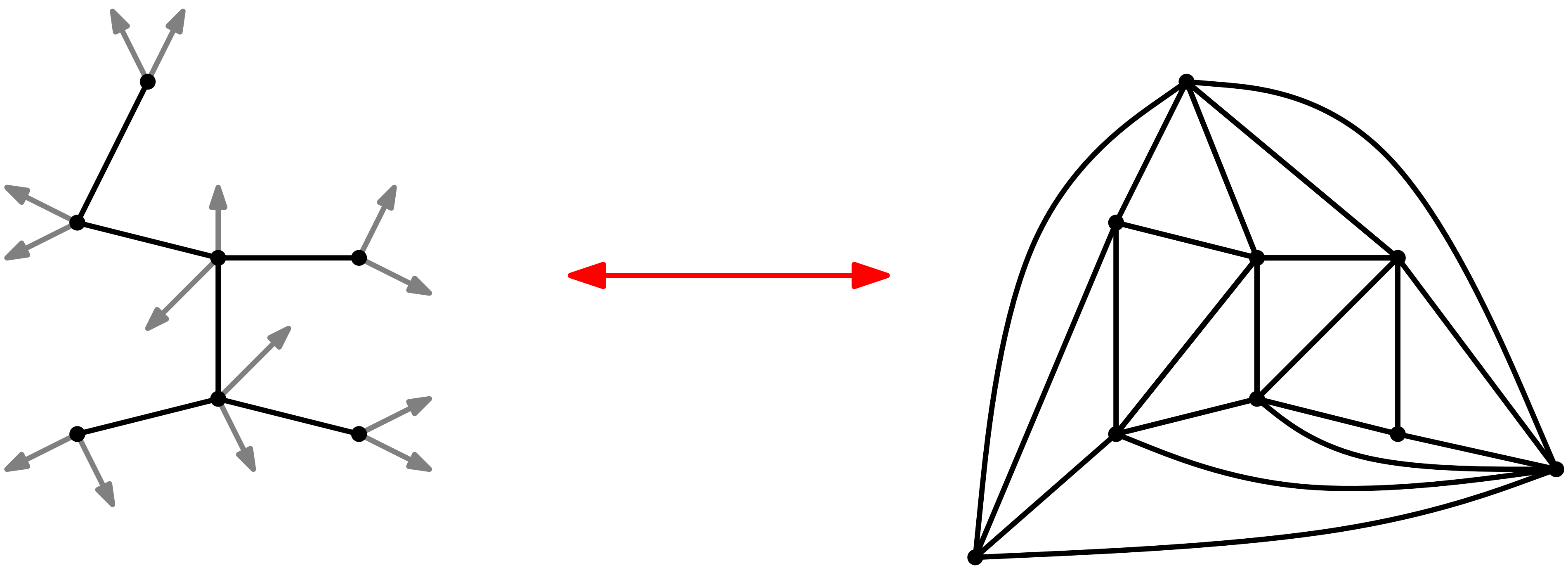
add two fans of triangles
and an edge



Theorem Closure is a one-to-one correspondence between 2-leaf trees with n nodes and marked triangulations with $n+2$ vertices.

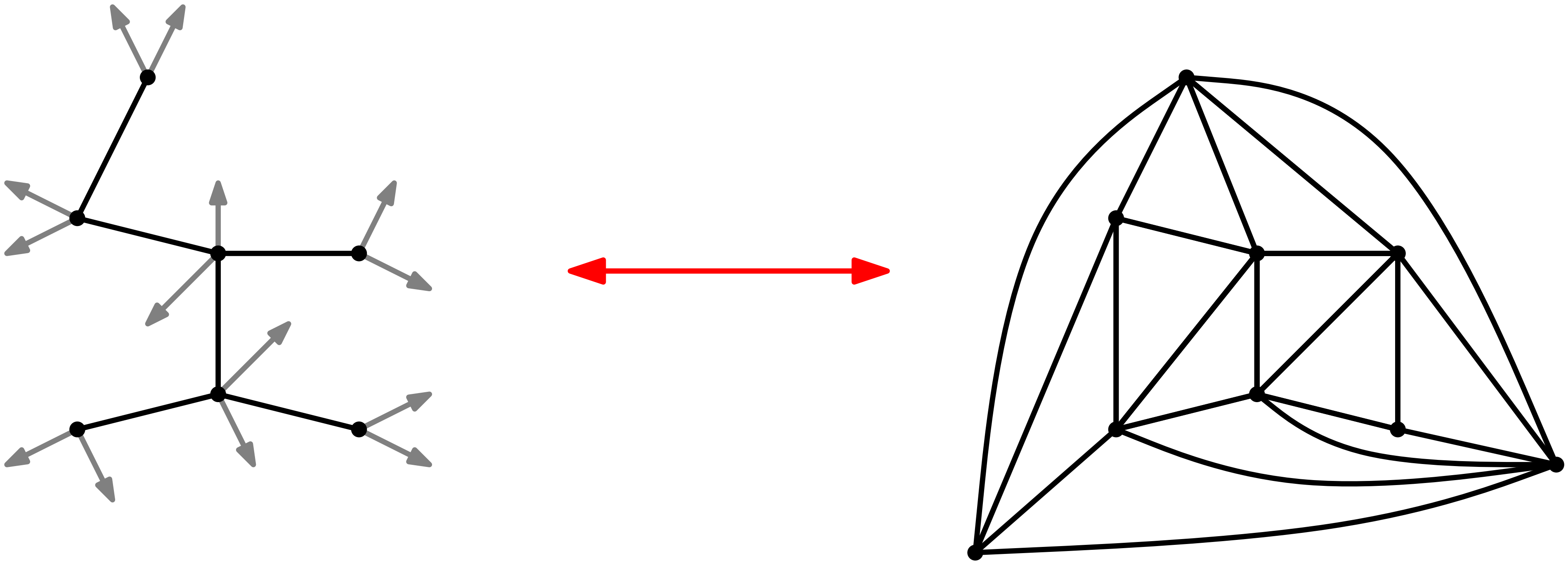
From trees to triangulations

Theorem (Poulalhon-S. 2004) Closure is a one-to-one correspondence between 2-leaf trees with n nodes and marked triangulations with $n+2$ vertices.

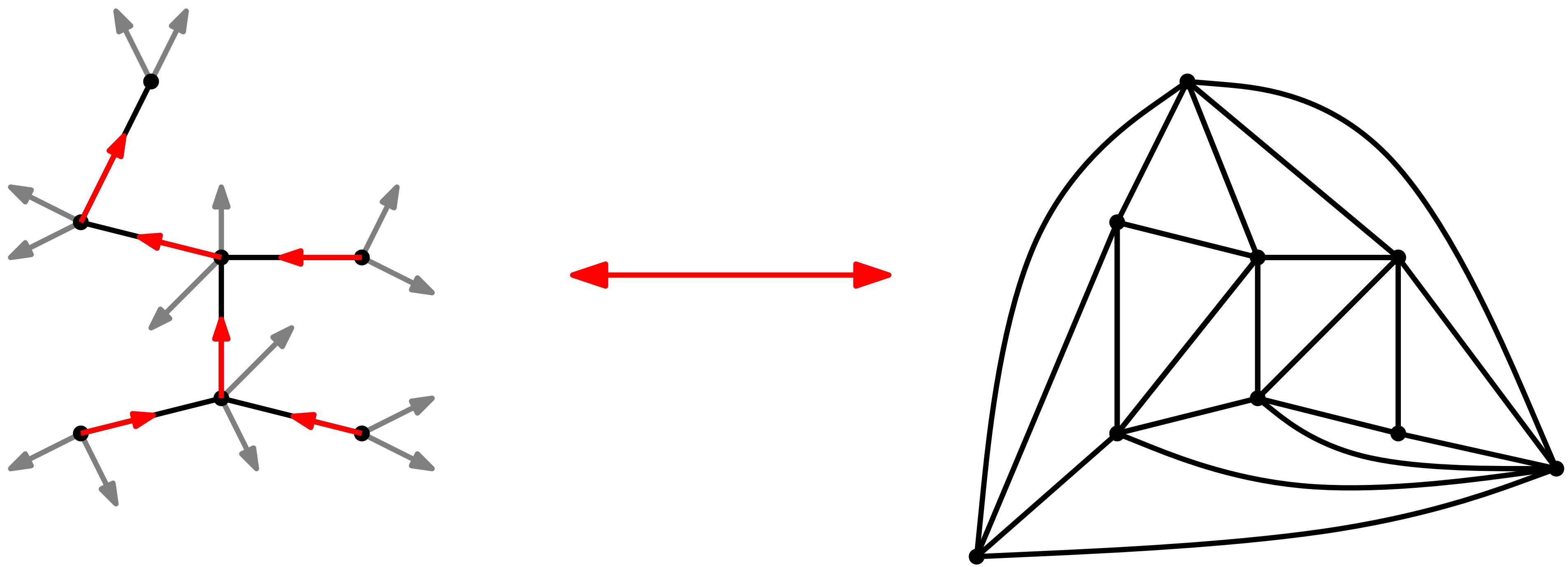


Corollary The number of triangulations with $n+2$ vertices is $\frac{2(4n-3)!}{n!(3n-1)!}$

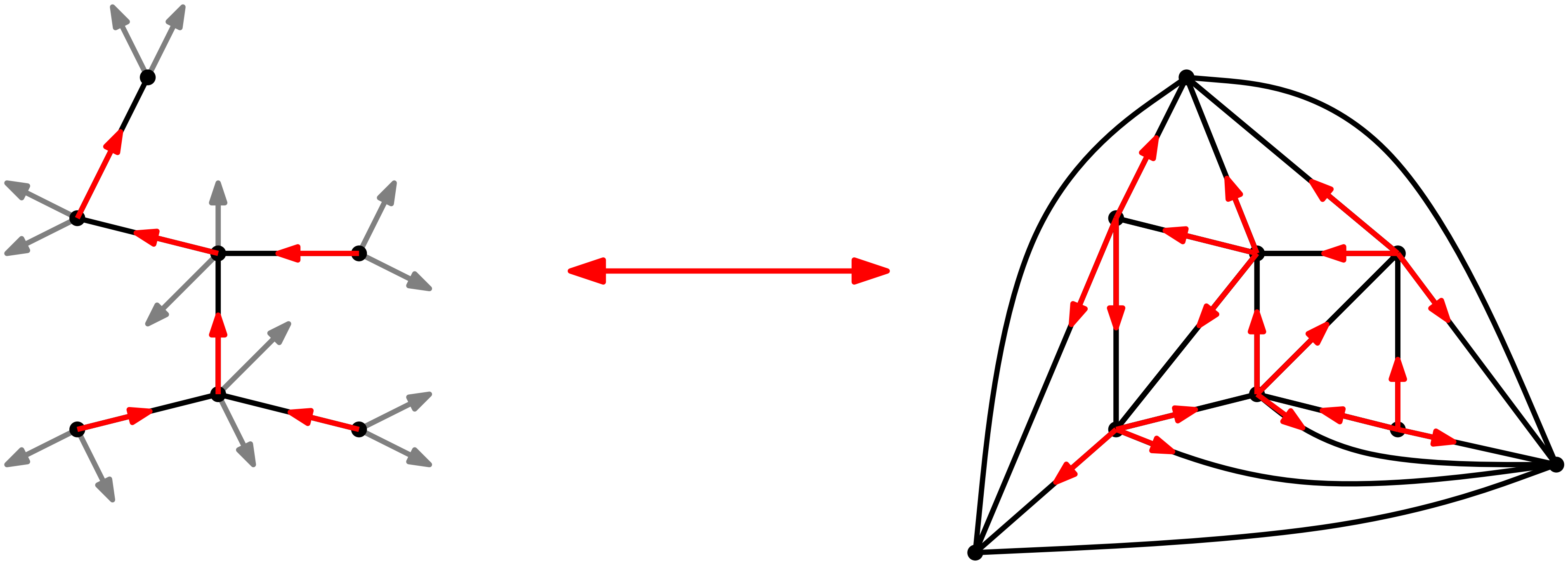
Some hints about the machinery



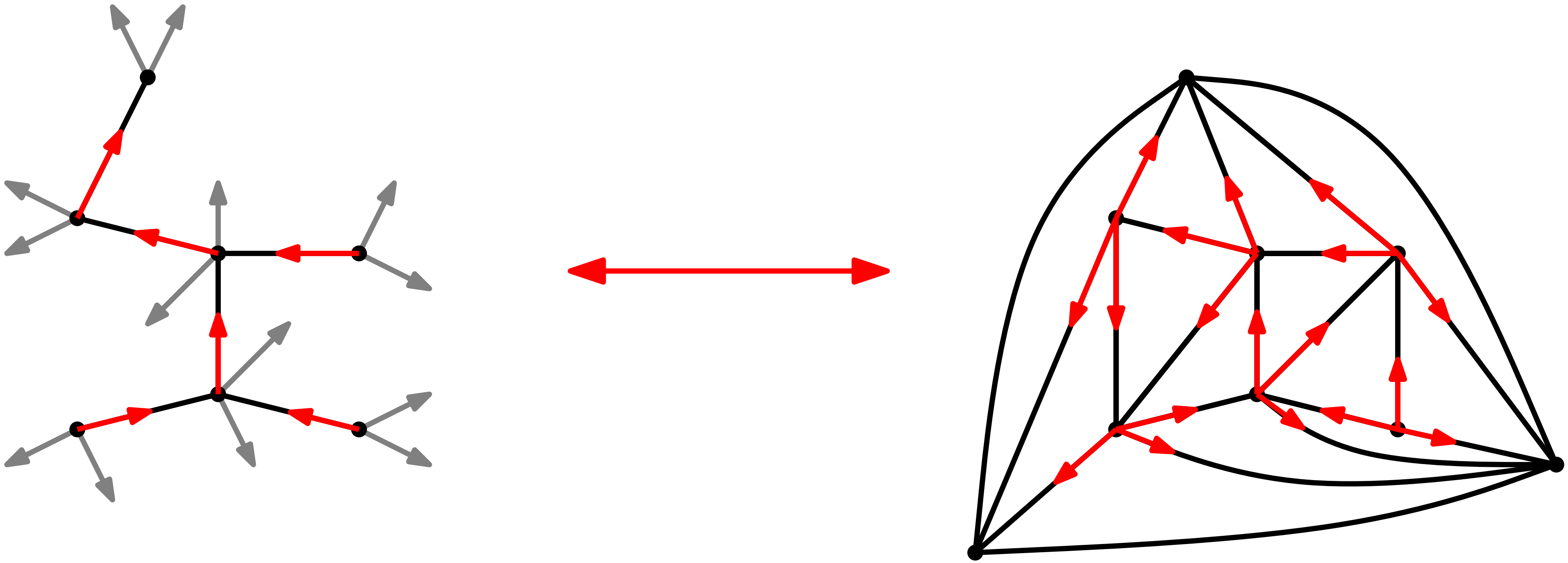
Some hints about the machinery



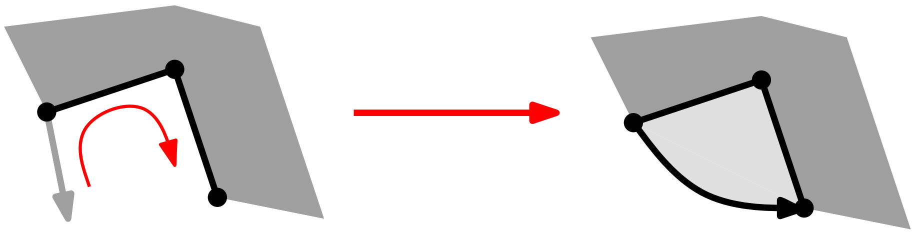
Some hints about the machinery



Some hints about the machinery



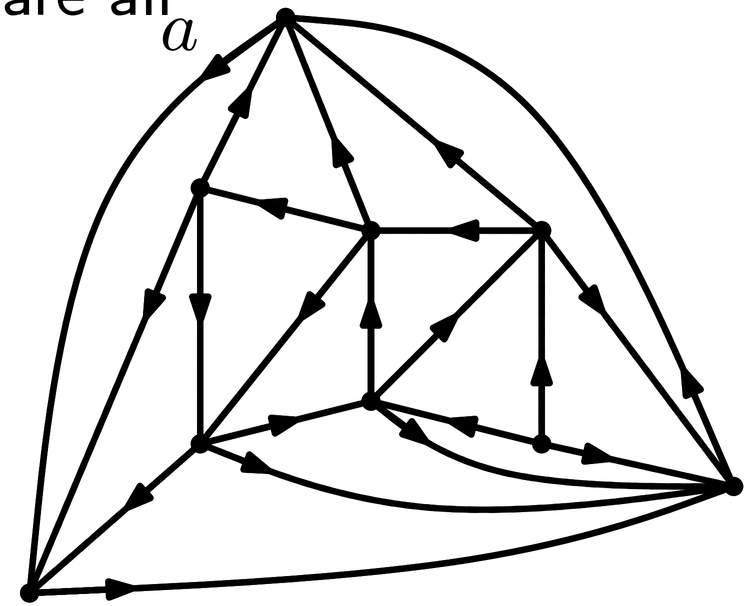
Lemma. Closure endows the triangulation with an orientation without clockwise cycles.



Indeed all faces are closed by counterclockwise edges.

Some hints about the machinery

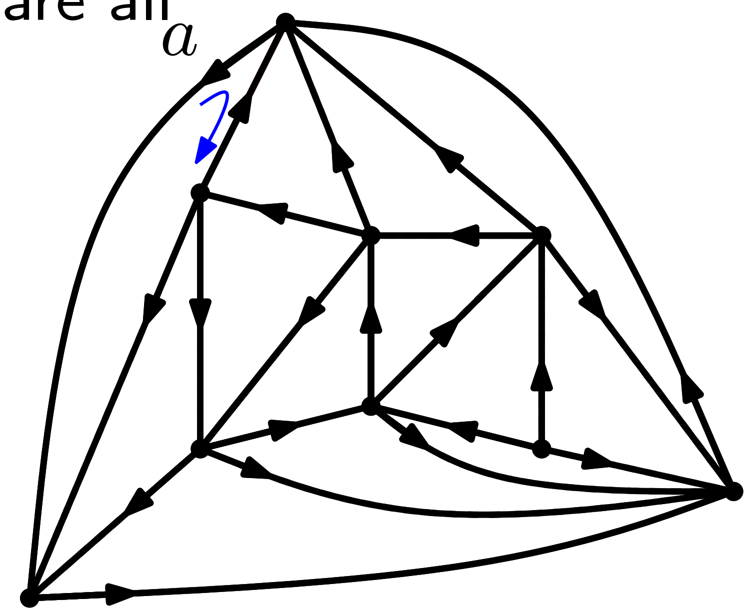
Lemma (Poulalhon-Schaeffer 2004/ Bernardi 2005.)
A planar map endowed with an accessible orientation without clockwise cycles admits a unique spanning tree such that external edges are all counterclockwise.



Lemma (Poulalhon-Schaeffer 2004.)
This tree can be recovered by depth first search traversal.

Some hints about the machinery

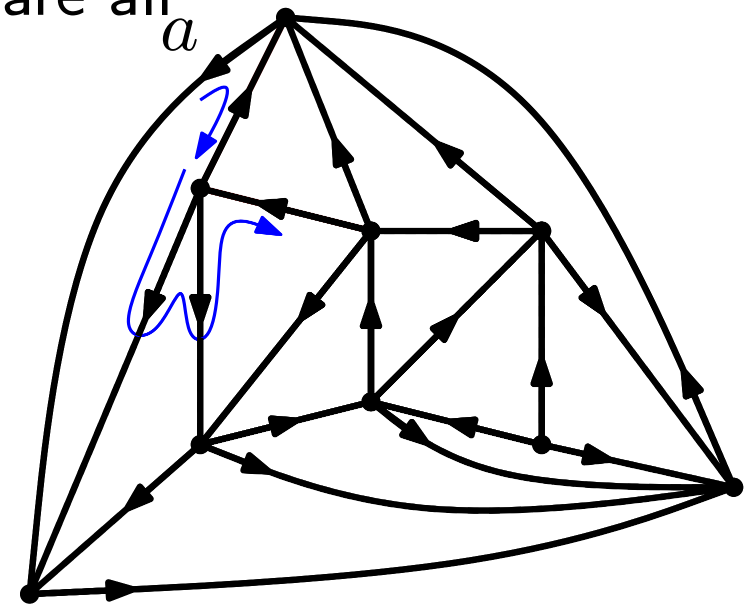
Lemma (Poulalhon-Schaeffer 2004/ Bernardi 2005.)
A planar map endowed with an accessible orientation without clockwise cycles admits a unique spanning tree such that external edges are all counterclockwise.



Lemma (Poulalhon-Schaeffer 2004.)
This tree can be recovered by depth first search traversal.

Some hints about the machinery

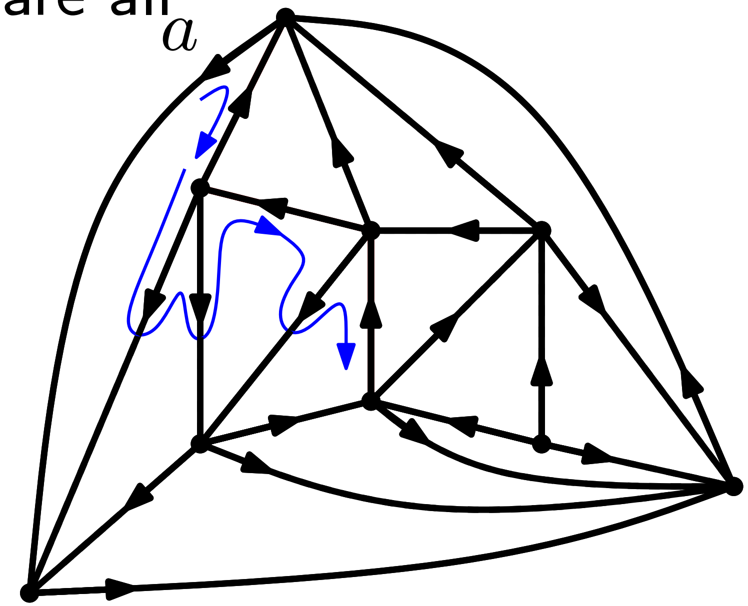
Lemma (Poulalhon-Schaeffer 2004/ Bernardi 2005.)
A planar map endowed with an accessible orientation without clockwise cycles admits a unique spanning tree such that external edges are all counterclockwise.



Lemma (Poulalhon-Schaeffer 2004.)
This tree can be recovered by depth first search traversal.

Some hints about the machinery

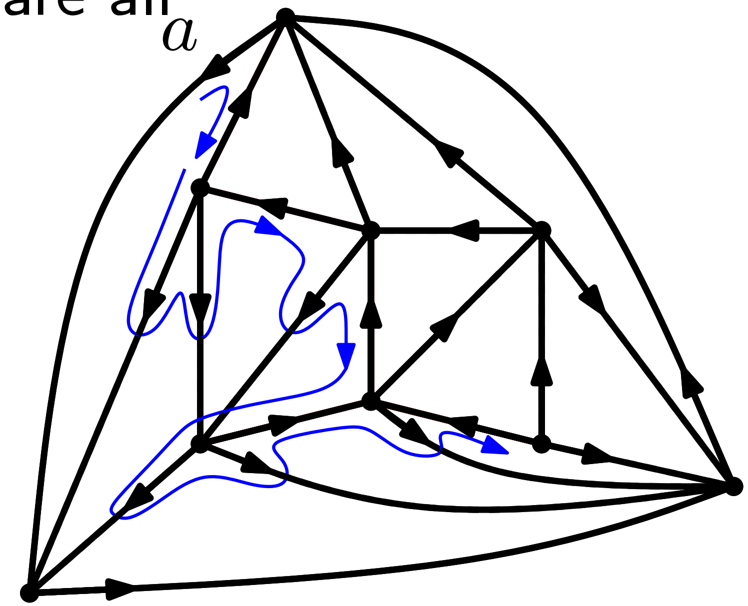
Lemma (Poulalhon-Schaeffer 2004/ Bernardi 2005.)
A planar map endowed with an accessible orientation without clockwise cycles admits a unique spanning tree such that external edges are all counterclockwise.



Lemma (Poulalhon-Schaeffer 2004.)
This tree can be recovered by depth first search traversal.

Some hints about the machinery

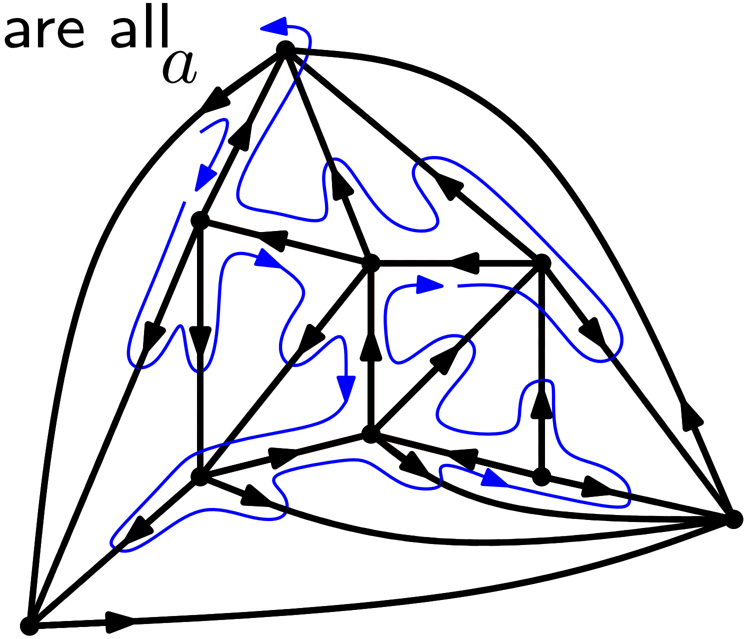
Lemma (Poulalhon-Schaeffer 2004/ Bernardi 2005.)
A planar map endowed with an accessible orientation without clockwise cycles admits a unique spanning tree such that external edges are all counterclockwise.



Lemma (Poulalhon-Schaeffer 2004.)
This tree can be recovered by depth first search traversal.

Some hints about the machinery

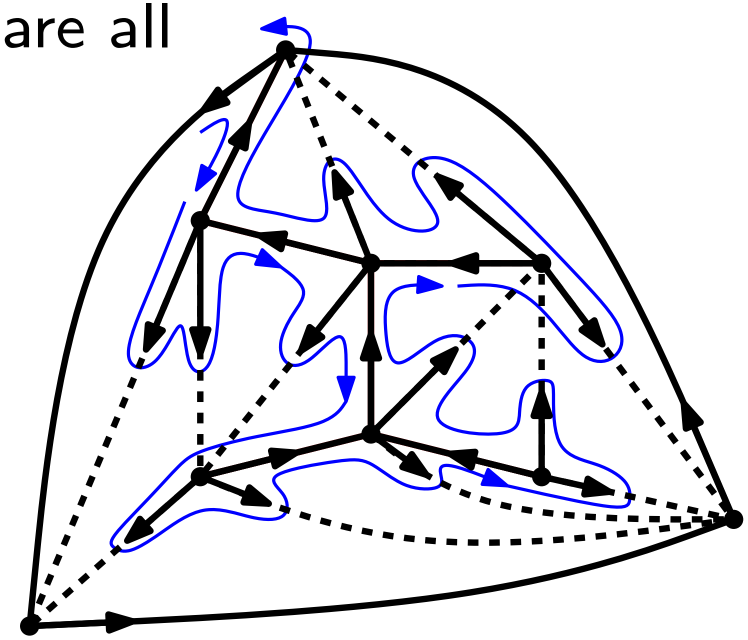
Lemma (Poulalhon-Schaeffer 2004/ Bernardi 2005.)
A planar map endowed with an accessible orientation without clockwise cycles admits a unique spanning tree such that external edges are all counterclockwise.



Lemma (Poulalhon-Schaeffer 2004.)
This tree can be recovered by depth first search traversal.

Some hints about the machinery

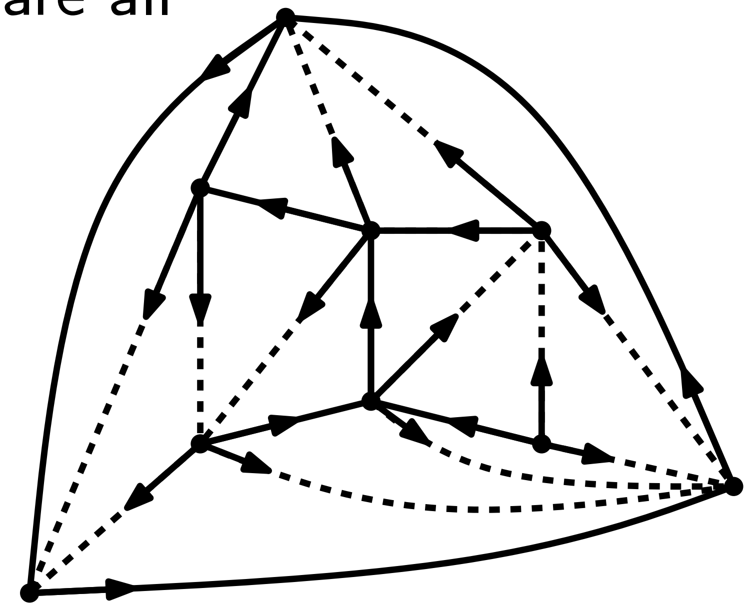
Lemma (Poulalhon-Schaeffer 2004/ Bernardi 2005.)
A planar map endowed with an accessible orientation without clockwise cycles admits a unique spanning tree such that external edges are all counterclockwise.



Lemma (Poulalhon-Schaeffer 2004.)
This tree can be recovered by depth first search traversal.

Some hints about the machinery

Lemma (Poulalhon-Schaeffer 2004/ Bernardi 2005.)
A planar map endowed with an accessible orientation without clockwise cycles admits a unique spanning tree such that external edges are all counterclockwise.



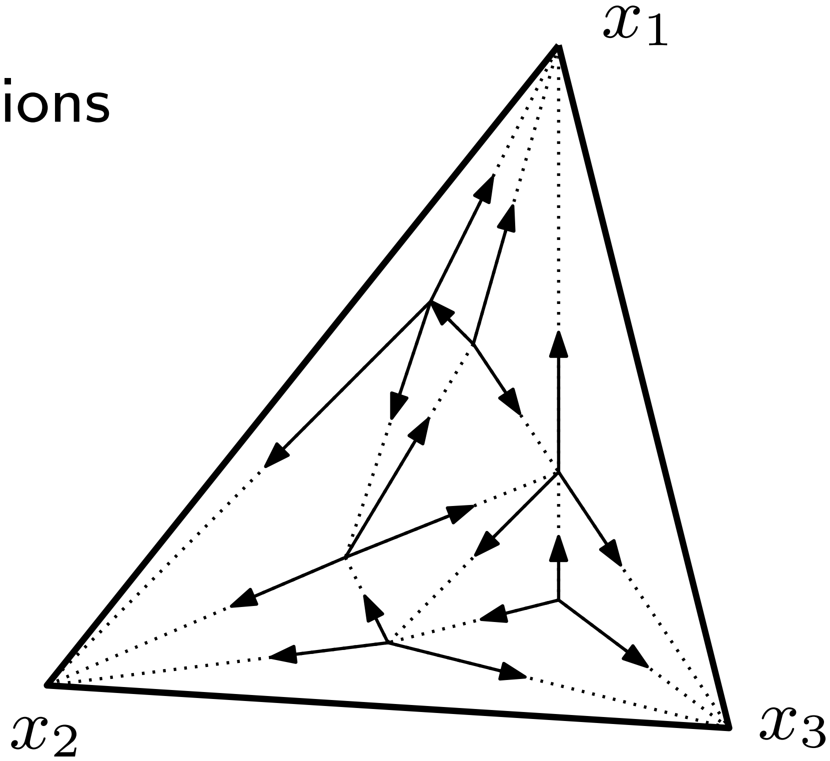
Lemma (Poulalhon-Schaeffer 2004.)
This tree can be recovered by depth first search traversal.

Corollary The closure is a bijection between 2-leaf trees with n nodes and triangulations with a minimal accessible 3-orientation.

Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

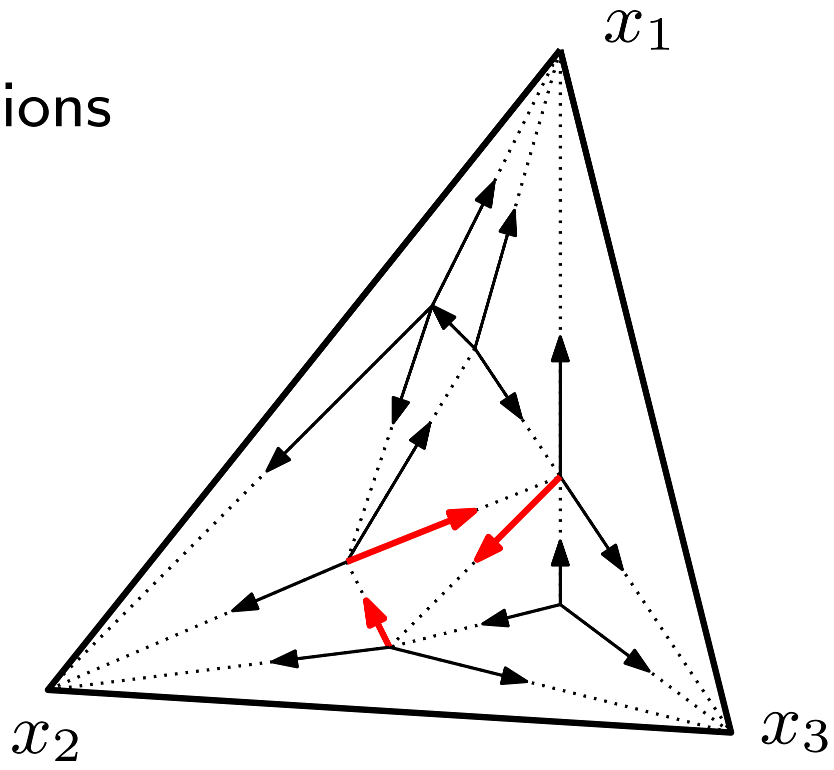
Define a partial order on the 3-orientations of a triangulation by circuit reversal.



Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

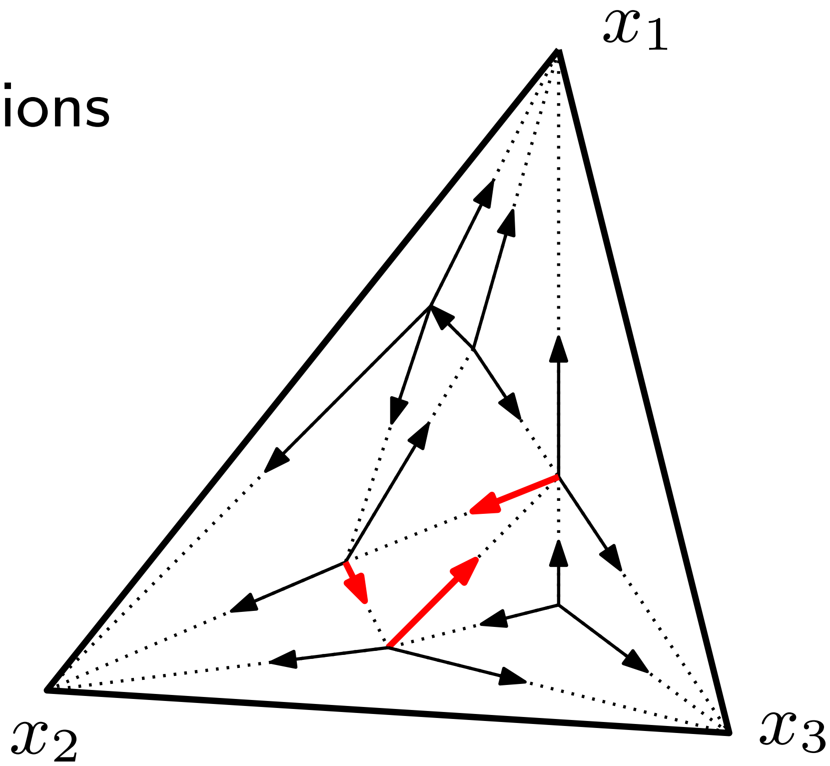
Define a partial order on the 3-orientations of a triangulation by circuit reversal.



Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

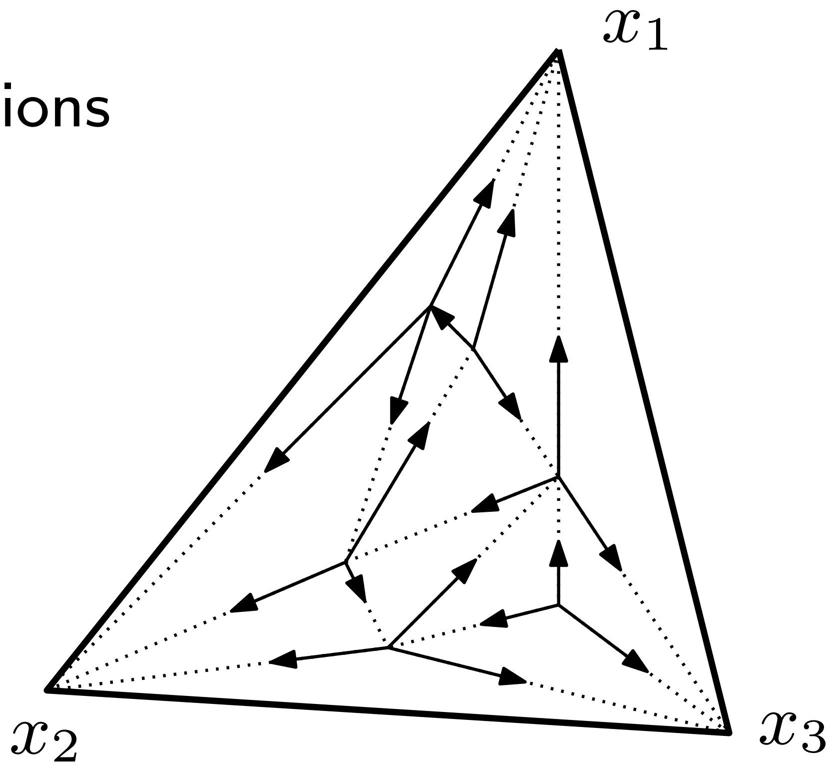
Define a partial order on the 3-orientations of a triangulation by circuit reversal.



Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

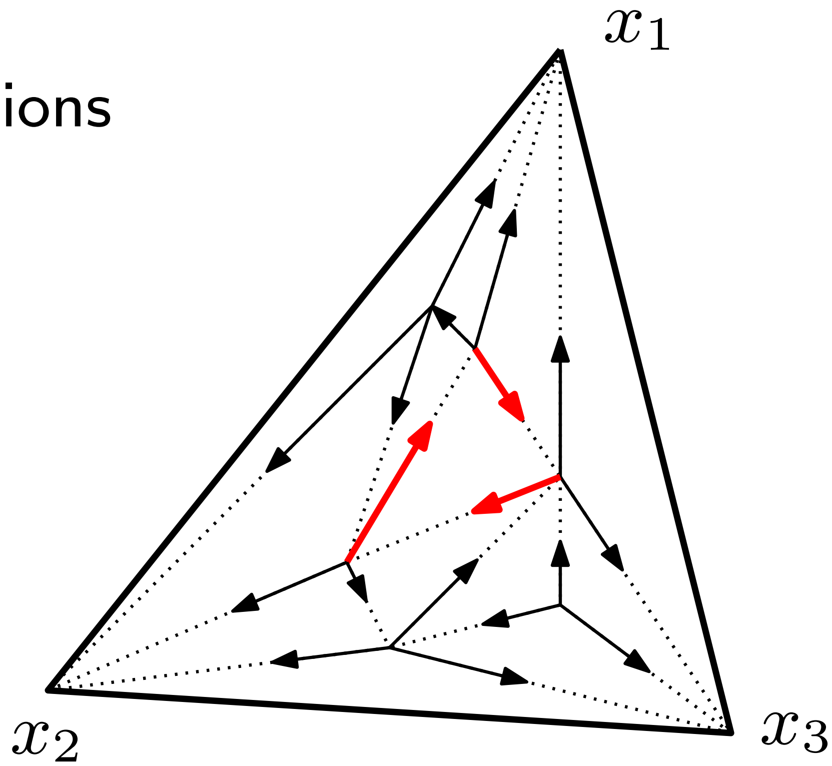
Define a partial order on the 3-orientations of a triangulation by circuit reversal.



Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

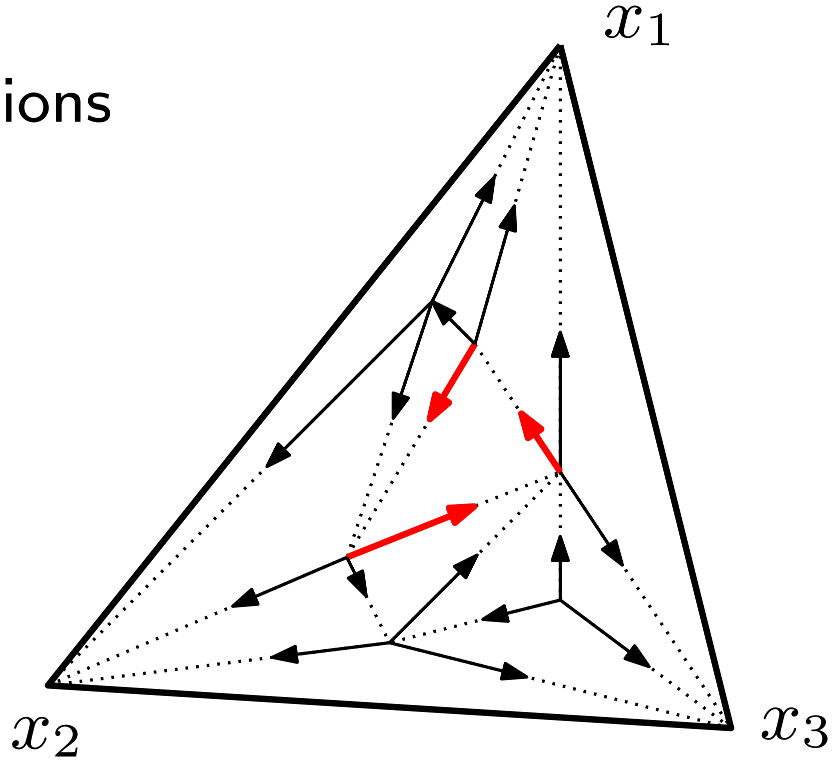
Define a partial order on the 3-orientations of a triangulation by circuit reversal.



Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

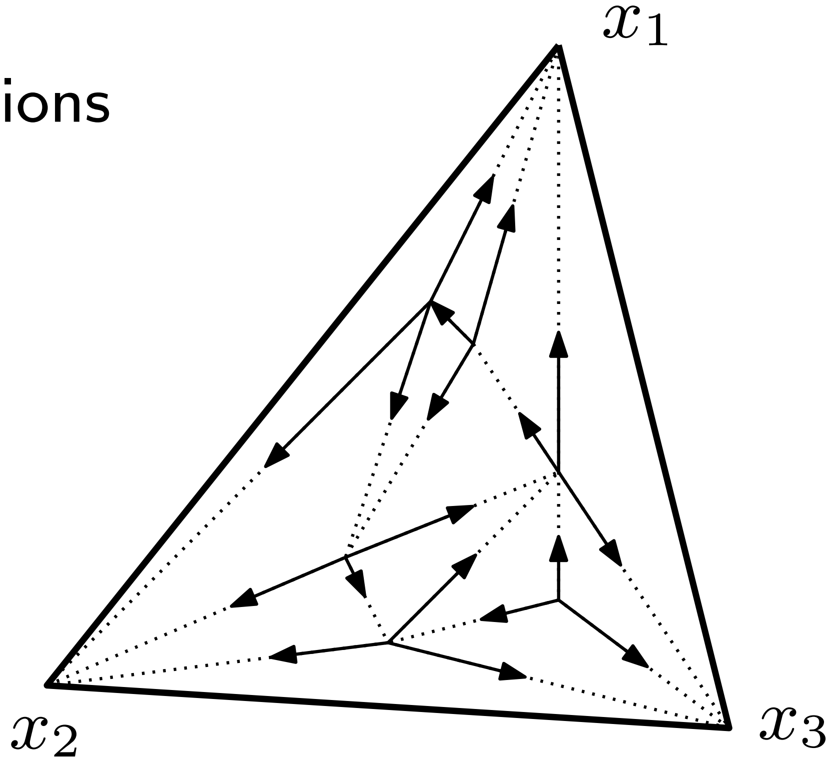
Define a partial order on the 3-orientations of a triangulation by circuit reversal.



Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

Define a partial order on the 3-orientations of a triangulation by circuit reversal.

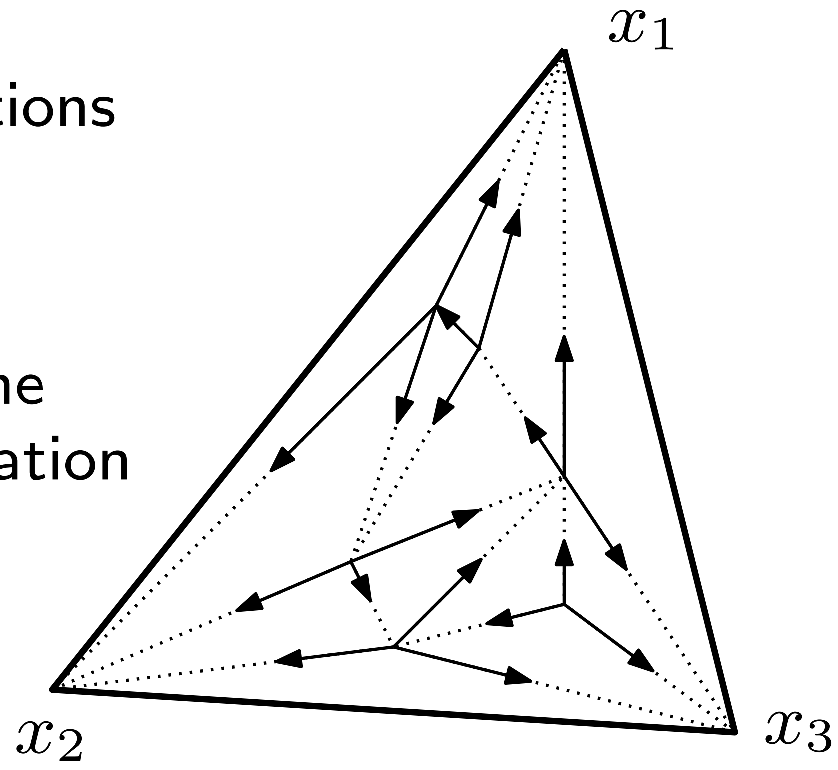


Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

Define a partial order on the 3-orientations of a triangulation by circuit reversal.

Theorem (Ossona de Mendez 94)
Circuit reversal defines a lattice, and the minimal element is the unique 3-orientation without clockwise cycle.

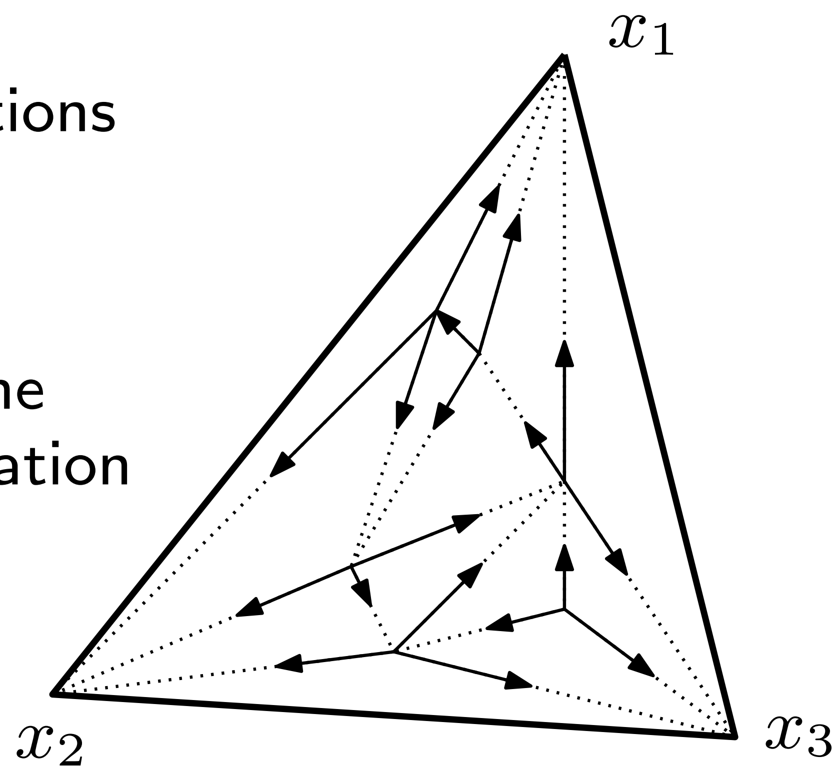


Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

Define a partial order on the 3-orientations of a triangulation by circuit reversal.

Theorem (Ossona de Mendez 94)
Circuit reversal defines a lattice, and the minimal element is the unique 3-orientation without clockwise cycle.



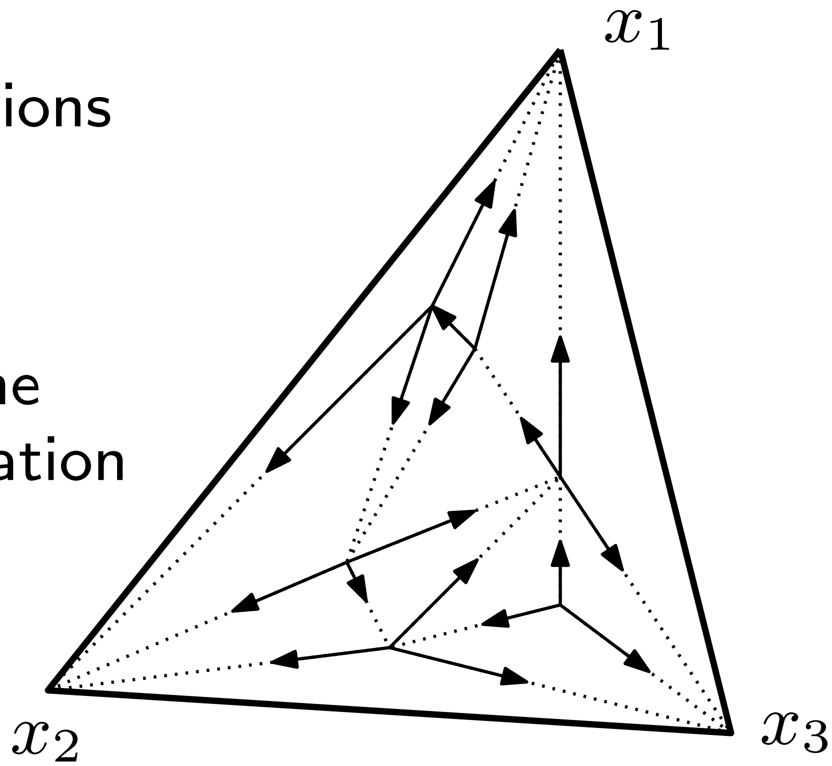
Corollary Every planar triangulation has a unique accessible 3-orientation without clockwise cycle.

Some hints about the machinery

Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

Define a partial order on the 3-orientations of a triangulation by circuit reversal.

Theorem (Ossona de Mendez 94)
Circuit reversal defines a lattice, and the minimal element is the unique 3-orientation without clockwise cycle.

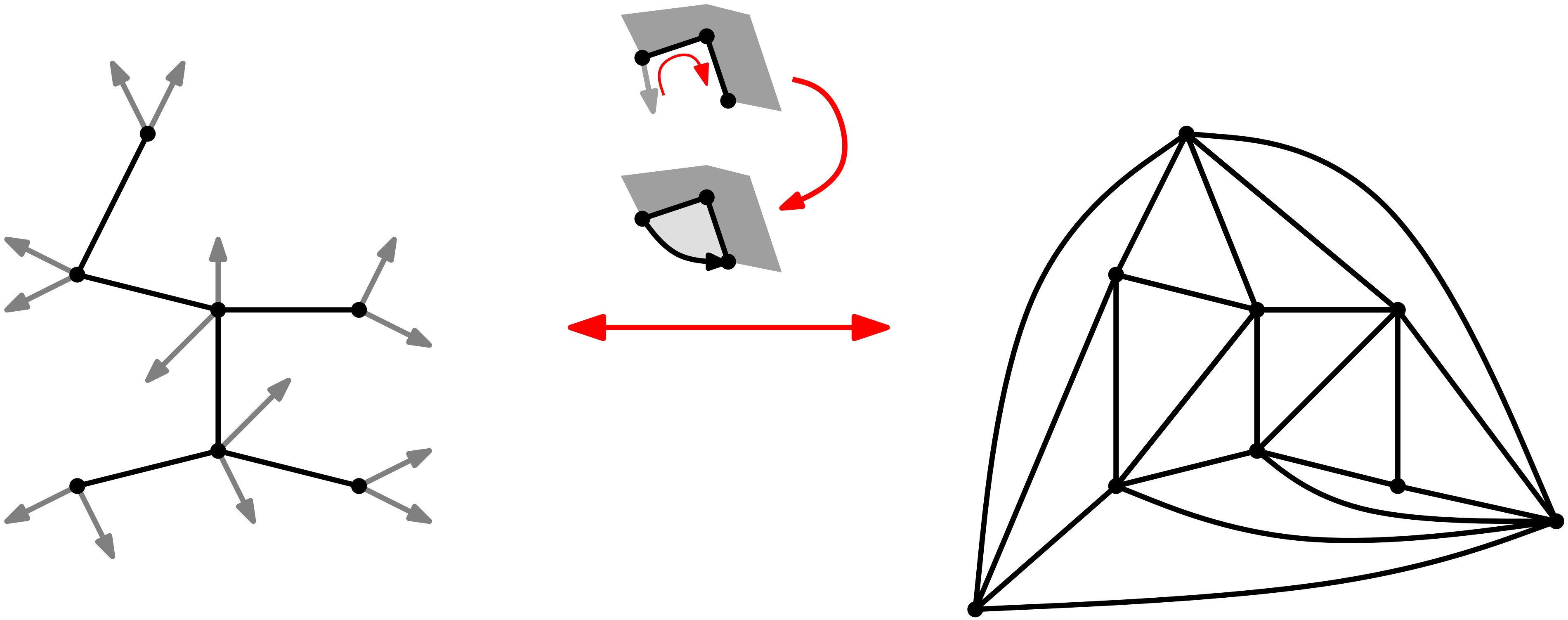


Corollary Every planar triangulation has a unique accessible 3-orientation without clockwise cycle.

Corollary The previous closure is a bijection between 2-leaf trees with n nodes and planar triangulations with $n+2$ vertices.

Summary

Theorem Closure is a one-to-one correspondence between 2-leaf trees with n nodes and marked triangulations with $n+2$ vertices.

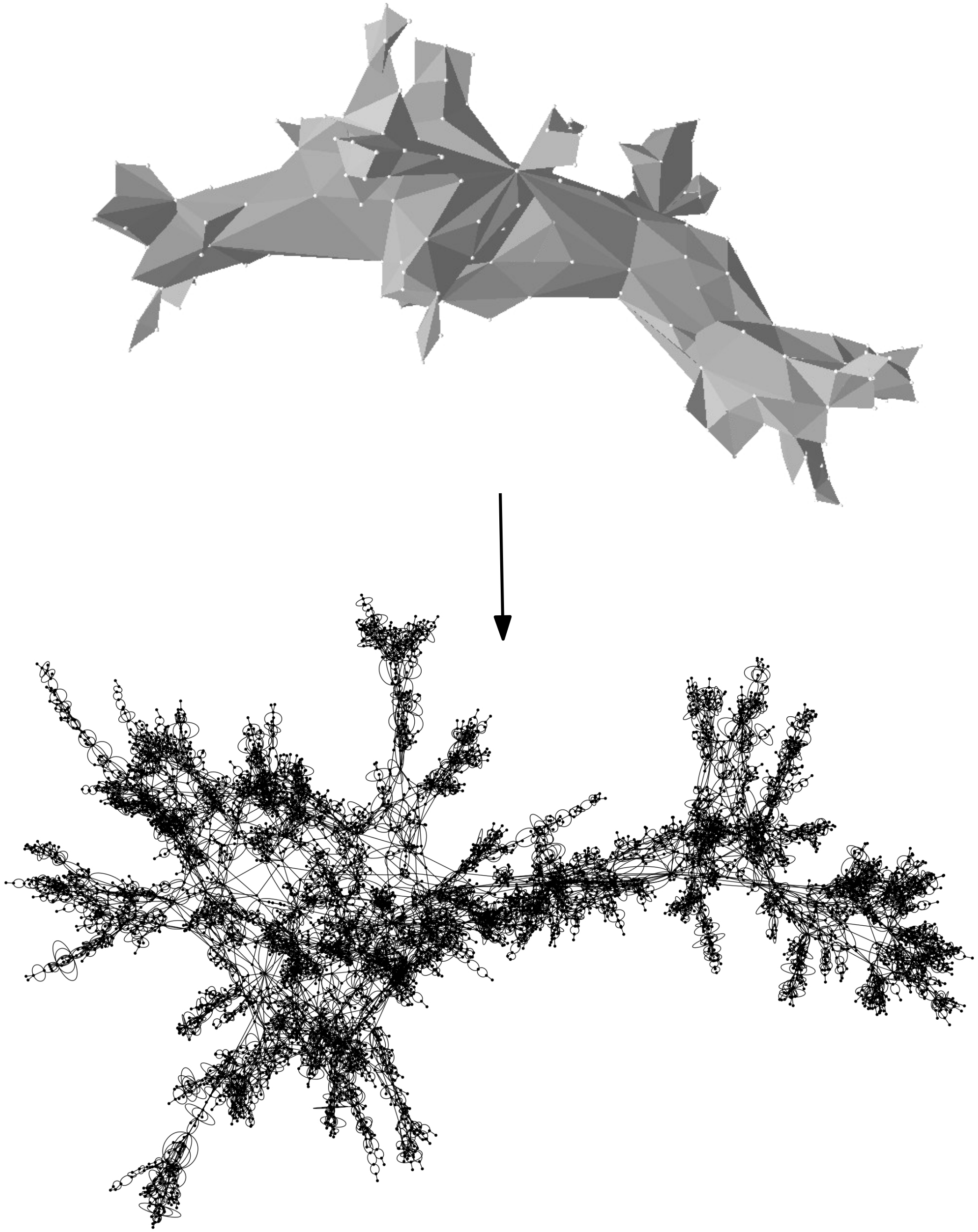


Bonus The closure edges form almost geodesics toward the root.

This allows to study the number of vertices at distance r from the root.

Triangulations converge to the Brownian map

Theorem (Albenque, Adderio-Berry 2013) Upon rescaling edge length by a factor $n^{1/4}$, uniform random triangulations of size n converge to the Brownian map of Legall in the sense of Gromov-Hausdorf



The meaning of this statement will be explained in Le Gall's talk who will discuss the proof of his earlier analog result for $2g$-angulations and for triangulations with loops and multiple edges.

Partial conclusion

"Many" exact enumeration results on planar (or higher genus maps).

Quite a number of them involve algebraic gf.

As shown by Eynard, this corresponds to a peculiar (non fundamental he would say) property of their spectral curve.

Yet it corresponds to cases which we can solve "even more explicitely" by combinatorial tools: by revealing hidden tree structure, we prove that these models are in some sense $\mathbb{N}$-algebraic.

The machinery based on orientation without clockwise cycle is very general.

For some not completely clear reasons, the revealed tree structures also allows to study geodesics in the corresponding maps, and to prove convergence of the rescaled surfaces to the Brownian map.

Counting maps and triangulations

Trees, independence, algebraic generating series

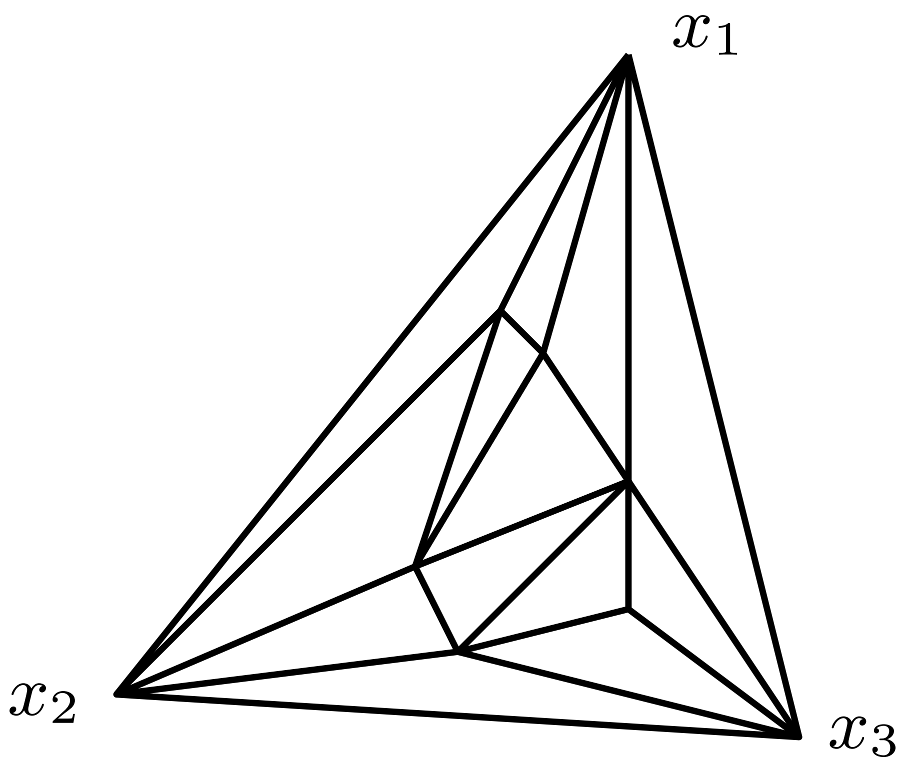
Stack triangulations

General 2d triangulations

Realizers

Realizers as a toy model on triangulations

Let T be a triangulation with boundary $\{x_1, x_2, x_3\}$.



$I = \{$internal vertices$\}$.

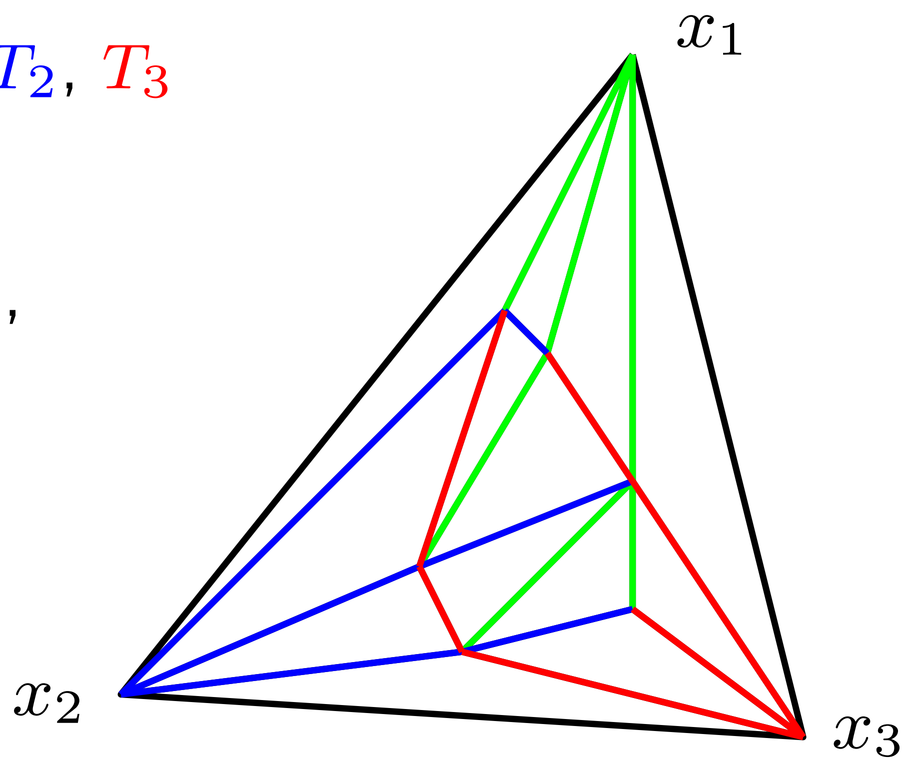
(here $|I| = 6$)

Realizers as a toy model on triangulations

Let T be a triangulation with boundary $\{x_1, x_2, x_3\}$.

A Schnyder wood is a partition T_1, T_2, T_3 of the internal edges of T such that:

i) T_i is a spanning tree of $I \cup \{x_i\}$,



$I = \{\text{internal vertices}\}$.

(here $|I| = 6$)

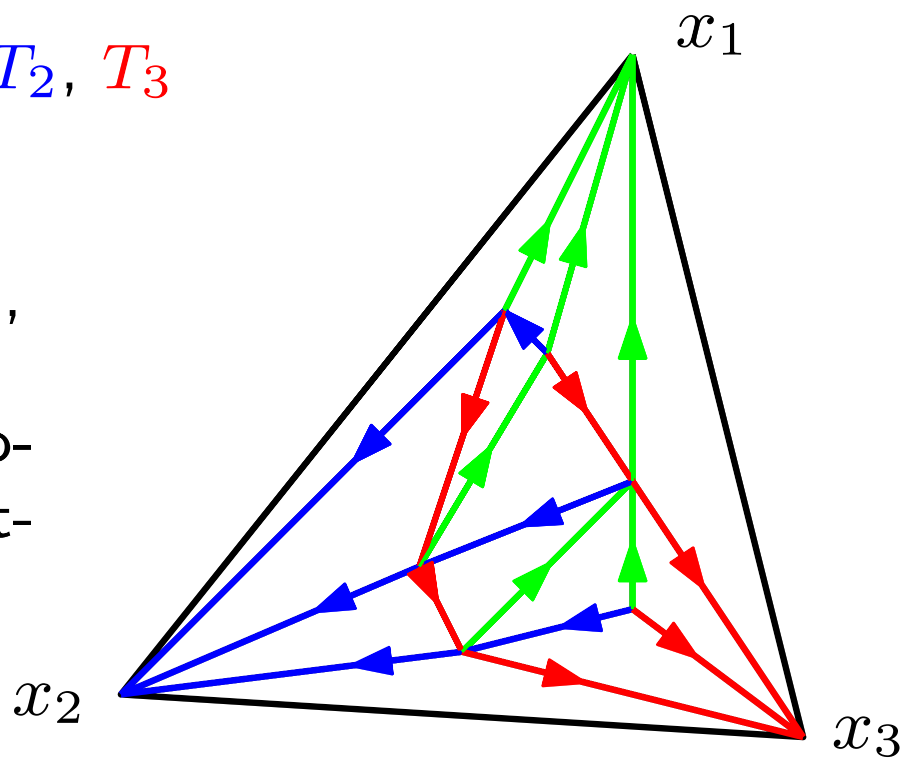
Realizers as a toy model on triangulations

Let T be a triangulation with boundary $\{x_1, x_2, x_3\}$.

A Schnyder wood is a partition T_1, T_2, T_3 of the internal edges of T such that:

i) T_i is a spanning tree of $I \cup \{x_i\}$,

Upon orienting edges of each tree toward its root, each vertex has 1 outgoing edge of each color, and



$I = \{\text{internal vertices}\}$.

(here $|I| = 6$)

Realizers as a toy model on triangulations

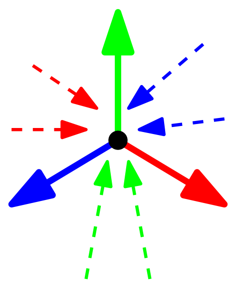
Let T be a triangulation with boundary $\{x_1, x_2, x_3\}$.

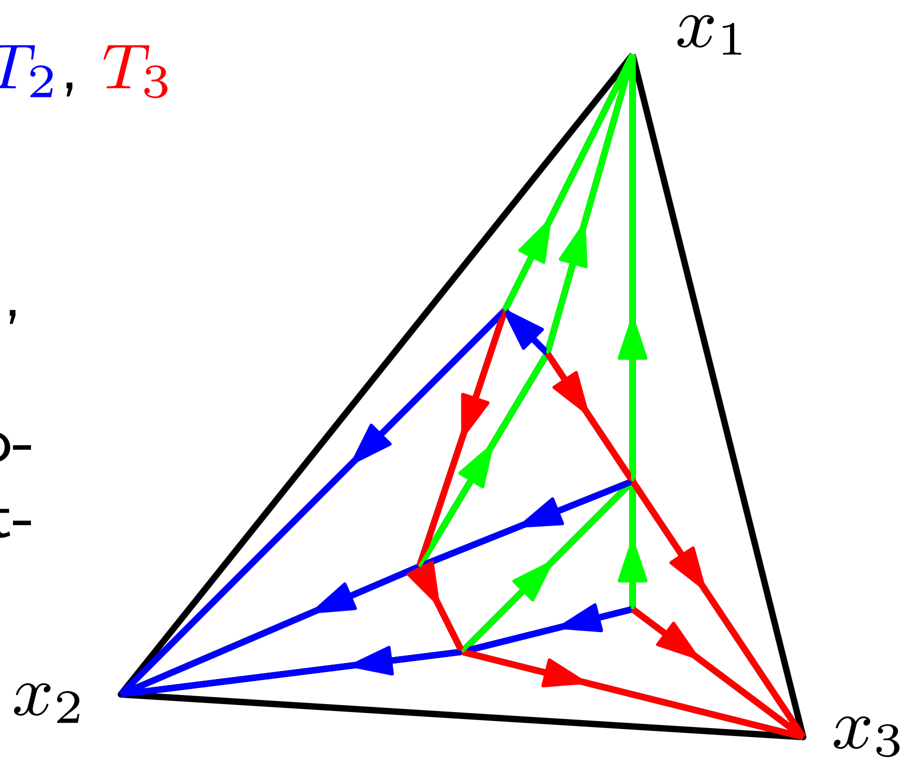
A Schnyder wood is a partition T_1, T_2, T_3 of the internal edges of T such that:

i) T_i is a spanning tree of $I \cup \{x_i\}$,

Upon orienting edges of each tree toward its root, each vertex has 1 outgoing edge of each color, and

ii)Colors must satisfy the local Schnyder rule:





$I = \{\text{internal vertices}\}$.

(here $|I| = 6$)

Realizers as a toy model on triangulations

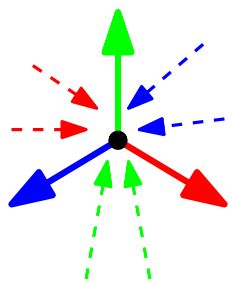
Let T be a triangulation with boundary $\{x_1, x_2, x_3\}$.

A Schnyder wood is a partition T_1, T_2, T_3 of the internal edges of T such that:

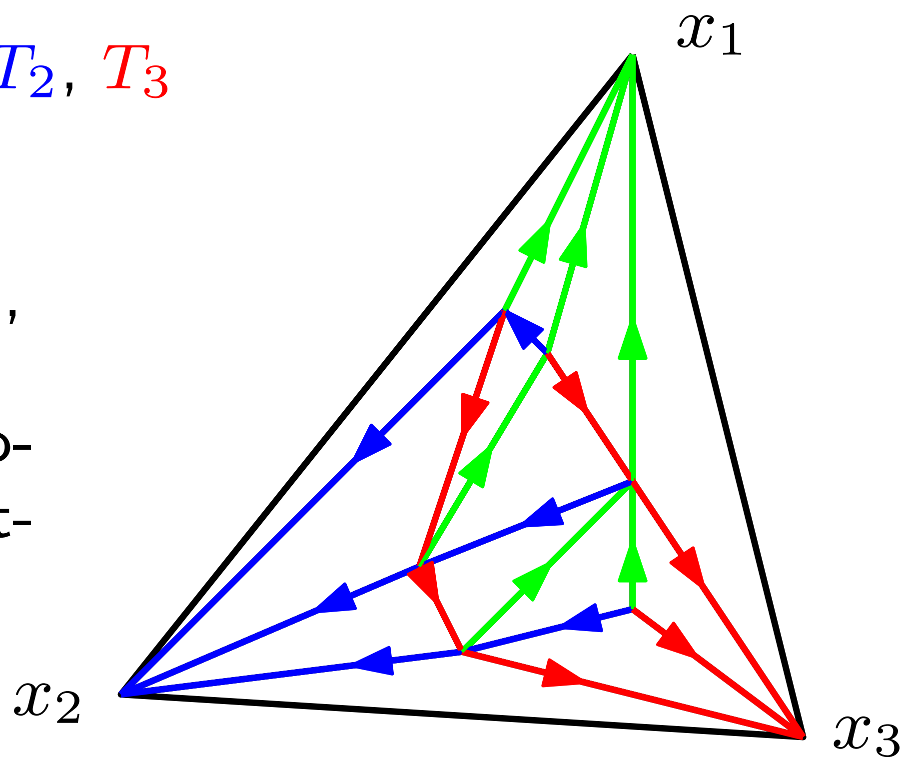
i) T_i is a spanning tree of $I \cup \{x_i\}$,

Upon orienting edges of each tree toward its root, each vertex has 1 outgoing edge of each color, and

ii)Colors must satisfy the local Schnyder rule:



A realizer is a triangulation endowed with a Schnyder wood.

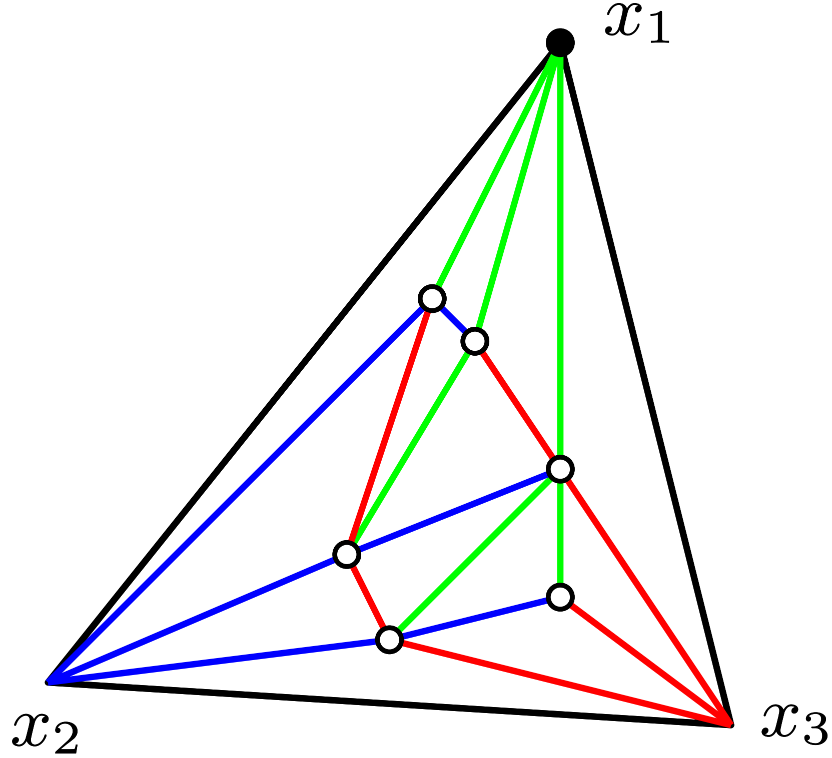


$I = \{$internal vertices$\}$.

(here $|I| = 6$)

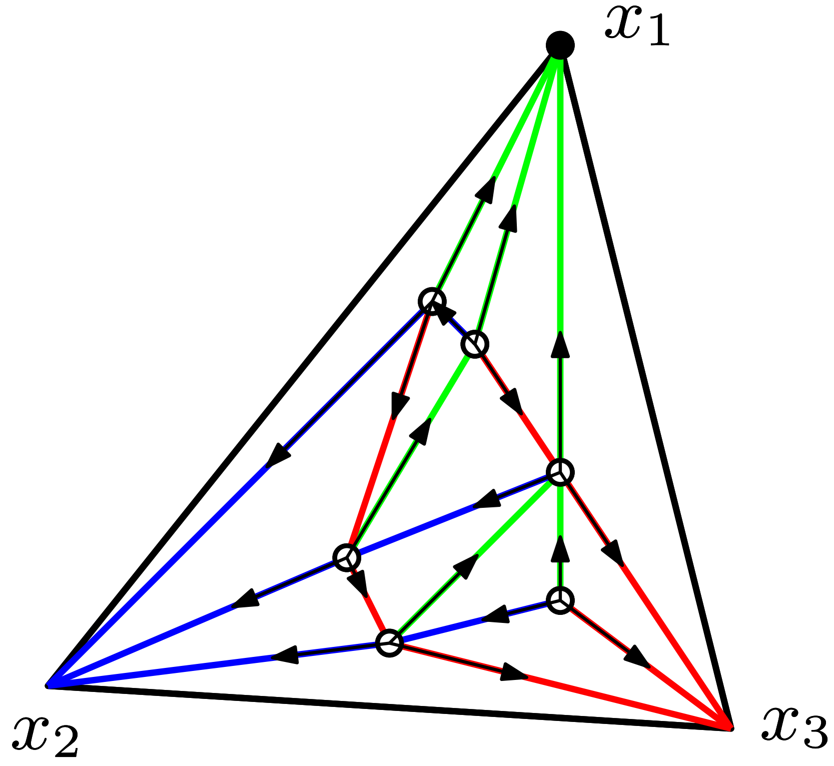
Realizers as a toy model on triangulations

A Schnyder wood induces a 3-orientation of the triangulation



Realizers as a toy model on triangulations

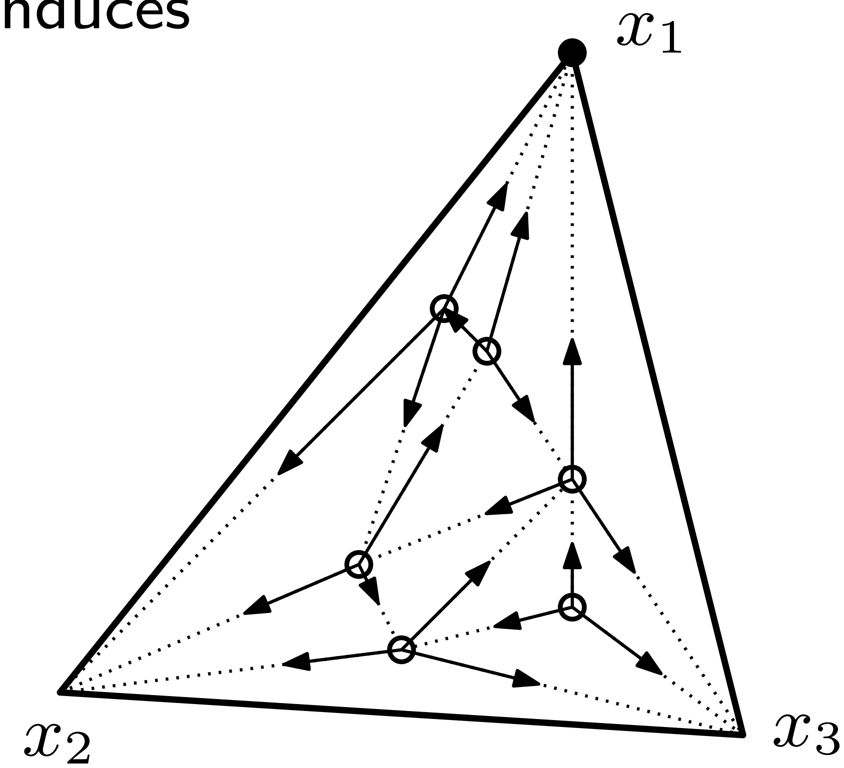
A Schnyder wood induces a 3-orientation of the triangulation



Realizers as a toy model on triangulations

A Schnyder wood induces a 3-orientation of the triangulation

Lemma Conversely a 3-orientation induces a unique Schnyder wood.

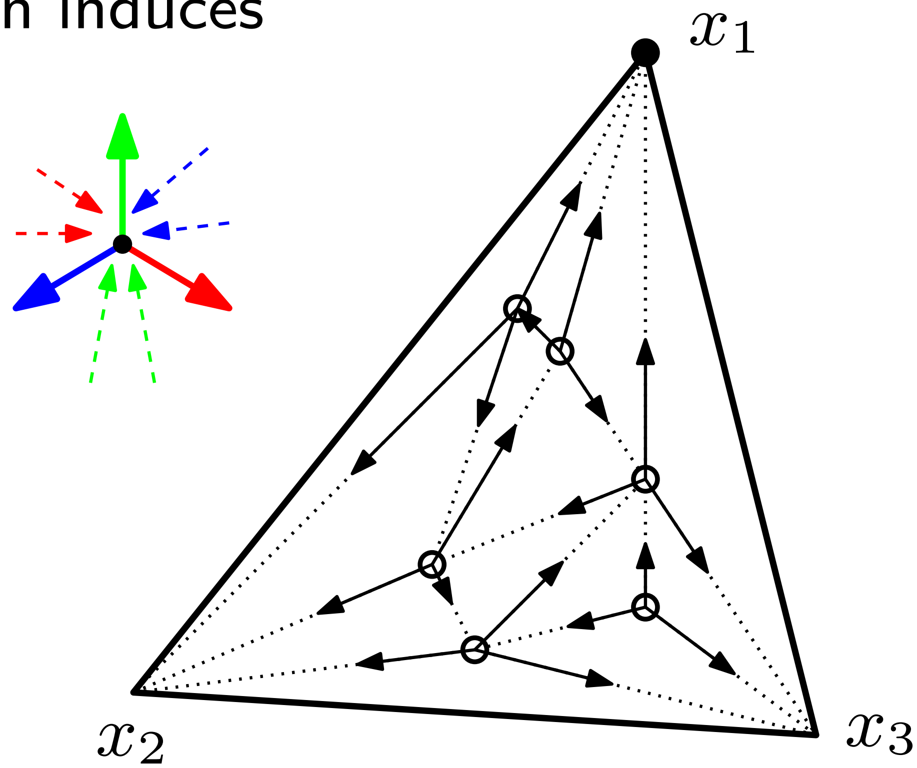


Realizers as a toy model on triangulations

A Schnyder wood induces a 3-orientation of the triangulation

Lemma Conversely a 3-orientation induces a unique Schnyder wood.

proof: use the rule and check no contradiction can arise

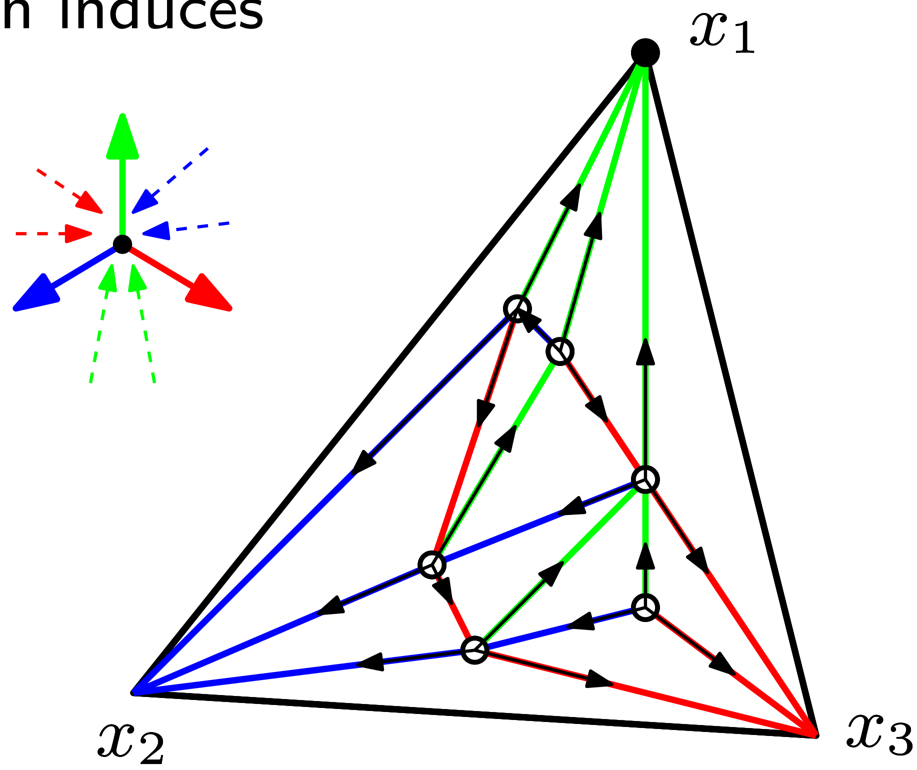


Realizers as a toy model on triangulations

A Schnyder wood induces a 3-orientation of the triangulation

Lemma Conversely a 3-orientation induces a unique Schnyder wood.

proof: use the rule and check no contradiction can arise

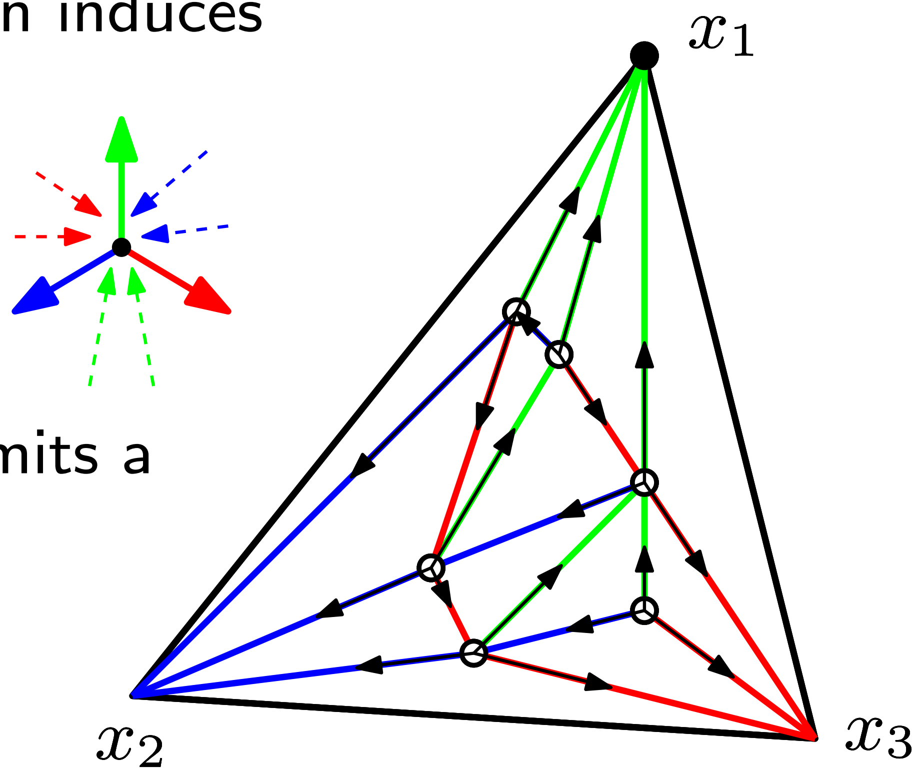


Realizers as a toy model on triangulations

A Schnyder wood induces a 3-orientation of the triangulation

Lemma Conversely a 3-orientation induces a unique Schnyder wood.

proof: use the rule and check no contradiction can arise



Corollary Every triangulation admits a Schnyder wood.

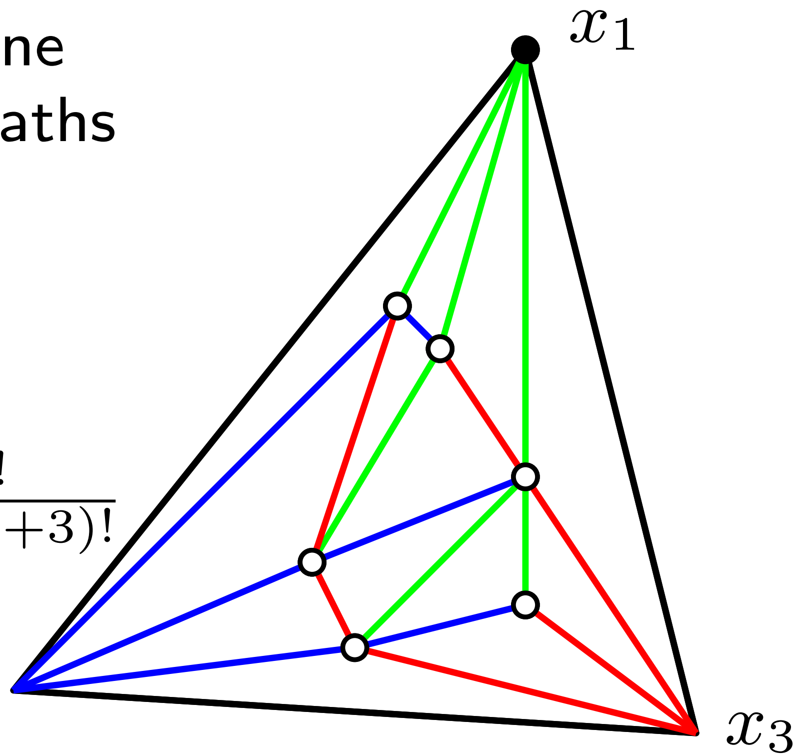
Enumeration of realizers

Realizers can be counted exactly:

Theorem (Bonichon 2003) Realizers of size $n+3$ vertices are in one-to-one correspondence with pairs of Dyck paths of length $2n$.

Corollary The number of realizers of size $n+3$ is

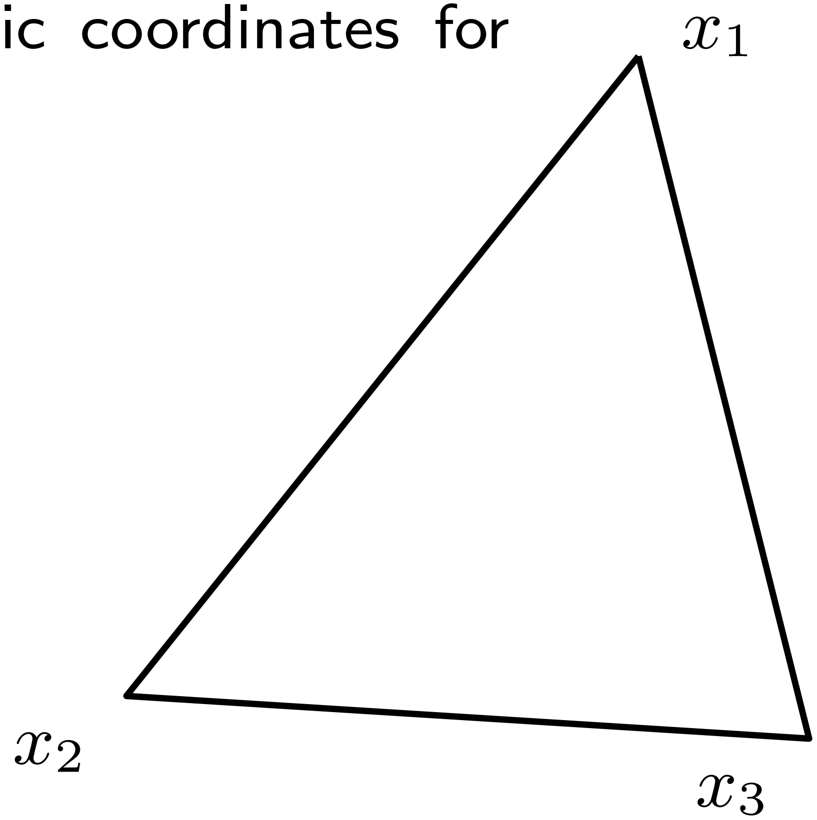
$$C_{n+2}C_n - C_{n+1}^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$



Realizers can thus be viewed as a curious exactly solvable model on triangulations.

Schnyder's drawing algorithms

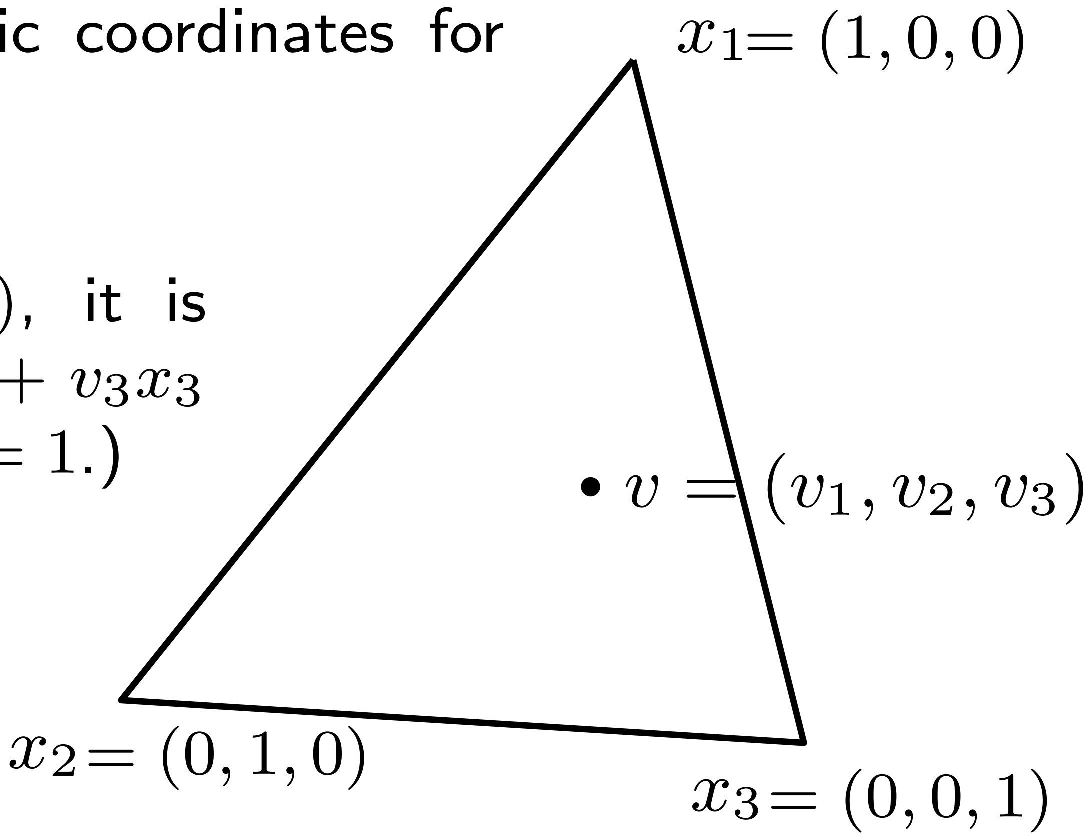
We give arbitrary positions to the 3 points x_1, x_2 and x_3, and we want barycentric coordinates for internal points.



Schnyder's drawing algorithms

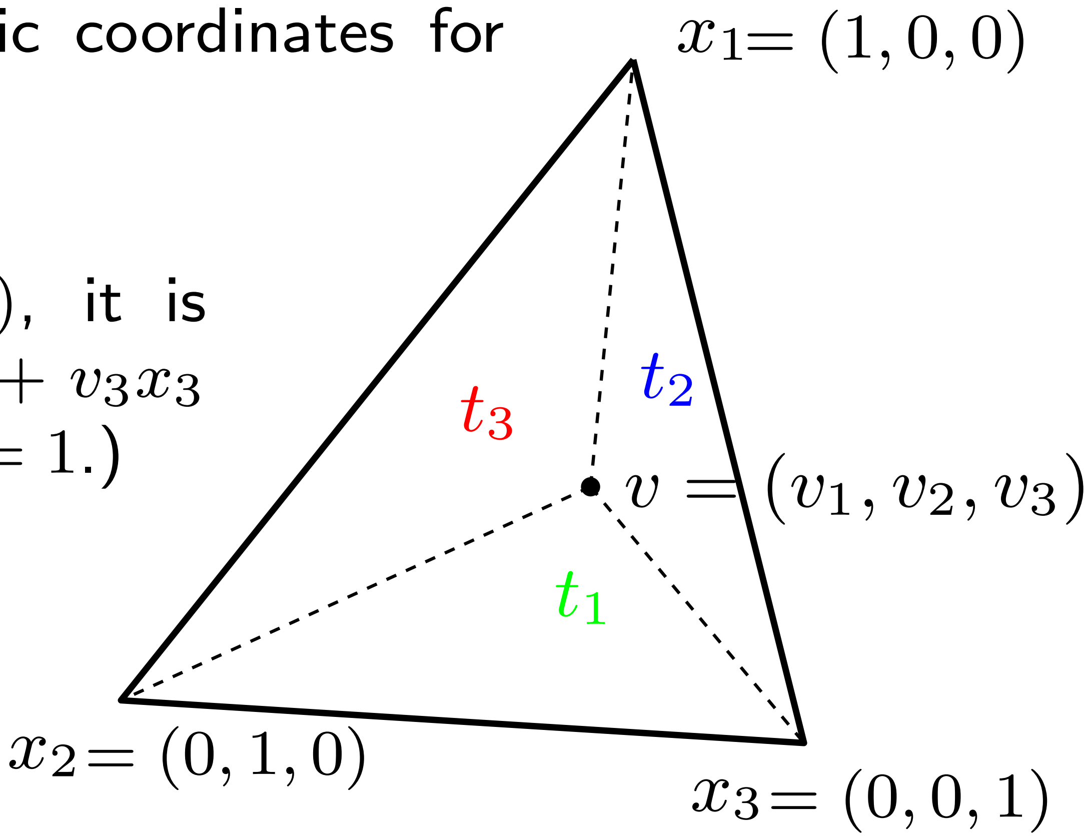
We give arbitrary positions to the 3 points x_1, x_2 and x_3, and we want barycentric coordinates for internal points.

If v has coordinates (v_1, v_2, v_3), it is placed in position $v_1x_1 + v_2x_2 + v_3x_3$ (Assuming $v_i \in (0,1)$ and $\sum v_i = 1$.)



Schnyder's drawing algorithms

We give arbitrary positions to the 3 points x_1, x_2 and x_3, and we want barycentric coordinates for internal points.

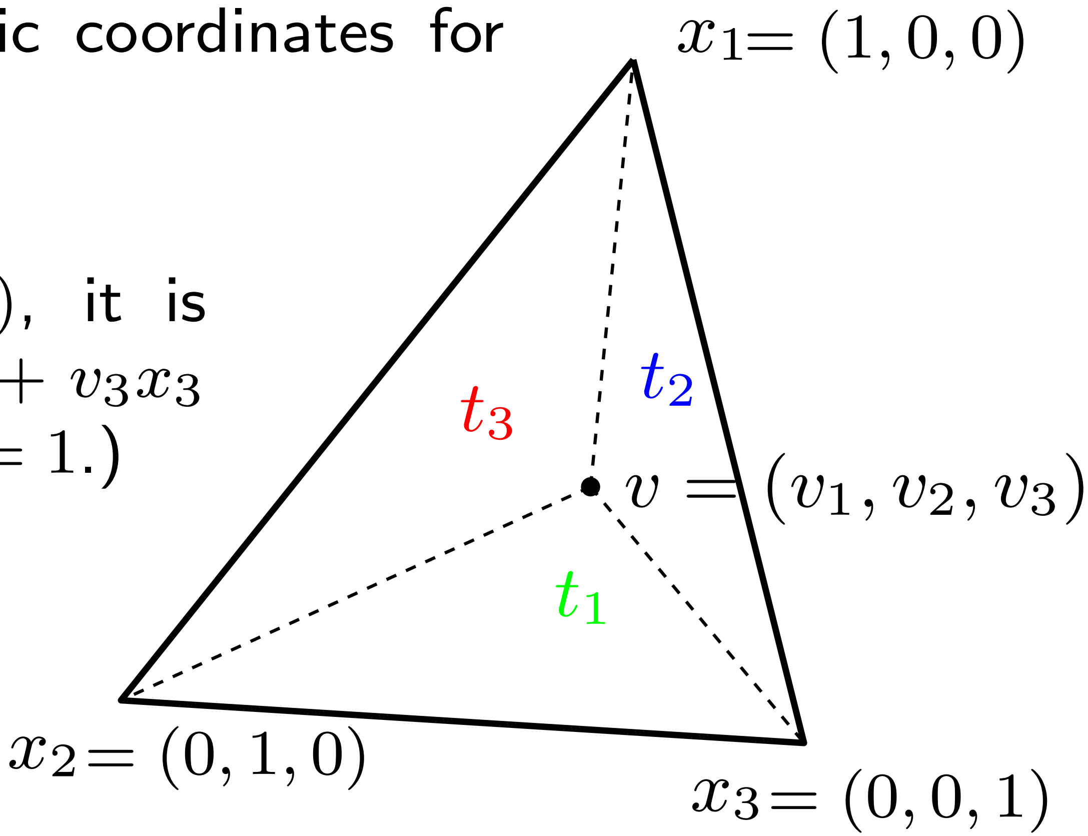


If v has coordinates (v_1, v_2, v_3), it is placed in position $v_1 x_1 + v_2 x_2 + v_3 x_3$ (Assuming $v_i \in (0, 1)$ and $\sum v_i = 1$.)

and Coordinate v_i corresponds to the area of Triangle t_i.

Schnyder's drawing algorithms

We give arbitrary positions to the 3 points x_1, x_2 and x_3, and we want barycentric coordinates for internal points.



If v has coordinates (v_1, v_2, v_3), it is placed in position $v_1x_1 + v_2x_2 + v_3x_3$ (Assuming $v_i \in (0, 1)$ and $\sum v_i = 1$.)

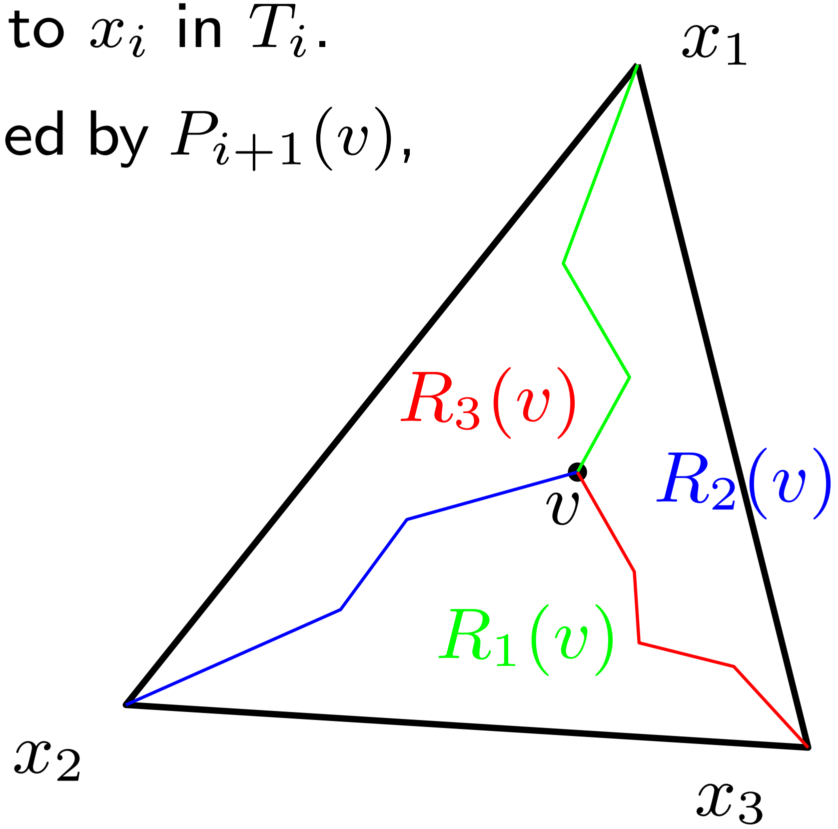
and Coordinate v_i corresponds to the area of Triangle t_i.

We shall use a combinatorial analog to the areas of the 3 triangles to make canonical drawings of triangulations

Schnyder's drawing algorithms

Assume we have a Schnyder forest:

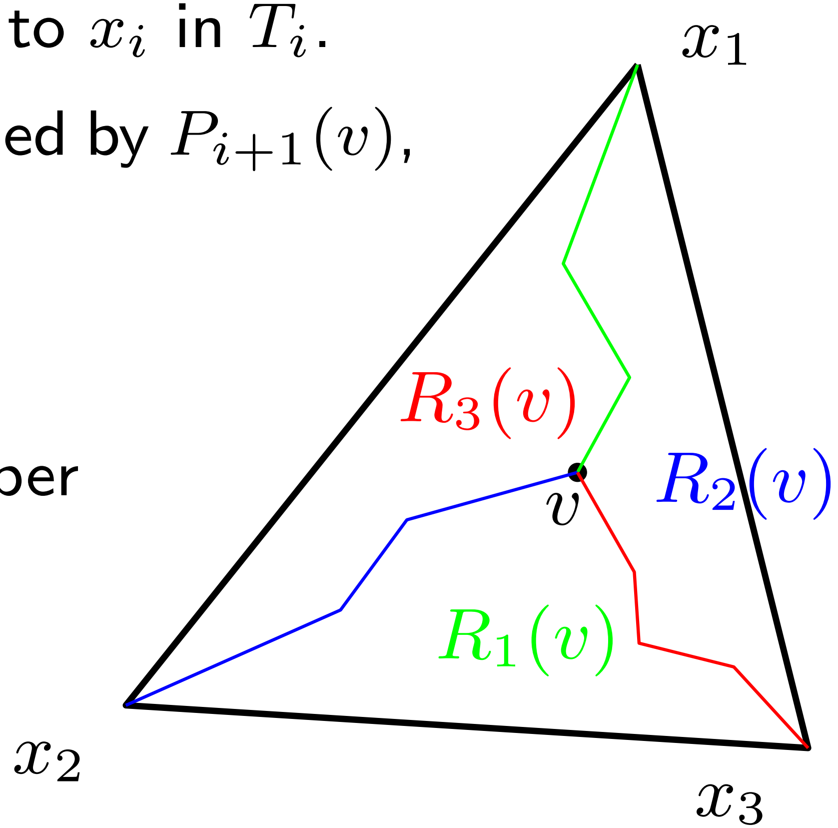
- Let $P_i(v)$ the path from v to x_i in T_i.
- Let $R_i(v)$ the region bounded by $P_{i+1}(v)$, $P_{i+2}(v)$ and (x_{i+1}, x_{i+2}).



Schnyder's drawing algorithms

Assume we have a Schnyder forest:

- Let $P_i(v)$ the path from v to x_i in T_i.
- Let $R_i(v)$ the region bounded by $P_{i+1}(v)$, $P_{i+2}(v)$ and (x_{i+1}, x_{i+2}).



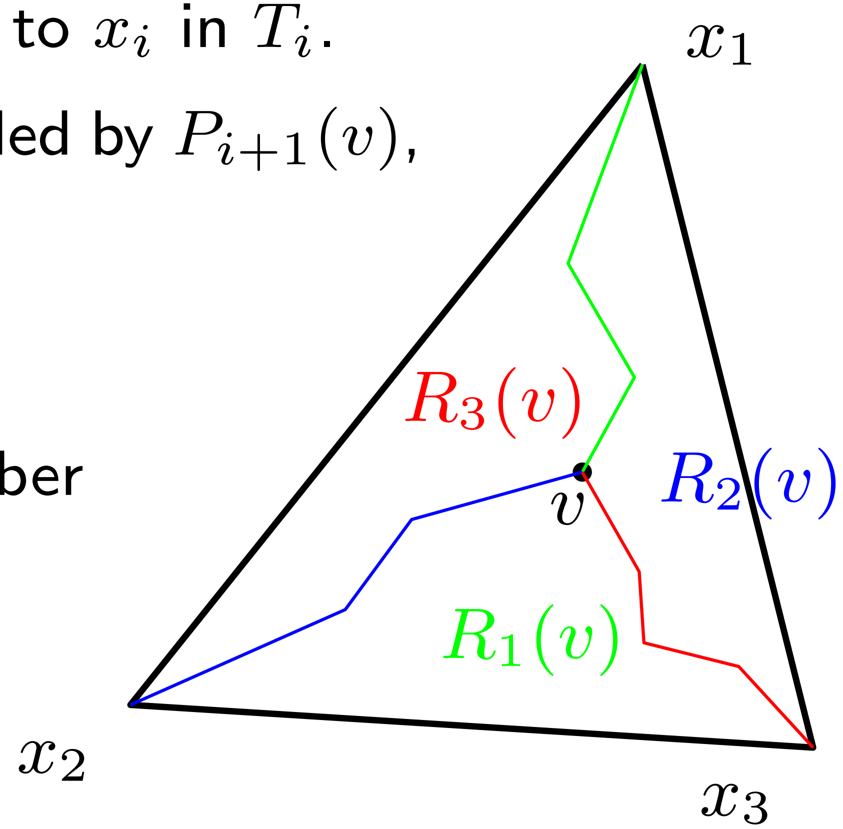
The combinatoiral analog of the triangle areas is given by the number of faces included in each regions:

$$V_i(v) = \frac{|R_i(v)|}{|T|}$$

Schnyder's drawing algorithms

Assume we have a Schnyder forest:

- Let $P_i(v)$ the path from v to x_i in T_i.
- Let $R_i(v)$ the region bounded by $P_{i+1}(v)$, $P_{i+2}(v)$ and (x_{i+1}, x_{i+2}).



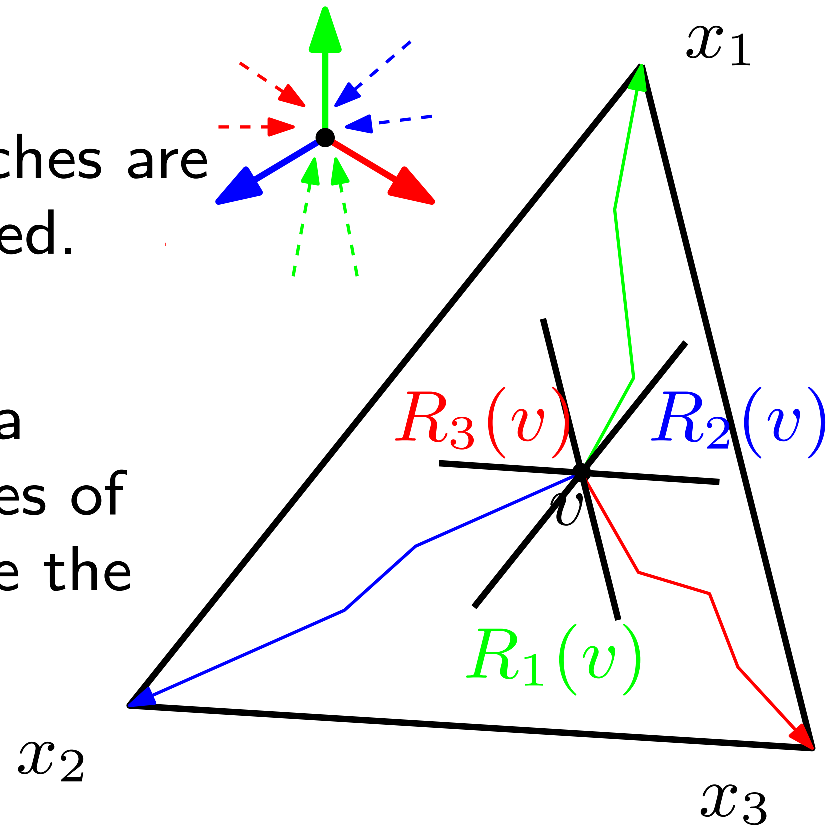
The combinatoiral analog of the triangle areas is given by the number of faces included in each regions:

$$V_i(v) = \frac{|R_i(v)|}{|T|}$$

Theorem (Schnyder) The drawing of T with straight lines with each vertice v at its barycentric coordinate $(V_1(v), V_2(v), V_3(v))$ is planar, whatever the original placement of x_1, x_2, x_3 (non aligned).

Schnyder's drawing algorithms

A scketch of proof



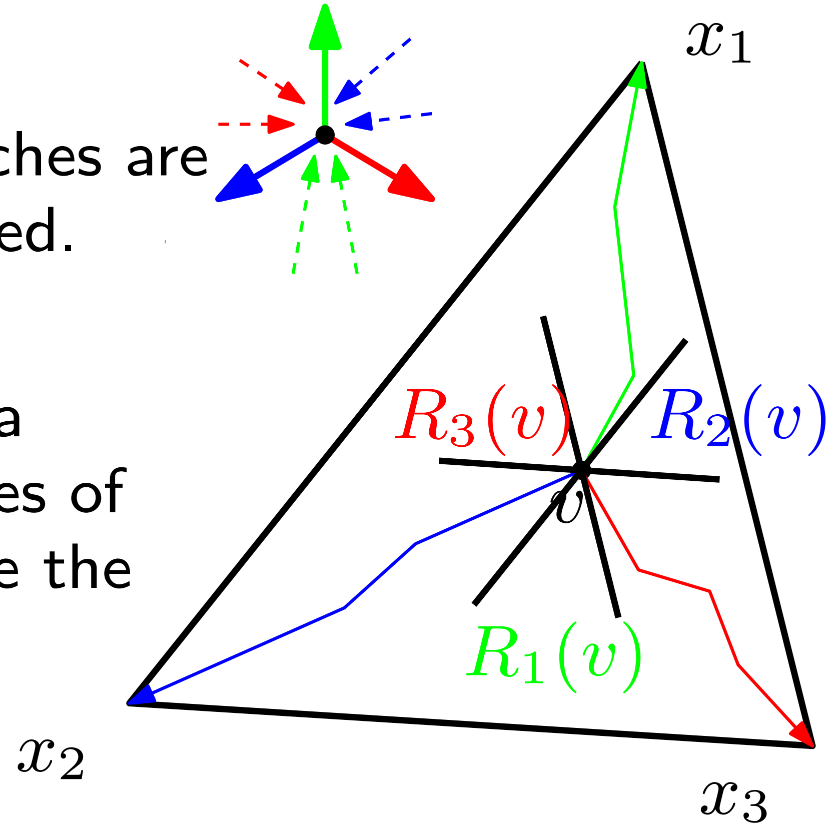
we only need to show that branches are geometrically oriented as expected.

Lemma. In the neigborhood of a vertex, the parallels to the 3 sides of the triangle (x_1, x_2, x_3) separate the 6 type of edges.

Schnyder's drawing algorithms

A scketch of proof

we only need to show that branches are geometrically oriented as expected.



Lemma. In the neigborhood of a vertex, the parallels to the 3 sides of the triangle (x_1, x_2, x_3) separate the 6 type of edges.

This allows to prove that each region $R_i(v)$ of the new drawing contains the same vertices than the $R_i(v)$ of the initial drawing.

If an edge crosses one of the 3 edges emanating from v, it intersects 2 different regions in the new drawing and thus also in the old one. This contradicts planarity of the original picture.

□

Some questions to conclude

About realizers:

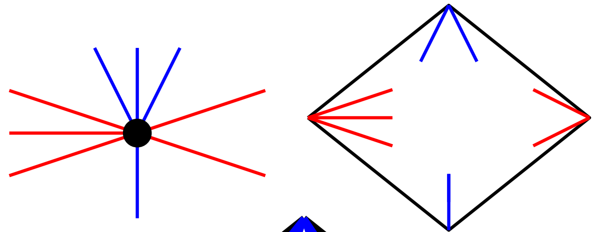
In which class of universality do realizers fall? (*eg* what is the central charge of the underlying toy model?)

What is the Hausdorf dimension of realizers?

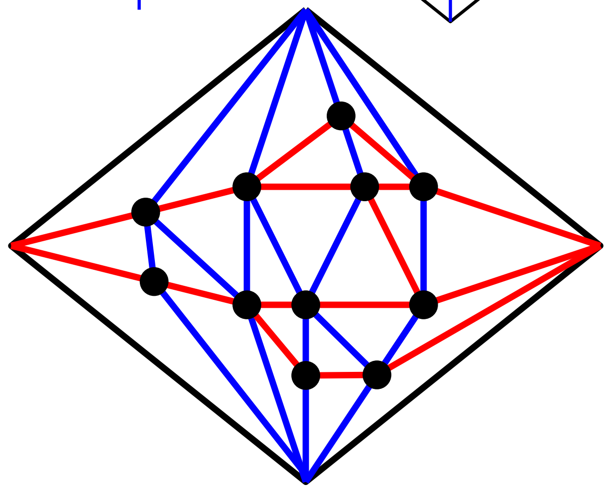
Has the Schnyder drawing of a realizer any physical relevance?

About a variant called transversal structures:

Definition: A tranversal structure of a triangulation of a square is a partition of edges such that locally:



Lemma: A triangulation of the square admits a transversal structure if and only if it contains no separating 3-cycle.



Is this related with the local causal structure introduced by Loll?

Merci de votre attention